RATIOS OF VOLUMES AND FACTORIZATION THROUGH $\ell_\infty$

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Abstract

We develop a method effective in the computation of projection constants $\lambda(X)$ for $n$-dimensional normed, as well as quasi-normed spaces $X$. We show that for every centrally symmetric compact body $K$ in $\mathbb{R}^n$ there exist a parallelotope $P$ and a zonoid $Z$ such that $Z \subset K \subset P$, and $\left( \frac{\text{vol}_n(P)}{\text{vol}_n(Z)} \right)^{1/n} \leq \lambda(X)$, where $\lambda(X)$ is the projection constant of the quasi normed space $X = (\mathbb{R}^n, \|\cdot\|)$ having $K$ as its unit ball. Thus, it follows easily that the projection constant of the quasi-normed space $\ell_p^n (0 < p < 1)$ is equivalent to $n^{1/p-1/2}$. This improves the estimate due to T. Peck. The method also yields easily the projection constants of the quasi-normed Schatten classes $s_p^n (0 < p < 1)$, consisting of operators on $\ell_p^n$, and other examples, including classical examples of normed spaces.

In addition, we establish relations between various other known parameters associated with the geometry of normed spaces, such as volume-ratio, the $\ell_\infty$ norm of operators between quasi-normed spaces, and others. We prove that if $X$ is a normed space of dimension $n$ then the unit ball $B$ of $X$ and $B^*$ of $X^*$, contain zonoids $Z_1$ and $Z_2$ respectively, such that $\left( \frac{\text{vol}_n(B)}{\text{vol}_n(Z_1)} \frac{\text{vol}_n(B^*)}{\text{vol}_n(Z_2)} \right)^{1/n} \leq c g l_2(X)$, where $c$ is an absolute constant and $g l_2(X)$ is the Gordon-Lewis parameter of $X$.

We show also that if $K$ is the convex hull of $m$ points in $\mathbb{R}^n$, $n \leq m$, then there is a simplex $\Delta$ inside $K$ satisfying the inequality $\left( \frac{\text{vol}_n(K)}{\text{vol}_n(\Delta)} \right)^{1/n} \leq c \sqrt{\ln \frac{2m}{n}} \left( \frac{\text{vol}_n(\Delta)}{\text{vol}_n(\Delta)} \right)^{1/n}$.

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Introduction

The projection constant \( \lambda(X) \) of a finite dimensional normed space \( X \) is often difficult to compute, but it plays an important role in the classical and in the local theory of Banach spaces. We extend its definition in a natural way so as to include the class of quasi-normed spaces as well, and we present a new method for getting a lower bound for \( \lambda(X) \) in terms of ratios of volumes. This bound allows us for example to to obtain easily the right asymptotic estimate for \( \lambda(\ell_p^n) \) in the case \( 0 < p < 1 \) and dispenses of the logarithmic factor in the estimate obtained by Peck ([Pe]) who used some involved probabilistic method. The method applies also for the Schatten classes \( s_p^n \) \((0 < p \leq 1)\) of operators on \( \ell_2^n \).

Given a centrally symmetric body \( K \) in \( \mathbb{R}^n \) we can endow \( \mathbb{R}^n \) with the quasi-norm defined by

\[
\|x\| = \inf \{ a > 0; \ x \in aK \},
\]

and set \( E = (\mathbb{R}^n, \| \cdot \|) \) to be the \( n \)-dimensional quasi-normed space with \( K \) as its unit ball. Let \( B = B_X \) be the unit ball of a given Banach space \( X \). We define the volume ratio \( \text{vr}(E, X) \), sometimes denoted also by \( \text{vr}(K, B) \), to be

\[
\text{vr}(E, X) = \inf \left( \frac{\text{vol}_n(K)}{\text{vol}_n(T(B))} \right)^{1/n}
\]

where the infimum ranges over all onto linear maps \( T : X \to \mathbb{R}^n \) satisfying \( T(B) \subset K \). We define the external volume ratio \( \text{evr}(E, X) \), denoted sometimes also by \( \text{evr}(K, B) \), to be

\[
\text{evr}(E, X) = \inf \left( \frac{\text{vol}_n(T(B))}{\text{vol}_n(K)} \right)^{1/n}
\]

where the infimum ranges over all onto linear maps \( T : X \to \mathbb{R}^n \) such that \( T(B) \supset K \). For \( 0 < p \leq \infty \), let \( \ell_p^n \) be the space \( \mathbb{R}^n \) equipped with the quasi-norm

\[
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}
\]

and let \( B_p^n \) be its unit ball:

\[
B_p^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n; \ \sum_{i=1}^{n} |x_i|^p \leq 1 \}.
\]

Let \( Q_n = [-1, 1]^n = B_\infty^n \) be the unit cube of \( \mathbb{R}^n \) and \( C_n = B_1^n \) be its polar body:

\[
C_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n; \ \sum_{i=1}^{n} |x_i| \leq 1 \}.
\]

The ratio \( \text{evr}(K, B_\infty^n) = \text{evr}(K, Q_n) \), known also as the cubic ratio of \( K \), was studied by various authors (see [B1], [Ge], [PS]) in relation with the classical volume ratio.
$\nu_{r}(K, B_{2}^{2})$. The **zonoid ratio**, which is in our notation $\nu_{r}(K, B_{t\infty})$, was also studied in [B1].

We prove that if $K$ is the unit ball of a quasi-normed space $X$, then

$$\nu_{r}(K, Q_{m}t322044_{n}) \nu_{r}(K, B_{t\infty}) \leq \lambda(X)$$

geometrically this means that there exist a parallelotope $P$ and a zonoid $Z$ such that $Z \subset K \subset P$, and $\left(\frac{\text{vol}_{n}(P)}{\text{vol}_{n}(Z)}\right)^{1/n} \leq \lambda(X)$, and we study various relations between the above mentioned quantities and other parameters associated with centrally symmetric, and not necessarily convex, bodies $K$.

