LOGARITHMIC SYSTOLIC GROWTH FOR HYPERBOLIC SURFACES IN EVERY GENUS

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Abstract. More than thirty years ago, Brooks and Buser–Sarnak constructed sequences of closed hyperbolic surfaces with logarithmic systolic growth in the genus. Recently, Liu and Petri showed that such logarithmic systolic lower bound holds for every genus (not merely for genera in some infinite sequence) using random surfaces. In this article, we show a similar result through a more direct approach relying on the original Brooks/Buser–Sarnak surfaces.

1. Introduction

The systole of a closed hyperbolic surface $S$, denoted by $\text{sys}(S)$, is the length of a shortest (noncontractible) closed geodesic of $S$. By Mumford’s compactness theorem, the systole function achieves a maximum among all closed hyperbolic surfaces of a given genus. By considering the area of a disk of radius $\frac{1}{2} \text{sys}(S)$, one easily sees (see [6, Lemma 5.2.1]) that the systole of every closed hyperbolic surface $S$ of genus $g$ satisfies the upper bound

$$\text{sys}(S) \leq 2 \log(4g - 2).$$

As far as lower bounds are concerned, Brooks [4] and Buser–Sarnak [1] constructed a sequence of closed hyperbolic surfaces $S_{g_p}$ of genus $g_p$ with

$$g_p = (p^3 - p)p + 1$$

obtained as congruence coverings of a fixed arithmetic surface $S$ defined from a division quaternion algebra $A$ such that

$$\text{sys}(S_{g_p}) \geq 4 \log(p)$$

where $p$ is any odd prime and $\nu$ is fixed and depends only on $A$; see [4, Lemma 3.1] and [1, Eq. (4.3)]. This immediately leads to the following logarithmic systolic growth

$$\text{sys}(S_{g_p}) \geq \frac{4}{3} \log(g_p) - C$$

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where \( C \) is a positive constant depending only on \( A \), but not on \( p \).

Combining the upper and lower bounds (1.1) and (1.4), we obtain

\[
\frac{4}{3} \leq \limsup_{g \to \infty} \max_{S \in M_g} \frac{\sys(S)}{\log g} \leq 2
\]

where \( M_g \) is the moduli space of closed hyperbolic surfaces of genus \( g \). See also Brooks [5], Gromov [9] and Schmutz Schaller [19].

The Brooks/Buser–Sarnak construction was generalized in [12] to principal congruence covers of any closed arithmetic surface. Maﬁ-isumi [14] proved that the multiplicative constant \( \frac{4}{3} \) in (1.4) is optimal for these congruence covers. Other sequences of closed hyperbolic surfaces with arbitrarily large genus satisfying a logarithmic growth have been obtained in [17] and [18] from graphs of large (logarithmic) girth/systole. Still, these constructions provide families of surfaces of fairly sparse genera.

Recently, Liu and Petri [13] showed using a probabilistic method that for every \( c < \frac{2}{3} \), there exists a constant \( C > 0 \) such that for every genus \( g \geq 2 \) (not merely for genera in some infinite sequence), there exists a closed hyperbolic surface \( S_g \) of genus \( g \) with

\[
\sys(S_g) \geq c \log(g) - C.
\]

It follows that

\[
\liminf_{g \to \infty} \max_{S \in M_g} \frac{\sys(S)}{\log g} \geq \frac{2}{9}.
\]

Note that their argument relies on an appropriate model of random surfaces and does not provide an explicit construction.

In the present article, we show a similar result, see Theorem 3.1, without using probabilistic arguments (albeit with a worse constant). Instead, we use a more direct approach relying on the original surfaces of Brooks/Buser–Sarnak.

