

Convergence of a semi-discrete scheme for the stochastic Korteweg–de Vries equation.

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Abstract

In this article, we prove the convergence of a semi-discrete scheme applied to the stochastic Korteweg–de Vries equation driven by an additive and localized noise. It is the Crank–Nicholson scheme for the deterministic part and is implicit. This scheme was used in previous numerical experiments on the influence of a noise on soliton propagation [8, 9]. Its main advantage is that it is conservative in the sense that in the absence of noise, the L^2 norm is conserved. The proof of convergence uses a compactness argument in the framework of L^2 weighted spaces and relies mainly on the path-wise uniqueness in such spaces for the continuous equation. The main difficulty relies in obtaining a priori estimates on the discrete solution. Indeed, contrary to the continuous case, Ito formula is not available for the discrete equation.

1 Introduction

The Korteweg–de Vries equation models the propagation of weakly nonlinear dispersive waves in various fields: plasma physics, surfaces waves on the top of an incompressible irrotational inviscid fluid, beam propagation, . . .

When this equation is forced by a random force of white noise type, it models for instance the propagation of waves in a plasma and has been investigated in the physical literature in [5, 15, 21, 22]. More recently, some numerical experiments have been devoted to this equation in order to understand the influence of a noise on the propagation and interaction of solitons in [8, 9, 20]. The stochastic Korteweg–de Vries equation has also been studied mathematically and various well posedness results have been obtained [1, 3, 4, 17]. It has been proved that there exists a unique pathwise solution for a space-time white noise, provided it is spatially localized.

The equation can be written as

$$(1.1) \quad \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = \dot{\xi}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

where $u(x, t)$ is a real-valued stochastic process defined on $\mathbb{R} \times \mathbb{R}^+$ and $\dot{\xi} = \frac{\partial \xi}{\partial t}$ with ξ a real-valued Gaussian process. In the case of a space-time white noise, its correlation

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function is given by

$$\mathbb{E} \xi(t, x) \xi(s, y) = \delta_{x-y}(s \wedge t).$$

In this article, we are interested in the numerical analysis of a semi-discrete scheme for this equation. This scheme is given by

$$(1.2) \quad u_n^{k+1} - u_n^k + \Delta t \left(\partial_x^3 u_n^{k+1/2} + \frac{1}{2} \partial_x \left(u_n^{k+1/2} \right)^2 \right) = \sqrt{\Delta t} \chi^{k+1}.$$

The time step is $\Delta t = T/(n+1) > 0$ and $\sqrt{\Delta t} \chi^{k+1} = \xi((k+1)\Delta t) - \xi(k\Delta t)$ is the noise increment. The unknown is thus approximated at the discrete time $k\Delta t$ by u_n^k . In (1.2), we have used the notation $u_n^{k+1/2} = (u_n^k + u_n^{k+1})/2$.

In [8], [9], boundary conditions were used on a bounded interval since it is impossible to discretize an equation on the real line. Moreover, a spatial discretization based on finite elements was introduced. Here we concentrate our investigation to the time discretization. The scheme (1.2) was chosen because it is conservative for the deterministic part. This is very important in order to make sure that numerical dissipation does not affect the results. Recall that the L^2 norm and energy are conserved quantities for equation (1.1). The L^2 norm is also conserved for the scheme (1.2). The problem is that a conservative scheme is necessarily implicit and the numerical analysis of implicit schemes for stochastic partial differential equations is often very difficult. Moreover, we have to deal with the mathematically complex structure of the Korteweg–de Vries equation.

Let us recall that existence and uniqueness results for the initial value problem associated to this equation are very difficult. The main reason is that the linear part of Korteweg–de Vries equation defines a unitary group — the Airy one parameter group $\{e^{it\partial_x^3}\}_{t \in \mathbb{R}}$ — in $H^s(\mathbb{R})$, $s \in \mathbb{R}$. In [3], the method developed by Bourgain and further improved by Kenig, Ponce and Vega has been generalized to the stochastic case. It allows to work in negative Sobolev spaces and thus is able to handle spatially rough noises. However, it seems very difficult to adapt this method to treat our scheme and for the moment we are not able to analyze its convergence in the case of a spatially uncorrelated noise. Here, we use another framework introduced in [12] and used in the stochastic case in [17]. It gives existence and uniqueness in $L^2(\mathbb{R})$ -weighted spaces thanks to a local smoothing effect of the Airy group. In [17], the Gaussian process ξ is a Wiener process on $L^2(\mathbb{R})$, with a covariance operator $\Phi\Phi^*$ of finite trace in a localized $L^2(\mathbb{R})$ -based space, i.e. in L_w^2 where

$$L_w^2 = \left\{ (1 + x_+)^{3/8} u \in L^2(\mathbb{R}) \right\}.$$

In fact, a little more smoothness is required if the solution is constructed by a fixed point argument. In this setting, the numerical noise in (1.2) is given by

$$\chi^{k+1} = \frac{\Phi W((k+1)\Delta t) - \Phi W(k\Delta t)}{\sqrt{\Delta t}}, \quad 0 \leq k \leq n-1,$$

where W is a cylindrical Wiener process on $L^2(\mathbb{R})$.

Two type of results are expected in the numerical analysis of a partial differential equation: convergence of the sequence of approximation under weak regularity assumptions on the data and order of convergence with more regularity. Here, we study the first aspect which in our opinion is more important for the stochastic Korteweg–de Vries equation which has been simulated with very irregular data in [8, 9, 20]. In these articles, the noise was

white in both time and space. However, the framework chosen here does not allow to treat such a rough noise. Future work will deal with the second aspect.

The first difficulty we have to deal with is the problem of the existence of an adapted solution u^{k+1} for a given u^k and χ^{k+1} . In the deterministic case (with a vanishing right hand side), the existence (and uniqueness) of the solution at each time step requires some smallness condition on the time step depending for example on the initial data. In the stochastic case, such a condition on the time step would be random and too restrictive. A remedy for this problem has been given in [18, 19] where it is proposed to truncate the noise when an implicit scheme is used. However, in our numerical experiments, we never encountered any problem and always were able to solve (1.2). Thus, we do not introduce this truncation. In fact, there always exists at least one solution of the semi-discrete equation which is adapted. However, we do not know whether it is unique.

Our proof of convergence relies on a compactness method and a lemma due to Gyöngy and Krylov [14]. This lemma is very useful and allows to get convergence in probability in the original probability space provided tightness of laws of the approximating sequence and uniqueness of solution of the continuous equation can be shown. This latter point follows from [12] as shown in [17]. Tightness results from a priori estimates in adequate functional spaces. In fact, using the local smoothing effect mentioned above and standard arguments, tightness is proved as soon as we get an appropriate mean square bound, uniformly with respect to the time step, in the weighted L^2 spaces. The main problem is to get a bound on the moments of the $L^2(\mathbb{R})$ norm. In the continuous case, an estimate of the $L^2(\mathbb{R})$ norm is easy to obtain and results from Ito formula but this tool is not available in the discrete case.

This difficulty always appears when working with a numerical scheme applied to stochastic partial differential equation with non Lipschitz non linear term. In [13], this is overcome thanks to a truncation of this non linear term. The same idea has been used in [2] for the stochastic non linear Schrödinger equation. We think that this argument is not completely satisfactory. Indeed, it is important for the numerical experiments to know that the numerical solutions are bounded uniformly with respect to the time step and to get explicit bounds in terms of the data. This property is a form of stability and shows that the scheme has a good behavior. When the noise is not additive as in [13] or [2], it seems that this property is very difficult to prove. We use here a trick which is specific to the additive noise and consists in subtracting an appropriate form of the noise term *at each time step* and in working with a shifted unknown. More precisely, for some real $\alpha > 0$, we consider a sequence $\{z_\alpha^k\}_k$ defined for any k by

$$z_\alpha^{k+1} + \alpha \frac{\Delta t}{2} z_\alpha^{k+1} - \frac{\Delta t}{2} \partial_x^3 z_\alpha^{k+1} = \sqrt{\Delta t} \chi^{k+1},$$

we have removed the dependance in n for the sake of simplicity. Let us note that it is not the linear part of the semi-discrete solution. Writing the $L^2(\mathbb{R})$ -norm of u^{k+1} in function of the $L^2(\mathbb{R})$ -norm of $u^k + z_\alpha^{k+1}$, we derive the appropriate uniform bound in $L^2(\mathbb{R})$. The key point is to choose a random α and to prove precise estimate on $\{z_\alpha^k\}_k$. The natural idea would be to use this auxiliary process with $\alpha = 0$, however it is easy to see that the proof of L^2 a priori estimates fails with this choice. The problem is technical and we need that (z_α^k) is small path-wise, this is the reason why we have to choose a random α . In fact, this argument can be considered as a discrete version of an idea used in [6], [7] in the context of the Burgers equation. Note that the independance of u^k and z_α^{k+1} will also plays a crucial role.

Another argument used in order to surround the problem of lack of Ito formula consists in plugging in the noise term of the L^2 -norm evolution equation at the step k the integrated equation from 0 to step k in order to express u at time $(k+1)\Delta t$ in function of u at time $k\Delta t$ and a martingale term only. This provides additional and useful powers of Δt and allows us to get the wanted estimates thanks to martingale inequalities. Let us point out that this method is not specific to the additive nature of the equation.

The paper is organized as follows. In Section 2 we recall after some notations the deterministic framework of uniqueness and the theorem of existence and uniqueness of the solution of the initial value problem obtained in [17]. Then we describe our numerical scheme and state our main result, i.e. the convergence of the sequence of approximation $\{u_n\}_{n \geq 0}$ in various topologies. In Section 3, we give the details of the proof of convergence of the scheme.

