THE SINGULARITY SPECTRUM OF THE INVERSE OF COOKIE-CUTTERS

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Abstract. Gibbs measures μ on cookie-cutter sets are the archetype of multifractal measures on Cantor sets. In this article we compute the singularity spectrum of the inverse measure of μ. Such a measure is discrete (it is constituted only by Dirac masses), it satisfies a multifractal formalism, and its L^q-spectrum possesses one point of non differentiability. The results rely on heterogeneous ubiquity theorems.

1. Introduction

Gibbs measures μ on cookie-cutter sets are the archetype of multifractal measures on Cantor sets. The singularity spectrum of such a measure μ has been obtained in one of the pioneer papers on multifractal analysis [24]. Then, many papers have extended this study for Gibbs measures or Birkhoff averages on conformal repellers (see for instance [22, 21, 7, 16]). As noticed in [19], a very natural object associated with a Gibbs measure on a cookie-cutter set is its inverse measure ν, which belongs to the class of purely discrete measures. The multifractal nature of the measure ν cannot be described only by the techniques developed for the study of measures generated by a multiplicative procedure and enjoying self-similarity properties. The aim of this work is to perform the multifractal analysis for the measure ν, and to prove that ν also obeys some multifractal formalism. It appears that the L^q-spectrum $\tau_\nu(q)$ of ν is analytic except at the point q equal to the Hausdorff dimension of the cookie-cutter set, where it is non-differentiable. This non-differentiability is reminiscent of the phase transition phenomenon occurring in the thermodynamic formalism setting. The multifractal analysis of the discrete measure ν is explained by a subtle combination between the fine local structure of Gibbs measures and the distribution of the jump points of ν. The results rely on the notion of *heterogeneous ubiquity* introduced in [2, 4].

Subsequently, taking the inverse of Gibbs measures on cookie-cutter sets provides us with a very natural way of generating new objects obeying the multifractal formalism, though being of very different nature than Gibbs measures. The relationship between the spectra of μ and ν will be given.

We start by recalling the definition of a cookie-cutter set.

For every $n \geq 0$, let $\Sigma_n = \{0,1\}^n$, where Σ_0 contains only the empty word $\emptyset$. Also, let $\Sigma^* = \bigcup_{n\geq 0} \Sigma_n$ and let $\Sigma = \{0,\dots,1\}^{\mathbb{N}^*}$ be the set of infinite words on the alphabet $\{0,1\}$. The concatenation operation from $\Sigma^* \times (\Sigma^* \cup \Sigma)$ to $(\Sigma^* \cup \Sigma)$ is denoted $\cdot$.

The notations $\overline{0}$ and $\overline{1}$ stand for the infinite words in Σ whose letters are respectively all equal to 0 and 1.

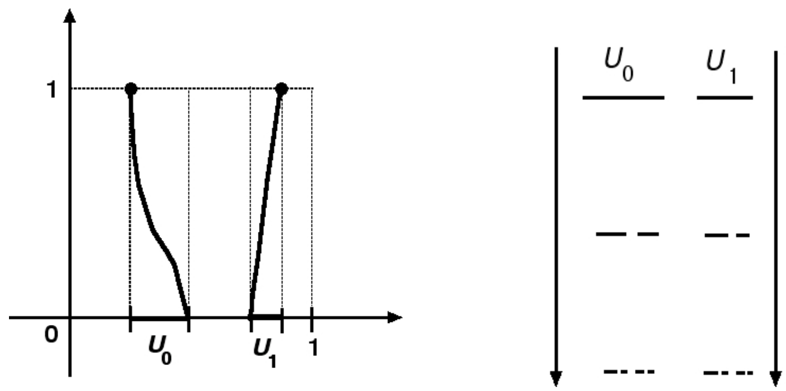

FIGURE 1. **Left:** The mapping $T : U_0 \cup U_1 \to [0,1]$, **Right:** First three steps of the construction of the cookie-cutter set.

For $w = w_1 w_2.... \in \Sigma$ and $n \geq 1$, $w|n := w_1 w_2 ... w_n$ is the projection of w on Σ_n and $w|\infty = w$. For $w \in \Sigma_n$, the cylinder with root w is $[w] = \{x \in \Sigma : x|n = w\}$. Given two words of infinite length $(w_1, w_2) \in \Sigma^2$, one defines $w_1 \wedge w_2$ as $w_1|n_0$, where $n_0 = \sup\{n \geq 1 : w_1|n = w_2|n\}$. We adopt the convention that $\inf \emptyset = 0$ and $w|0$ is the empty word $\emptyset$.

The length of any element w of Σ_n is equal to n and is denoted by $|w|$.

The set Σ is endowed with the shift operation denoted $\sigma : w = w_1 w_2..... \in \Sigma \longmapsto \sigma(w) = w_2 w_3... \in \Sigma$.

A *cookie-cutter set* is defined as follows (see [24] for instance). Let U_0 and U_1 be two compact disjoint subintervals of $[0,1]$. For $i \in \{0,1\}$ consider $T_i : U_i \to [0,1]$, a $C^{1+\gamma}$ ($\gamma > 0$) diffeomorphism such that $|T_i'| > 1$, and denote by $g_i = (T_i)^{-1}$ the inverse of T_i.

We denote by T the mapping from $U_0 \cup U_1$ to $[0,1]$ whose restrictions to U_0 and U_1 are respectively T_0 and T_1.

Then consider the subset X of $[0,1]$, called the cookie-cutter set associated with T, given by

$$X = \bigcap_{n \geq 1} \bigcup_{w \in \Sigma_n} g_{w_1} \circ \cdots \circ g_{w_n}([0,1]).$$

By construction $T^{-1}(X) = X$, and X is the unique compact set satisfying this equation. The cookie-cutter set we work with is naturally associated with a dyadic tree, but up to technical modifications, the rest of the paper is easily adapted to cookie-cutter sets constructed with more than two contractions.

For any finite word $w \in \Sigma^*$, we introduce the sets

$$X_w = g_{w_1} \circ \cdots \circ g_{w_{|w|}}(X).$$

There is a natural identification between Σ and X via the homeomorphism

$$\pi : \begin{array}{ccc} \Sigma & \to & X \\ w & \mapsto & \lim_{n\to\infty} g_{w|1} \circ \cdots \circ g_{w|n}(0). \end{array}$$

The dynamical systems (Σ, σ) and (X, T) are thus topologically conjugate.

In order to define a Gibbs measure on the cookie-cutter set X, we eventually need a Hölder continuous function $\varphi : X \to \mathbb{R}$. As usual, for $x \in X$ and $n \geq 1$, $S_n\varphi(x)$ stands for $\sum_{k=0}^{n-1} \varphi(T^k(x))$, the n-th Birkhoff sum of φ at x.

We recall the definition of the topological pressure and Gibbs measure associated with φ.

Definition 1.1. *The topological pressure $P(\varphi)$ of φ is the unique real number such that for some constant $C \geq 1$ we have*

$$\text{for all } n \in \mathbb{N},\ \ C^{-1}\exp(nP(\varphi)) \leq \sum_{w\in\Sigma_n} \sup_{x\in X_w} \exp(S_n\varphi(x)) \leq C\exp(nP(\varphi)). \tag{1.1}$$

The Gibbs measure μ_φ on (X, T) associated with φ is the unique ergodic measure such that for some constant $C > 0$

$$\forall\, w \in \Sigma^*,\ \forall\, x \in X_w,\ \ C^{-1} \leq \frac{\mu_\varphi(X_w)}{\exp\big(S_{|w|}\varphi(x) - |w|P(\varphi)\big)} \leq C. \tag{1.2}$$

This measure μ_φ has a unique extension to a probability Borel measure on $[0, 1]$ that we also denote by μ_φ.

The existence of the topological pressure as well as the existence and uniqueness of μ_φ are parts of the thermodynamic formalism theory. Proofs of these assertions can be found in [9]. We can assume that $P(\varphi) = 0$. Clearly the latter can always be achieved by replacing φ by $\varphi - P(\varphi)$, if necessary.

The next proposition states some classical results for the topological pressure (see [24, 13] for proofs and related references). Let us first introduce, for $(q, t) \in \mathbb{R}^2$, the mappings

$$\psi = -\log|T'| : X \to \mathbb{R} \quad \text{and} \quad \psi_{q,t} = q\psi - t\varphi : X \to \mathbb{R}. \tag{1.3}$$

Proposition 1.2.
1. *The mapping $(q, t) \mapsto P\big(q\psi - t\varphi\big)$ is strictly decreasing in q and strictly increasing in t.*
2. *For every $q \in \mathbb{R}$, there exists a unique $\theta(q) \in \mathbb{R}$ such that $P(\psi_{q,\theta(q)}) = P\big(q\psi - \theta(q)\varphi\big) = 0$.*
3. *The mapping $q \mapsto \theta(q)$ is analytic and concave.*

Let us now introduce the inverse measure of μ_φ, that we are going to focus on for the rest of the paper. Let F_m denote the distribution function of a probability measure m on $[0, 1]$: for $x \in [0, 1]$, $F_m(x) = m([0, x])$.

Definition 1.3. *The inverse measure ν of μ_φ is the unique Borel probability measure on $[0, 1]$ such that*

$$\forall\, x \in [0, 1], \quad F_\nu(x) = \sup\{t \in [0, 1] : F_{\mu_\varphi}(t) \leq x\}. \tag{1.4}$$

Investigations of inverse measures in terms of multifractals were first carried out in [19], where the authors point out the interest of such studies from the multifractal viewpoint.