**Notation.**

If $I$ is a finite set, we shall denote by $|I|$ its cardinality, and by $M_{n,m}$ the set of all matrices $A = [a_{ij}]_{i=1,\ldots,n, j=1,\ldots,m}$ with real entries consisting of $n$ rows and $m$ columns. For $I \subset \{1, 2, \ldots, n\}$ and $J \subset \{1, 2, \ldots, m\}$, let $A_{IJ} = [a_{ij}]_{i \in I, j \in J}$. We will make use of the following Cauchy-Binet formula: If $A \in M_{n,m}$ and $B \in M_{m,n}$, and if $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$, $1 \leq n \leq m$, then

$$\det(AB) = \sum_{I \subset M, |I| = n} \det(A_{NI}) \det(B_{IN}).$$

Let $v_{n} = \pi^{n}/\Gamma(1 + \frac{n}{2})$ denote the volume of the Euclidean ball $B_{2}^{n}$ of $\mathbb{R}^{n}$; then $v_{n}^{1/n} \sim \sqrt{\frac{2\pi e}{n}}$. If $K$ is a centrally symmetric body (not necessarily convex) in $\mathbb{R}^{n}$, we note that by our definition

$$\nu_{r}(K, \ell_{1}^{n}) = \min \left\{ \left(\frac{\text{vol}_{n}(K)}{\text{vol}_{n}(C)}\right)^{1/n} ; C \subset K \text{ is the symmetric convex hull of } n \text{ points} \right\},$$

and

$$\nu_{r}(K, B_{t\infty}) = \min \left\{ \left(\frac{\text{vol}_{n}(K)}{\text{vol}_{n}(Z)}\right)^{1/n} ; Z \subset K \text{ is a zonoid} \right\}.$$

The cubic ratio of $K$ is

$$\nu_{r}(K, Q_{n}) = \min \left\{ \left(\frac{\text{vol}_{n}(P)}{\text{vol}_{n}(K)}\right)^{1/n} ; P \text{ is a parallelotope containing } K \right\},$$

and the classical volume ratio of $K$ is

$$\nu_{r}(K, B_{2}^{n}) = \min \left\{ \left(\frac{\text{vol}_{n}(K)}{\text{vol}_{n}(D)}\right)^{1/n} ; D \subset K \text{ is an ellipsoid} \right\}.$$

Since $B_{2}^{n}$ is a zonoid and $\ell_{1}^{n}$ has uniformly bounded volume ratio, it is clear that

$$\nu_{r}(E, \ell_{2}) \leq \nu_{r}(E, \ell_{1}^{n}) \leq \nu_{r}(E, \ell_{1}^{n}) \nu_{r}(\ell_{1}^{n}, \ell_{2}^{n}) \sim \sqrt{\frac{2e}{\pi}} \nu_{r}(E, \ell_{1}^{*}).$$

**Ratios of Volumes and factorization through $\ell_{\infty}$**

The following proposition is essentially known ([B1], [Ge], [PS]).
Proposition 1. Let $K$ be a centrally symmetric convex body in $\mathbb{R}^n$ and $K^c$ be its polar body with respect to the ordinary scalar product denoted by $\langle , \rangle$. Suppose that $u_i \in \mathbb{R}^n$, $\langle u_i, u_i \rangle = 1$ and $c_i > 0$, $i = 1, \ldots, m$ satisfy

$$\sum_{i=1}^{m} c_i < u_i, x > u_i = x$$

for every $x \in \mathbb{R}^n$. Then

(i) If $u_i \in K^c$, $i = 1 \ldots, m$, there exists a parallelepiped $P$ such that $K \subset P$ and

$$\left( \frac{\text{vol}_n(P)}{\text{vol}_n(Q_n)} \right)^{1/n} \leq \sqrt{e} \left( \frac{\text{vol}_n(Q_n)}{\text{vol}_n(C_n)} \right)^{1/n} = 2\sqrt{e}.$$

(ii) If $u_i \in K$, $i = 1 \ldots, m$, there exists a cross-polytope $C$ such that $C \subset K$ and

$$\left( \frac{\text{vol}_n(C)}{\text{vol}_n(Q_n)} \right)^{1/n} \geq \frac{1}{\sqrt{e}} \left( \frac{\text{vol}_n(Q_n)}{\text{vol}_n(C_n)} \right)^{1/n} \sim 2\sqrt{e}.$$

Proof: If $C \in \mathcal{M}_{n,m}$ is the matrix whose columns are the coordinates of the vectors $\sqrt{c_i} u_i$, $1 \leq i \leq m$, in the canonical basis of $\mathbb{R}^n$, we have $C^* C = I_n$, where $I_n$ denotes the identity on $\mathbb{R}^n$ and thus $\sum_{i=1}^{m} c_i = n$. It follows from the Cauchy-Binet identity that

$$1 = \sum_{I \subset \{1, \ldots, m\}, |I| = n} \left( \prod_{i \in I} c_i \right) \left( \det(u_i)_{i \in I} \right)^2 \leq \left( \begin{array}{c} m \nonumber \end{array} \right) \max_{I \subset \{1, \ldots, m\}, |I| = n} \left( \prod_{i \in I} \det(u_i)_{i \in I} \right)^2 \left( \sum_{I \subset \{1, \ldots, m\}, |I| = n} \prod_{i \in I} c_i \right)^2.$$

Now, since $\sum_{i=1}^{m} c_i = n$, we get by Newton's inequality

$$1 \leq \left( \begin{array}{c} m \nonumber \end{array} \right) n^n \max_{I \subset \{1, \ldots, m\}, |I| = n} \left( \det(u_i)_{i \in I} \right)^2.$$

It follows that

$$\max_{I \subset \{1, \ldots, m\}, |I| = n} |\det(u_i)_{i \in I}|^{1/n} \geq \frac{1}{\sqrt{e}}.$$

In case (i) for some $I \subset \{1, \ldots, m\}, |I| = n$, the parallelepiped $P = \{ x \in \mathbb{R}^n; |< x, u_i >| \leq 1$ for every $i \in I \}$ satisfies the required properties. Case (ii) follows from (i) by replacing $K^c$ with $K$ and taking $C = P^o$. \qed

The following results relate $\text{vr}(K, B_2^n)$ to $\text{evr}(K, Q_n)$. It is a direct consequence of the preceding proposition.