2 Main result and notations.

Let X be any Banach space and I any interval in \mathbb{R} . We shall denote by $L^p(I, X)$, $1 \leq p \leq \infty$, the space of functions which are Böchner-integrable from I to X . When $X = \mathbb{R}$, we simply use the notation $L^p(I)$. We shall also denote by $\mathcal{C}([0, T], X)$ (resp. $\mathcal{C}^\beta([0, T], X)$) the space of continuous (resp. Hölder continuous with exponent β) functions from $[0, T]$ to X . We shall denote by $\|\cdot\|_Y$ the norm on a Banach space Y . In the case of $Y = L^2(\mathbb{R})$, we will denote $\|\cdot\|$ (resp. (\cdot, \cdot)) the L^2 -norm (resp. the L^2 -inner product).

Given any non negative number σ , the Sobolev space $H^\sigma(\mathbb{R})$ is defined as the space of tempered distributions u such that

$$\int_{\mathbb{R}} (1 + \xi^2)^\sigma |\mathcal{F}u(\xi)|^2 d\xi < \infty$$

where \mathcal{F} is the Fourier transform. We shall also use the local Sobolev spaces $H_{loc}^\sigma(\mathbb{R})$ consisting of the tempered distributions which belong to $H^\sigma(\mathbb{R})$ when multiplied by a smooth and compactly supported function. For σ negative, $H^\sigma(\mathbb{R})$ (resp. $H_{loc}^\sigma(\mathbb{R})$) is the topological dual space of $H^{-\sigma}(\mathbb{R})$ (resp. $H_{loc}^{-\sigma}(\mathbb{R})$).

The Sobolev space in the time variable $W^{\alpha,p}([0, T], X)$ with $\alpha > 0$ and $1 \leq p \leq \infty$ is defined as the space of functions u such that

$$\iint_{[0, T]^2} \frac{|u(t) - u(s)|_X^p}{|t - s|^{1+\alpha p}} dt ds < +\infty.$$

Let Φ be a linear operator from $L^2(\mathbb{R})$ into a Hilbert space H , Φ is said to be *Hilbert-Schmidt* if the following term is finite:

$$|\Phi|_{\mathcal{L}_2^0(L^2(\mathbb{R}), H)}^2 \stackrel{\text{def}}{=} \sum_{i \geq 0} |\Phi e_i|_H^2 < +\infty.$$

It is classical that the sum on the right hand side does not depend on the choice of the Hilbertian basis $\{e_i\}_{i \geq 0}$ of H . We denote by $\mathcal{L}_2^0(L^2(\mathbb{R}), H)$ the space of Hilbert-Schmidt operators from $L^2(\mathbb{R})$ to H . When $H = H^\sigma(\mathbb{R})$ for $\sigma > 0$, we shall write $\mathcal{L}_2^0(L^2(\mathbb{R}), H^\sigma(\mathbb{R})) = \mathcal{L}_2^{0,\sigma}$ and for $\sigma = 0$, we will simply use the notation \mathcal{L}_2 .

Let $T > 0$ and let us now consider the stochastic Korteweg–de Vries equation with an additive noise written in the following Ito form

$$(2.3) \quad du + \left(\partial_x^3 u + \frac{1}{2} \partial_x (u^2) \right) dt = \Phi dW,$$

for $x \in \mathbb{R}$, $t \in [0, T]$ (where we have used the notation ∂_x^k for the k^{th} partial derivative with respect to x) with the initial condition

$$(2.4) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$

Here $\{W(t)\}_{t \in [0, T]}$ denotes a cylindrical Wiener process on $L^2(\mathbb{R})$ adapted to a given filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We work in weighted L^2 spaces and assume that the initial data satisfies

$$(2.5) \quad (1 + x_+)^{3/8} u_0 \in L^2(\mathbb{R}).$$

Correspondingly, we assume that Φ is a linear operator from $L^2(\mathbb{R})$ into itself and that moreover

$$(2.6) \quad (1 + x_+)^{3/8} \Phi \in \mathcal{L}_2.$$

We also need the weak regularity assumption on the noise

$$(2.7) \quad \Phi \in \mathcal{L}_2^{0, \varepsilon}$$

for some $\varepsilon > 0$.

Let us recall the result of existence and uniqueness of the stochastic process u solution of (2.3)–(2.4) in the framework of L^2 weighted spaces (see [17]).

Theorem 2.1 *Let $T > 0$ and u_0 such that (2.5) holds. Let $\{W(t)\}_{t \in [0, T]}$ be a cylindrical Wiener process adapted to a given filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under the assumptions (2.6) and (2.7), there exists a unique stochastic process u which is a global mild solution of (2.3)–(2.4) such that*

$$(1 + x_+)^{3/8} u \in L^\infty([0, T], L^2(\mathbb{R})), \quad a.s..$$

Moreover, we can show that $u \in \mathcal{C}(0, T, L_w^2) \cap L^2([0, T], H_{loc}^1(\mathbb{R}))$, a.s..

Let us now describe our numerical scheme. From now on, we assume that Φ and u_0 satisfies the assumptions of Theorem 2.1. In the sequel, following the notations of Theorem 2.1, we consider a fixed positive T and we set for each integer n , $\Delta t = \frac{T}{n+1}$. We also set for any integer k such that $0 \leq k \leq n$:

$$\chi^{k+1} = \Phi \frac{W((k+1)\Delta t) - W(k\Delta t)}{\sqrt{\Delta t}}.$$

We seek for an approximation u^k of $u(k\Delta t)$ for k such that $0 \leq k \leq n$. Our semi-discrete scheme is described by the following finite difference equation:

$$(2.8) \quad \frac{u^{k+1} - u^k}{\Delta t} + \partial_x^3 u^{k+1/2} + \frac{1}{2} \partial_x (u^{k+1/2})^2 = \frac{\chi^{k+1}}{\sqrt{\Delta t}},$$

where we have set

$$u^{k+1/2} = \frac{1}{2}(u^k + u^{k+1}), \quad 0 \leq k \leq n.$$

The deterministic part is second order and has the property that the L^2 norm is conserved. We need the following notion.

Definition 2.2 (pathwise solution) *We say that a function u_n in $L^\infty([0, T], L^2(\mathbb{R}))$ is a solution of the difference equation (2.8) where $\Delta t = T/(n+1)$, if u_n is constant on each interval $[k\Delta t, (k+1)\Delta t]$, equal to u_n^k , and if the sequence $\{u_n^k\}_{0 \leq k \leq n}$ satisfies (2.8).*

The scheme is implicit and, at each time step, a fixed point method or a Newton iteration is used to find u^{k+1} . In the numerical experiments in [8], [9], we used the Newton iteration and we never encountered any problem with its convergence. Note that this convergence is not implied by any theoretical arguments. We do not even know if the solution is uniquely defined. However, arguing as in [2] section 3.1, it is possible to show the existence of a pathwise solution of (2.8) according to the latter definition using Galerkin approximations and a measurable selection theorem. We now state the main result of this paper.

Theorem 2.3 *Let u_0 and Φ as in Theorem 2.1. Let n_0 such that $T/(n_0+1) \leq 1$, then any sequence $\{u_n\}_{n \geq n_0}$ of solutions of the numerical scheme (2.8) converges to the solution u of (2.3)–(2.4) given by Theorem 2.1. The previous convergence holds in probability in $L^2(0, T, H_{loc}^s(\mathbb{R}))$ for any $s < 1$ and in $L^2(\Omega; L^\infty([0, T], L^2(\mathbb{R})))$.*

The proof of Theorem 2.3 is the object of next section. We will use a compactness method together with the following lemma taken from [14]. This lemma allows to get the convergence of the approximation scheme in probability in any space in which these approximations are tight. As it can be seen the uniqueness of the solution of the continuous equation is crucial.

Lemma 2.4 *Let $\{Z_n\}_{n \geq 0}$ be a sequence of random elements on a Polish space E endowed by its borel σ -algebra. Then $\{Z_n\}_{n \geq 0}$ converges in probability to an E -valued random element if and only if from every pair of subsequences $\{(Z_{n_k}, Z_{m_k})\}_{k \geq 0}$, one can extract a subsequence which converges weakly to a random element supported on the diagonal $\{(x, y) \in E \times E, x = y\}$.*

The proof of convergence of the numerical scheme also relies on the following well-known compactness argument whose proof is based on a classical compact embedding theorem (see [16], Theorem 5.2 p. 61), the Ascoli-Arzelà theorem, and on diagonal extraction.

Lemma 2.5 *Let $T > 0$, $\alpha > 0$, $\beta > 0$. Let \mathcal{A} be a set of distributions u such that*

(i) \mathcal{A} is bounded in $L^2([0, T], H_{loc}^1(\mathbb{R})) \cap W^{\alpha, 2}([0, T], H_{loc}^{-2}(\mathbb{R}))$;

(ii) \mathcal{A} is bounded in $C^\beta([0, T], H_{loc}^{-2}(\mathbb{R}))$.

Then \mathcal{A} is relatively compact in $L^2([0, T], H_{loc}^s(\mathbb{R})) \cap C([0, T], H_{loc}^{-s'}(\mathbb{R}))$, for any $s < 1$ and $s' > 2$.

Finally, we introduce a weight function used in the proof of Theorem 2.1. The idea of working with weighted spaces is motivated by results of uniqueness of the Cauchy problem in the deterministic case (see [12]). We consider a weight function h satisfying the following properties.

Hypothesis 1

$$(2.9) \quad h' \leq c_1 \max(1, h),$$

$$(2.10) \quad h''^2 \leq c_2 h h',$$

$$(2.11) \quad h''' \leq c_3 h.$$

where the c_i 's denote positive constants.

It is easy to construct such function (see [12]).