Our aim is to study the local scaling properties of ν. Let us recall how the local regularity of any positive Borel measure m on $[0, 1]$ is described. For $x \in [0, 1]$, the pointwise Hölder exponent of m at x is given by

$$h_m(x) = \liminf_{r\to 0^+} \frac{\log m(B(x, r))}{\log(r)}.$$

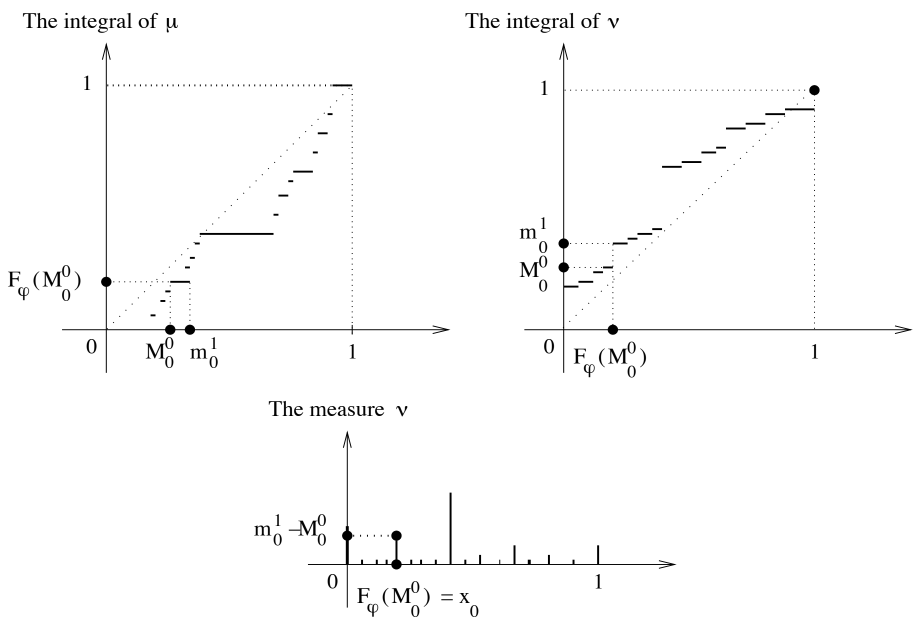

FIGURE 2. **Top Left:** the integral of a Gibbs measure μ_φ associated with a cookie-cutter set, **Top Right:** the inverse of the integral of μ_φ, and **Bottom:** the discrete measure ν. The first two graphs are symmetric with respect to the first diagonal.

Performing the multifractal analysis of m consists in computing the Hausdorff dimensions of the level sets of h_m, i.e. the sets

$$E_m(h) = \{x \in [0,1] : h_m(x) = h\}\,,\ h \geq 0.$$

We note $d_m(h) = \dim E_m(h)$ the singularity spectrum of m, where dim stands for the Hausdorff dimension (see [13] for instance for the definition of the dimension), and $\dim \emptyset = -\infty$ by convention. The knowledge of the singularity spectrum provides us with informations on the geometric distribution of the singularities of the measure m.

Recall that if g is a function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$, then its Legendre transform g^* is defined for any $\alpha \in \mathbb{R}$ by

$$g^*(\alpha) = \inf_{q\in\mathbb{R}} (\alpha q - g(q)) \ \in \mathbb{R} \cup \{-\infty\}. \tag{1.5}$$

In [24], the multifractal analysis of μ_φ is performed. Recall that for any positive Borel measure μ on $[0,1]$, the L^q-spectrum $\tau_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$\tau_\mu(q) = \liminf_{r\to 0} \frac{\log \sup \{\sum_i \mu(B_i)^q\}}{\log(r)},$$

the supremum being taken over all families of disjoint closed intervals B_i of radius r with centers in supp (μ) (see [10, 8, 20, 21, 23] for definitions and study of multifractal formalisms).

With our notations, the result of [24] is the following.

Theorem 1.4. *For every $q \in \mathbb{R}$, $\tau_{\mu_\varphi}(q) = -\theta^{-1}(-q)$, and the measure μ_φ satisfies the multifractal formalism, i.e. for every $h \geq 0$, $d_{\mu_\varphi}(h) = \tau^*_{\mu_\varphi}(h)$.*

The singularity spectrum of μ_φ can thus be written as

$$d_{\mu_\varphi}(h) = \inf\{qh + \theta^{-1}(-q) : q \in \mathbb{R}\}. \tag{1.6}$$

The main goal of this paper will be to derive similar multifractal statements for the inverse measure ν of μ_φ. These main results are summarized in the following theorem.

Theorem 1.5. *Let X be a cookie-cutter set, and let μ_φ be a Gibbs measure on X associated with a Hölder continuous potential φ.*

Let ν be the inverse measure of μ_φ.

(1) *The measure ν is discrete (i.e. it can be written as an infinite sum of Dirac masses), and its singularity spectrum d_ν satisfies:*
   - *for all $h \in [0, \theta'(\dim X)]$, $d_\nu(h) = h \cdot \dim X$,*
   - *for all $h > \theta'(\dim X)$ such that $\theta^*(h) > 0$, $d_\nu(h) = \theta^*(h)$,*

   In addition, $E_\nu(h) = \emptyset$ for all $h > \theta'(\dim X)$ such that $\theta^*(h) < 0$.

(2) *The L^q-spectrum of ν is given by*

$$\text{for every } q \in \mathbb{R}, \quad \tau_\nu(q) = \min(\theta(q), 0).$$

(3) *The measure ν obeys the multifractal formalism: for every $h > 0$ such that $\tau^*_\nu(h) > 0$, one has $d_\nu(h) = \tau^*_\nu(h)$.*

Let $h_{\max} := \sup\{h : \tau^*_\nu(h) \geq 0\}$. The attentive reader has noticed that Theorem 1.5 does not include the case $\tau^*_\nu(h_{\max}) = 0$, i.e. when the singularity spectrum touches 0 at the right end of its decreasing part. We discuss this point in Section 4.5.

Note that there is an interesting relationship between the singularity spectrum of μ_φ and the spectrum of ν. This can be expressed in terms of their Legendre transforms:

$$\tau_\nu(q) = -(\tau_{\mu_\varphi})^{-1}(-q) \quad \text{when } q \leq \dim X. \tag{1.7}$$

The main difficulty here is that the measure ν is constituted only by Dirac masses. Subsequently, the multifractal analysis of ν is not entirely based on the study of auxiliary Gibbs measures as in the case of μ_φ [24]. This measure ν is an example of discrete measures whose multifractal analysis requires other mathematical tools, namely ubiquity results (see Section 3.3).

The singularity spectrum of ν is composed of two parts: a linear part with slope $\dim X$, and a strictly concave part. The linear part is essentially due to the presence of Dirac masses, and the concave part follows from standard multifractal arguments involving auxiliary (diffuse) Gibbs measures. As claimed above, the L^q-spectrum of ν possesses a point of non-differentiability as soon as the graph of the function $q \mapsto \theta(q)$ does not have a horizontal tangent when it reaches 0 (see Figure 3).

Other examples of infinite homogeneous [1, 17, 14] or heterogeneous [3, 5] sums of Dirac masses have been studied. Note that the measure ν does not belong to the class of discrete measures considered in these papers, except when μ_φ is the measure of maximal entropy associated with a system where the mappings g_0 and g_1 are affine maps with same contraction ratio. Moreover, compared to the measures

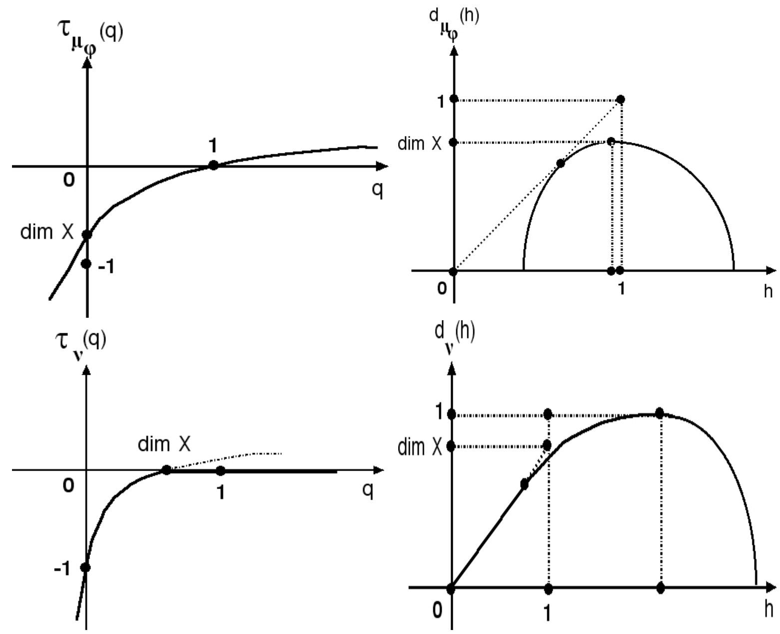

FIGURE 3. **Top Left:** A typical L^q-spectrum for a Gibbs measure μ_φ on a cookie-cutter set X and **Top Right:** the typical singularity spectrum of μ_φ. **Bottom Left:** A typical L^q-spectrum for the inverse measure ν, and **Bottom Right:** the typical singularity spectrum of ν. The two graphs on the left hand-side are related via the formula (1.7). The graphs on the right hand-side are the Legendre transforms of the graphs of the left hand-side.

appearing in [1, 17, 14, 3, 5], an additional important property is that ν is naturally associated with a dynamical system.