Corollary 2. ([B2],[Ge],[PS]) Let $K$ be a convex symmetric convex body, then

$$\text{vr}(Q_n, B_2^n) \leq \text{evr}(K, Q_n) \text{ vr}(K, B_2^n) \leq \sqrt{e} \text{ vr}(Q_n, B_2^n),$$

where $\text{vr}(Q_n, B_2^n) = \frac{2}{\sqrt{n}} \approx \frac{2}{\sqrt{2\pi}}$.\n
Proof: The left-hand side inequality follows from the definition. For the right hand-side, we may suppose that the Euclidean ball is the maximal volume ellipsoid inside $K$, that is the John ellipsoid of $K$; then by [J], both assumptions (i), (ii) of Proposition 1 are satisfied. The result follows. \qed

The next corollary is an improvement of an estimate due independently to many authors ([BF],[BP], [C], [GI], and B. Maurey in [Pi1]).
**Corollary 3.** There exists a constant \( c > 0 \) such that if \( x_1, \ldots, x_m \in \mathbb{R}^n \), \( m \geq n \) and \( K = \text{conv}(\pm x_i, 1 \leq i \leq m) \), then

\[
\left( \frac{\text{vol}_n(K)}{\text{vol}_n(K')} \right)^{1/n} \leq c \sqrt{\ln \left( \frac{2m}{n} \right)}, \quad \max_{I \subseteq \{1, \ldots, m\}, |I| = n} \left( \frac{\text{vol}_n(\text{conv}(\pm x_i, i \in I))}{\text{vol}_n(K)} \right)^{1/n}.
\]

Proof: Let \( A \in \mathcal{M}_{n,n} \) be a matrix such that the minimal volume ellipsoid containing the body \( K' = AK \) is the Euclidean ball. Then again the assumptions of Proposition 1,(ii) are satisfied. Thus there exists a cross-polytope \( C \subseteq K' \) such that \( \left( \frac{\text{vol}_n(C)}{\text{vol}_n(K')} \right)^{1/n} \geq \frac{1}{\sqrt{e}} \left( \frac{\text{vol}_n(C)}{\text{vol}_n(K')} \right)^{1/n} \). It follows that

\[
\left( \frac{\text{vol}_n(K)}{\text{vol}_n(K')} \right)^{1/n} = \left( \frac{\text{vol}_n(K)}{\text{vol}_n(K')} \right)^{1/n} \leq \left( \frac{\text{vol}_n(K)}{\text{vol}_n(C)} \right)^{1/n} \leq \sqrt{e} \left( \frac{\text{vol}_n(K)}{\text{vol}_n(C)} \right)^{1/n}.
\]

But since \( K' \subseteq B^n_2 \) is the convex hull of \( Ax_1, \ldots, Ax_m \), it follows from [BP], [BF], [CS], [GI] or [Pi1] that for some constant \( d > 0 \), independent of \( n \) and \( m \), we have

\[
\left( \frac{\text{vol}_n(K)}{\text{vol}_n(K')} \right)^{1/n} \leq d \sqrt{\ln \left( \frac{2m}{n} \right)} \left( \frac{\text{vol}_n(C)}{\text{vol}_n(K')} \right)^{1/n}.
\]

Combining the preceding inequalities, we get our estimate. \( \square \)

**Remark.** If the convex body \( K \) is not supposed to be centrally symmetric, then Proposition 1 can be generalized in both cases if we replace the parallelogram \( P \) and the cross-polytope \( C \) by a simplex, and \( Q_n \) and \( C_n \) by the regular simplices circumscribed to \( B^n_2 \) and inscribed in \( B^n_2 \). Observe also that Corollary 3 can be generalized as follows: if \( K = \text{conv}(x_1, \ldots, x_m) \), then

\[
\left( \frac{\text{vol}_n(K)}{\text{vol}_n(K')} \right)^{1/n} \leq d \sqrt{\ln \left( \frac{2m}{n} \right)} \max \{ \left( \frac{\text{vol}_n(\Delta)}{\text{vol}_n(K')} \right)^{1/n} \},
\]

for some constant \( d > 0 \) independent of \( n, m > n + 1 \) and \( x_1, \ldots, x_m \in \mathbb{R}^n \).

If \( E \) is a subspace of \( \mathbb{R}^n \), then \( P_E \) will denote the orthogonal projection onto \( E \).

**Lemma 4.** Let \( B \) be a symmetric body in \( \mathbb{R}^n \) (not necessarily convex). Let \( 1 \leq k \leq n \leq m \) and \( T = VU \), with \( T \in \mathcal{M}_{n,n} \), \( V \in \mathcal{M}_{n,m} \) and \( U \in \mathcal{M}_{m,n} \), rank\( (T) = k \) and \( UB \subseteq \|U\|Q_m \), where \( \|U\| > 0 \). Then

\[
\frac{\text{vol}_k(TB)}{\text{vol}_k(P_{\ker(T)\perp}B)} \leq \frac{k!}{4^k} \sqrt{n \choose k} \lambda_k(B^n) \|U\|^k \max_{\dim(E) = k} \text{vol}_k(P_EVQ_m).
\]
\[
\sqrt{\binom{n}{k}} \|U\|^k \max_{\dim(E) = k} \vol_k(P_{E V Q_m}) \left( \min_{\dim(E) = k} \left( \evr(P_E B, Q_k)^k \vol_k(P_E B) \right) \right)^{-1}.
\]

where \(B^o = \{ y \in \mathbb{R}^n; \langle x, y \rangle \leq 1, \text{ for all } x \in B \} \) and for a subset \(C\) of \(\mathbb{R}^n\), \(\lambda_k(C) =: \max\{\vol_k(\conv(\pm x_1, \ldots, \pm x_k)); x_1, \ldots, x_k \in C\}\).

Proof: By standard linear algebra methods, we have

\[
\vol_k(TB) = \left( \sum_{|I| = |J| = k} (\det(T_{IJ}))^2 \right)^{1/2} \vol_k(P_{\ker(T) \perp B}).
\]

For \(I, J \subset \{1, \ldots, n\}\) with \(|I| = |J| = k\) define \(t_{IJ} = \det(T_{IJ})\) and similarly for \(u_{IJ}\) and \(v_{IJ}\). Then

\[
t_{IJ} = \sum_{K \subset \{1, \ldots, m\}, |K| = k} v_{IK} u_{KJ},
\]

so that

\[
\sum_{|I| = |J| = k} t_{IJ}^2 = \sum_{|I| = |J| = k} \left( \sum_{|K| = |I| = k} v_{IK} u_{KJ} v_{IL} u_{LJ} \right)
\]

\[
= \sum_I \left( \sum_{K,L} v_{IK} v_{IL} \left( \sum_J u_{KJ} u_{LJ} \right) \right) \leq \left( \max_K \sum_{J} u_{KJ}^2 \right) \left( \sum_K \left( \sum_{|K|} v_{IK} \right)^2 \right)
\]

For \(I \subset \{1, \ldots, n\}\), \(\text{card}(I) = k\), let \(\Pi_I : \mathbb{R}^n \to \mathbb{R}^I\) denote the orthogonal projection; we have

\[
\sum_{|K| = k} |v_{IK}| = 2^{-k} \vol_k(\Pi_I V Q_m).
\]

