Covering numbers and "low $M^*$-estimate" for quasi-convex bodies. *

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Abstract

This article gives estimates on covering numbers and diameters of random proportional sections and projections of quasi-convex bodies in $\mathbb{R}^n$. These results were known for the convex case and played an essential role in development of the theory. Because duality relations can not be applied in the quasi-convex setting, new ingredients were introduced that give new understanding for the convex case as well.

1. Introduction and notation.

Let $|\cdot|$ be on $\mathbb{R}^n$. Let $D$ be an ellipsoid associated with this norm. Denote $A = \frac{1}{n} \int_{S^{n-1}} \sqrt{\sum_{i=1}^{k} x_i^2} \, d\sigma(x)$, where $\sigma$ is the normalized rotation invariant measure on the euclidean sphere $S^{n-1}$. Then $A = A(n,k) < 1$ and $A \to 1$ as $n,k \to \infty$. For any star-body $K$ in $\mathbb{R}^n$ define $M_K = \int_{S^{n-1}} \| x \| \, d\sigma(x)$, where $\| x \|$ is the gauge of $K$. Let $M_K^0$ be $M_K^0$, where $K^0$ is the polar of $K$. For any subsets $K_1, K_2$ of $\mathbb{R}^n$ denote by $N(K_1,K_2)$ the smallest number $N$ such that there are $N$ points $y_1, \ldots, y_N$ in $K_1$ such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Recall that a body $K$ is called quasi-convex if there is a constant $c$ such that $K + K \subset cK$, and given a $p \in (0, 1]$ a body $K$ is called $p$-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to $K$.

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Note that for the gauge \( \| \cdot \| = \| \cdot \|_K \) associated with the the quasi-convex (p-convex) body \( K \) the following inequality holds for any \( x, y \in \mathbb{R}^n \)

\[
\| x + y \| \leq C \max\{ \| x \|, \| y \| \} \quad (\| x + y \|_p \leq \| x \|_p + \| y \|_p).
\]

In particular, every p-convex body \( K \) is also quasi-convex one and \( K + K \subset 2^{1/p} K \). A more delicate result is that for every quasi-convex body \( K \) \((K + K \subset cK)\) there exists a q-convex body \( K_0 \) such that \( K \subset K_0 \subset 2cK \), where \( 2^{1/q} = 2c \). This is Aoki-Rogers theorem ([KPR], [R], see also [K], p.47). In this note by a body we always mean a compact star-body, i.e. a body \( K \) satisfying \( tK \subset K \) for all \( t \in [0, 1] \).

Let us remind of the so-called "low \( M^*\)-estimate" result.

**Theorem 1** Let \( \lambda > 0 \) and \( n \) be large enough. Let \( K \) be a centrally-symmetric convex body in \( \mathbb{R}^n \) and \( \| \cdot \| \) be the gauge of \( K \). Then there exists a subspace \( E \) of \( (\mathbb{R}^n, \| \cdot \|) \) such that \( \dim E = \lfloor \lambda n \rfloor \) and for any \( x \in E \) the following inequality holds

\[
\| x \| \geq \frac{f(\lambda)}{M^*_K} |x|
\]

for some function \( f(\lambda) \), \( 0 < \lambda < 1 \).

*Remark.* Inequality of this type was first proved in [M1] with very poor dependence on \( \lambda \) and then improved in [M2] to \( f(\lambda) = C(1 - \lambda) \). It was later shown ([PT]), that one can take \( f(\lambda) = C\sqrt{1 - \lambda} \) (for different proofs see [M3] and [G]).

By duality this theorem is equivalent to the following theorem.

**Theorem 1'** Let \( \lambda > 0 \) and \( n \) be large enough. For every centrally-symmetric convex body \( K \) in \( \mathbb{R}^n \) there exists an orthogonal projection \( P \) of rank \( \lfloor \lambda n \rfloor \) such that

\[
PD \subset \frac{M_K}{f(\lambda)} PK.
\]

In this note we will extend both theorems to quasi-convex, not necessary central-symmetric bodies. Because duality arguments can not be applied to a non-convex body these two theorems become different statements. Also "\( M^*_K \)" should be substituted by an appropriate quantity not involving duality. Note that by avoiding the use of convexity assumption we in fact simplified proof also for a convex case.