3 Convergence of the Crank–Nicholson scheme

3.1 $L^2(\mathbb{R})$ -a priori estimates

Let n be a positive integer and $\Delta t = T/(n+1)$. Let $u_n = \{u^k\}_{0 \leq k \leq n}$ be an adapted solution of the following scheme

$$(3.12) \quad u^{k+1} = u^k - \Delta t \partial_x^3 u^{k+\frac{1}{2}} - \frac{\Delta t}{2} \partial_x \left(u^{k+\frac{1}{2}} \right)^2 + \sqrt{\Delta t} \chi^{k+1},$$

$$(3.13) \quad u^0 = u_0.$$

where we have set $\chi^{k+1} = \frac{1}{\sqrt{\Delta t}} (\Phi W(t_{k+1}) - \Phi W(t_k))$. Note that χ^{k+1} is a gaussian random variable with covariance operator $\Phi \Phi^*$. Therefore, by (2.7), it takes values in $H^\varepsilon(\mathbb{R})$.

Let $\alpha > 0$. For each integer k , we define the auxiliary random variable z_α^{k+1} by

$$(3.14) \quad z_\alpha^{k+1} + \alpha \frac{\Delta t}{2} z_\alpha^{k+1} - \frac{\Delta t}{2} \partial_x^3 z_\alpha^{k+1} = \sqrt{\Delta t} \chi^{k+1}.$$

We need in the sequel the following estimates on the random variables z_α^{k+1} .

Lemma 3.1 *For any $\varepsilon \in]0, 1[$, $\delta > 0$ there exists positive constants $\kappa_i > 0$, $i = 1, 2, 3, 4$ independent on n such that for any $0 \leq k \leq n$, the following inequalities hold:*

$$(3.15) \quad \|z_\alpha^{k+1}\|^2 \leq \kappa_1 \Delta t \|\chi^{k+1}\|^2, \quad a.s.,$$

$$(3.16) \quad \|\partial_x z_\alpha^{k+1}\|^2 \leq \kappa_2 (\Delta t)^{1/3} \|\chi^{k+1}\|^2, \quad a.s.,$$

$$(3.17) \quad \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})}^2 \leq \kappa_3 \alpha^{-\varepsilon/3} \|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^2, \quad a.s.,$$

$$(3.18) \quad \|z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \leq \kappa_4 (\Delta t)^{2/3-\delta} \|\chi^{k+1}\|, \quad a.s..$$

Proof.

Fourier transform applied to (3.14) leads to

$$(3.19) \quad \mathcal{F}(z_\alpha^{k+1})(\xi) = \frac{\sqrt{\Delta t}}{1 + \frac{\alpha \Delta t}{2} - i \frac{\xi^3 \Delta t}{2}} \mathcal{F}(\chi^{k+1}).$$

so that (3.15) follows easily from Plancherel equality. The proof of (3.16) uses similar arguments.

In order to obtain (3.17), we use the following well-known Sobolev estimate. For any $\varepsilon > 0$, there exists a positive constant c_ε such that

$$\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \leq c_\varepsilon \|z_\alpha^{k+1}\|_{H^{(3+\varepsilon)/2}(\mathbb{R})} = c_\varepsilon (\|z_\alpha^{k+1}\|^2 + \|\partial_x^3 z_\alpha^{k+1}\|_{H^{\varepsilon/2}(\mathbb{R})}^2)^{1/2}.$$

The first term in the right hand side is bounded again thanks to Plancherel equality and (3.19):

$$\begin{aligned} \|z_\alpha^{k+1}\|^2 &= \Delta t \int \left[\left(1 + \frac{\alpha \Delta t}{2}\right)^2 + \frac{\Delta t^2 \xi^6}{4} \right]^{-1} |\mathcal{F}(\chi^{k+1})(\xi)|^2 d\xi \\ &\leq \alpha^{-1} \int |\mathcal{F}(\chi^{k+1})(\xi)|^2 d\xi \\ &= \alpha^{-1} \|\chi^{k+1}\|^2, \end{aligned}$$

since $\alpha \Delta t \left[\left(1 + \frac{\alpha \Delta t}{2}\right)^2 + \frac{\Delta t^2 \xi^6}{4} \right]^{-1} \leq 1$. To bound the second term, we first note that

$$\begin{aligned} \Delta t |\xi|^{3+\varepsilon} \left[\left(1 + \frac{\alpha \Delta t}{2}\right)^2 + \frac{\Delta t^2 \xi^6}{4} \right]^{-1} &\leq (\Delta t)^{\varepsilon/3} |\xi|^{2\varepsilon} \sup_{x \geq 0} x^{1-\varepsilon/3} \left(\left(1 + \frac{\alpha \Delta t}{2}\right)^2 + x^2 \right)^{-1} \\ &\leq c(\varepsilon) (\Delta t)^{\varepsilon/3} |\xi|^{2\varepsilon} \left(1 + \frac{\alpha \Delta t}{2}\right)^{-(1+\varepsilon/3)} \\ &\leq c(\varepsilon) \alpha^{-\varepsilon/3} |\xi|^{2\varepsilon}. \end{aligned}$$

Then, again by Plancherel equality and (3.19),

$$\begin{aligned} \|\partial_x^3 z_\alpha^{k+1}\|_{H^{\varepsilon/2}(\mathbb{R})}^2 &= \Delta t \int |\xi|^{3+\varepsilon} \left[\left(1 + \frac{\alpha \Delta t}{2}\right)^2 + \frac{\Delta t^2 \xi^6}{4} \right]^{-1} |\mathcal{F}(\chi^{k+1})(\xi)|^2 d\xi \\ &\leq c(\varepsilon) \alpha^{-\varepsilon/3} \int |\xi|^{2\varepsilon} |\mathcal{F}(\chi^{k+1})(\xi)|^2 d\xi \\ &\leq c(\varepsilon) \alpha^{-\varepsilon/3} \|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}. \end{aligned}$$

Inequality (3.17) follows. The proof of (3.18) uses similar arguments. ■

We are now able to prove the following L^2 -estimates.

Proposition 3.2 *Let n_0 an integer such that $T/(n_0 + 1) \leq 1$. Then there exists positive constants $\kappa_i > 0$, $i = 5, 6, 7$, independent on n and which is a non decreasing function of its arguments such that*

$$(3.20) \quad \max_{0 \leq k \leq n} \mathbb{E} \|u_n^k\|^2 \leq \kappa_5 \left(T, \mathbb{E} \|u_0\|^2, \|\Phi\|_{\mathcal{L}_2^{0,\varepsilon}} \right),$$

$$(3.21) \quad \max_{0 \leq k \leq n} \mathbb{E} \|u_n^k\|^4 \leq \kappa_6 \left(T, \mathbb{E} \|u_0\|^4, \|\Phi\|_{\mathcal{L}_2^{0,\varepsilon}} \right),$$

$$(3.22) \quad \mathbb{E} \max_{0 \leq k \leq n} \|u_n^k\|^2 \leq \kappa_7 \left(T, \mathbb{E} \|u_0\|^4, \|\Phi\|_{\mathcal{L}_2^{0,\varepsilon}} \right),$$

for any $n \geq n_0$.

Proof of (3.20)

Taking the difference of (3.14) with (3.12) gives

$$(3.23) \quad u^{k+1} - v = -\Delta t \partial_x^3 \left(\frac{v + u^{k+1}}{2} \right) - \frac{\Delta t}{2} \partial_x \left(\frac{v + u^{k+1}}{2} - \frac{z_\alpha^{k+1}}{2} \right)^2 + \alpha \frac{\Delta t}{2} z_\alpha^{k+1},$$

where we have set

$$(3.24) \quad v = u^k + z_\alpha^{k+1}.$$

Multiplying (3.23) by $(v + u^{k+1})$, we obtain after integration on \mathbb{R} and suitable integrations by parts:

$$(3.25) \quad \begin{aligned} \|u^{k+1}\|^2 &\leq \|v\|^2 + \frac{\Delta t}{8} \|v + u^{k+1}\|^2 \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \\ &\quad + \frac{\Delta t}{4} \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \|z_\alpha^{k+1}\| \|v + u^{k+1}\|. \\ &\quad + \alpha \frac{\Delta t}{2} \|z_\alpha^{k+1}\| \|v + u^{k+1}\|. \end{aligned}$$

The last two terms of inequality (3.25) can be majorized as follows

$$\frac{\Delta t}{4} \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \|z_\alpha^{k+1}\| \|v + u^{k+1}\| \leq \frac{\eta}{2} \Delta t \left\| \frac{v + u^{k+1}}{2} \right\|^2 + \Delta t C_1(\eta) \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})}^2 \|z_\alpha^{k+1}\|^2,$$

and

$$\alpha \frac{\Delta t}{2} \|z_\alpha^{k+1}\| \|v + u^{k+1}\| \leq \frac{\eta}{2} \Delta t \left\| \frac{v + u^{k+1}}{2} \right\|^2 + \Delta t C_2(\eta) \alpha^2 \|z_\alpha^{k+1}\|^2$$

where we have used the well-known inequality $|ab| \leq \eta a^2 + C_\eta b^2$ for any positive real a, b and any $\eta > 0$. Plugging these inequalities into (3.25) leads to

$$(3.26) \quad \begin{aligned} \left(1 - \frac{\Delta t}{4} \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} - \frac{\eta \Delta t}{2} \right) \|u^{k+1}\|^2 &\leq \left(1 + \frac{\Delta t}{4} \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} + \frac{\eta \Delta t}{2} \right) \|v\|^2 \\ &\quad + \Delta t \left(C_1(\eta) \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})}^2 + C_2(\eta) \alpha^2 \right) \|z_\alpha^{k+1}\|^2. \end{aligned}$$

Owing to Lemma 3.1 (estimate (3.17)) and given $\eta > 0$, we can find $\alpha = \alpha(k, \omega) = \kappa_3^{3/\varepsilon} \|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^{6/\varepsilon} / (2\eta)^{6/\varepsilon}$ such that

$$\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \leq 2\eta, \quad a.s..$$