Indeed, \(Q_m = \sum_{j=1}^m [-e_j, e_j]\), hence \(\Pi_I V Q_m = \sum_{j=1}^m [-\Pi_I v_j, \Pi_I v_j] \subset \Pi_I(\mathbb{R}^n) = \mathbb{R}^k\), where \(\{v_j\}_{j=1}^m\) denote the columns of the matrix \(V\); and now it is well known that if \(Z = \sum_{j=1}^m [-z_j, z_j]\) is a zonotope in \(\mathbb{R}^k\) then

\[
\vol_k(Z) = 2^k \sum_{J \subset \{1, \ldots, m\}, |J| = k} |\det(z_j; j \in J)|.
\]

It follows that

\[
\sum_I \left( \sum_{|K|} |v_{IK}| \right)^2 \leq \binom{n}{k} \left(2^{-k} \max_{\dim(E) = k} \vol_k(P_{E V Q_m}) \right)^2.
\]

Observe also that since \(U(B) \subset \|U\| Q_m\), the rows \(U_1, \ldots, U_m\) of the matrix \(U\) are vectors of \(\mathbb{R}^n\) which satisfy \(U_i \in \|U\| B^o\) for \(1 \leq i \leq m\). We have then for \(K \subset \{1, \ldots, m\}\), \(|K| = k\),

\[
\left( \sum_{J \subset \{1, \ldots, n\}, |J| = k} u_{KJ}^2 \right)^{1/2} = \frac{k!}{2^k} \vol_k(\conv(\pm U_i; \ i \in K)).
\]

This may require an explanation: Let \(\{z_i\}_{i=1}^k\) be \(k\) vectors in \(\mathbb{R}^n\) and \(C = \conv(\pm z_i; 1 \leq i \leq k)\). Denote by \(Z \in \mathcal{M}_{k,n}\) the matrix with \(z_i\)'s as rows. Then there is an orthogonal
matrix \( \Lambda \in \mathcal{M}_{n,n} \) such that \( Z \Lambda = [W, 0] \), where \( W \in \mathcal{M}_{k,k} \) is the matrix with rows \( \{w_i\}_{i=1}^k \) and 0 denotes the zero matrix in \( \mathcal{M}_{k,n-k} \). Obviously, \( WW^* = ZZ^* \), and we obtain that

\[
\text{vol}_k(C) = \text{vol}_k(\text{conv}(\pm w_i, 1 \leq i \leq k)) = \text{vol}_k(W(C_k)) = \frac{2^k}{k!} |\text{det}(W)|
\]

\[
= \frac{2^k}{k!} \sqrt{\text{det}(WW^*)} = \frac{2^k}{k!} \sqrt{\text{det}(ZZ^*)} = \frac{2^k}{k!} \left( \sum_{J \subseteq \{1, \ldots, n\}, |J| = k} (\text{det}(Z_{KJ}))^2 \right)^{1/2},
\]

where \( K = \{1, \ldots, k\} \).

It follows that

\[
\left( \max_K \sum_J u_{KJ}^2 \right)^{1/2} \leq \frac{k!}{2^k} \|U\|^k \lambda_k(B^o).
\]

Finally, we have by duality

\[
\frac{k!}{2^k} \lambda_k(B^o) = 2^k \left( \min_{\dim(E) = k} \left( \text{evr}(P_E B, Q_k)^k \text{vol}_k(P_E B) \right) \right)^{-1}.
\]

The lemma follows. \(\square\)

**Lemma 5.** Let \( K \) be a symmetric body in \( \mathbb{R}^n \) (not necessarily convex). Let \( 1 \leq n \leq m \) and \( T = VU \), with \( T \in \mathcal{M}_{n,n} \), \( V \in \mathcal{M}_{n,m} \) and \( U \in \mathcal{M}_{m,n} \), with \( \text{rank}(T) = n \) and \( U(K) \subset \|U\|Q_m \), where \( \|U\| > 0 \). Then

\[
\text{evr}(K, Q_n) \cdot |\text{det}(T)|^{1/n} \leq \|U\| \left( \frac{\text{vol}_n(V(Q_m))}{\text{vol}_n(K)} \right)^{1/n}.
\]

Proof: Apply Lemma 4 with \( k = n \). \(\square\)

Let now \( E \) and \( F \) be two \( n \)-dimensional quasi-normed spaces with unit balls \( B_E \) and \( B_F \) respectively, and for \( T \in L(E, F) \) define

\[
\gamma_{\infty}(T) = \inf \{\|U\| |\|V\|\}
\]

where the infimum is taken over all the factorizations \( T = VU \), \( U \in L(E, \ell_\infty) \), \( V \in L(\ell_\infty, F) \), and if \( B_\infty \) denotes the unit ball of \( \ell_\infty \), \( \|U\| = \inf\{a > 0; U(B_E) \subset aB_\infty \} \) and \( \|V\| = \inf\{b > 0; V(B_\infty) \subset bB_F \} \).

For a centrally symmetric body \( K \) in \( \mathbb{R}^n \), if \( E \) is the quasi-normed space such that \( B_E = K \), we define the projection constant of \( E \) or, of \( K \), to be

\[
\lambda(K) = \lambda(E) = \gamma_{\infty}(I)
\]

where \( I : E \rightarrow E \) denotes the identity.
Theorem 6. Let $T \in \mathcal{L}(E, F)$, where $E = (\mathbb{R}^n; \| \cdot \|_E)$, and $F = (\mathbb{R}^n; \| \cdot \|_F)$ are $n$-dimensional quasi-normed spaces. Then,

$$\text{evr}(E, \ell^n_\infty) \text{vr}(F, \ell_\infty) | \det T |^{1/n} \leq \gamma_\infty(T) \left( \frac{\text{vol}_n(B_F)}{\text{vol}_n(B_E)} \right)^{1/n}.$$  

Proof: By Lemma 5, taking $K = B_E$, we obtain that for any factorization $T = VU$ through $\ell^n_\infty$

$$\text{evr}(E, \ell^n_\infty) | \det(T) |^{1/n} \leq \|U\| \left( \frac{\text{vol}_n(V(Q_m))}{\text{vol}_n(B_E)} \right)^{1/n}.$$  

Thus, if $Z$ is the zonoid $\|V\|^{-1}V(Q_m)$, then $Z \subset B_F$. \hfill \Box

Corollary 7. If $E$ is an $n$-dimensional quasi-normed space, with unit ball $B_E$, then

$$\lambda(E) \geq \text{evr}(E, \ell^n_\infty) \text{vr}(E, \ell_\infty)$$

$$= \max \left\{ \left( \frac{\text{vol}_n(P)}{\text{vol}_n(Z)} \right)^{1/n} ; \text{ Z zonoid, P parallelotope, } Z \subset B_Z \subset P \right\}.$$  

Remarks.
1) It is easy to prove that, under the hypothesis of the preceding corollary, we have

$$\text{evr}(E, \ell^n_\infty) \leq \inf \left\{ \|U\| \left( \frac{\text{vol}_n(V(B_\infty))}{\text{vol}_n(B_E)} \right)^{1/n} \right\}$$

where the infimum is taken over all linear operators $U : X \to \ell_\infty$ and $V : \ell_\infty \to X$ such that $VU$ is the identity on $\mathbb{R}^n$.