2. A Reformulation of the Problem - Some Remarks.

2.1. **Identifying the intensities of the Dirac masses of ν.** We assume that $\sup U_0 < \inf U_1$. For $w \in \Sigma^*$ (including $w = \emptyset$) we introduce the real numbers

$$m_w^0 = \min X_{w\cdot 0} = \pi(w\cdot \overline{0}) \quad \text{and} \quad M_w^0 = \max X_{w\cdot 0} = \pi(w\cdot 0\cdot \overline{1}),$$
$$m_w^1 = \min X_{w\cdot 1} = \pi(w\cdot 1\cdot \overline{0}) \quad \text{and} \quad M_w^1 = \max X_{w\cdot 1} = \pi(w\cdot \overline{1}).$$

Obviously, for every integer $n \geq 1$,

$$X \subset \bigcup_{w\in\Sigma_n} [m_w^0, M_w^0] \cup [m_w^1, M_w^1].$$

Let us also introduce the mapping

$$F_\varphi : \quad X \to [0,1], \qquad t \mapsto \mu_\varphi([0,t]).$$

For every $w \in \Sigma^*$, let

$$x_w = F_\varphi(M_w^0) \quad \text{(this includes the case } x_\emptyset = F_\varphi(\max(X_{0\cdot\overline{1}}))\text{)}. \tag{2.1}$$

We prove that ν is a discrete measure whose Dirac masses are located at the family of points x_w, where w ranges in Σ^*.

Proposition 2.1. *With the notations above, we have:*

$$\nu = m^0_\emptyset \cdot \delta_0 + \sum_{w\in\Sigma^*} (m^1_w - M^0_w)\cdot \delta_{x_w} + (1-M^1_\emptyset)\cdot\delta_1. \tag{2.2}$$

Proof. Denote the right hand side of (2.2) by ρ.

For every $w\in\Sigma^*\backslash\{\emptyset\}$, $m^1_w - M^0_w$ represents the length of the interval located between $X_{w\cdot 0}$ and $X_{w\cdot 1}$. Recall that for every $x\in[0,1]$, $\nu([0,x]) = \inf\{t : \mu_\varphi([0,t]) > x\}$. Hence, if $x < x_w < y$, then $\nu([0,y]) - \nu([0,x]) \geq m^1_w - M^0_w$. Since this holds for every (x,y) for which $x<x_w<y$, we deduce that ν has a Dirac mass with weight greater than $m^1_w - M^0_w$ at x_w. The same arguments also work for the Dirac masses at 0, $x_\emptyset$ and at 1.

Subsequently, $\nu-\rho$ is still a positive Borel measure on $[0,1]$.

Moreover, $\rho([0,1]) = m^0_\emptyset + \sum_{w\in\Sigma^*}(m^1_w - M^0_w) + (1-M^1_\emptyset) = \max X + 1 - M^1_\emptyset = 1$, and by construction $\nu([0,1])$ is also equal to 1. We conclude that $\nu=\rho$.

2.2. **A tractable reformulation of (2.2).** If $w\in\Sigma^*$, then we denote

$$I_w = F_\varphi(X_w)\setminus F_\varphi(\max X_w). \tag{2.3}$$

By construction I_w is an interval, since the support of μ_φ (the derivative of F_φ) restricted to the interval $[\min(X_w),\max(X_w)]$ is X_w, and μ_φ is atomless. The families of intervals $\mathcal{F}_n = \{I_w\}_{w\in\Sigma_n}$, $n\geq 1$, form a nested grid of $[0,1)$, since supp $(\mu_\varphi)\subset X$ and $\bigcup_{w\in\Sigma_n} X_w = X$.

Now we make full use of the structure of the construction in order to slightly transform ν into a measure enjoying the same multifractal nature as ν. This will make the calculations easier. For this, note that for $w\in\Sigma^*$ we have

$$[M^0_\emptyset, m^1_\emptyset] = T^{|w|}([M^0_w, m^1_w]). \tag{2.4}$$

Here $T^{|w|}$ is the $|w|$-th iterate of T. Due to the definition (1.3), for $w\in\Sigma_n$ and $x\in X_w$ we have $S_n\psi(x) = -\log|(T^n)'(x)|$.

We will often use the standard bounded distortion principle, for which we refer the reader to [24] or to [13], Chapter 4, Propositions 4.1 and 4.2. This principle yields a control, uniform in n, of the variations of the Birkhoff sums $S_n\psi(x)$ inside each X_w, $w\in\Sigma_n$. Specifically, there exists a constant $C>1$ such that for every $w\in\Sigma^*$, for every $(x,y)\in(X_w)^2$, $C^{-1}\leq |X_w|\cdot\exp(-S_n\psi(x))\leq C$ and

$$C^{-1}|T^{|w|}(x) - T^{|w|}(y)| \leq \frac{|x-y|}{|X_w|} \leq C|T^{|w|}(x) - T^{|w|}(y)|. \tag{2.5}$$

Let us introduce the mapping $\widetilde{\psi} : \Sigma^*\to\mathbb{R}$

$$\widetilde{\psi}(w) = \sup_{x\in X_w} S_{|w|}\psi(x). \tag{2.6}$$

This yields

$$C^{-1}\leq |X_w|\cdot\exp(\widetilde{\psi}(w))\leq C. \tag{2.7}$$

Applying (2.5) to $x=M^0_w$ and $y=m^1_w$, and using (2.4), we obtain that there is a constant $C>0$ such that for all $w\in\Sigma^*$,

$$C^{-1}(m^1_\emptyset - M^0_\emptyset) \leq \frac{m^1_w - M^0_w}{\exp(\widetilde{\psi}(w))} \leq C(m^1_\emptyset - M^0_\emptyset). \tag{2.8}$$

Since ψ is Hölder (T is supposed to belong to $C^{1+\gamma}$), we can consider μ_ψ, the Gibbs measure on X associated with ψ. Then, (1.2) holds for (ψ, μ_ψ) instead of (φ, μ_φ). There exists another constant $C > 0$ such that for all $w \in \Sigma^*$, for every $x \in X_w$,

$$C^{-1} \leq \frac{\mu_\psi(X_w)}{\exp\left(S_{|w|}\psi(x) - |w|P(\psi)\right)} \leq C.$$

This implies that for all $w \in \Sigma^*$ (using our definition for $\widetilde{\psi}(w)$)

$$C^{-1} \leq \frac{\exp(|w|P(\psi))\mu_\psi(X_w)}{\exp(\widetilde{\psi}(w))} \leq C.$$

Combining (2.8) and the last estimate, we see that there is a constant $C > 0$ such that for every $w \in \Sigma^*$,

$$C^{-1} \leq \frac{m_w^1 - M_w^0}{\exp(|w|P(\psi))\mu_\psi(X_w)} \leq C.$$

With μ_ψ can be associated the unique Borel measure $\mu_{\psi,\varphi}$ on $[0,1]$ such that $\mu_{\psi,\varphi}(I_w) = \mu_\psi(X_w)$, that is to say $\mu_{\psi,\varphi} = \mu_\psi \circ F_\varphi^{-1}$. The measure $\mu_{\psi,\varphi}$ has its support equal to $[0,1]$, while the support of μ_ψ is equal to X. The last inequalities can thus be rewritten

$$C^{-1} \leq \frac{m_w^1 - M_w^0}{\exp(|w|P(\psi))\,\mu_{\psi,\varphi}(I_w)} \leq C. \tag{2.9}$$

Remembering (2.2), we conclude that the discrete measure ν is equivalent to any of the following discrete measures:

$$\nu_1 = \sum_{w\in\Sigma^*} \exp(\widetilde{\psi}(w)) \cdot \delta_{x_w} + m \cdot \delta_0 + (1-M)\cdot \delta_1, \tag{2.10}$$

$$\nu_2 = \sum_{w\in\Sigma^*} \exp(|w|P(\psi))\,\mu_{\psi,\varphi}(I_w) \cdot \delta_{x_w} + m \cdot \delta_0 + (1-M)\cdot \delta_1. \tag{2.11}$$

It appears that ν_1 and ν_2 are more convenient to work with than (2.2). We are going to perform the multifractal analysis of these measures, that we again denote by ν for ease of notation.

3. Some definitions and tools for the multifractal analysis

If m is a positive Borel measure on $\mathbb{R}$, then the lower Hausdorff dimension of m is defined as

$$\dim(m) = \inf\{\dim E : m(E) > 0\}.$$

We refer the reader to [13] for more details and for the definition of Hausdorff measures and dimension.

3.1. **Definitions of exponents, singularity spectrum and approximation degree.**

Definition 3.1. *For $w \in \Sigma^*$, we set*

$$\begin{aligned} \lambda_w &= 2|I_w| \quad (= 2\mu_\varphi(X_w)), \\ \chi(w) &= \log|I_w|, \\ \widetilde{\alpha}(w) &= \frac{\widetilde{\psi}(w)}{\chi(w)} \quad \textit{when } |w| \geq 1. \end{aligned}$$

From (2.7), there exist $0 < \alpha_0 \leq \alpha_1 < \infty$ such that

$$\forall\ w \in \Sigma^*, \quad \alpha_0 \leq \widetilde{\alpha}(w) \leq \alpha_1. \tag{3.1}$$

Definition 3.2. *For $x \in [0,1)$ and $n \geq 1$, $w_n(x)$ stands for the unique element w of Σ_n such that $x \in I_w$. One then defines*

$$\alpha_n(x) = \widetilde{\alpha}(w_n(x)) = \frac{\widetilde{\psi}(w_n(x))}{\log |I_{w_n(x)}|} \quad \textit{and} \quad \alpha(x) = \liminf_{n\to\infty} \alpha_n(x).$$

Let us now focus on the family $\{(x_w, \lambda_w)\}_{w\in\Sigma}$. By construction (recall Formula (2.1)), for every $n \geq 1$, we have $[0,1] \subset \bigcup_{w\in\Sigma_n} B(x_w, \lambda_w/2)$. Hence any real number $x \in [0,1]$ is covered by infinitely many intervals $B(x_w, \lambda_w/2)$. The family $\{(x_w, \lambda_w)\}_{w\in\Sigma}$ is referred to as a *ubiquitous system*.

