On the Euclidean sections of some Banach and operator spaces

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1 Introduction.

Following the studies of Milman and Schechtman ([M-S1] [M-S2]) and of [G-G-M] and [G], we investigate here the "large" Euclidean sections of centrally symmetric convex bodies in $\mathbb{R}^n$, or equivalently, the Banach-Mazur distance of subspaces with "big dimension" of a finite dimensional normed space to an Euclidean space. We give first a general result about subspaces of a normed spaces which possesses a system of vectors satisfying a $(C,s)$-estimate (see the definition below), and apply these results to give sharp estimates of the distance to $\ell_s^k$ of $k$-dimensional subspace of $\ell_n^m$ for $q > 2$.

We treat then the same problem for subspaces of some normed spaces of operators from $\mathbb{R}^n$ to $\mathbb{R}^n$, and in particular of Schatten classes, for $q \geq 2$. These results are obtained mainly by the use of Gaussian operators ([G]), and so we obtain random subspaces.

Let $E$ be a $n$-dimensional normed space. We say that a family $u_1, \ldots, u_N$ of vectors of $E$, with $N \leq n$, satisfies a $(C,s)$-estimate for $C > 0$ and $s > 0$, if for all $(t_i^*)_{i=1}^N \in \mathbb{R}^N$ and all $m = 1, \ldots, N$, one has

$$\frac{C}{m^{1/s}} \left( \sum_{i=1}^m (t_i^*)^2 \right)^{1/2} \leq \| \sum_{i=1}^N t_i u_i \| \leq \left( \sum_{i=1}^N t_i^2 \right)^{1/2},$$

where $(t_i^*)_{i=1}^N$ denotes the decreasing rearrangement of the sequence $(|t_i|)_{i=1}^N$.

By a result of Bourgain and Szarek [B-S], there exists a constant $C > 0$ such that for any $n$, any $n$-dimensional normed space contains a sequence $u_1, \ldots, u_N$, with $N \geq \frac{n}{2}$, satisfying a $(C,2)$-estimate. We shall be interested here with $s \geq 2$. It is easy to see that for $q \geq 2$, $\ell_q^n$ satisfies a $(1,s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q'}$. It may be also observed that if we define $s' > 0$ by $\frac{1}{s'} = \frac{1}{s} - \frac{1}{\ln(n)}$, and if $(u_1, \ldots, u_N)$ satisfies a $(C,s)$-estimate, then it satisfies also a $(C/e, s')$-estimate; so one can restrict the study to the case when
s \leq \ln(n)$. Finally, we denote by $d(E, F)$ the Banach-Mazur distance between two normed spaces $E$ and $F$.

Let us recall the following estimates for the norm of Gaussian operators: if $E$ is a Banach space and $(v_j)_{j=1}^N \in E$, we define a Gaussian operator $G_\omega : \ell^2_2 \to E$ by

$$G_\omega = \sum_{i=1}^k \sum_{j=1}^N g_{ij}(\omega) e_i \otimes v_j : \ell^2_2 \to E,$$

where $(e_1, \ldots, e_k)$ denotes the canonical basis of $\ell^2_k$ and $g_{ij}$ are pairwise independent real Gaussian random variables for $1 \leq i \leq k, 1 \leq j \leq N$. We have the following inequalities [G] :

$$\mathbb{E} \left[ \sum_{j=1}^N g_j v_j \right] - a_k \sup_{1 \leq j \leq N} t_j^2 = 1 \quad \sum_{j=1}^N t_j v_j \leq \mathbb{E} \inf_{|x|=1} ||G_\omega(x)|| \tag{2}$$

and

$$\mathbb{E} \sup_{|x|=1} ||G_\omega(x)|| \leq \mathbb{E} \left[ \sum_{j=1}^N g_j v_j \right] + a_k \sup_{1 \leq j \leq N} t_j^2 = 1 \quad \sum_{j=1}^N t_j v_j \tag{3}$$

with

$$a_k = \sqrt{2 \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}} \leq \sqrt{k}.$$

2 Euclidean sections of Banach spaces.

The main result of this part is

**Theorem 1** Let $E$ be a $n$-dimensional normed space, and for $n \geq N \geq n/2$, let $(u_i)_{i=1}^N \in E$ satisfy a $(C, s)$-estimate for $s > 2$ and $C > 0$. Let $q$ satisfy $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. Then for some universal constants $c_i, d_i$, $1 \leq i \leq 3$, and for all integers $k$, $1 \leq k \leq N$, there exists a $k$-dimensional subspace $F^k$ of $E$ such that

(i) If $k \leq \frac{1}{2} \left( \mathbb{E} \left[ \sum_{j=1}^N g_j u_j \right] \right)^2$, then $d(F^k, \ell^2_2) \leq 3$.