We deduce from (3.15) that

$$(3.27) \quad \|u^{k+1}\|^2 \leq \|v\|^2 \left(\frac{1 + \eta \Delta t}{1 - \eta \Delta t} \right) + (\Delta t)^2 C(\omega, k), \quad a.s.,$$

with

$$C(\omega, k) = \kappa_1 \|\chi^{k+1}\|^2 (C_1(\eta) 4\eta^2 + \alpha(\omega, k)^2 C_2(\eta)).$$

This latter term can be rewritten in the form

$$(3.28) \quad C(\omega, k) = C_3 \|\chi^{k+1}\|^2 + C_4 \|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^{12/\varepsilon+2},$$

for some constants $C_3 > 0$ and $C_4 > 0$ depending only on η and ε . Let us note that since $\Delta t = T/(n+1) < 1$, it is easy to see that

$$\frac{1 + \eta \Delta t}{1 - \eta \Delta t} \leq e^{\frac{2\eta}{1-\eta} \Delta t}.$$

Finally, since u^k and z_α^{k+1} are independent (see (3.14)), taking the expectation of (3.27) with $\eta = 1/2$ yields

$$\begin{aligned}\mathbb{E}\|u^{k+1}\|^2 &\leq \mathbb{E}\|u^k\|^2 e^{4\Delta t} + \mathbb{E}\|z_\alpha^{k+1}\|^2 e^{4\Delta t} + \Delta t^2 \left(C_3 \mathbb{E}\|\chi^{k+1}\|^2 + C_4 \mathbb{E}\|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^{12/\varepsilon+2} \right) \\ &\leq \mathbb{E}\|u^k\|^2 e^{4\Delta t} + \kappa_1 \Delta t e^{4\Delta t} \mathbb{E}\|\chi^{k+1}\|^2 + \Delta t^2 \left(C_3 \mathbb{E}\|\chi^{k+1}\|^2 + C_4 \mathbb{E}\|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^{12/\varepsilon+2} \right) \\ &\leq \mathbb{E}\|u^k\|^2 e^{4\Delta t} + C_5 (\|\Phi\|_{\mathcal{L}_2^{0,\varepsilon}}^{12/\varepsilon+2} + 1),\end{aligned}$$

where we have used again (3.15). Eventually, the discrete Gronwall inequality leads to (3.20).

Proof of (3.21)

We take the square of inequality (3.27) and obtain

$$\begin{aligned}\|u^{k+1}\|^4 &\leq e^{c\Delta t} \|v\|^4 + 2(\Delta t)^2 C(\omega, k) \|v\|^2 e^{c\Delta t} + (\Delta t)^4 C(\omega, k)^2, \\ (3.29) \quad &\leq e^{c\Delta t} \|v\|^4 + 4(\Delta t)^2 C(\omega, k) \left(\|u^k\|^2 + \|z_\alpha^{k+1}\|^2 \right) e^{c\Delta t} + (\Delta t)^4 C(\omega, k)^2, \quad a.s.,\end{aligned}$$

where $C(\omega, k) > 0$ is given by (3.28). Due to (3.24), the term $\|v\|^4$ can be split into several parts:

$$\begin{aligned}\|v\|^4 &= \|u^k\|^4 + \|z_\alpha^{k+1}\|^4 + 4|(u^k, z_\alpha^{k+1})|^2 + 2\|u^k\|^2 \|z_\alpha^{k+1}\|^2 \\ &\quad + 4\|z_\alpha^{k+1}\|^2 (u^k, z_\alpha^{k+1}) + 4\|u^k\|^2 (u^k, z_\alpha^{k+1}).\end{aligned}$$

The expectation of the last term in (3.30) vanishes since again u^k and z_α^{k+1} are independent. We deduce

$$\begin{aligned}(3.30) \quad \mathbb{E}\|v\|^4 &\leq \mathbb{E}\|u^k\|^4 + \mathbb{E}\|z_\alpha^{k+1}\|^4 + 6\mathbb{E}\left(\|u^k\|^2 \|z_\alpha^{k+1}\|^2\right) + 4\mathbb{E}\left(\|u^k\| \|z_\alpha^{k+1}\|^3\right) \\ &\leq \mathbb{E}\|u^k\|^4 + C\Delta t \left\{ \|\Phi\|_{\mathcal{L}_2}^4 + \|\Phi\|_{\mathcal{L}_2}^2 \mathbb{E}\|u^k\|^2 + \|\Phi\|_{\mathcal{L}_2}^3 \mathbb{E}\|u^k\| \right\}\end{aligned}$$

by (3.15). It follows

$$\begin{aligned}\mathbb{E}\|u^{k+1}\|^4 &\leq e^{c\Delta t} \mathbb{E}\|u^k\|^4 + C\Delta t \left\{ \|\Phi\|_{\mathcal{L}_2}^4 + \|\Phi\|_{\mathcal{L}_2}^2 \mathbb{E}\|u^k\|^2 + \|\Phi\|_{\mathcal{L}_2}^3 \mathbb{E}\|u^k\| \right\} \\ &\quad + C\Delta t \left\{ \mathbb{E}\|u^k\|^2 \mathbb{E}C(\omega, k) + \mathbb{E}\left(\|z_\alpha^{k+1}\|^2 C(\omega, k)\right) + \mathbb{E}C(\omega, k)^2 \right\}.\end{aligned}$$

Eventually, the estimates (3.15),(3.17),(3.20), the Gaussianity of χ^{k+1} and the discrete Gronwall Lemma give (3.21).

Proof of (3.22)

We rewrite the equation

$$u^{k+1} = u^k + \Delta t \partial_x^3 u^{k+1/2} + \frac{\Delta t}{2} \partial_x \left(u^{k+1/2} \right)^2 + \sqrt{\Delta t} \chi^{k+1}$$

and take the scalar product in $L^2(\mathbb{R})$ with $u^{k+1/2}$ to get

$$(3.31) \quad \left(u^{k+1/2}, u^{k+1} - u^k \right) = \sqrt{\Delta t} \left(\chi^{k+1}, u^{k+1/2} \right).$$

Let us note that the stochastic term in the right hand side is not a martingale increment since the integrand $u^{k+1/2}$ is not adapted. In order to get rid of this problem, we use the discrete ‘‘mild’’ equation:

$$(3.32) \quad u^{k+1} = S_n u^k - \frac{\Delta t}{2} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x \left(u^{k+1/2} \right)^2 + \sqrt{\Delta t} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \chi^{k+1}.$$

where

$$S_n = \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \left(I - \frac{\Delta t}{2} \partial_x^3 \right).$$

Merging (3.32) and (3.31) leads to

$$\begin{aligned} \|u^{k+1}\|^2 - \|u^k\|^2 &= \sqrt{\Delta t} \left(\chi^{k+1}, u^k + S_n u^k \right) + \sqrt{\Delta t} \left(\chi^{k+1}, u^{k+1} - S_n u^k \right) \\ &= \sqrt{\Delta t} \left(\chi^{k+1}, u^k + S_n u^k \right) + \Delta t \left(\left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \chi^{k+1}, \chi^{k+1} \right) \\ &\quad - \frac{\Delta t^{3/2}}{2} \left(\left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x \left(u^{k+1/2} \right)^2, \chi^{k+1} \right). \end{aligned}$$

Now, the $\sqrt{\Delta t}$ - term is a martingale increment involving an adapted term. Next, we sum from $\ell = 0$ to $\ell = k - 1$ with $k \leq n$ and obtain

$$\begin{aligned} \|u^k\|^2 - \|u^0\|^2 &= M(t_k) + \Delta t \sum_{\ell=0}^{k-1} \left(\left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \chi^{\ell+1}, \chi^{\ell+1} \right) \\ &\quad - \frac{\Delta t}{2} \sum_{\ell=0}^{k-1} \left(\sqrt{\Delta t} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x \left(u^{\ell+1/2} \right)^2, \chi^{\ell+1} \right), \end{aligned}$$

where

$$M(t) = \int_0^t (F_n(s), \Phi \, dW(s)),$$

and

$$F_n(s) = u_n(s) + S_n u_n(s).$$

Since

$$(3.33) \quad \left\| \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 1, \quad \left\| \sqrt{\Delta t} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x \right\|_{\mathcal{L}(L^1(\mathbb{R}), L^2(\mathbb{R}))} \leq C,$$

with $C > 0$ independent from n , we derive

$$\max_{0 \leq k \leq n} \|u^k\|^2 \leq \|u^0\|^2 + \sup_{t \in [0, T]} |M(t)| + \Delta t \sum_{\ell=0}^{n-1} \|\chi^{\ell+1}\|^2 + C \frac{\Delta t}{2} \sum_{\ell=0}^{n-1} \|u^{\ell+1/2}\|^2 \|\chi^{\ell+1}\|, \quad \text{a.s.}$$

The second inequality in (3.33) is obtained thanks to Plancherel inequality and to the fact that the Fourier transform of a function in $L^1(\mathbb{R})$ is bounded.

Taking the expectation of the previous inequality leads to

$$\mathbb{E} \max_{0 \leq k \leq n} \|u^k\|^2 \leq \mathbb{E} \|u_0\|^2 + \mathbb{E} \sup_{t \in [0, T]} |M(t)| + T \|\Phi\|_{\mathcal{L}_2}^2 + CT/2 \left(\max_{0 \leq k \leq n} \mathbb{E} \|u^k\|^4 \right)^{1/2} \|\Phi\|_{\mathcal{L}_2}.$$

We conclude thanks to (3.21) and a classical martingale inequality. \blacksquare

3.2 L^2 -weighted *a priori* estimates

We will first establish the following pathwise estimate concerning $\{z_\alpha^{k+1}\}_k$ given by (3.14).