2) By Lemma 5, if $E, F$ are $n$-dimensional, then

$$\sup_{T : E \to F, T \neq 0} \frac{| \det(T) |^{1/n}}{\gamma_\infty(T)} \leq \frac{1}{\text{evr}(E, \ell^n_\infty) \text{vr}(F, \ell_\infty) \left( \frac{\text{vol}_n(B_F)}{\text{vol}_n(B_E)} \right)^{1/n}}.$$  

In particular it follows that

$$\text{evr}(E, \ell^n_\infty) \text{evr}(F, \ell^n_\infty) \text{vr}(E, \ell_\infty) \text{vr}(F, \ell_\infty) \leq \inf_{T \in \mathcal{L}(E, F)} \left\{ \gamma_\infty(T) \gamma_\infty(T^{-1}) \right\}.$$  

Let us recall now some definitions; if $X$ is a normed space, we define the following quantities:

1) The Gordon-Lewis constant (according to G. Pisier [Pi1]) $g_{l_2}(X)$ is the least constant $C$ such that for any operator $T \in \mathcal{L}(X, l_2)$,

$$\gamma_1(T) \leq C \pi_1(T).$$

2) The Gordon-Lewis constant $g_{l}(X)$ of $X$ (see [GL]), is the least constant $C$ such that for any normed space $Y$ and any operator $T \in \mathcal{L}(X, Y)$,

$$\gamma_1(T) \leq C \pi_1(T).$$
3) The weak Gordon-Lewis constant (according to K. Ball [B1], a different definition was introduced in [P13]) \( \text{wrgl}_2(X) \) of an \( n \)-dimensional space \( X \) is the least constant \( K \) such that for any \( T \in L(X, \ell_2) \)

\[
\left( \vol_n(T(B_X)) \right)^{1/n} \leq \frac{2K}{n} \gamma_1(T) .
\]

(For the definition of the ideal norm \( \gamma_p(T) \) the reader may refer to the books [Kô], [Pie], [Pi2], [Tj]). It was proved in [B2] that for some constant \( c > 0 \), independent of \( X \),

\[
(*) \quad \text{wrgl}_2(X) \leq c \min\{ \text{gl}_2(X), \vr(X, \ell^{\dim(X)}_2) \} .
\]

and moreover, if \( X \) is finite dimensional,

\[
\text{wrgl}_2(X) \sim \vr(X, \ell_\infty)
\]

in the sense that there exist absolute constants \( c_1 \) and \( c_2 > 0 \), such that,

\[
(**) \quad c_1 \vr(X, \ell_\infty) \leq \text{wrgl}_2(X) \leq c_2 \vr(X, \ell_\infty) .
\]

To see better how these numbers are related, let us define also the local unconditional constant of \( X \), \( \chi_u(X) \) ([GL]): this is the least constant \( C \) such that for any finite-dimensional subspace \( F \subset X \) there exists a Banach space \( U \) with a finite unconditional basis constant \( \chi(U) \), and operators \( A \in L(F, U) \) and \( B \in L(U, X) \) with \( BA = i_F \) (the inclusion of \( F \) into \( X \)), and satisfying

\[
\|A\| \|B\| \chi(U) \leq C .
\]

Of course if \( E \) is finite-dimensional, \( \chi_u(E) = \chi_u(E^*) \), and clearly subspaces of \( L_1 \), and quotients of \( L_\infty \), have finite l.u.s.t. constants. We see then, by results of [GL] and inequalities \((*)\) and \((**)\) that for some absolute constants \( c \) and \( d > 0 \), we have

\[
(***) \quad c \leq d \text{wrgl}_2(X) \leq \text{gl}_2(X) \leq \text{gl}(X) \leq \chi_u(X) \leq \chi(X) \leq \sqrt{\dim(X)} .
\]

The following result improves inequality \((*)\):

**Corollary 8.** There is an absolute positive constant \( c \) such that for every finite dimensional normed space \( F \), if \( F^* \) denotes the dual of \( F \), we have

\[
\vr(F, \ell_\infty) \vr(F^*, \ell_\infty) \sim \text{wrgl}_2(F) \text{wrgl}_2(F^*) \leq c \min(\text{gl}_2(F), \text{gl}_2(F^*)) .
\]

**Proof.** Take \( E \) in Theorem 6 to be the space \( \ell_2^n \), then \( \text{evr}(E, \ell_\infty) = \left( \frac{2^n}{\vol_n(B_{\ell_2^n})} \right)^{1/n} \sim \sqrt{n} . \)

Now, multiplying the inequality of Theorem 6 by \( \left( \vol_n(B_{F^*}) \right)^{1/n} \), and using Santaló’s inequality, we have that

\[
\text{wrgl}_2(F) \left( \frac{2^n}{\vol_n(B_{F^*})} \right)^{1/n} \leq \frac{c}{n} \gamma_\infty(T) .
\]
The inequality
\[ wrgl_2(F) \leq c \cdot gl_2(F) \]
follows now immediately from the fact that \( T^* \in L(F^*, l_2^*) \), and that \( \gamma_\infty(T) = \gamma_1(T^*) \leq gl_2(F^*) \pi_1(T^*) \), and the definition of \( wrgl_2(F^*) \). For the second inequality replace \( F \) by \( F^* \). We use then (**) \( \square \)

**Remarks:**

a) In particular, since \( gl_2(F^*) \leq \sqrt{\dim(F)} \) and \( \lambda(F^*) \leq \sqrt{\dim(F)} \), it follows from Corollaries 7 and 8 that if \( wrgl_2(F^*) \sim \sqrt{\dim(F)} \), then \( \text{evr}(F^*, \ell_\infty^{\dim(F)}) \sim 1 \), i.e. there are a cross-polytope \( C \) contained in \( B_F \) (see the comments before Proposition 10) such that \( \left( \frac{\omega_n(B_F)}{\text{vol}_n(C)} \right)^{1/n} \sim 1 \); in ‘volume sense’ \( B_F \) is equivalent to a cross-polytope; moreover both \( \lambda(F^*) \), \( gl_2(F^*) \) and \( gl_2(F) \) are then asymptotically equivalent to \( \sqrt{\dim(F)} \).

b) If \( gl_2(F) \sim 1 \), which happens for example when \( F \) is ‘well’ complemented in a Banach lattice, then both \( B_F \) and \( B_{F^*} \) are in ‘volume sense’ equivalent to zonoids.