Given a ubiquitous system, two approximation degrees of any real number $x \in [0,1]$ by the family $\{(x_w, \lambda_w)\}_{w\in\Sigma}$ can be defined as follows.

Definition 3.3. *For $x \in (0,1) \setminus \{x_w : w \in \Sigma^*\}$, the approximation degrees ξ_x and $\widetilde{\xi}_x$ of x by the ubiquitous system $\{(x_w, \lambda_w)\}_{w\in\Sigma^*}$ are defined as*

$$\xi_x = \limsup_{n\to\infty} \Big(\sup_{w\in\Sigma_n} \frac{\log |x - x_w|}{\log \lambda_w} \Big) \quad \textit{and} \quad \widetilde{\xi}_x = \limsup_{n\to\infty} \frac{\log |x - x_{w_n(x)}|}{\log \lambda_{w_n(x)}}.$$

Due to the covering property of the ubiquitous system $\{(x_w, \lambda_w)\}_{w\in\Sigma}$, one obviously has $\xi_x \geq \widetilde{\xi}_x \geq 1$, for every $x \in [0,1]$.

3.2. **Auxiliary Gibbs measures.** Recall the definitions of ψ, $\psi_{q,t}$ and θ given in Proposition 1.2 .

Definition 3.4. *For every $q \in \mathbb{R}$, we set $\mu_q = \mu_{\psi_{q,\theta(q)},\varphi} := \mu_{\psi_{q,\theta(q)}} \circ F_\varphi^{-1}$.*

Proposition 3.5. *For each $q \in \mathbb{R}$, the following hold.*

1. *The set $\{x \in (0,1) : \alpha(x) = \theta'(q)\}$ is of full μ_q-measure.*
2. *We have $\dim(\mu_q) \geq \theta^*(\theta'(q))$.*

The second statement in the proposition implies that if a set E has a Hausdorff dimension less than $\theta^*(\theta'(q))$, then $\mu_q(E) = 0$. This will turn out to be useful in Proposition 4.9, where we will derive lower bounds for the Hausdorff dimensions of the level sets $E_\nu(h)$.

Proof. By construction of $\theta(q)$, $P(\psi_{q,\theta(q)}) = 0$. Hence there exists $C_q > 0$ such that for all $w \in \Sigma^*$, for every $x \in X_w$,

$$C_q^{-1} \leq \frac{\mu_{\psi_{q,\theta(q)}}(X_w)}{\exp(S_{|w|}\psi_{q,\theta(q)}(x))} = \frac{\mu_{\psi_{q,\theta(q)}}(X_w)}{\exp(qS_{|w|}\psi(x) - \theta(q)S_{|w|}\varphi(x))} \leq C_q.$$

Consequently, using (2.3), (2.6) and the fact that $\mu_{\psi_{q,\theta(q)}}(X_w) = \mu_q(I_w)$, it follows for $C_q > 0$ some further constant that

$$C_q^{-1} \leq \frac{\mu_q(I_w)}{\exp(q\widetilde{\psi}(w))|I_w|^{-\theta(q)}} \leq C_q, \ \forall\ w \in \Sigma^* \tag{3.2}$$

Let $n \geq 1$ and $\varepsilon > 0$. We prove that $\{x \in (0,1) : \alpha(x) = \theta'(q)\}$ is of full μ_q-measure. Let us set $\alpha = \theta'(q)$, and let $\delta > 0$. Let K_n be defined by $K_n =$

$\Big\{x \in (0,1) : \widetilde{\psi}(w_n(x)) \geq (\alpha-\varepsilon)\log|I_{w_n(x)}|\Big\}$. We have

$$\begin{aligned} \mu_q(K_n) &= \mu_q\left(\Big\{x \in (0,1) : \exp(\delta\widetilde{\psi}(w_n(x)))|I_{w_n(x)}|^{-\delta(\alpha-\varepsilon)} \geq 1\Big\}\right) \\ &\leq \int_0^1 \exp\big(\delta\widetilde{\psi}(w_n(x))\big)|I_{w_n(x)}|^{-\delta(\alpha-\varepsilon)} d\mu_q(x) \\ &= \sum_{w\in\Sigma_n} \exp\big(\delta\widetilde{\psi}(w)\big)|I_w|^{-\delta(\alpha-\varepsilon)}\mu_q(I_w) \\ &\leq C_q \sum_{w\in\Sigma_n} \exp\big((\delta+q)\widetilde{\psi}(w)\big)|I_w|^{-(\theta(q)+\delta(\alpha-\varepsilon))} \\ &\leq C'_q \sum_{w\in\Sigma_n} \sup_{x\in X_w} \exp\big((\delta+q)S_n\psi(x) - (\theta(q)+\delta(\alpha-\varepsilon))S_n\varphi(x)\big) \\ &\leq C''_q \exp(n\gamma_q), \end{aligned}$$

where $C'_q, C''_q > 0$ are constants, and $\gamma_q = P\big((q+\delta)\psi - (\theta(q)+\delta(\alpha-\varepsilon))\varphi\big)$. Now, since the function θ is right-differentiable at q and $\alpha = \theta'(q)$, for δ small enough we have $\theta(q)+\delta(\alpha-\varepsilon) < \theta(q+\delta)$. Consequently, applying part 1. of Proposition 1.2, since by part 2. of Proposition 1.2, $P((\delta+q)\psi - \theta(q+\delta)\varphi) = 0$, we get $\gamma_q < 0$. Therefore, using the Borel-Cantelli Lemma, it follows that $\mu_q(\limsup_{n\to+\infty} K_n) = 0$. Hence, for μ_q-almost all $x \in [0,1]$, we have that $\widetilde{\psi}(w_n(x)) < (\alpha-\varepsilon)\log|I_{w_n(x)}|$ for n large enough. This holds for any $\varepsilon > 0$, and thus we have that μ_q-almost every $x \in [0,1]$ verifies $\alpha(x) \geq \alpha$.

The converse inequality is obtained using a similar argument, and this is left to the reader. This shows that $\mu_q(\{x : \alpha(x) = \theta'(q)\}) = 1$, and the first part of the proposition follows.

Note that by the bounded distortion principle, the ratio $|I_{w_n(x)}|/|I_{w_{n+1}(x)}|$ is bounded from below and above by constants independent of $n \geq 1$ and $x \in [0,1]$. To get the second part of the Proposition, we only have to prove that

$$\mu_q\text{-almost everywhere,}\quad \liminf_{n\to+\infty} \frac{\log\mu_q(I_{w_n(x)})}{\log|I_{w_n(x)}|} \geq \theta^*(\theta'(q)).$$

Indeed, in this case the *mass distribution principle* [13] allows to conclude that $\dim\mu_q \geq \theta^*(\theta(q))$.

For this, note that (3.2) implies

$$\left|\frac{\log\mu_q(I_{w_n(x)})}{\log|I_{w_n(x)}|} - q\frac{\widetilde{\psi}(w_n(x))}{\log|I_{w_n(x)}|} + \theta(q)\right| \leq \frac{\log C_q}{|\log(|I_{w_n(x)}|)|}.$$

For μ_q-almost every x, we proved before that $\lim_{n\to+\infty} \frac{\widetilde{\psi}(w_n(x))}{\log|I_{w_n(x)}|} = \alpha$. Since $\alpha = \theta'(q)$, we obtain that

$$\lim_{n\to+\infty} \frac{\log\mu_q(I_{w_n(x)})}{\log|I_{w_n(x)}|} = q\alpha - \theta(q) = q\theta'(q) - \theta(q).$$

By definition of the Legendre transform, we have that $q\theta'(q) - \theta(q) = \theta^*(\theta'(q))$, and hence the result follows.

3.3. **An heterogeneous ubiquity Theorem.** Let $q_0 := \dim X$ denote the unique solution of $\theta(q_0) = 0$. Let $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers which tend to 0 for n tending to infinity. For $\xi \geq 1$ we then introduce the limsup set

$$S(\xi, \widetilde{\varepsilon}) = \bigcap_{N\geq 1} \bigcup_{n\geq N} \bigcup_{\substack{w\in\Sigma_n: \\ \theta'(q_0)-\varepsilon_n \leq \alpha(w) \leq \theta'(q_0)+\varepsilon_n}} B(x_w, \lambda_w^\xi). \tag{3.3}$$

Using the definition of $\alpha(w)$ and the measure μ_{q_0} (for which there exists a constant $C_{q_0} > 0$ such that $C_{q_0}^{-1}|X_w|^{1/q_0} \leq \mu_{q_0}(I_w) \leq C_{q_0}|X_w|^{1/q_0}$ for every $w \in \Sigma^*$), one can rewrite $S(\xi, \widetilde{\varepsilon})$ as

$$S(\xi, \widetilde{\varepsilon}) = \bigcap_{N\geq 1} \bigcup_{n\geq N} \bigcup_{\substack{w\in\Sigma_n: \\ |I_w|^{q_0(\theta'(q_0)+\varepsilon_n)} \leq \mu_{q_0}(B(x_w,\lambda_w)) \leq |I_w|^{q_0(\theta'(q_0)-\varepsilon_n)}}} B(x_w, \lambda_w^\xi), \tag{3.4}$$

where the sequence $\widetilde{\varepsilon}$ has been slightly modified to take into account the constant C_{q_0}.