Proposition 3.3 *Let $h(x) = (1 + x_+)^{3/4}$ and n_0 such that $\Delta t \leq 1$. Then there exists a determinist constant $\kappa_8 > 0$ independent of n such that the following inequality holds almost surely:*

$$(3.34) \quad \|h^{1/2}z_\alpha^{k+1}\|^2 \leq \kappa_8 \Delta t \|h^{1/2}\chi^{k+1}\|^2, \quad a.s..$$

Proof.

Eq. (3.14) leads to $z_\alpha^{k+1} = \sqrt{\Delta t}Q\chi^{k+1}$ where $Q = ((1 + \alpha\Delta t)I - \Delta t/2\partial_x^2)^{-1}$. It is easy to see that Q is a bounded convolution operator from $L^2(\mathbb{R})$ into itself which can be expressed by

$$Qf(x) = \int_{\mathbb{R}} \rho(x-y)f(y) dy,$$

where $\rho = \mathcal{F}^{-1}(r)$ with $r(\xi) = (1 + \alpha\Delta t/2 + i\Delta t\xi^3/2)^{-1}$. The main difficulty here is that h and Q do not commute. We surround this problem with the following pointwise inequality:

$$|h^{1/2}(x)\rho(x-y)h^{-1/2}(y)| \leq |\rho(x-y)|(1 + (x-y)_+/2), \quad \text{for all } x, y,$$

whose proof based on the Mean Value Theorem essentially exploits the fact that $h' \leq 1$ and $h \geq 1$. We have then

$$(3.35) \quad \begin{aligned} \int h(x)(z_\alpha^{k+1})^2 dx &= \Delta t \int \left(\int h^{1/2}(x)\rho(x-y)\chi^{k+1}(y) dy \right)^2 dx, \\ &\leq \Delta t \int \left(\int |\rho(x-y)|(1 + (x-y)_+/2)h^{1/2}(y)\chi^{k+1}(y) dy \right)^2 dx, \end{aligned}$$

where we have used the previous inequality. Then a Cauchy-Schwartz inequality with respect to y applied to the right hand side of (3.35) followed by the Fubini theorem yields

$$(3.36) \quad \|h^{1/2}z_\alpha^{k+1}\|^2 \leq \Delta t (\|\rho\|_{L^1(\mathbb{R})} + \|x\rho\|_{L^1(\mathbb{R})}/2)^2 \|h^{1/2}\chi^{k+1}\|^2.$$

Eventually, the explicit computation of ρ based on the partial fraction decomposition of $r(\xi)$, and whose details are left to the reader, allows us to bound in (3.36) both $\|\rho\|_{L^1(\mathbb{R})}$ and $\|x\rho\|_{L^1(\mathbb{R})}$ independently of n and ω . \blacksquare

We now exploit the smoothing property of the linear part in weighted spaces.

Proposition 3.4 *Let n_0 be an integer such that $T/(n_0 + 1) \leq 1/4$. For any $n \geq n_0$, let $u_n \in L^\infty(0, T, L^2(\mathbb{R}))$ be a solution of the finite difference equation (3.12)–(3.13). Let $h : \mathbb{R} \rightarrow \mathbb{R}_+$ be an increasing regular function such that (2.9)–(2.11) holds. Then, there exists positives constants κ_i , $i = 9, 10$, non-decreasing functions of theirs arguments and independent of n such that the following inequalities hold:*

$$(3.37) \quad \max_{0 \leq k \leq n} \mathbb{E} \|h^{1/2}u_n^k\|^2 \leq \kappa_9 \left(T, \mathbb{E} \|h^{1/2}u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2}\Phi\|_{L_2^0} \right),$$

$$(3.38) \quad \Delta t \mathbb{E} \sum_{k=0}^{n-1} \|h^{1/2}\partial_x u_n^{k+1/2}\|^2 \leq \kappa_9 \left(T, \mathbb{E} \|h^{1/2}u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2}\Phi\|_{L_2^0} \right),$$

$$(3.39) \quad \mathbb{E} \max_{0 \leq k \leq n} \|h^{1/2}u_n^k\|^2 \leq \kappa_{10} \left(T, \mathbb{E} \|h^{1/2}u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2}\Phi\|_{L_2^0} \right).$$

Remark 3.5 The bounds (3.37)–(3.39) with h replaced by $(1+x_+)^{3/4}$ can be obtained by working first with a sequence of smooth weighting factors $\{h_n\}$ such that (2.9)–(2.11) hold uniformly with respect to n and such that h_n tends to $(1+x_+)^{1/4}$ uniformly in $\mathcal{C}(\mathbb{R}, \mathbb{R}_+)$ (see [12] p. 1397).

Remark 3.6 Since h in Proposition 3.4 is an increasing regular function on \mathbb{R} , its derivative h' is bounded below by a positive constant on any bounded closed interval on \mathbb{R} . Hence estimate (3.38) implies the following local regularity

$$\Delta t \mathbb{E} \sum_{k=0}^{n-1} \|\partial_x u_n^{k+1/2}\|_{L^2(-R,R)}^2 \leq \kappa \left(R, T, \mathbb{E} \|h^{1/2} u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2} \Phi\|_{\mathcal{L}_2} \right).$$

Proof of (3.37) and (3.38)

Let $u_n = \{u^k\}_{0 \leq k \leq n}$ be a solution of (3.12)–(3.13). Since no confusion is possible, we omit to write the dependance on n in this proof. We rewrite equation (3.12) in the following form as in the proof of Proposition 3.2:

$$(3.40) \quad u^{k+1} = v - \Delta t \partial_x^3 \left(\frac{v + u^{k+1}}{2} \right) - \frac{\Delta t}{2} \partial_x \left(\frac{v + u^{k+1}}{2} - \frac{z_\alpha^{k+1}}{2} \right)^2 + \alpha \frac{\Delta t}{2} z_\alpha^{k+1},$$

where z_α^{k+1} is defined by (3.14) for some $\alpha > 0$ which will be precised later and where we have set again

$$(3.41) \quad v = u^k + z_\alpha^{k+1}.$$

Multiplying (3.40) by $h(v + u^{k+1})/2$, we obtain after integration on \mathbb{R} :

$$(3.42) \quad \begin{aligned} \frac{1}{2} \|h^{1/2} u^{k+1}\|^2 &= \frac{1}{2} \|h^{1/2} v\|^2 - \Delta t \left(\partial_x^3 \left(\frac{v + u^{k+1}}{2} \right), h \left(\frac{v + u^{k+1}}{2} \right) \right) \\ &\quad - \frac{\Delta t}{2} \left(\partial_x \left(\frac{v + u^{k+1}}{2} - \frac{z_\alpha^{k+1}}{2} \right)^2, h \left(\frac{v + u^{k+1}}{2} \right) \right) \\ &\quad + \alpha \frac{\Delta t}{2} \left(z_\alpha^{k+1}, h \left(\frac{v + u^{k+1}}{2} \right) \right). \end{aligned}$$

Let us note that

$$\left(\partial_x^3 \left(\frac{v + u^{k+1}}{2} \right), h \left(\frac{v + u^{k+1}}{2} \right) \right) = \frac{3}{2} \|h^{1/2} \partial_x \left(\frac{v + u^{k+1}}{2} \right)\|^2 - \frac{1}{2} \|h^{3/2} \left(\frac{v + u^{k+1}}{2} \right)\|^2,$$

and

$$\frac{1}{2} \left(h \left(\frac{v + u^{k+1}}{2} \right), \partial_x \left(\frac{v + u^{k+1}}{2} \right)^2 \right) = -\frac{1}{3} \int_{\mathbb{R}} h' \left(\frac{v + u^{k+1}}{2} \right)^3 dx.$$

Merging the last two equalities into (3.42), we obtain after some suitable integration by parts the following inequality:

$$(3.43) \quad \begin{aligned} \frac{1}{2} \|h^{1/2} u^{k+1}\|^2 + \frac{3\Delta t}{8} \|h^{1/2} \partial_x (v + u^{k+1})\|^2 &\leq \frac{1}{2} \|h^{1/2} v\|^2 + \frac{\Delta t}{24} \left| \int_{\mathbb{R}} h' (v + u^{k+1})^3 dx \right| \\ &+ \frac{\Delta t}{8} \left(1 + c_3 + \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \right) \|h^{1/2} (v + u^{k+1})\|^2 \\ &+ c_1 \frac{\Delta t}{8} \|z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \|v + u^{k+1}\|^2 + \frac{\Delta t}{16} \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \|h^{1/2} z_\alpha^{k+1}\|^2 \\ &+ \alpha^2 \frac{\Delta t}{8} \|h^{1/2} z_\alpha^{k+1}\|^2, \end{aligned}$$

where we have used several times the assumptions (2.9)–(2.11) together with the inequality: for any non negative reals a, b , we have

$$(3.44) \quad 2ab \leq a^2 + b^2.$$

The cubic term of the right hand side of (3.43) is estimated owing to the following lemma.