As we shall see, the estimate given by Corollary 7 for \( \lambda(K) \) can provide a good information about its real value; however, we ignore whether it is a sharp estimate. The weaker estimate: \( \text{evr}(F, \ell_\infty^{\dim(F)}) \leq \lambda(F) \) is not sharp, as it is shown by the following example.

**Example:** There is a \( n \)-dimensional subspace \( F \) of \( \ell_\infty^m \) such that
\[ \lambda(F) \sim \sqrt{n} \text{ and } \text{evr}(F, \ell_\infty^m) \leq \sqrt{2e}, \]
(for another example of the same type, see [B1]). In order to show this we first prove:

a) Let \( E \) be a \( n \)-dimensional subspace of \( \ell_\infty^m \) with \( B_E \) as unit ball. Then
\[ \text{evr}(B_E, Q_n) \leq \left( \sqrt{\binom{m}{n}} \right)^{1/n} \leq \sqrt{\frac{me}{n}}. \]

Let \( P_E \) be the orthogonal projection of \( \mathbb{R}^m \) onto \( E \). Then \( B_E \) can be described as follows:
\[ B_E = \{ x \in E; \ |x, P_Ee_i| \leq 1, \text{ for } 1 \leq i \leq m \}. \]

Let \( u_1, \ldots, u_n \) be an orthonormal basis of \( E \), and set \( P_E = \sum_{i=1}^n \sum_{j=1}^n \mu_{ij} e_i \otimes u_j \). Denote by \( M \in \mathcal{M}_{m,n} \) the matrix \( (\mu_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \) which represents \( P_E \). Since \( P_E \) is an orthogonal projection, the matrix \( M^*M \in \mathcal{M}_{n,n} \) is the identity on \( E \). Therefore by the Cauchy-Binet formula,
\[ 1 = \det(M^*M) = \sum_{I \subset \{1, \ldots, m\}, |I| = n} \left( \det(M_{IN}) \right)^2, \]
where \( N = \{1, \ldots, n\} \). The matrix \( M_{IN} \) represents the operator \( P_E|\text{span}\{e_i, i \in I\} \) which maps \( e_i \) to \( P_E(e_i) = \sum_{j=1}^n \mu_{ij} u_j \) for every \( i \in I \). Hence, denoting by \( C^I \) the cross-polytope \( \text{conv}(\pm P_E(e_i), i \in I) \), we obtain
\[ \text{vol}_n(C^I) = \frac{2^n}{n!} |\det(M_{IN})|. \]
It follows that
\[ \sum_{|I|=n} \left( \frac{n!}{2^n} \text{vol}_n(C^I) \right)^2 = 1. \]

Therefore there exists \( J \subset \{1, \ldots, m\}, |J| = n \) such that
\[ \text{vol}_n(C^J) \geq \frac{2^n}{n! \sqrt{\binom{m}{n}}}. \]

Now the parallelootope \( Q^J = \{ x \in E; | < x, P_E e_i > | \leq 1 \text{ for } i \in J \} \) contains \( B_E \), and moreover since \( (Q^J)^o = C^J \), we have that \( \text{vol}_n(Q^J) \text{vol}_n(C^J) = 4^n/n! \), from which it follows that
\[ (\text{vol}_n(Q^J))^{1/n} \leq 2 \left( \frac{m}{n} \right)^{1/n} \leq 2 \sqrt{\frac{\text{me}}{n}}. \]

But it follows from [V] that \( \text{vol}_n(B_E) \geq \text{vol}_n(Q_n) = 2^n \). Therefore we have
\[ \text{evr}(B_E, Q_n) \leq \sqrt{\frac{\text{me}}{n}}. \]

b) Under the same hypothesis as in a), if \( i : \ell_2^m \to \ell_\infty^m \) denotes the identity map, and if \( i_E \) denotes its restriction to \( E \), we have
\[ \|i_E\| \geq \sqrt{\frac{n}{m}}. \]

In fact, if we set \( u_i = \sum_{j=1}^m u_{ij} e_j, 1 \leq i \leq n \), then
\[ i_E = \sum_{i=1}^n u_i \otimes u_i = \sum_{i=1}^n \sum_{j=1}^m u_{ij} u_i \otimes e_j, \]
so that
\[ \|i_E\|^2 = \max_{a_1^2 + \ldots + a_n^2 \leq 1} \left( \max_{j=1, \ldots, m} \left( \sum_{i=1}^n a_i u_{ij} \right)^2 \right) = \max_{j=1, \ldots, m} \left( \max_{a_1^2 + \ldots + a_n^2 \leq 1} \left( \sum_{i=1}^n a_i u_{ij} \right)^2 \right) \]
\[ = \max_{j=1, \ldots, m} \sum_{i=1}^n u_{ij}^2 \geq \frac{1}{m} \sum_{j=1}^m \sum_{i=1}^n u_{ij}^2 = \frac{n}{m}. \]

c) Let now \( F \) be a \( n \)-dimensional subspace of \( \ell_\infty^m \) and let \( q_F : \ell_\infty^m \to \ell_\infty^m/F \) be the quotient mapping. If \( P \) is any linear projection of \( \ell_\infty^m \) onto \( F \), set \( E = i^{-1}(\ker P) \) and \( Q = I_\infty - P \), where \( I_\infty \) denotes the identity mapping on \( \ell_\infty^m \). If we set
\[ r_Q(q_F y) = Q y \quad \text{for every } y \in \ell_\infty^m, \]
we get a mapping \( r_Q : \ell^m_\infty / F \to \ker P \) such that \( r_Q q_F \) is the identity on \( \ker P \), hence \( i_E = r_Q q_F i_E \), and moreover \( \|r_Q\| = \|Q\| = \|I_\infty - P\| \), so that
\[
\|i_E\| \leq \|I_\infty - P\| \|q_F i_E\|.
\]
Let now \( m = 2n \); then it follows from a result of Kashin ([K1]) that for some constant \( c > 0 \), independent of \( n \), there exists an \( n \)-dimensional subspace \( F \) of \( \ell^2_\infty \) satisfying with the previous notation
\[
\|q_F i\| \leq \frac{c}{\sqrt{n}}.
\]
Applying b), we get
\[
\frac{1}{\sqrt{2}} = \sqrt{\frac{n}{2n}} \leq \|i_E\| \leq \|I_\infty - P\| \|q_F i_E\| \leq \|I_\infty - P\| \|q_F i\| \leq (1 + \|P\|) \frac{c}{\sqrt{n}}
\]
for every projection \( P : \ell^m_\infty \to F \), with \( i^{-1}(\ker P) = E \). It follows that
\[
\lambda(F) \geq \frac{\sqrt{n}}{c\sqrt{2}} - 1,
\]
but by a) applied to \( F \), we have
\[
\operatorname{evr}(B_F, Q_n) \leq \sqrt{2e}.
\]