Limsup sets of the form (3.4) arise in *ubiquity* theory. We present here a short version of ubiquity and heterogeneous ubiquity theorems, adapted to our context. Given $(x_n)_{n\geq 1}$ a sequence of real numbers in $[0, 1]$, and $(\lambda_n)_{n\geq 1}$ a sequence of positive real numbers tending to 0 when n tends to infinity, the classical ubiquity results are concerned with the computations of limsup sets of the form

$$A(\xi) = \bigcap_{N\geq 1} \bigcup_{n\geq N} B(x_n, \lambda_n^\xi),$$

where $\xi > 1$ is a contraction exponent. The classical ubiquity theorems, proved in [11, 18], state the following: As soon as the set $A(1)$ is of full Lebesgue measure, we have $\dim A(\xi) \geq 1/\xi$ for every $\xi \geq 1$.

Now suppose that $\{(x_n, \lambda_n)\}_{n\geq 1} = \{(x_w, \lambda_w)\}_{w\in\Sigma^*}$. Although the sets $S(\xi, \widetilde{\varepsilon})$ defined in (3.3) are very similar to the sets $A(\xi)$, the Hausdorff dimension of the sets $S(\xi, \widetilde{\varepsilon})$ cannot be reached by such theorems. This is due to the fact that the measure playing a key role is not the Lebesgue measure any more, but the monodimensional measure μ_{q_0}.

The notion of heterogeneous ubiquity, introduced in [2, 3, 4], has been developed to be able to estimate the Hausdorff dimension of limsup sets like the $S(\xi, \widetilde{\varepsilon})$. This is a crucial point in the following, since such limsup sets arise naturally when performing the multifractal analysis of the inverse measure ν of μ_φ. Proposition 3.6 has been obtained in [4], Theorem 2.2 (case $\rho = 1$).

We now give the assumptions needed to obtain sharp lower bounds for the Hausdorff dimension of $S(\xi, \widetilde{\varepsilon})$. In order to apply Theorem 2.2 of [4], three properties are required on the measure μ_{q_0} and the system $\{(x_w, \lambda_w)\}_{w\in\Sigma^*}$:

- The measure μ_{q_0} is quasi-Bernoulli [10] with respect to the grid constituted by the intervals $\{I_w\}_{w\in\Sigma^*}$, i.e. there is a constant $C > 1$ such that for every words $w, w' \in \Sigma^*$,

$$\begin{cases} C^{-1}\mu_{q_0}(I_w)\mu_{q_0}(I_{w'}) \leq \mu_{q_0}(I_{ww'}) \leq C\mu_{q_0}(I_w)\mu_{q_0}(I_{w'}) \\ C^{-1}|I_w| \cdot |I_{w'}| \leq |I_{ww'}| \leq C|I_w| \cdot |I_{w'}|. \end{cases}$$

- The measure μ_{q_0} is monofractal, in the sense that

$$\mu_{q_0}\text{-almost everywhere,} \quad \lim_{n\to+\infty} \frac{\log \mu_{q_0}(I_{w_n(x)})}{\log |I_{w_n(x)}|} = \theta^*(\theta'(q_0)) = q_0\theta'(q_0). \tag{3.5}$$

  Note that since $\theta(q_0)=0$, we have $\theta^*(\theta'(q_0)) = q_0\theta'(q_0)$.
- We have

$$\mu_{q_0}\Big(\limsup_{n\to+\infty} \ \big\{B(x_w,\lambda_w/2) : w\in\Sigma_n\big\}\Big) = 1,$$

  since, by construction, the system $\{(x_w,\lambda_w)\}_{w\in\Sigma^*}$ enjoys the covering property: $\limsup_{n\to+\infty}\big\{B(x_w,\lambda_w/2) : w\in\Sigma_n\big\} = [0,1]$.

Then, according to the definitions of [2, 4], the system of points $\{(x_w,\lambda_w)\}_{w\in\Sigma^*}$ forms *an heterogenous ubiquitous system* with respect to μ_{q_0} and its almost sure Hölder exponent $\theta^*(\theta'(q_0))$.

Theorem 2.2 of [4] states that, as soon as $\mu_{q_0}(A(1)) = \|\mu_{q_0}\|$, there exists $\widetilde{\varepsilon}$ such that $\dim S(\xi,\widetilde{\varepsilon}) \geq \dim(\mu_{q_0})/\xi$ for every $\xi\geq 1$. In our context, this yields the following proposition.

Proposition 3.6. *There is a positive sequence $\widetilde{\varepsilon}$ converging to 0 at ∞ such that for every $\xi\geq 1$, there exists a positive Borel measure m_ξ such that $m_\xi(S(\xi,\widetilde{\varepsilon}))>0$ and* $\dim(m_\xi) \geq \dfrac{q_0\theta'(q_0)}{\xi}$.

More on heterogeneous ubiquity can be found in [4, 12, 6].

4. Proof of Theorem 1.5

4.1. **A first result on the local regularity analysis of ν.** For the following proposition the reader might like to recall Definition 3.3.

Proposition 4.1. *Let $x\in[0,1]$. If $x\in\{x_w : w\in\Sigma^*\}$, then $h_\nu(x)=0$, otherwise*

$$\frac{\alpha(x)}{\xi_x} \leq \frac{\alpha(x)}{\widetilde{\xi}_x} \leq h_\nu(x) \leq \alpha(x).$$

Here, if $\xi_x = +\infty$ then $\alpha(x)/\xi(x) := 0$. Likewise if $\widetilde{\xi}_x = +\infty$, then $\alpha(x)/\widetilde{\xi}(x) := 0$.

Proof. Let $x\in[0,1]\setminus\{x_w : w\in\Sigma^*\}$ and $r>0$.

- Obviously, $\xi_x \geq \widetilde{\xi}_x$, so the left inequality is trivial.
- Let $n_{x,r} = \min\{n : \exists\ w\in\Sigma_n,\ x_w\in B(x,r)\}$. Let $w(x,r) = w_{n_{x,r}}(x)$ be the unique word of $\Sigma_{n_{x,r}}$ such that $x_{w(x,r)}\in B(x,r)$. Indeed, if two such words w and w' exists (without lost of generality we can assume that $x_w < x_{w'}$), then by construction there is another word w'' such that $|w''| < n_{x,r}$ and $x_w < x_{w''} < x_{w'}$. This contradicts the minimality of $n_{x,r}$. This implies that, using (2.10), $\nu(B(x,r)) \geq \exp\big(\widetilde{\psi}(w(x,r))\big)$.

To find an upper bound for $\nu(B(x,r))$, we use the form (2.11) of ν. Note that the minimality of $n_{x,r}$ yields that we necessarily have

$$x_{w(x,r)} \in B(x,r) \subset I_{w(x,r)}. \tag{4.1}$$

Hence, for every $n > n_{x,r}$, we get

$$\sum_{v\in\Sigma_n :\, x_v\in B(x,r)} \exp(|v|P(\psi))\mu_{\psi,\varphi}(I_v) \leq \exp(nP(\psi))\mu_{\psi,\varphi}(I_{w(x,r)}).$$

Since $P(\psi) < 0$, we deduce combing (2.8) and (2.9) that

$$\begin{aligned} \nu(B(x,r)) &\leq \exp\big(\widetilde{\psi}(w(x,r))\big) + \Big(\sum_{n>n_{x,r}} \exp(nP(\psi))\Big)\mu_{\psi,\varphi}(I_{w(x,r)}) \\ &\leq C\exp\big(\widetilde{\psi}(w(x,r))\big). \end{aligned}$$

Summarizing the above, we now have that there exists a constant $C \geq 1$ such that

$$C^{-1}\exp\big(\widetilde{\psi}(w(x,r))\big) \leq \nu(B(x,r)) \leq C\exp\big(\widetilde{\psi}(w(x,r))\big). \tag{4.2}$$

Fix now $\varepsilon > 0$. By definition of $\widetilde{\xi}_x$, for r small enough we have $r \geq |x - x_{w(x,r)}| \geq (2|I_{w(x,r)}|)^{\widetilde{\xi}_x+\varepsilon}$. Moreover, again for r small enough, $\exp\big(\widetilde{\psi}(w(x,r))\big) \leq |I_{w(x,r)}|^{\alpha(x)-\varepsilon}$ by definition of $\alpha(x)$.

These estimates yield

$$\nu(B(x,r)) \leq C\exp\big(\widetilde{\psi}(w(x,r))\big) \leq Cr^{\frac{\alpha(x)-\varepsilon}{\widetilde{\xi}_x+\varepsilon}},$$

and thus, by letting r tend to zero, it follows that $h_\nu(x) \geq (\alpha(x)-\varepsilon)/(\widetilde{\xi}_x+\varepsilon)$.

• Finally, for the right inequality, let $(n_j)_{j\geq1}$ be an increasing sequence of integers such that $\exp\big(\widetilde{\psi}(w_{n_j}(x))\big) \geq |I_{w_{n_j}(x)}|^{\alpha(x)+\varepsilon}$ for all $j \geq 1$. Such a sequence exists by definition of $\alpha(x)$. By construction, $x_{w_j} \in B(x, 2|I_{w_j}|)$ so that, using formula (2.10) for ν, we get $\nu(B(x, 2|I_{w_{n_j}(x)}|)) \geq |I_{w_{n_j}(x)}|^{\alpha(x)+\varepsilon}$ and $h_\nu(x) \leq \alpha(x)+\varepsilon$.

Since the previous estimates hold for all $\varepsilon > 0$, we obtain the desired result.

4.2. **The singularity sets of ν.** In view of Proposition 4.1, it is natural to introduce the following property for real numbers of $[0,1]$.