Lemma 3.7 *There exists a positive constant $\kappa > 0$ such that for any regular function φ ,*

$$(3.45) \quad \left| \int_{\mathbb{R}} h' \varphi^3 dx \right| \leq \kappa \|h^{1/2} \varphi\|^{3/2} \|\varphi\| \left\{ \|h^{1/2} \varphi\|^{1/2} + \|h^{1/2} \partial_x \varphi\|^{1/2} \right\}.$$

Proof:

Thanks to the Hölder inequality, we have

$$\left| \int_{\mathbb{R}} h' \varphi^3 dx \right| \leq \|h^{1/2} \varphi\|_{L^4(\mathbb{R})}^2 \|\varphi\|.$$

Then the continuous Sobolev embedding of $L^4(\mathbb{R})$ into $H^{1/4}(\mathbb{R})$ followed by a classical interpolation inequality leads to the inequality:

$$\|h^{1/2} \varphi\|_{L^4(\mathbb{R})}^2 \leq c_0 \|h^{1/2} \varphi\|^{3/2} \|h^{1/2} \varphi\|_{H^1(\mathbb{R})}^{1/2}.$$

Assertion (2.10) concerning the weight function h allows to estimate the $H^1(\mathbb{R})$ -norm by

$$(3.46) \quad \|h^{1/2} \varphi\|_{H^1(\mathbb{R})}^2 \leq \|h^{1/2} \varphi\|^2 + \frac{c_2}{2} \|h^{1/2} \varphi\|^2 + 2 \|h^{1/2} \partial_x \varphi\|^2.$$

Plugging (3.46) into the former inequality leads to the result. ■

By Lemma 3.7 with φ replaced by $v + u^{k+1}$, (3.43) becomes

$$(3.47) \quad \begin{aligned} & \|h^{1/2} u^{k+1}\|^2 + \frac{3\Delta t}{4} \|h^{1/2} \partial_x (v + u^{k+1})\|^2 \leq \|h^{1/2} v\|^2 \\ & + \frac{\Delta t}{4} \|h^{1/2} (v + u^{k+1})\|^2 \left(1 + c_3 + \|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \right) \\ & + \kappa \frac{\Delta t}{12} \|v + u^{k+1}\|^{5/2} \left\{ \|h^{1/2} (v + u^{k+1})\|^{1/2} + \|h^{1/2} \partial_x (v + u^{k+1})\|^{1/2} \right\} \\ & + c_1 \frac{\Delta t}{4} \|z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} \|v + u^{k+1}\|^2 + \frac{\Delta t}{8} \left(\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} + 2\alpha^2 \right) \|h^{1/2} z_\alpha^{k+1}\|^2 \end{aligned}$$

Finally, by Young inequality, for any $a, b, \gamma > 0$, we can find a constant C_γ such that

$$ab \leq \gamma a^4 + C_\gamma b^{4/3}.$$

Choosing successively

$$(a, b, \gamma) = \left(\|h^{1/2} (v + u^{k+1})\|^{1/2}, \frac{\kappa}{12} \|v + u^{k+1}\|^{5/2}, 1/4 \right)$$

and

$$(a, b, \gamma') = \left(\|h^{1/2} \partial_x (v + u^{k+1})\|^{1/2}, \frac{\kappa}{12} \|v + u^{k+1}\|^{5/2}, 3/8 \right),$$

and using several times again the inequality (3.44) in order to collect terms in $\|h^{1/2}u^{k+1}\|$ and $\|h^{1/2}v\|$ leads to

$$\begin{aligned}
& \|h^{1/2}u^{k+1}\|^2(1 - \Delta t(1 + \frac{c_3}{2} + \frac{1}{2}\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})})) + \frac{3\Delta t}{2} \|h^{1/2}\partial_x u^{k+1/2}\|^2 \\
& \leq \|h^{1/2}v\|^2(1 + \Delta t(1 + \frac{c_3}{2} + \frac{1}{2}\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})})) + C\Delta t\|v + u^{k+1}\|^{10/3} \\
& + c_1\frac{\Delta t}{4}\|z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})}\|v + u^{k+1}\|^2 + \frac{\Delta t}{8}(\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} + 2\alpha^2)\|h^{1/2}z_\alpha^{k+1}\|^2 + \frac{3\Delta t}{4}\|\partial_x z_\alpha^{k+1}\|^2,
\end{aligned}
\tag{3.48}$$

where $C = C_\gamma + C_{\gamma'} > 0$.

Let $\eta = 2(1 + c_3/2) \geq 2$. We now choose $\alpha = \alpha(\omega, k) = (\kappa_3)^{3/\varepsilon}\|\chi^{k+1}\|_{H^\varepsilon(\mathbb{R})}^{6/\varepsilon}/\eta^{6/\varepsilon}$ in (3.17) such that

$$\|\partial_x z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})} = \eta, \quad \text{a.s.}$$

For $\Delta t \leq 1/(2\eta) \leq 1/4$, we have $1 - \eta\Delta t \geq 1/2 > 0$ and $(1 + \eta\Delta t)/(1 - \eta\Delta t) \leq e^{4\eta\Delta t}$. Therefore, we get

$$(3.49) \quad \|h^{1/2}u^{k+1}\|^2 + 3\Delta t \|h^{1/2}\partial_x u^{k+1/2}\|^2 \leq \|h^{1/2}v\|^2 e^{4\eta\Delta t} + \Delta t C(\omega, k),$$

with

$$\begin{aligned}
C(\omega, k) &= 2C\|v + u^{k+1}\|^{10/3} + \frac{c_1}{2}\|z_\alpha^{k+1}\|_{L^\infty(\mathbb{R})}\|v + u^{k+1}\|^2 + \frac{1}{4}(\eta + 2\alpha^2(\omega, k))\|h^{1/2}z_\alpha^{k+1}\|^2 \\
&+ \frac{3}{4}\|\partial_x z_\alpha^{k+1}\|^2.
\end{aligned}$$

Owing to (3.15), (3.16), (3.18) and (3.34), $C(\omega, k)$ can be rewritten as

$$\begin{aligned}
C(\omega, k) &= C_1\left(\|u^{k+1/2}\|^4 + \|\chi^{k+1}\|^4\right) + C_2\|\chi^{k+1}\|\left(\|u^{k+1/2}\|^2 + \|\chi^{k+1}\|^2\right) \\
&+ C_3\|h^{1/2}\chi^{k+1}\|^2\left(1 + \|\chi^{k+1}\|^{12/\varepsilon}\right) + C_4\|\chi^{k+1}\|^2,
\end{aligned}$$

where the C_i , $i = 1, \dots, 4$ are positive constants which depend only on η and ε . We conclude as in the proof of estimate (3.20) owing to (2.6), (2.7), (3.20), (3.21) and the Gaussianity of the noise. \blacksquare

Proof of (3.39)

Expanding the term $\|h^{1/2}v\|^2$ in (3.49) yields first

$$\begin{aligned}
(3.50) \quad \|h^{1/2}u^{k+1}\|^2 &\leq e^{4\eta\Delta t}\left(\|h^{1/2}u_k\|^2 + \|h^{1/2}z_\alpha^{k+1}\|^2\right) + \Delta t C(\omega, k) \\
&+ 2\sqrt{\Delta t}e^{4\eta\Delta t}\left(h^{1/2}u^k, h^{1/2}Q\chi^{k+1}\right),
\end{aligned}$$

where the operator Q has been introduced in the proof of (3.34). Here the $\sqrt{\Delta t}$ -term is a martingale increment. This will allow us to bound the expectation of the time supremum of the $\{u^k\}$. Indeed, after summation with respect to k , Ineq. (3.50) leads to

$$\begin{aligned}
(3.51) \quad \|h^{1/2}u^k\|^2 &\leq e^{4\eta k\Delta t}\|h^{1/2}u_0\|^2 + \Delta t\sum_{\ell=0}^{k-1}e^{4\eta(k-\ell)\Delta t}\left(\kappa_8\|h^{1/2}\chi^{k+1}\|^2 + C(\omega, k)\right) \\
&+ 2e^{4\eta k\Delta t}M(t_k),
\end{aligned}$$

where we have used again (3.34) and where $M(t)$ denotes the stochastic integral

$$M(t) = \int_0^t \left(h^{1/2} u_n(s), h^{1/2} Q \Phi dW(s) \right).$$

Eventually, taking the expectation of the maximum with respect to k of (3.51) leads to

$$(3.52) \quad \mathbb{E} \max_{0 \leq k \leq n} \|h^{1/2} u^k\|^2 \leq C(T) \left(\mathbb{E} \|h^{1/2} u_0\|^2 + \Delta t \sum_{\ell=0}^{k-1} \left(\kappa_8 \mathbb{E} \|h^{1/2} \chi^{k+1}\|^2 + \mathbb{E} C(\omega, k) \right) \right. \\ \left. + \mathbb{E} \sup_{t \in [0, T]} |M(t)| \right).$$

We conclude again thanks to (2.6), (2.7), (3.20), (3.21) and a classical martingale inequality.

■

3.3 $\mathcal{C}(0, T)$ -a priori estimates

In order to go further in the compactness method, we need estimates on the modulus of continuity of the approximation. Classically, we introduce a piecewise interpolation of u_n . Let $\{v_n\}_n$ be the sequence of $L^2(\mathbb{R})$ -valued processes defined by

$$v_n(t) = \frac{(t_{k+1} - t)u_n^{k-1} + (t - t_k)u_n^k}{t_{k+1} - t_k}, \quad \forall t \in [t_k, t_{k+1}[, \quad k \geq 1,$$

and

$$v_n(t) = u_n^0, \quad \forall t \in [0, \Delta t[.$$

Let us note that, thanks to the shift in k , for each integer n , v_n is a continuous stochastic adapted process. Moreover the previous inequalities derived on $\{u_n\}_n$ are easily proved to hold also for $\{v_n\}_n$.

Corollary 3.8 *There exists positive constants $\kappa_i = \kappa_i(T, \mathbb{E} \|h^{1/2} u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2} \Phi\|_{L_2^0})$, $i = 11, 12, 13$ such that*

$$(3.53) \quad \mathbb{E} \sup_{t \in [0, T]} \|v_n(t)\|^2 \leq \kappa_{11},$$

$$(3.54) \quad \int_0^T \mathbb{E} \|\partial_x v_n(t)\|_{L^2(-R, R)}^2 dt \leq \kappa_{12}(R), \quad \forall R > 0,$$

$$(3.55) \quad \mathbb{E} \sup_{t \in [0, T]} \|h^{1/2} v_n(t)\|^2 \leq \kappa_{13},$$

for any weight function h satisfying assumptions (2.9)–(2.11).

The following estimates are useful to show the tightness of the laws of $\{v_n\}$.