**Theorem 9.** For every \( 0 < p \leq 1 \) there exists a constant \( c(p) > 0 \) such that for every integer \( n \), we have
\[
c(p) n^{\frac{1}{p} - \frac{1}{2}} \leq \operatorname{evr}(\ell^n_p, \ell^n_\infty) \leq \lambda(\ell^n_p) \leq n^{\frac{1}{p} - \frac{1}{2}}.
\]

Proof: Observe that a parallelootope contains the unit ball \( B^n_p \) of \( \ell^n_p \), \( 0 < p \leq 1 \), if and only if it contains \( C_n = B^n_1 \). Therefore, since \( \operatorname{vr}(C_n, B^n_2) \) is bounded and hence from Corollary 2, \( \operatorname{evr}(C_n, Q_n) \sim \sqrt{n} \), it follows from Corollary 7
\[
\lambda(\ell^n_p) \geq \operatorname{evr}(B^n_p, Q_n) = \operatorname{evr}(C_n, Q_n) \left( \frac{\text{vol}_n(C_n)}{\text{vol}_n(B^n_1)} \right)^{1/n} \geq c(p) \sqrt{n} n^{-1 + \frac{1}{p}} = c(p) n^{\frac{1}{p} - \frac{1}{2}}.
\]
The upper estimate is trivial, since the distance between \( \ell^n_p \) and \( \ell^n_1 \) is \( n^{1/p - 1} \), and \( \lambda(\ell^n_1) < \sqrt{n} \).

**Remark.** The preceding lower estimate for \( \lambda(\ell^n_p) \) has been obtained in [Pe], with an extra multiplicative \( \ln(n) \), using a much more involved proof.
We also observe that easy calculation of \( \operatorname{evr}(\ell^n_p, Q_n) \) yields the known asymptotic estimates of the projection constants for all values of \( p \geq 1 \).

For \( 0 < p \leq +\infty \), let \( s^n_p \) be the \( n^2 \)-dimensional space of all real \([n \times n]\) matrices \( A \) equipped with the quasi-norm
\[
\|A\|_p = \left( \sum_{i=1}^{n} \lambda_i^p \right)^{1/p},
\]
where \( (\lambda_1, \ldots, \lambda_n) \) are the eigenvalues of \( (A^* A)^{1/2} \), and let \( S^n_p \) be the unit ball of \( s^n_p \).
Theorem 10. For every $0 < p \leq 1$, there exists a positive constant $a(p)$ such that

$$a(p)n^{1/p} \leq \text{evr}(S^n_p, Q_{n^2}) \leq \lambda(s^n_p) \leq n^{1/p}.$$  

Proof: By Corollary 8 of [S], for some constant $d(p) > 0$, $(\text{vol}_{n^2}(S^n_p))^{1/n^2} \sim d(p)n^{-\left(\frac{1}{2} + \frac{1}{p}\right)}$ (the proof of [S] considers only the case $p \geq 1$, but it is easily seen that it yields this estimate for $0 < p \leq 1$). As in Theorem 9, we have

$$\text{evr}(S^n_p, Q_{n^2}) = \text{evr}(S^n_1, Q_{n^2})\left(\frac{\text{vol}_{n^2}(S^n_1)}{\text{vol}_{n^2}(S^n_p)}\right)^{1/n^2} \geq d(p)\text{evr}(S^n_1, Q_{n^2})n^{-1+\frac{1}{p}}.$$  

But $S^n_1 \subset S^n_2 \subset \sqrt{n} S^n_1$ and $\text{vr}(S^n_1, B^n_{2^2}) \leq \left(\frac{\text{vol}_{n^2}(S^n_1)}{\text{vol}_{n^2}(S^n_2)}\right)^{1/n^2} \sqrt{n} \leq c_1$. Hence, by Corollary 2, \text{evr}(S^n_1, Q_{n^2}) \geq c_2 n$, from which the lower estimates follow. For the upper estimate, observe that

$$\lambda(s^n_1) \leq \lambda(s^n_p) d(s^n_p, s^n_1) \leq n^{1/p} n^{\frac{1}{p} - 1} = n^{\frac{1}{p}}$$

where $d(\ldots, \ldots)$ denotes here the Banach-Mazur distance. \hfill \Box

Remark  

If $K$ is a centrally symmetric convex body in $\mathbb{R}^n$, let

$$K^o = \{x \in \mathbb{R}^n; <x, y> \leq 1 \text{ for every } x \in K\}$$

be its polar body; Then for some absolute constant $c > 0$,

$$\frac{2}{\pi} \leq \frac{\text{evr}(K, Q_n)}{\text{vr}(K^o, C_n)} \leq c.$$  

Indeed,

$$\frac{\text{evr}(K, Q_n)}{\text{vr}(K^o, C_n)} = \inf_{P \subset K} \sup_{C \subset K^o} \left(\frac{\text{vol}_n(P)}{\text{vol}_n(K)} \frac{\text{vol}_n(C)}{\text{vol}_n(K^o)}\right)^{1/n}$$

$$\geq \inf_{P \subset K} \left(\frac{\text{vol}_n(P)}{\text{vol}_n(K)} \frac{\text{vol}_n(P^o)}{\text{vol}_n(K^o)}\right)^{1/n} = \left(\frac{4^n}{n! \text{vol}_n(K) \text{vol}_n(K^o)}\right)^{1/n},$$

and by Santalò’s inequality we get

$$\frac{\text{evr}(K, Q_n)}{\text{vr}(K^o, C_n)} \geq \left(\frac{4^n}{n! n^{2n}}\right)^{1/n} \geq \frac{2}{\pi}.$$  

On the other hand, by the inverse Santalò’s inequality (see [BM] or [Pi2])

$$\frac{\text{evr}(K, Q_n)}{\text{vr}(K^o, C_n)} \leq \sup_{C \subset K^o} \left(\frac{\text{vol}_n(C)}{\text{vol}_n(K) \text{vol}_n(K^o)}\right)^{1/n} \leq c.$$  

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If we suppose that $K$ or $K^\circ$ is a zonoid, we have using [R], or [GMR], that
\[ \text{vol}_n(K) \text{vol}_n(K^\circ) \geq \frac{4^n}{n!}, \]
so that
\[ \frac{\text{evr}(K, Q_n)}{\text{vr}(K^\circ, C_n)} \leq 1. \]

**Proposition 11.** Let $Z$ be a zonoid in $\mathbb{R}^n$. Then $\text{cr}(Z, B^n_1) \geq \text{vr}(Q_n, C_n) \geq \sqrt{n}/e$.