Definition 4.2. *Let $\alpha > 0$, $\xi \geq 1$ and $\varepsilon > 0$.*

A real number $x \in [0,1]$ is said to satisfy the property $\mathcal{P}(\alpha,\xi,\varepsilon)$ if there exists an increasing sequence of positive integers $(n_j)_{j\geq1}$ such that for every $j \geq 1$, there exists $w \in \Sigma_{n_j}$ such that $x \in B(x_w, \lambda_w^{\xi-\varepsilon})$ and $\widetilde{\alpha}(w) \in [\alpha-\varepsilon, \alpha+\varepsilon]$.

We say that x is approximated at degree $\xi - \varepsilon$ by the couple (x_w, λ_w) when $x \in B(x_w, \lambda_w^{\xi-\varepsilon})$. Hence a real number x satisfies $\mathcal{P}(\alpha,\xi,\varepsilon)$ when x is approximated at degree $\xi-\varepsilon$ by some couples among the family $\{(x_w,\lambda_w)\}_{w\in\Sigma^*}$, those couples being selected according to the value of $\widetilde{\alpha}(w)$ (we impose that $\widetilde{\alpha}(w) \in [\alpha-\varepsilon, \alpha+\varepsilon]$).

Definition 4.3. *For $h > 0$ one sets*

$$F(h) = \left\{ x \in (0,1) : \left\{ \begin{array}{l} \forall\, \varepsilon > 0, \quad \exists\, \alpha > 0, \quad \exists\, \xi \geq 1 \textit{ such that} \\ \alpha/\xi \leq h+\varepsilon \quad \textit{and} \quad \mathcal{P}(\alpha,\xi,\varepsilon) \textit{ holds} \end{array} \right. \right\}.$$

We explore the relationship between the singularity sets $E_\nu(h)$ and our sets $F(h)$.

Proposition 4.4. *Let $h > 0$.*

1. *One has $F(h) \subset \bigcup_{h'\leq h} E_\nu(h')$.*
2. *One has $E_\nu(h) \subset F(h)$.*

Proof. 1. Let $x \in F(h)$. Then fix $\varepsilon \in (0, 1/2)$, as well as $\alpha > 0$ and $\xi \geq 1$ such that x satisfies $\mathcal{P}(\alpha,\xi,\varepsilon)$. Let $(n_j)_{j\geq1}$ be an increasing sequence of integers such that for every $j \geq 1$, there exists $w_j \in \Sigma_{n_j}$ such that $x \in B(x_{w_j}, (\lambda_{w_j})^{\xi-\varepsilon})$ and $\widetilde{\alpha}(w_j) \in [\alpha-\varepsilon, \alpha+\varepsilon]$. Recall that $\widetilde{\alpha}(w_j) = \frac{\widetilde{\psi}(w_j)}{\log|I_{w_j}|}$.

Taking $r_j = (\lambda_{w_j})^{\xi-\varepsilon}$ for $j \geq 1$, we obtain using (2.10) that

$$\nu(B(x,r_j)) \geq \exp(\widetilde{\psi}(w_j)) \geq (\lambda_{w_j})^{\widetilde{\alpha}(w_j)} \geq r_j^{(\alpha+\varepsilon)/(\xi-\varepsilon)}.$$

This implies that

$$h_\nu(x) \leq \frac{\alpha+\varepsilon}{\xi-\varepsilon} \leq \frac{\alpha}{\xi}\frac{\xi}{\xi-\varepsilon} + 2\varepsilon \leq h + O(\varepsilon).$$

Consequently, letting ε tend to 0 yields $h_\nu(x) \leq h$.

2. Fix $x \in E_\nu(h)$ and $\varepsilon > 0$. By definition there is a sequence $(r_j)_{j\geq 1}$ of positive real numbers decreasing to 0 such that for all $j \geq 1$, $\nu(B(x,r_j)) \geq (r_j)^{h+\varepsilon}$. Then (4.2) yields

$$\exp\left(\widetilde{\psi}(w(x,r_j))\right) \geq C^{-1}(r_j)^{h+\varepsilon},$$

which is equivalent to

$$|I_{w(x,r_j)}|^{\widetilde{\alpha}(w(x,r_j))} \geq C^{-1}(r_j)^{h+\varepsilon}.$$

Thus, writing $|x - x_{w(x,r_j)}| = (2|I_{w(x,r_j)}|)^{\xi_j} = (\lambda_{w(x,r_j)})^{\xi_j}$, we get

$$|I_{w(x,r_j)}|^{\widetilde{\alpha}(w(x,r_j))} \geq C^{-1}(2|I_{w(x,r_j)}|)^{\xi_j(h+\varepsilon)}. \tag{4.3}$$

since $|x - x_{w(x,r_j)}| \leq r_j$.

Recall that $B(x,r_j) \subset I_{w(x,r_j)}$ by (4.1). This implies that $\xi_j \geq 1$. Moreover, $\limsup_{j\to\infty} \xi_j < \infty$: if this limsup is $+\infty$, then combining (3.1) and Part 1. of the current Proposition, we would have $h_\nu(x) = 0$.

Consequently, there exists $(\alpha,\xi) \in \mathbb{R}_+^* \times [1,\infty)$ and an increasing sequence of integers, $(j_k)_{k\geq 1}$, such that simultaneously:

$$\begin{array}{rcl} |\alpha - \widetilde{\alpha}(w(x,r_{j_k}))| & \leq & \varepsilon, \\ |\xi - \xi_{j_k}| & \leq & \varepsilon, \\ (\widetilde{\alpha}(w(x,r_{j_k})), \xi_{j_k}) & \longrightarrow & (\alpha,\xi) \quad \text{as} \quad k \to +\infty. \end{array}$$

Notice that (4.3) implies that $\alpha/\xi \leq h + \varepsilon$.
Since these properties hold for all $\varepsilon > 0$, $x \in F(h)$.

Corollary 4.5. *For every $h > 0$, $E_\nu(h) = F(h) \setminus \bigcup_{h'<h} F(h')$.*

Proof. This follows directly from the combination of Parts 1. and 2. of Proposition 4.4.

4.3. **Upper bound for the dimensions of the singularity sets.**

It is known that, for any positive Borel measure ν with a bounded support, we always have the upper bound $\dim E_\nu(h) \leq \tau_\nu^*(h)$ (see [10] for instance). Using the definition of the Legendre transform (1.5), in order to find an upper bound for $\dim E_\nu(h)$, we need a lower bound for τ_ν.

Proposition 4.6. *For every $q \in \mathbb{R}$, we have $\tau_\nu(q) \geq \min(\theta(q), 0))$.*

This immediately yields the desired upper bound for the spectrum. Indeed, using the definition of the Legendre transform (1.5), we get that $\tau_\nu^*(h) = h \cdot \dim X$ if $h \in [0, \theta'(\dim X)]$, and $\tau_\nu^*(h) = \theta^*(h)$ if $h > \theta'(\dim X)$.

In the next Section, we prove that $\dim E_\nu(h) \geq \tau_\nu^*(h)$ for every $h \geq 0$. By applying the inverse Legendre transform to the inequality claimed by Proposition 4.6, we get the equality $\tau_\nu(q) = \min(\theta(q), 0))$, as stated in Theorem 1.5.

Proof. Let $r > 0$ and consider $\mathcal{B} = \{B_i\}$, a packing of $[0,1]$ by disjoint closed intervals B_i of radius r.

• First fix $q < 0$. Due to the bounded distortion principle (which rules the size of the elements of the grid $\{I_w\}_{w\in\Sigma^*}$), there exists a constant $C > 0$ depending on φ only, such that each B_i contains an interval I_{w_i} satisfying $|I_{w_i}| \geq r/C$.

Using the formula (2.11) for ν, we have that $\nu(B_i) \geq \exp(|w_i|P(\psi))\mu_{\psi,\varphi}(I_{w_i})$ and since $q < 0$, we get $\nu(B_i)^q \leq \exp(q|w_i|P(\psi))\mu_{\psi,\varphi}(I_{w_i})^q$. Formula (3.2) yields

$$\nu(B_i)^q \leq C_q \mu_q(I_{w_i}) |I_{w_i}|^{\theta(q)}.$$

Finally, since $\theta(q) < 0$ and $|I_{w_i}| \geq r/C$, we obtain

$$\nu(B_i)^q \leq C_q C^{-q} r^{\theta(q)} \mu_q(I_{w_i}).$$

Consequently,

$$\sum_{B_i\in\mathcal{B}} \nu(B_i)^q \leq C_q C^{-q} r^{\theta(q)}$$

independently of the choice of the r-packing $\mathcal{B} = \{B_i\}$. This yields $\tau_\nu(q) \geq \theta(q)$.

• Secondly fix $q \in (0, \dim X)$. Let $M = \sup_{t\in I_0\cup I_1} |T'(t)|$. Let $S_r = \{w \in \Sigma^* : 2r \leq |I_w| \leq 2r(M+1)\}$. We also set $n_r = \max\{|w| : w \in S_r\}$ and $n_r' = \min\{|w| : w \in S_r\}$. Since the mapping T is C^1 and $|T'| > 1$, we have $n_r = O(|\log(r)|)$ and $n_r' = O(|\log(r)|)$.

For $w \in S_r$, I_w meets at most $M+1$ balls B_i of the packing $\mathcal{B}$. Reciprocally, each B_i is included in the union of at most two intervals belonging to S_r, say I_w and $I_{w'}$ (and possibly by the singleton $\{1\}$).