**Theorem 1.1.** For every \( c < \frac{19}{120} \), there exists a constant \( C > 0 \) such that for every genus \( g \geq 2 \), there exists a closed hyperbolic surface \( S_g \) of genus \( g \) with

\[
\sys(S_g) \geq c \log(g) - C.
\]

Actually, Buser and Sarnak [1] considered a similar question for the systole of the Jacobian of a Riemann surface of large genus after showing that there exists a sequence of Riemann surfaces of arbitrarily large genus whose square of the Jacobian systole has a logarithmic growth in the genus. To show this holds in every genus, they mention that one could seek to vary the parameters in the definition of the quaternion
algebra $A$ and to vary the congruence groups in the arithmetic construction, but as they point out this seems rather complicated; see [1, p. 47]. Instead, they rely on the circle method from additive analytic number theory and Fay’s gluing of Riemann surfaces [7, §III] to construct Riemann surfaces whose Jacobian decomposes as the product of Jacobians of lower dimension up to a small error term (see [1, p. 47]), which is enough for their purposes. As mentioned in [16, p. 127], it seems likely (though this would require some argument) that the homological systole of the underlying hyperbolic surfaces thus-obtained have a logarithmic growth. It is however unclear whether this construction yields an explicit homological systolic lower bound (as we have not been able to fill in all the details) and no such explicit bound for the Jacobian systole is given in [1] (only for the surfaces constructed as congruence covers of an arithmetic surface). Note also that, though the homological systole of these surfaces is likely to have a logarithmic growth, their systole is equally likely to go to zero. However, since for hyperbolic surfaces of a fixed genus, the maximum of the systole is equal to the maximum of the homological systole by Parlier [15], this would not be an obstacle to deriving a logarithmic systolic growth in every genus.

To prove Theorem 1.1, we rely on a more flexible construction using CAT($-1$) metrics rather than the more rigid hyperbolic metrics. This change of setting is not restrictive for our purpose since the maximum of the systole on CAT($-1$) surfaces of fixed genus is achieved by a hyperbolic metric according to [10]. Along the way, we apply the prime gap theorem to control the growth of the genus in Brooks/Buser–Sarnak’s congruence surfaces. As a result, we obtain an explicit asymptotic systolic lower bound.

Recent advances in systolic geometry include [8] and [11].

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2. Preliminaries

For future reference, let us recall some classical formulas in hyperbolic geometry. The area and the perimeter of a disk $D(r)$ of radius $r$
in the hyperbolic plane are given by

\[ \text{area } D(r) = 4\pi \sinh^2 \left( \frac{r}{2} \right) \]  \hspace{1cm} (2.1)\\
\[ \text{length } \partial D(r) = 2\pi \sinh(r) \]  \hspace{1cm} (2.2)

and the area of a closed hyperbolic surface \( S \) of genus \( g \) is given by

\[ \text{area}(S) = 4\pi(g - 1). \]  \hspace{1cm} (2.3)

Due to the arithmetic nature of the construction of the Brooks/Buser–Sarnak surfaces, we will also need the following prime gap theorem.

**Theorem 2.1** (Prime gap theorem [3]). There exist \( \lambda \geq 1 \) and \( \theta \in (0, 1) \) such that the gap between any prime \( p \) and the next prime \( p' \) satisfies

\[ p' - p \leq \lambda p^\theta. \]  \hspace{1cm} (2.4)

For this bound, we can take \( \theta = \frac{21}{40} \approx 0.525 \).

**Remark 2.2.** For \( p \) large enough, the multiplicative constant \( \lambda \geq 1 \) can be taken arbitrarily close to 1; see [3].

3. **Logarithmic systolic growth**

We will use the notation \( A_g \simeq B_g \) for \( A_g = B_g(1 + o(1)) \) as \( g \) goes to infinity (similarly with \( A_g \lesssim B_g \) and \( A_g \gtrsim B_g \)).

**Theorem 3.1.** For every \( g \geq 2 \), there exists a closed hyperbolic surface \( S_g \) of genus \( g \) with

\[ \text{sys}(S_g) \gtrsim \frac{1 - \theta}{3} \log(g) \]  \hspace{1cm} (3.1)

where \( \theta \in (0, 1) \) is the exponent in (2.4).