**Proof:** Let us observe first that
\[ \text{vr}(Q_n, C_n) = \left( \max \{ |\det(\theta_{ij}, 1 \leq i, j \leq n)|; |\theta_{ij}| \leq 1 \} \right)^{1/n} \geq \frac{\sqrt{n}}{e}. \]
Indeed, any cross-polytope $C \subset Q_n$ has the form $\text{conv}(\pm \sum_{j=1}^n \theta_{ij} e_j, 1 \leq i \leq n)$, for some choice of the $n \times n$ matrix $\Theta = (\theta_{ij})$ and clearly
\[ \text{vol}_n(C) = \frac{2^n}{n!} |\det(\Theta)| = \frac{\text{vol}_n(Q_n)}{n!} |\det(\Theta)|. \]
By Hadamard’s inequality, $|\det(\Theta)| \leq \prod_{j=1}^n \left( \sum_{j=1}^n \theta_{ij}^2 \right)^{1/2} \leq n^{n/2}$, hence
\[ \text{vr}(Q_n, C_n) \geq \left( \frac{n!}{n^{n/2}} \right)^{1/n} \geq \frac{\sqrt{n}}{e}. \]

Since a zonoid can be approximated by zonotopes in the Hausdorff metric, we may reduce to the latter case and suppose that $Z = \sum_{j=1}^m [-z_j, z_j]$, for some $z_j \in \mathbb{R}^n$, $1 \leq j \leq m$ and $n \leq m$.

Let $A \in M_{n,m}$ be the matrix with the coordinates of $z_j$, $1 \leq j \leq m$ in the canonical basis of $\mathbb{R}^n$ as columns. If $x_1, \ldots, x_n$ are points in $Z$, then they have the form $x_i = \sum_{j=1}^m \theta_{ij} z_j$, with $\theta_{ij} \in [-1, 1]$, so letting $C = \text{conv}(\pm x_1, \ldots, \pm x_n) \subset Z$, and denoting by $L = [x_1, \ldots, x_n] \in M_{n,n}$ the corresponding matrix, and by $\Theta \in M_{m,n}$ the matrix with entries $\theta_{ji}$ in the $i$-th row and $j$-th column, for $1 \leq i \leq m$, $1 \leq j \leq n$, we have that $L = A\Theta$. By the Cauchy-Binet formula,
\[ \det(L) = \sum_{I \subset \{1, \ldots, m\}, |I| = n} \det(A_{NI}) \det(\Theta_{IN}). \]
But $\text{vol}_n(Z) = 2^n \left( \sum_{I \subset \{1, \ldots, m\}, |I| = n} |\det(A_{NI})| \right)$, and hence
\[ 2^n |\det(L)| \leq \text{vol}_n(Z) \max_{|I| = n} |\det(\Theta_{IN})| \leq (n!)^{\frac{\text{vol}_n(Z)}{\text{vr}(Q_n, C_n)n}}. \]
Therefore
\[ \left( \frac{\text{vol}_n(Z)}{\text{vol}_n(C)} \right)^{1/n} = \left( \frac{\text{vol}_n(Z)}{\frac{2^n}{n!} |\det(L)|} \right)^{1/n} \geq \text{vr}(Q_n, C_n) \geq \frac{\sqrt{n}}{e}. \]
Remarks.

1) The estimate \( \vr(Q_n, C_n) \geq \frac{(n!)^{1/n}}{\sqrt[n]{n}} \) is sharp, in the case when \( n = 2^k, k = 1, 2, \ldots \) (use Walsh matrix). For an upper estimate of \( \vr(K, C_n) \), valid for every convex symmetric body \( K \), observe that the quantity

\[
\max_{x_1, \ldots, x_n \in K} \det(x_1, \ldots, x_n)
\]

decreases under Steiner symmetrisation of \( K \) (see for instance [M]). It follows that

\[
\vr(K, C_n) \leq \vr(B_2^n, C_n) = \left( \frac{n!}{2^n} \right)^{1/n} \sim \sqrt[2n]{\frac{n}{2e}}.
\]

This estimate was proved in [K2], up to a multiplicative constant.

2) Proposition 11 allows to give an easy geometric proof of the following result, which is also a consequence of the fact, originally due to Bourgain and Milman [BM], that the set of all finite-dimensional subspaces \( \{F\} \) of an infinite-dimensional normed space of cotype 2, have uniformly bounded volume ratios \( \vr(F, \ell_2^{\dim(F)}) \) (see also [Pi2], [Tj], and [GK] for the general quasi-normed case); this applies in particular for \( \ell_1 \) which has cotype 2: In this case, we see that every zonoid \( Z \) in \( \mathbb{R}^n \) satisfies \( \vr(Z^o, B_2^n) \leq e \sqrt{\frac{n}{2}} \). Indeed, by Corollary 2, the remarks preceding Proposition 11 and Proposition 11 itself, we have successively

\[
\vr(Z^o, B_2^n) \leq \sqrt{e} \sqrt{\frac{2n}{\pi}} \cdot \frac{1}{\vr(Z^o, Q_n)} \leq \sqrt{\frac{2n}{\pi}} \cdot \frac{\pi/2}{\vr(Z, C_n)} \leq \sqrt{\frac{n}{2}} \cdot \frac{e}{\sqrt{n}} \leq e \sqrt{\frac{n}{2}}.
\]

It was proved by K. Ball ([B2]), using more involved arguments, that if \( Z \) is a zonoid in \( \mathbb{R}^n \), one has always

\[
\vr(Z^o, B_2^n) \leq \vr(C_n, B_2^n) \sim \sqrt{\frac{2e}{\pi}}.
\]

It may be observed that finding the exact maximum of \( \vr(K, Q_n) \) over all the centrally symmetric convex bodies \( K \) in \( \mathbb{R}^n \) is still an open problem for \( n \geq 3 \) (see [Ba], where it is solved for \( n = 2 \)).

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