2. **Main results.**

The following theorem is an extension of Theorem 1'.

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Theorem 2 Let $\lambda > 0$ and $n$ be large enough ($n > c/(1-\lambda)^2$). For any $p$-convex body $K$ in $\mathbb{R}^n$ there exists an orthogonal projection $P$ of the rank $[\lambda n]$ such that

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+p/p}} PK,$$

where $A_p = \text{const} \frac{\ln(2/p)}{p}$.

Remark. Also, this result is new for the convex non-symmetric case. To appreciate the strength of this inequality apply it to the standard simplex $S$ inscribed in $D$. Then $M_S \approx \sqrt{n \cdot \log n}$ and therefore for every $\lambda < 1$ there are $\lambda n$-dimensional projections containing euclidean ball of radius $\approx 1/\sqrt{n \cdot \log n}$. At the same time $S$ contains only a ball of radius $1/n$.

The proof of Theorem 2 is based on the next three lemmas. The first one was proved by W.B.Johnson and J.Lindenstrauss in [JL]. The second one was proved in [PT] for centrally-symmetric convex bodies and is the dual form of Sudakov minoration theorem.

Lemma 1 There is an absolute constant $c$ such that if $\epsilon > \sqrt{c/k}$ and $N \leq 2e^{2k/c}$, then for any set of points $y_1, \ldots, y_N \in \mathbb{R}^n$ and any orthogonal projection $P$ of rank $k$

$$\mu \left( \{ U \in O_n \mid \forall j : A(1-\epsilon)\sqrt{k/n} |y_j| \leq |PUy_j| \leq A(1+\epsilon)\sqrt{k/n} |y_j| \} \right) > 0.$$ 

Lemma 2 Let $K$ be a body such that $K + K \subset aK$. Then

$$N(D,tK) \leq 2e^{8n(aM_K/t)^2}.$$ 

Proof: M. Talagrand gave a direct simple proof of this lemma for a convex case ([LT], pp. 82-83). One can check that his proof does not use symmetry and convexity of the body and produces estimate $N(D,tB) \leq 2e^{2n(aM_B/t)^2}$ for every body $B$, such that $B - B \subset aB$.

Now for a body $K$, satisfying $K + K \subset aK$ denote $B = K \cap -K$.

Then $B - B \subset aB$ and $M_B \leq 2M_K$, since

$$\| x \|_B = \max (\| x \|_K, \| x \|_{-K}) \leq \| x \|_K + \| x \|_{-K}.$$ 

Thus

$$N(D,tK) \leq N(D,tB) \leq 2e^{2n(2aM_K/t)^2}.$$ 

□
**Lemma 3** Let $B$ be a body, $K$ be a $p$-convex body, $r \in (0, 1)$, $\{x_i\} \subset rB$ and $B \subset \bigcup(x_i + K)$. Then $B \subset t_rK$, where $t_r = \frac{1}{(1-r^p)^{1/p}}$.

**Proof:** Obviously $t_r = \max\{\|x\|_K \mid x \in B\}$. Since $B \subset \bigcup(x_i + K)$, for any point $x$ in $B$ there are points $x_0$ in $rB$ and $y$ in $K$ such that $x = x_0 + y$. Then by maximality of $t_r$ and $p$-convexity of $K$ we have $t_r^p \leq r^p t_r^p + 1$. That proves the lemma. □

**Remark.** Somewhat similar argument was used by N. Kalton in dealing with $p$-convex sets.

**Proof of Theorem 2:**
Any $p$-convex body $K$ satisfies $K + K \subset aK$ with $a = 2^{1/p}$. By Lemma 2 we have

$$N = N(D, tK) \leq 2 \cdot \exp\left(2^{1+2/p} n(M_K/t)^2\right),$$

i.e. there exist points $x_1, ..., x_N$ in $D$, such that

$$D \subset \bigcup_{i=1}^N (x_i + tK).$$

Denote $c_p = 2^{1+2/p}$. Let $t$ and $\varepsilon$ satisfy

$$c_pm\left(\frac{M_K}{t}\right)^2 \leq \frac{\varepsilon^2 k}{c}$$

and $\varepsilon > \sqrt{c/k}$ for $c$ being the constant from Lemma 1.