**Remark 3.2.** With the value of \( \theta \) given in Theorem 2.1, the coefficient \( \frac{1 - \theta}{3} \) can be taken equal to \( \frac{19}{120} = 0.18333... \geq \frac{1}{7} \).

**Proof.** For every odd prime \( p \), consider a closed hyperbolic surface \( S_{gp} \) of genus \( gp \) satisfying \([1,3]\). Let \( r \) and \( d \) be two positive reals (to be determined later) such that \( r + d < \frac{1}{4} \text{sys}(S_{gp}) \). Consider a maximal collection \((D_i)_{1 \leq i \leq N}\) of disjoint disks \( D_i \) of radius \((r+d)\) in \( S_{gp} \). Denote by \( x_i \) the center of \( D_i \). Since the collection of disks \( D_i \) is maximal, the disks \( 2D_i \) of radius \( 2(r+d) \) centered at \( x_i \) cover \( S_{gp} \). Since their radius is less than \( \frac{1}{2} \text{sys}(S_{gp}) \), the disks \( 2D_i \) can be isometrically embedded in the hyperbolic plane onto a disk \( 2D \) of radius \( 2(r+d) \). It follows that the number \( N \) of points \( x_i \) satisfies

\[ N \cdot \text{area}(2D) \geq \text{area}(S_{gp}). \]
By the relations (2.1) and (2.3), we immediately obtain
\[ N \geq \frac{g_p - 1}{\sinh^2(r + d)}. \] (3.2)

Now, remove 2k disks \( D(x_i, r) \) from \( S_{g_p} \) with \( k \leq \frac{N}{2} \), and pairwise glue together the boundary components of the resulting surface. This gives rise to a closed CAT\((-1)\) surface \( \bar{S}_{g_p+k} \) of genus \( g_p + k \).

Before showing that the surface \( \bar{S}_{g_p+k} \) satisfies the logarithmic systolic growth of (3.1), we will first show that every genus large enough can be attained as the genus of a surface \( \bar{S}_{g_p+k} \). For that, for \( p \) large enough, we want \( g_p + \lfloor \frac{N}{2} \rfloor \) to be at least \( g_p' \) to recover every genus between \( g_p \) and \( g_p' \), when \( k \) runs between 0 and \( \frac{N}{2} \) (recall that \( p' \) is the prime right after \( p \)). By (3.2), it is enough to have
\[ \frac{g_p - 1}{2 \sinh^2(r + d)} \geq g_p' - g_p + 1. \] (3.3)

Since \( g_p = (p^3 - p)\nu + 1 \), the genus gap \( g_p' - g_p \) satisfies
\[
\begin{align*}
g_p' - g_p + 1 &= (p'^3 - p^3)\nu + (p' - p)\nu + 1 \\
&\leq 3\nu\lambda p^{2+\theta} + 3\nu\lambda^2 p^{1+2\theta} + \nu\lambda^3 p^{3\theta} + \nu\lambda^p + 1 \\
&\leq 8\nu\lambda^3 p^{2+\theta} + 1
\end{align*}
\] (3.4)

after making use of the bound \( p' \leq p + \lambda p^\theta \) (see Theorem 2.1), expanding the power, and bounding each term \( p^{1+2\theta} \), \( p^{3\theta} \) and \( p^\theta \) by \( p^{2+\theta} \).

Thus, by (3.3), (1.2) and (3.4), it is enough to have
\[
\sinh^2(r + d) \leq p^{1-\theta} \frac{(1 - p^{-2})\nu}{2(8\nu\lambda^3 + p^{-2-\theta})}.
\]

Since \( \sinh(r + d) \leq \frac{e^{r+d}}{2} \) and \( p \geq 3 \), it is enough to have
\[
r + d \leq \frac{1 - \theta}{2} \log(p) - C(\lambda)
\] (3.5)

where \( C(\lambda) = \frac{1}{2} \log \left( \frac{2(8\nu\lambda^3 + p^{-2-\theta})}{(1-3^{-2})\nu^2} \right) \). Note that this condition immediately implies that
\[
r + d \leq \log(p) \leq \frac{1}{4} \text{sys}(S_{g_p})
\]
for \( p \) large enough, as previously required.