The subadditivity of the application $t \geq 0 \mapsto t^q$ yields $\nu(B_i)^q \leq \nu(\{1\}) + \nu(I_w)^q + \nu(I_{w'})^q$ if $1 \in B_i$, and $\nu(B_i)^q \leq \nu(I_w)^q + \nu(I_{w'})^q$ otherwise. Thus

$$\begin{aligned}\sum_{B_i\in\mathcal{B}} \nu(B_i)^q &\leq \nu(\{1\}) + (M+1) \sum_{n=n_r'}^{n_r} \sum_{w\in S_r\cap\Sigma_n} \nu(I_w)^q \\ &\leq \nu(\{1\}) + (M+1) \sum_{n=0}^{n_r} \sum_{w\in \widetilde{S}_r\cap\Sigma_n} \nu(I_w)^q,\end{aligned}$$

where $\widetilde{S}_r=\{w\in\Sigma^*: 2r\le |I_w|\}\supset S_r$. Since $q\in(0,\dim X)$ and $\dim X<1$, we have

$$\begin{aligned}
\nu(I_w)^q &= \Big(\sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})+\sum_{n'\ge|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})\Big)^q\\
&\le \sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})^q+\Big(\sum_{n'\ge|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})\Big)^q\\
&\le \sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})^q+\Big(\sum_{n'\ge|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\mu_{\psi,\varphi}(I_v)\exp(n'P(\psi))\Big)^q\\
&\le \sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})^q+\mu_{\psi,\varphi}(I_w)^q\Big(\sum_{n'\ge|w|}\exp(n'P(\psi))\Big)^q\\
&= \sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})^q+K_q\exp(q|w|P(\psi))\mu_{\psi,\varphi}(I_w)^q
\end{aligned}$$

where we used the formula (2.11) for ν and $K_q=(1-\exp(qP(\psi)))^{-q}$. Consequently,

$$\sum_{B_i\in\mathcal{B}}\nu(B_i)^q\le\nu(\{1\})+(M+1)(A_1+A_2),$$

where

$$\begin{aligned}
A_1 &= \sum_{n=0}^{n_r}\sum_{w\in\widetilde{S}_r\cap\Sigma_n}\sum_{n'<|w|}\sum_{\substack{v\in\Sigma_{n'},\\ x_v\in I_w}}\nu(\{x_v\})^q\\
\text{and}\qquad A_2 &= \sum_{n=0}^{n_r-1}\sum_{w\in\widetilde{S}_r\cap\Sigma_n}K_q\exp(q|w|P(\psi))\mu_{\psi,\varphi}(I_w)^q.
\end{aligned}$$

Obviously

$$A_2\le K_q n_r\sum_{n=0}^{n_r}\sum_{w\in\widetilde{S}_r\cap\Sigma_n}\exp(q|w|P(\psi))\mu_{\psi,\varphi}(I_w)^q.$$

Then, a simple reordering of the terms in A_1 yields

$$A_1\ \le\ \sum_{n=0}^{n_r-1}\sum_{v\in\Sigma_n}\left(\#\left\{w\in\widetilde{S}_r\cap\bigcup_{p=n+1}^{n_r}\Sigma_p: x_v\in I_w\right\}\right)\nu(\{x_v\})^q.$$

Each cardinality is by construction less than n_r. Hence

$$A_1\le n_r\sum_{n=0}^{n_r-1}\sum_{v\in\widetilde{S}_r\cap\Sigma_n}\nu(\{x_v\})^q\le n_r\sum_{n=0}^{n_r-1}\sum_{v\in\widetilde{S}_r\cap\Sigma_n}\exp(q|v|P(\psi))\mu_{\psi,\varphi}(I_v)^q.$$

Finally, formula (3.2) yields

$$\begin{aligned}\sum_{B_i\in\mathcal{B}} \nu(B_i)^q &\le \nu(\{1\}) + (M+1)(K_q+1)n_r \sum_{n=0}^{n_r} \sum_{w\in\widetilde{S}_r\cap\Sigma_n} |I_w|^{\theta(q)}\mu_q(I_w)\\ &\le \nu(\{1\}) + (M+1)(K_q+1)n_r \sum_{n=0}^{n_r} r^{\theta(q)}\|\mu_q\|\\ &\le \nu(\{1\}) + (M+1)(K_q+1)(n_r+1)^2 r^{\theta(q)},\end{aligned}$$

where we used that $r \le |I_w|$ for $w \in \widetilde{S}_r$ and $\theta(q) < 0$. The estimate on n_r finally gives that

$$\sum_{B_i\in\mathcal{B}} \nu(B_i)^q \le O(r^{\theta(q)}|\log(r)|^2).$$

The above upper bound is independent of the packing $\mathcal{B}$. This yields $\tau_\nu(q) \ge \theta(q)$.

• Finally, let $q \ge \dim X$. Since ν is discrete, we necessarily have $\tau_\nu(q) = 0$ for every $q \ge 1$. In addition, if $q = \dim X$, then $\tau_\nu(q) \ge \theta(q) = 0$. The concavity of τ_ν then implies that $\tau_\nu(q) = 0$ for every $q \ge \dim X$.

4.4. **Lower bounds for the dimensions of the singularity sets.**

In view of Definition 4.2 and Corollary 4.5, in order to find lower bounds for the Hausdorff dimensions of the level sets $E_\nu(h)$, it is natural to introduce the following limsup sets: For every $\alpha, \varepsilon > 0$ and $\xi \ge 1$, set

$$G(\alpha,\xi,\varepsilon) = \bigcap_{N\ge1} \bigcup_{n\ge N} \bigcup_{w\in\Sigma_n:\,\alpha-\varepsilon\le\widetilde{\alpha}(w)\le\alpha+\varepsilon} B(x_w, \lambda_w^\xi).$$

It is obvious that, if $x \in F(h)$, then for every $\varepsilon > 0$, there exists α and $\xi \ge 1$ such that $\alpha + \xi \le h + \varepsilon$ and x satisfies $\mathcal{P}(\alpha,\xi,\varepsilon)$. In particular, x belongs to $G(\alpha,\xi,\varepsilon)$. Our aim here is to control the Hausdorff dimension of the sets $G(\alpha,\xi,\varepsilon)$ for the following reason: If the Hausdorff dimension of $G(\alpha,\xi,\varepsilon)$ is proved to be strictly less than $\tau^*_{\mu_q}(\tau'_\mu(q))$, then $G(\alpha,\xi,\varepsilon)$ is μ_q-negligible.

Proposition 4.7. *There exists $C > 0$, depending on θ only, such that for $\varepsilon > 0$ small enough, for all $\alpha > 0$ and $\xi \ge 1$,*

$$\dim G(\alpha,\xi,\varepsilon) \le C\varepsilon + \frac{\max(\theta^*(\alpha-\varepsilon), \theta^*(\alpha+\varepsilon))}{\xi}.$$

When the right hand-side of the above inequality is negative, the set $G(\alpha,\xi,\varepsilon)$ is empty.

Proof. Fix $\alpha, \varepsilon > 0$, $\xi \ge 1$, and for $N \ge 1$ let $\delta_N = \sup_{w\in\Sigma_N} \lambda_w^\xi$. For any $s > 0$ and $\delta > 0$, we denote by $\mathcal{H}^s_\delta$ the s-Hausdorff pre-measure computed using coverings by sets of diameter less than δ [13]. By construction,

$$\mathcal{H}^s_{\delta_N}(G(\alpha,\xi,\varepsilon)) \le \sum_{n\ge N} \sum_{\substack{w\in\Sigma_n:\\ \alpha-\varepsilon\le\widetilde{\alpha}(w)\le\alpha+\varepsilon}} 2^s(\lambda_w)^{s\xi}.$$

• Suppose that $\alpha+\varepsilon\le\theta'(0)$. If $q\ge 0$, using that $\widetilde{\alpha}(w)=\frac{\widetilde{\psi}(w)}{\chi(w)}$, $\widetilde{\psi}(w)<0$ and $\chi(w)=\log|I_w|<0$, we have for some constant $C>0$ that

$$\begin{aligned}\mathcal{H}^s_{\delta_N}(G(\alpha,\xi,\varepsilon)) &\le C\sum_{n\ge N}\sum_{\substack{w\in\Sigma_n:\\ q\widetilde{\psi}(w)\ge q(\alpha+\varepsilon)\chi(w)}}(\lambda_w)^{s\xi}\\ &\le C\sum_{n\ge N}\sum_{w\in\Sigma_n}\exp(q\widetilde{\psi}(w)-q(\alpha+\varepsilon)\chi(w))\exp(s\xi\chi(w)).\end{aligned}$$

Now take $s=\big(\eta+\theta^*(\alpha+\varepsilon)\big)/\xi$ with $\eta>0$. We deduce that

$$\mathcal{H}^s_{\delta_N}(G(\alpha,\xi,\varepsilon))\le C\sum_{n\ge N}\sum_{w\in\Sigma_n}\exp\big(q\widetilde{\psi}(w)-(q(\alpha+\varepsilon)-\eta-\theta^*(\alpha+\varepsilon))\chi(w)\big). \tag{4.4}$$

We know that, $n\ge N$ being fixed, the second sum in the previous double sum is comparable to

$$\exp\Big(nP(\psi_{q,t_{q,\alpha,\varepsilon}})\Big),\quad\text{where } t_{q,\alpha,\varepsilon}=q(\alpha+\varepsilon)-\eta-\theta^*(\alpha+\varepsilon).$$

Since $\alpha+\varepsilon\le\theta'(0)$, we can choose $q\ge 0$ such that $\theta^*(\alpha+\varepsilon)=(\alpha+\varepsilon)q-\theta(q)-\gamma_q$, with $0\le\gamma_q\le\eta/2$. This implies that

$$P(\psi_{q,t_{q,\alpha,\varepsilon}})\le P(\psi_{q,(\theta(q)-\eta/2)})<0.$$

Using (4.4), this implies that $\lim_{N\to\infty}\mathcal{H}^s_{\delta_N}(G(\alpha,\xi,\varepsilon))=0$. Consequently, $\dim G(\alpha,\xi,\varepsilon)\le\big(\eta+\theta^*(\alpha+\varepsilon)\big)/\xi$. Since this holds for all $\eta>0$, we have the desired conclusion.