Applying Lemma 1 we obtain that there exist an orthogonal projection $P$ of rank $k$ such that

$$PD \subset \bigcup(Px_i + tP) \quad \text{and} \quad |Px_i| \leq (1 + \varepsilon)\sqrt{\frac{k}{n}} |x_i|.$$

Let $\lambda = k/n$. Denote $r = (1 + \varepsilon)\sqrt{\lambda}$. Lemma 3 gives us

$$PD \subset trPK \quad \text{for} \quad t = \frac{\sqrt{c_p} M_K}{\varepsilon \sqrt{\lambda}} \quad \text{and} \quad \varepsilon^2 > \frac{c}{\lambda n}, \quad r < 1.$$

Choose

$$\varepsilon = \frac{1 - \sqrt{\lambda}}{2 \sqrt{\lambda}}.$$
Then for \( n \) large enough we get

\[
PD \subset \frac{A_p M_K}{(1 - \lambda)^{1 + \frac{1}{p}}} PK,
\]

for \( A_p = \text{const} \frac{\ln(2/p)}{p} \). This completes the proof. \( \square \)

Theorem 2 can be formulated in the global form.

**Theorem 2'** Let \( K \) be a \( p \)-convex body in \( \mathbb{R}^n \). Then there is an orthogonal operator \( U \) such that

\[
D \subset A'_p M_K(K + UK),
\]

where \( A'_p = \text{const} \frac{\ln(2/p)}{p} \).

This theorem can be proved independently, but we show how it follows from Theorem 2.

**Proof of Theorem 2':** It follows from the proof of Theorem 2 that actually the measure of such projections is large. So we can choose two orthogonal subspaces \( E_1, E_2 \) of \( \mathbb{R}^n \) such that \( \dim E_1 = \lfloor n/2 \rfloor \), \( \dim E_2 = \lceil (n + 1)/2 \rceil \) and

\[
P_i D \subset A''_p M_K P_i K,
\]

where \( P_i \) is the projection on the space \( E_i \) (\( i = 1, 2 \)). Denote \( I = \text{id}_{\mathbb{R}^n} = P_1 + P_2 \) and \( U = P_1 - P_2 \). So \( P_1 = 1/2(I + U) \) and \( P_2 = 1/2(I - U) \). Then \( U \) is an orthogonal operator and for any \( x \in D \) we have

\[
x = P_1 x + P_2 x \subset 1/2 A''_p M_K(I + U)K + 1/2 A''_p M_K(I - U)K \subset A''_p M_K \left( \frac{K + K}{2} + A''_p M_K \frac{UK - UK}{2} \right) = A'_p M_K(K + UK).
\]

That proves Theorem 2'. \( \square \)

Let us complement Lemma 2 by mentioning how covering number \( N(K, tD) \) can be estimated. In the convex case this estimate is given by Sudakov inequality, using quantity \( M^* \). More precisely, if \( K \) is a centrally-symmetric convex body, then

\[
N(K, tD) \leq 2e^{cn(M^*/t)^2}.
\]

Of course, using duality for a non-convex setting leads to a weak result, and we suggest below a substitution for quantity \( M^* \).
For two quasi-convex bodies $K, B$ define the following number
\[ M(K, B) = \frac{1}{|K|} \int_K \| x \|_B \, dx, \]
where $|K|$ is volume of $K$, and $\| x \|_B$ is the gauge of $B$. Such numbers are considered in [MP1], [MP2] and [BMMP].

**Lemma 4** Let $K$ be $p$-convex body and $B$ be a body. Assume $B - B \subset aB$. Then
\[ N(K, tB) \leq 2e^{(cn/p)(aM(K, B)/t)^p}, \]
where $c$ is an absolute constant.