Proposition 3.9 *There exists $\beta > 0$, $\gamma \in]0, 1/2[$ such that for any $R > 0$, there exists positive constants $\kappa_i = \kappa(R, T, \mathbb{E} \|h^{1/2} u_0\|^2, \mathbb{E} \|u_0\|^4, \|h^{1/2} \Phi\|_{L_2^0})$, $i = 14, 15$ such that*

$$(3.56) \quad \mathbb{E} \left(\|v_n\|_{C^\beta(0, T, H^{-2}(-R, R))}^2 \right) \leq \kappa_{14}(R),$$

$$(3.57) \quad \mathbb{E} \left(\|v_n\|_{W^{\gamma, 2}(0, T, H^{-2}(-R, R))}^2 \right) \leq \kappa_{15}(R).$$

Proof.

The idea for proving these estimates is to write

$$(3.58) \quad v_n(t) = u_0 - \int_0^t \sum_{k=1}^n \left(\partial_x^3 u_n^{k+1/2} + \frac{1}{2} \partial_x \left(u_n^{k+1/2} \right)^2 \right) \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds + \Phi w_n(t),$$

where $\{w_n(t)\}$ is a continuous, piecewise linear adapted stochastic process on $[0, T]$ such that $w_n(t_k) = W(t_{k-1})$ for any $k \leq n$, i.e.

$$(3.59) \quad w_n(t) = \frac{(t_{k+1} - t)W(t_{k-1}) + (t - t_k)W(t_k)}{\Delta t},$$

for any $t \in [t_k, t_{k+1}[$. The last term of (3.58) is estimated thanks to the characterization of the Sobolev spaces $W^{\gamma, 2p}(0, T, L^2(\mathbb{R}))$ (see e.g. Lemma 2.1 in [10]) and we get the bound (3.57) in $W^{\gamma, 2p}(0, T, H^{-2}(-R, R))$ for any $\gamma < 1/2$ and any integer p . Estimate (3.56) is obtained using, for p large enough, the Sobolev embedding of $W^{\gamma, 2p}(0, T, H^{-2}(-R, R))$ into $C^\beta(0, T, H^{-2}(-R, R))$ when $0 < \beta < \gamma - 1/(2p)$. \blacksquare

3.4 Passage to the limit and conclusion.

We first state a tightness result for the sequence $\{v_n\}_{n \geq 0}$ and then conclude thanks to the pathwise uniqueness and Lemma 2.4. Let us denote for some $\gamma \in]0, 1/2[$, $\beta > 0$, $s < 1$ and $s' > 2$ the following spaces

$$X_{\gamma, \beta}(T) = W^{\gamma, 2}(0, T, H_{loc}^{-2}(\mathbb{R})) \cap L^2(0, T, H_{loc}^1(\mathbb{R})) \cap C^\beta(0, T, H_{loc}^{-2}(\mathbb{R})),$$

and

$$Y_{s, s'}(T) = L^2(0, T, H_{loc}^s(\mathbb{R})) \cap C(0, T, H_{loc}^{-s'}(\mathbb{R})).$$

We can state now the tightness result.

Proposition 3.10 *For any $T > 0$, $s < 1$ and $s' > 2$, the family of laws $\{\mathcal{L}(v_n)\}_{n \geq 0}$ on $X_{\gamma, \beta}(T)$ is tight in $Y_{s, s'}(T)$, for any $s < 1$ and $s' > 2$.*

Proof.

Each probability measure $\mathcal{L}(v_n)$ is inner regular since both spaces $X_{\gamma, \beta}(T)$ and $Y_{s, s'}(T)$ are Fréchet spaces, thus metrizable, complete and separable. Therefore, in order to apply Prokhorov criterion of tightness, it is sufficient to prove that for any $\varepsilon > 0$, there exists a compact K_ε of $Y_{s, s'}(T)$ such that

$$\mathcal{L}(v_n)(K_\varepsilon) \geq 1 - \varepsilon,$$

for any $n \in \mathbb{N}$. Thus, for any $\varepsilon > 0$, let B_ε be the following subset of $X_{\gamma, \beta}(T)$:

$$B_\varepsilon = \bigcap_{k \in \mathbb{N}} \left\{ |\varphi|_{L^2(0, T, H^1(-k, k))}^2 + |\varphi|_{W^{\gamma, 2}(0, T, H^{-2}(-k, k))}^2 + |\varphi|_{C^\beta(0, T, H^{-2}(-k, k))}^2 \leq \frac{2^k}{\varepsilon} (C_8(k) + C_9(k) + C_{10}(k)) \right\}$$

We then take K_ε as the closure of B_ε in $Y_{s,s'}(T)$. Then (3.54)–(3.57) together with the Bienaymé–Chebyshev inequality and Lemma 2.5 yield the result. \blacksquare

In order to use Lemma 2.4 and get convergence in probability in the suitable spaces, we use a slightly different formulation of the previous Proposition whose proof is straightforward.

Proposition 3.11 *For any $T > 0$, $s < 1$ and $s' > 2$, for any pair of subsequence (n_k, m_k) , the family of laws $\{\mathcal{L}(v_{n_k}, v_{m_k}, W)\}_{k \geq 0}$ is tight in*

$$(Y_{s,s'}(T))^2 \times \mathcal{C}(0, T, H_{loc}^{-2}).$$

Owing to Proposition 3.11, we now apply Skorohod’s embedding Theorem to some subsequence $\{(v_{n_\ell}, v_{m_\ell}, W)\}$. Then there exists subsequences, which are denoted by the same, $\{n_\ell\}$ and $\{m_\ell\}$, a sequence of continuous stochastic processes $\{(\tilde{v}_{n_\ell}, \hat{v}_{m_\ell}, W_\ell)\}$ and a triplet of stochastic processes $(\tilde{v}, \hat{v}, \tilde{W})$ together with a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0})$, such that the corresponding joint laws are equal:

$$(3.60) \quad \mathcal{L}(\tilde{v}_{n_\ell}, \hat{v}_{m_\ell}, W_\ell) = \mathcal{L}(v_{n_\ell}, v_{m_\ell}, W),$$

for any integer ℓ and such that

$$(3.61) \quad (\tilde{v}_{n_\ell}, \hat{v}_{m_\ell}) \longrightarrow (\tilde{v}, \hat{v}), \quad W_\ell \longrightarrow \tilde{W}, \quad \tilde{\mathbb{P}} \text{ a.s.}$$

as ℓ tends to $+\infty$, respectively in $(Y_{s,s'}(T))^2$ and $\mathcal{C}(0, T, H_{loc}^{-2})$.

We now have to take the limit in (3.58) almost surely in the weak sense of distributions. It is the deterministic part of the proof.

We first note that both $\{\tilde{v}_{n_\ell}\}$ and $\{\hat{v}_{m_\ell}\}$ satisfies (3.58)–(3.59), in the weak sense of distributions, with W replaced by W_ℓ . This is true thanks to the following observations. First, the scalar product of the deterministic part of (3.58) (the drift) with any test function φ is continuous from $Y_{s,s'}(T)$ into $\mathcal{C}(0, T)$. In fact, this follows from the arguments used in the passage to the limit in the weak sense below. Second, the time discretization (3.59) of the stochastic integral allows to say that the same is true for the stochastic part of (3.58), i.e. the scalar product of this stochastic part with any test function is continuous from $\mathcal{C}(0, T, H^{-2}(-R, R))$ into $\mathcal{C}(0, T)$. We deduce

$$(3.62) \quad \begin{aligned} \hat{v}_{m_\ell}(t) &= u_0 - \int_0^t \sum_{k=1}^{m_\ell} \left(\partial_x^3 \hat{v}_{m_\ell}(t_{k+1/2}) + \frac{1}{2} \partial_x (\hat{v}_{m_\ell}(t_{k+1/2}))^2 \right) \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds \\ &\quad - \sum_{k=1}^{m_\ell} \Phi \frac{(t_{k+1} - t)W_\ell(t_{k-1}) + (t - t_k)W_\ell(t_k)}{\Delta t} \mathbf{1}_{[t_k, t_{k+1}[}(t) \end{aligned}$$

We are now able to let ℓ go to infinity in (3.62). We recall from (3.61) that $\hat{v}_{m_\ell} \rightarrow \hat{v}$ almost surely in $L^2(0, T, H_{loc}^s) \cap \mathcal{C}(0, T, H_{loc}^{-2})$ for any $s < 1$. It follows that

$$\sum_{k=0}^{m_\ell} \int_0^{\cdot} \partial_x^3 \hat{v}_{m_\ell}(t_{k+1/2}) \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds \longrightarrow \int_0^{\cdot} \partial_x^3 \hat{v}(s) \, ds,$$

$\tilde{\mathbb{P}}$ -almost surely in $L^\infty(0, T, H_{loc}^{s-3}(\mathbb{R}))$.