Now, in order to show that the surface \( \bar{S}_{g_p+k} \) satisfies the logarithmic systolic growth of (3.1), we will need the following lemma.
Lemma 3.3.  
\[ \text{sys}(\bar{S}_{g^p+k}) \geq \min\{\text{sys}(S_{g^p}), 2\pi \sinh(r), 2d\}. \]

**Proof.** By the circle length formula \((2.2)\), the closed geodesics of \(\bar{S}_{g^p+k}\) corresponding to a multiple of \(\partial D(x_i, r)\) are of length at least \(\text{length} \partial D(x_i, r) = 2\pi \sinh(r)\).

The closed geodesics of \(\bar{S}_{g^p+k}\) intersecting \(\partial D(x_i, r)\), but non homotopic to a multiple of \(\partial D(x_i, r)\), have to leave \(D(x_i, r+d)\), and thus, are of length at least \(2d\). Finally, the closed geodesics of \(\bar{S}_{g^p+k}\) which do not intersect \(\partial D(x_i, r)\) lie in the original surface \(S_{g^p}\), and thus, are of length at least \(\text{sys}(S_{g^p})\). This implies the desired systolic lower bound. \(\square\)

At this point, we can fix the values of \(r\) and \(d\) as follows. Take the real \(r\) so that \(2\pi \sinh(r) = 4 \log(p)\). That is, \(r = \text{arcsinh} \left( \frac{2}{\pi} \log(p) \right) \simeq \log \log(p)\).

Take also the real \(d\) so that the equality case is attained in the equality \((3.5)\). That is, \(d = \frac{1 - \theta}{2} \log(p) - r - C(\lambda) \simeq \frac{1 - \theta}{2} \log(p)\).

By Lemma 3.3, we deduce that \(\text{sys}(\bar{S}_{g^p+k}) \gtrsim (1 - \theta) \log(p)\).

For \(k \leq \lambda p^\theta\), we obtain that \(g^p + k \simeq \nu p^3\). Thus, \(\text{sys}(\bar{S}_{g^p+k}) \gtrsim \frac{1 - \theta}{3} \log(g^p + k)\).

Since every genus \(g\) large enough can be decomposed as \(g = g_p + k\), where \(p\) is an odd prime and \(k \leq \lambda p^\theta\), the maximum of the systole over all closed CAT\((-1)\) surfaces \(\bar{S}_g\) of genus \(g\) is roughly at least \(\frac{1 - \theta}{3} \log g\). But the maximum of the systole over all closed CAT\((-1)\) surfaces of genus \(g\) is attained by a hyperbolic metric; see \([10]\). Hence the result. \(\square\)

**Remark 3.4.** Our argument applies to surfaces with sparser genera than the ones of Brooks/Buser–Sarnak \(e.g.,\) for polynomial genera in primes in an arithmetic progression. Let \(a\) and \(q\) be two coprime numbers. Suppose that \(\{S_{g^p} \mid p \text{ prime with } p \equiv a (\text{mod } q)\}\) is a family of closed hyperbolic surfaces of genus \(g^p\) with

\[ \text{sys}(S_{g^p}) \geq C_1 \log(g^p) - C_2 \]

for some positive constants \(C_1\) and \(C_2\), where \(g_p = P(p)\) is polynomial in \(p\) as in \((1.2)\). We can apply the same arguments as in the proof of
Theorem 3.1 to construct a family of closed hyperbolic surfaces $S_g$ for every genus $g$ with a logarithmic systolic growth using the prime gap theorem version of [2, Theorem 3] (see the MathSciNet review of the article for a statement without misprints) instead of Theorem 2.1.

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