**Proof:** We follow the idea of M. Talagrand of estimating covering numbers in case $K = D$ ([LT], pp. 82-83, see also [BLM] Proposition 4.2). Denote the gauge of $K$ by $\| \cdot \|$ and the gauge of $B$ by $| \cdot |_B$. Define the measure $\mu$ by following
\[ d\mu = \frac{1}{A} e^{-\| x \|^p} \, dx, \text{ where } A \text{ is chosen such that } \int_{\mathbb{R}^n} d\mu = 1. \]
Let $L = \int_{\mathbb{R}^n} |x|_B \, d\mu$. Then $\mu\{|x|_B \leq 2L\} \geq 1/2$. Let $x_1, x_2, \ldots$ be a maximal set of points in $K$ such that $|x_i - x_j|_B \geq t$. So the sets $x_i + \frac{t}{d}B$ have mutually disjoint interiors. Let $y_i = \frac{a}{t} x_i$ for some $a$. Then, by $p$-convexity of $K$ and convexity of the function $e^t$, we have
\[ \mu\{y_i + aB\} = \frac{1}{A} \int_{bB} e^{-\| x + y_i \|^p} \, dx \geq \frac{1}{A} \int_{bB} e^{(\| x \|^p + \| y_i \|^p)} \, dx = \]
\[ = \frac{1}{A} e^{-\| y_i \|^p} \int_{bB} e^{-\| x \|^p} \, dx \geq e^{-(ba/t)p} \mu\{aB\}. \]
Choose $b = 2L$. Then $\mu\{aB\} \geq 1/2$ and, hence,
\[ N(K, tB) \leq 2e^{(2aL/|t|)^p}. \]
Now compute $L$. First, the normalization constant $A$ is equal
\[ A = \int_{\mathbb{R}^n} e^{-\| x \|^p} \, dx = \int_{\mathbb{R}^n} \int_0^\infty (-e^{-\| x \|^p})' \, dt \, dx = \int_0^\infty \int_0^\infty t^{p-1} e^{-p \| x \|^p} \, dx \, dt = \]

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\[ = \int_{\mathbb{R}^n} dx \int_0^\infty p t^{p+n-1} e^{-tp} dt = |K| \cdot \Gamma \left( 1 + \frac{n}{p} \right), \]

where \( \Gamma \) is the gamma-function. The remaining integral is

\[
\int_{\mathbb{R}^n} |x| B e^{-\|x\|^p} dx = \int_{\mathbb{R}^n} |x| B \left( e^{-\|x\|^p} \right) dx = \int_0^\infty \int_{\|x\| \leq t} \int_{\mathbb{R}^n} |x| B dx dt =
\]

\[
= \int_{\|x\| \leq 1} \int_{\mathbb{R}^n} |x| B dx \int_0^\infty p t^{p+n} e^{-tp} dt = |K| \cdot M(K, B) \cdot \Gamma \left( 1 + \frac{n+1}{p} \right). \]

Using Stirling's formula we get

\[ L \approx \left( \frac{n}{p} \right)^{1/p} M(K, B). \]

That proves the lemma. \( \square \)

**Remark.** An analogous lemma for \( p \)-smooth \((1 \leq p \leq 2)\) body \( K \) and convex centrally-symmetric body \( B \) was announced in [MP2]. Of course, the proof holds for all \( p > 0 \) and every quasi-convex centrally-symmetric body \( B \). More precisely the following lemma holds.

**Lemma 4’** Let \( K \) and \( B \) be bodies. Let \( B - B \subset aB \) and assume that for some \( p > 0 \) there is a constant \( c_p \) which depends only on \( p \) and body \( K \), such that

\[ \| x + y \|_K^p + \| x - y \|_K^p \leq 2 \cdot (\| x \|_K^p + c_p \| y \|_K^p) \text{ for all } x, y \in \mathbb{R}^n. \]

Then

\[ N(K, tB) \leq 2e^{cn(c_p/(aM(K,B)/t)^p)}, \]

where \( c \) is an absolute constant.

Lemma 4’ is an extension of Lemma 2 in the symmetric case. Indeed, since Euclidean space is a 2-smooth space, then in case \( K = D \) being an ellipsoid, we have \( c_2(D) = 1 \). By direct computation, \( M(D, B) = \frac{n}{n+1} M_B \). Thus,

\[ N(D, tB) \leq 2e^{cn(M_B/ct)^2}. \]

Define the following characteristic of \( K \),

\[ \bar{M}_K = \frac{1}{|K|} \int_K |x| dx. \]
Lemma 4 shows that for $p$-convex body $K$

$$N(K, tD) \leq 2e^{(cn/p)(2\bar{M}_K/t)^p}.$$  

The Theorem 3 follows from this estimate by arguments similar of that in [MP].