• When $\alpha-\varepsilon\ge\theta'(0)$, the same approach with $q<0$ yields the desired estimate.

• When $\alpha-\varepsilon\le\theta'(0)\le\alpha+\varepsilon$, for $\eta>0$ and $s=(\eta+\theta(0))/\xi$ we can write

$$\mathcal{H}^s_{\delta_N}(G(\alpha,\xi,\varepsilon))\le C\sum_{n\ge N}\sum_{w\in\Sigma_n}(\lambda_w)^{s\xi}\le C\sum_{n\ge N}\exp\big(nP(-(\eta+\theta(0))\varphi)\big),$$

which tends to as N tends to infinity. Thus $\dim G(\alpha,\xi,\varepsilon)\le\theta(0)/\xi$.

Now, since θ^* is differentiable near $\theta'(0)$, there exists a constant $C>0$, depending only on θ, such that $\theta(0)/\xi\le C\varepsilon+\max(\theta^*(\alpha-\varepsilon),\theta^*(\alpha+\varepsilon))/\xi$ when $\varepsilon>0$ is sufficiently small and $\alpha-\varepsilon\le\theta'(0)\le\alpha+\varepsilon$. The result follows.

Corollary 4.8. *For all $h>0$, $\dim F(h)\le h\cdot\dim X$.*

Proof. Let $x\in F(h)$, and $\varepsilon>0$. There exists α and $\xi\ge 1$ such that $\alpha+\xi\le h+\varepsilon$ and x satisfies $\mathcal{P}(\alpha,\xi,\varepsilon)$. It follows that x belongs to the set $G(\alpha,\xi,\varepsilon)$, whose Hausdorff dimension is bounded from above by the last Proposition 4.7. This yields that for every $\varepsilon>0$

$$\dim F(h)\le\sup_{\alpha,\xi:\,\alpha/\xi\le h+\varepsilon}C\varepsilon+\frac{\max(\theta^*(\alpha-\varepsilon),\theta^*(\alpha+\varepsilon))}{\xi}.$$

Letting ε tend to zero, yields

$$\dim F(h)\le\sup_{\alpha,\xi:\,\alpha/\xi\le h}\frac{\theta^*(\alpha)}{\xi}\le h\cdot\sup_{\alpha\ge 0}\frac{\theta^*(\alpha)}{\alpha}\le h\cdot\dim X.$$

The last inequality follows from the fact that the measure μ_φ and μ_ψ have their support included in X and from the definition of the Legendre transform (1.5).

Proposition 4.9.

(1) *If $h\ge\theta'(\dim X)$ and $\theta^*(h)\ge 0$, then $\dim E_\nu(h)\ge\theta^*(h)$.*

(2) *If* $h \in [0, \theta'(\dim X)]$, *then* $\dim E_\nu(h) \geq h \cdot \dim X$.

Proof. 1. If $h \geq \theta'(\dim X)$ and $\theta^*(h) \geq 0$, then $h \in \{\theta'(q) : q \geq \dim X\} \cup \{\lim_{q\to-\infty} \theta'(q)\}$. Proposition 4.1 shows that $E_\nu(h)$ contains the set

$$\begin{aligned}\widetilde{E}(h) &= \big\{x : \lim_{n\to\infty} \alpha_n(x) = h \text{ and } \widetilde{\xi}(x) = 1\big\} \\ &= \big\{x : \lim_{n\to\infty} \alpha_n(x) = h\big\} \setminus \bigcup_{m\geq 1} \{x : \lim_{n\to\infty} \alpha_n(x) = h \text{ and } \widetilde{\xi}(x) > 1 + 1/m\big\}.\end{aligned}$$

Consider first the case where $h = \theta'(q)$ for some $q \geq \dim X$.
Part 1. of Proposition 3.5 gives that $\mu_q(\big\{x : \lim_{n\to\infty} \alpha_n(x) = h\big\}) = 1$.
From Proposition 4.7, we easily deduce that for all $m \geq 1$,

$$\dim \big\{x : \lim_{n\to\infty} \alpha_n(x) = \theta'(q) \text{ and } \widetilde{\xi}(x) > 1 + 1/m\big\} < \theta^*(\theta'(q)) = \theta^*(h).$$

Then, Part 2. of Proposition 3.5 gives that the countable union of sets

$$\bigcup_{m\geq 1} \{x : \lim_{n\to\infty} \alpha_n(x) = h \text{ and } \widetilde{\xi}(x) > 1 + 1/m\}$$

is μ_q-negligible. Hence $\mu_q(\widetilde{E}(h)) = 1$. Finally, $\dim E_\nu(h) \geq \dim(\mu_q) \geq \theta^*(\theta'(q)) = \theta^*(h)$.

It remains us to treat the case of the right endpoint of the singularity spectrum $h_{\max} := \sup\{h \geq 0 : \tau_\nu^*(h) \geq 0\}$. We have $h_{\max} = \lim_{q\to-\infty} \theta'(q) < +\infty$. In this case there is no natural Gibbs measure supported by the set $\widetilde{E}_\nu(h_{\max}) = \{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max}\}$.

When $\tau_\nu^*(h_{\max}) > 0$, the approach used in [15] to achieve the multifractal analysis of Birkhoff averages associated with a continuous, but not necessarily Hölderian, potential on Σ allows to construct a Cantor set included in $\{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max}\}$ (usually called in this context a Moran set) and a measure $\mu_{-\infty}$ supported by this Cantor set, for which $\dim(\mu_{-\infty}) \geq \theta^*(h_{\max})$. The conclusion is then obtained as in the previous case.

We discuss the case $\tau_\nu^*(h_{\max}) = 0$ briefly in Section 4.5.

2. Let $h \in (0, \theta'(\dim X)]$ and write $h = \theta'(\dim X)/\xi$, with $\xi \geq 1$. Consider the set $S(\xi, \widetilde{\varepsilon})$ of formula (3.3) with a suitable sequence $\widetilde{\varepsilon}$ such that Proposition 3.6 can be applied. This provides us with a positive Borel measure m_ξ such that the following hold.

- $m_\xi(S(\xi, \widetilde{\varepsilon})) > 0$
- $m_\xi(E) = 0$ as soon as $\dim E < \dim(X) \dfrac{\theta'(\dim(X))}{\xi} = h \cdot \dim X$.

It follows from Proposition 4.4 that

$$S(\xi, \widetilde{\varepsilon}) \setminus \bigcup_{0<h'<h} F(h') \subset E_\nu(h).$$

By Corollary 4.8 we have $\dim F(h') \leq h' \cdot \dim X < h \cdot \dim X$, for all $0 < h' < h$. Hence $m_\xi(F(h')) = 0$ for all $h' < h$. Moreover, the set family $(F(h'))_{0<h'<h}$ is non decreasing.

Summarizing the above, we now have that $m_\xi(E_\nu(h)) > 0$, and hence $\dim E_\nu(h) \geq h \cdot \dim X$.

4.5. **The case $\tau_\nu^*(h_{\max}) = 0$.** When $\tau_\nu^*(h_{\max}) = 0$, by the same computations as in [15], it is possible to construct a Cantor set included in $\{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max}\}$ and a non-trivial measure $\mu_{-\infty}$ supported by this Cantor set. Obviously $\dim(\mu_{-\infty}) = 0$, but there is a gauge function $f : \mathbb{R}^+ \to \mathbb{R}^+$ naturally associated with $\mu_{-\infty}$, such that for every $\varepsilon > 0$, we have that $f(x) = o(x^\varepsilon)$, for x tending to 0 from above, and for every Borel set E,

$$\mu_\infty(E) \le f(|E|).$$

The difficulty lies in the fact that the argument of previous Section do not apply here. Indeed, every set $\{x : \lim_{n\to\infty} \alpha_n(x) = h_{\max} \text{ and } \widetilde{\xi}(x) > 1 + 1/m\}$ has also Hausdorff dimension 0, and these sets are not negligible *a priori* from the $\mu_{-\infty}$ viewpoint. Hence, we cannot conclude directly that $\mu_{-\infty}(\{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max} \text{ and } \widetilde{\xi}(x) = 1\}) = 1$.

This can certainly be circumvented by at least two methods:

- one could modify the construction of the Cantor set included in $\{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max}\}$ to ensure that this Cantor set is indeed included in $\{x : \lim_{n\to+\infty} \alpha_n(x) = h_{\max} \text{ and } \widetilde{\xi}(x) = 1\}$,
- one could replace the dimension argument (to neglect the sets $\{x : \lim_{n\to\infty} \alpha_n(x) = h_{\max} \text{ and } \widetilde{\xi}(x) > 1 + 1/m\}$) by an argument based on the study of generalized Hausdorff measure using gauge functions.

Both approaches, which are very technical, will certainly work. However, this would require some lengthy extra work, and therefore we do not go into further details here.

4.6. **The L^q-spectrum.** If $q \ge \dim X$, then $\tau_\nu(q) = 0 \le \theta(q)$. This follows from the observations that $\tau_\nu(\dim X) = \tau_\nu(1) = 0$ and that τ_ν is concave.

When $q \le \dim X$, we proved that $\tau_\nu^*(\theta'(q)) = \theta^*(\theta'(q))$. Taking the inverse Legendre transform yields that we have $\tau_\nu(q) = \theta(q)$.

Summarizing the above, we have $\tau_\nu(q) = \min(0, \theta(q))$ for every $q \in \mathbb{R}$.

Acknowledgments. The authors thank the anonymous referee for his valuable remarks and suggestions.

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