Also concerning the nonlinear term, one has for any $t \in [0, T]$ and for any positive R :

$$\left\| \sum_{k=0}^{m_\ell} \int_0^t (\partial_x \widehat{v}_{m_\ell}^2(t_{k+1/2}) - \partial_x \widehat{v}^2(s)) \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds \right\|_{H^{-1}(-R, R)} \leq I + II$$

where

$$I = \sum_{k=0}^{m_\ell} \int_0^t \|\partial_x \widehat{v}_{m_\ell}^2(t_{k+1/2}) - \partial_x \widehat{v}_{m_\ell}^2(s)\|_{H^{-1}(-R, R)} \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds,$$

and

$$II = \sum_{k=0}^{m_\ell} \int_0^t \|\partial_x \widehat{v}_{m_\ell}^2(s) - \partial_x \widehat{v}^2(s)\|_{H^{-1}(-R, R)} \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds.$$

It follows from the Gagliardo–Nirenberg inequality

$$\begin{aligned} & \|\widehat{v}_{m_\ell}(t_{k+1/2}) - \widehat{v}_{m_\ell}(s)\|_{L^4(-R, R)} \\ & \leq C_R \|\widehat{v}_{m_\ell}(t_{k+1/2}) - \widehat{v}_{m_\ell}(s)\|_{H^{-2}(-R, R)}^{1/4} \|\widehat{v}_{m_\ell}(t_{k+1/2}) - \widehat{v}_{m_\ell}(s)\|_{H^1(-R, R)}^{3/4}, \end{aligned}$$

that

$$\begin{aligned} & \|\partial_x \widehat{v}_{m_\ell}^2(t_{k+1/2}) - \partial_x \widehat{v}_{m_\ell}^2(s)\|_{H^{-1}(-R, R)} \\ & \leq \|\widehat{v}_{m_\ell}(t_{k+1/2}) - \widehat{v}_{m_\ell}(s)\|_{L^4(-R, R)} \|\widehat{v}_{m_\ell}(t_{k+1/2}) + \widehat{v}_{m_\ell}(s)\|_{L^4(-R, R)} \\ & \leq C_R \|\widehat{v}_{m_\ell}(t_{k+1/2}) - \widehat{v}_{m_\ell}(s)\|_{H^{-2}(-R, R)}^{1/4} \left(\|\widehat{v}_{m_\ell}(t_{k+1/2})\|_{H^1(-R, R)}^{7/4} + \|\widehat{v}_{m_\ell}(s)\|_{H^1(-R, R)}^{7/4} \right). \end{aligned}$$

Hence, we get the following bound

$$\begin{aligned} (3.63) \quad I & \leq C(R, T) \left(\int_0^T \|\widehat{v}_{m_\ell}(s)\|_{H^1(-R, R)}^{7/4} \, ds \right) \sup_{|t-s| \leq \Delta t} \|\widehat{v}_{m_\ell}(t) - \widehat{v}_{m_\ell}(s)\|_{H^{-2}(-R, R)}^{1/4} \\ & \leq C(R, T) \left(\int_0^T \|\widehat{v}_{m_\ell}(s)\|_{H^1(-R, R)}^{7/4} \, ds \right) \Delta t^{\beta/4} \|\widehat{v}_{m_\ell}\|_{\mathcal{C}^\beta(0, T, H^{-2}(-R, R))}^{1/4}, \end{aligned}$$

for any $\beta < 1/2$. The second term II is easier to estimate

$$(3.64) \quad II \leq 2 \left(\int_0^T \|\widehat{v}(s)\|_{H^s(-R, R)}^2 \, ds \right)^{1/2} \left(\int_0^T \|\widehat{v}_{m_\ell}(s) - \widehat{v}(s)\|_{H^s(-R, R)}^2 \, ds \right)^{1/2},$$

for some $s \in [1/4, 1[$. Eventually, it follows from (3.61), (3.63) and (3.64) that

$$\sum_{k=0}^{m_\ell} \int_0^\cdot \partial_x \widehat{v}_{m_\ell}^2(t_{k+1/2}) \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds \longrightarrow \int_0^\cdot \partial_x \widehat{v}^2(s) \, ds$$

$\widetilde{\mathbb{P}}$ -almost surely in $L^\infty(0, T, H_{loc}^{-1}(\mathbb{R}))$ as ℓ tends to infinity.

As concerns the stochastic term of (3.62), the very definition of Ito stochastic integral together with (3.61) shows that

$$\sum_{k=1}^{m_\ell} \int_0^\cdot \Phi \frac{W_\ell(t_k) - W_\ell(t_{k-1})}{\Delta t} \mathbf{1}_{[t_k, t_{k+1}[}(s) \, ds \longrightarrow \int_0^\cdot \Phi \, d\widetilde{W}(s),$$

$\widetilde{\mathbb{P}}$ -almost surely in $\mathcal{C}(0, T, L_{loc}^2)$. Eventually, we have shown that

Lemma 3.12 *The stochastic process \widehat{v} adapted to $\{\widetilde{\mathcal{F}}_t\}$, satisfies*

$$(3.65) \quad \widehat{v}(t) - u_0 + \int_0^t \left(\partial_x^3 \widehat{v}(s) + \frac{1}{2} \partial_x \widehat{v}^2(s) \right) ds = \int_0^t \Phi d\widetilde{W}(s), \quad t \in [0, T],$$

in the weak sense of distributions with

$$(1 + x_+)^{3/8} \widehat{v} \in L^\infty(0, T, L^2(\mathbb{R})), \quad \widetilde{\mathbb{P}}a.s.,$$

$$\partial_x \widehat{v} \in L^2(0, T, L^2_{loc}), \quad \widetilde{\mathbb{P}}a.s..$$

We now end the proof of Theorem 2.3. By Theorem 2.1, we know that the solution given by Lemma 3.12 is unique. Hence, we know that (v_{m_ℓ}, v_{n_ℓ}) tends to (u, u) in distribution, u given by Theorem 2.1. Then Lemma 2.4 implies that the whole sequence $\{v_n\}$ converges in probability $u \in Y_{s,s'}(T)$. Now, we show that the convergence holds in $L^2(\Omega; L^\infty(0, T, L^2(\mathbb{R})))$. First, it is clear that (3.53) and weak convergence implies that for any $t \in [0, T]$, we have

$$(3.66) \quad \sup_{t \in [0, T]} \mathbb{E} \|u(t)\|^2 \leq \liminf_n \sup_{t \in [0, T]} \mathbb{E} \|v_n(t)\|^2.$$

and

$$(3.67) \quad \mathbb{E} \sup_{t \in [0, T]} \|u(t)\|^2 \leq \liminf_n \mathbb{E} \sup_{t \in [0, T]} \|v_n(t)\|^2.$$

Moreover, it is not difficult to see that owing to the convexity of the $L^2(\mathbb{R})$ -norm, we have, for any $t \in [t_k, t_{k+1}[$, the following inequality

$$\|v_n(t)\|^2 \leq \|u_n^{k-1}\|^2 + \left(\frac{t - t_k}{\Delta t} \right) \left(\|u_n^k\|^2 - \|u_n^{k-1}\|^2 \right).$$

We deduce by (3.31) that for any $t \in [0, T]$, the following inequality holds

$$\|v_n(t)\|^2 \leq \|u_0\|^2 + \frac{2}{\sqrt{\Delta t}} \int_0^t \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1}[}(s) (u_n^{k-1/2}, \chi^k) ds.$$

As in the proof of Proposition 3.2 (estimate (3.22)), we split the stochastic integral into two parts:

$$\frac{2}{\sqrt{\Delta t}} \int_0^t \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1}[}(s) (u_n^{k-1/2}, \chi^k) ds = I + II,$$

where

$$I = \frac{1}{\sqrt{\Delta t}} \int_0^t \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1}[}(s) \frac{(u_n^k - S_n u_n^{k-1}, \Phi W(t_k) - \Phi W(t_{k-1}))}{\sqrt{\Delta t}},$$

and

$$II = \frac{1}{\sqrt{\Delta t}} \int_0^t \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1}[}(s) \frac{(u_n^{k-1} + S_n u_n^{k-1}, \Phi W(t_k) - \Phi W(t_{k-1}))}{\sqrt{\Delta t}}.$$

Plugging the semi-discrete mild formulation (3.32) in I yields

$$I = \int_0^t \sum_{k=1}^{n-1} \mathbf{1}_{[t_k, t_{k+1}[}(s) \left\{ -\frac{1}{2} \left(\Lambda_n \left((u_n^{k-1/2})^2 \right), \chi^k \right) + \left(\left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \chi^k, \chi^k \right) \right\} ds,$$

where

$$\Lambda_n = \sqrt{\Delta t} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x.$$

We have

$$\left\| \sqrt{\Delta t} \left(I + \frac{\Delta t}{2} \partial_x^3 \right)^{-1} \partial_x \right\|_{\mathcal{L}(L^1(\mathbb{R}), H^{-\varepsilon}(\mathbb{R}))} \leq C \Delta t^{\varepsilon/3}.$$

Therefore, using (3.33) for the second term,

$$\mathbb{E} \sup_{t \in [0, T]} \|I\| \leq C(T) \Delta t^{\varepsilon/3} \max_{k=1, \dots, n} \mathbb{E} \left(\|u_n^{k-1/2}\|^2 \|\chi^k\|_{H^\varepsilon} \right) + \sum_{k=0, \dots, n-1} \Delta t \mathbb{E} \|\chi^k\|^2.$$

Thanks to (3.20) and (3.21), we derive

$$\limsup_n \sup_{t \in [0, T]} \mathbb{E} \|I\| \leq \limsup_n \mathbb{E} \sup_{t \in [0, T]} \|I\| \leq T \|\Phi\|_{\mathcal{L}_2}^2.$$

Since the expectation of the second term II is zero, we deduce

$$\limsup_n \sup_{t \in [0, T]} \mathbb{E} \|v_n(t)\| \leq \|u_0\|^2 + T \|\Phi\|_{\mathcal{L}_2}^2 = \sup_{t \in [0, T]} \mathbb{E} \|u(t)\|^2.$$

Weak convergence, this inequality and (3.66) imply that v_n converges to u in $L^\infty(0, T; L^2(\Omega \times \mathbb{R}))$ strongly. It is then easy to use this convergence to show that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \left| II - 2 \int_0^t (u(s), dW(s)) \right| \right) \rightarrow 0$$

and we deduce

$$\limsup_n \mathbb{E} \sup_{t \in [0, T]} \|v_n(t)\| \leq \|u_0\|^2 + T \|\Phi\|_{\mathcal{L}_2}^2 + 2 \mathbb{E} \sup_{t \in [0, T]} \int_0^t (u(s), dW(s)) = \mathbb{E} \sup_{t \in [0, T]} \|u(t)\|^2.$$

Using (3.67) and weak convergence, the convergence in $L^2(\Omega; L^\infty(0, T; L^2(\mathbb{R})))$ follows.

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