**Theorem 3** Let $\lambda > 0$ and $n$ be large enough. Let $K$ be a $p$-convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of $K$. Then there exists subspace $E$ of $(\mathbb{R}^n, \| \cdot \|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds

$$\| x \| \geq \frac{(1 - \lambda)^{1/2+1/p}}{a_p \bar{M}_K} |x|,$$

where $a_p$ depends on $p$ only (more precisely $a_p = \text{const}^{\frac{\ln(2p)}{p}}$).

**Proof:** By Lemma 4 there are points $x_1, \ldots, x_N$ in $K$, such that $N < c_p n \left( \frac{\bar{M}_K}{t} \right)^p$ and for any $x \in K$ there exists some $x_i$ such that $|x - x_i| < t$. By Lemma 1 there exists an orthogonal projection $P$ on a subspace of dimension $\delta n$ such that for

$$c_p n \left( \frac{\bar{M}_K}{t} \right)^p < \frac{\varepsilon^2 \delta n}{c} \quad \text{and} \quad \varepsilon > \sqrt{\frac{c}{\delta n}},$$

we have

$$b|x_i| = (1 - \varepsilon)A\sqrt{\delta}|x_i| \leq |P x_i| \leq (1 + \varepsilon)A\sqrt{\delta}|x_i|$$

for every $x_i$. Let $E = \text{Ker}P$. Then $\dim E = \lambda n$, where $\lambda = 1 - \delta$. Take $x$ in $K \cap E$. There is $x_i$ such that $|x - x_i| < t$. Hence

$$|x| \leq |x - x_i| + |x_i| \leq t + \frac{|P x_i|}{b} = t + \frac{|P(x - x_i)|}{b} \leq$$

$$\leq t + \frac{|x - x_i|}{b} \leq t(1 + \frac{1}{b}) \leq \frac{\text{const} \cdot t}{(1 - \varepsilon) \sqrt{\delta}}.$$  

Therefore for $n$ large enough and

$$t = \left( \frac{\text{const} \cdot c_p}{\varepsilon^2 \delta} \right)^{1/p} \bar{M}_K$$

we get

$$\| x \| \geq \frac{\text{const} \cdot \varepsilon^2 (1 - \varepsilon)^{1/2+1/p}}{c_p^{1/p} \bar{M}_K} |x|.$$
To obtain our result take $\varepsilon$, say, equal to $1/2$.

As was noted in [MP2] in some cases $M_K << M^*$ and then Theorem 3 gives better estimate than Theorem 1 even for a convex body (in some range of $\lambda$). As an example, $K = B(\mathbb{R}^n)$, $M_K \leq c \cdot n^{-1/2}$, but $M_K^* \geq c \cdot n^{-1/2}(\log n)^{1/2}$ for some absolute constant $c$.

3. Additional remarks.

In fact, during the proof of Theorem 2 a more general fact was proved.

**Fact.** Let $D$ be an ellipsoid and $K$ be a $p$-convex body. Let

$$N(D, K) \leq e^{on}.$$ 

Denote for an integer $1 \leq k \leq n$ the ratio $\lambda = k/n$. Then for some absolute constant $c$ and

$$\gamma = c \sqrt{\alpha}, \quad k \in (\gamma^2 n, (1 - 2\gamma)^2 n)$$

there exists an orthogonal projection $P$ of rank $k$ such that

$$\left( \frac{p(1 - \sqrt{\lambda})/2}{e_k(D, K)} \right)^{1/p} PD \subset PK.$$ 

In terms of entropy numbers this means

$$\left( \frac{p(1 - \sqrt{k/n})/2}{e_k(D, K)} \right)^{1/p} PD \subset PK,$$

where $e_k(D, K) = \inf \{ \varepsilon > 0 \mid N(D, \varepsilon K) \leq 2^{k-1} \}$.

It is worth to point out that Theorem 2 can be obtained from this results.

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