(ii) If $\frac{1}{4} \left( \mathbb{E} \left[ \sum_{j=1}^N g_j u_j \right] \right)^2 \leq k \leq \frac{1}{2} \left( \mathbb{E} \left[ \sum_{j=1}^N g_j u_j \right] \right)^2$, then $d(F^k, \ell^2_2) \leq \frac{d_1 \sqrt{k}}{C \sqrt{q} n^{1/q}}$.

(iii) If $c_1 q e^{-q_n} \leq k \leq c_2 n$, $d(F^k, \ell^2_2) \leq \frac{d_2 k^{1/2 - 1/q}}{C \ln(1 + n/k)}$.
(iv) If $c_2 n \leq k \leq N$, then $d(F^k, \ell_2^k) \leq d_0 k^{1/s}$.

Moreover, the spaces $F^k$, $1 \leq k \leq N$, can be chosen randomly with high probability as subspaces of the linear span of $(u_i)_{i=1}^N$.

**Proof:**

Let $U = \text{span} \{u_1, \ldots, u_N\}$; we define a Gaussian operator $G_\omega : \ell_2^k \to U$ by

$$G_\omega = \sum_{i=1}^k \sum_{j=1}^N g_{ij}(\omega)e_i \otimes u_j.$$ 

Observe that $\sup_{|x|=1} \| \sum_{j=1}^N t_j u_j \| \leq 1$. Applying (2) and (3), we get

1. If $k \leq \frac{1}{4}(\mathbb{E} \| \sum_{j=1}^N g_j u_j \|)^2$, then

$$\mathbb{E} \sup_{|x|=1} \| G_\omega(x) \| \leq \left( 1 + \frac{a_k}{\mathbb{E} \| \sum_{j=1}^N g_j u_j \|} \right) \left( 1 - \frac{a_k}{\mathbb{E} \| \sum_{j=1}^N g_j u_j \|} \right) \leq 3.$$ 

So, there exists $\omega_0$ such that $\dim(\text{Im } G_{\omega_0}) = k$ and

$$\sup_{|x|=1} \| G_{\omega_0}(x) \| \leq \frac{1}{\mathbb{E} \| \sum_{j=1}^N g_j u_j \|} \| G_{\omega_0}(x) \| \leq 3.$$ 

Let $F^k = \text{Im } G_{\omega_0}$; then $\dim F^k = k$, $d(F^k, \ell_2^k) \leq 3$ and case (i) is proved (it is the classical Dvoretzky’s theorem).

2. In the other cases, one has $k \geq \frac{1}{4}(\mathbb{E} \| \sum_{j=1}^N g_j u_j \|)^2$ so that

$$\mathbb{E} \sup_{|x|=1} \| G_\omega(x) \| \leq 3\sqrt{k}.$$
For $1 \leq m \leq N$, in order to get a better lower bound for $\mathbb{E} \inf_{|x|=1} \|G_\omega(x)\|$, we define a new norm $\|y\|_m$ on $U$. For all $y \in U$, $y = \sum_{j=1}^N y_j u_j$, let

$$
\|y\|_m = \|\sum_{j=1}^N y_j u_j\|_m = \frac{C}{m^{1/s}} \left( \sum_{i=1}^m (y_i^*)^2 \right)^{1/2}.
$$

It is clear from (1) that $\|G_\omega(x)\| \geq \|G_\omega(x)\|_m$. By inequality (2) applied to $G_\omega : \ell^k_2 \to (U, \| \cdot \|_m)$, we get

$$
\mathbb{E} \inf_{|x|=1} \|G_\omega(x)\| \geq \mathbb{E} \inf_{|x|=1} \|G_\omega(x)\|_m
\geq \mathbb{E} \|\sum_{j=1}^N g_j u_j\|_m - a_k \sum_{1 \leq j \leq N} t_j^2 = 1 \|\sum_{j=1}^N t_j u_j\|_m
\geq \frac{1}{m^{1/s}} \left( C \mathbb{E} \left( \sum_{i=1}^m (g_i^*)^2 \right)^{1/2} - \sqrt{k} \right).
\geq m^{1/q} \left( C \sqrt{\ln(1 + \frac{N}{m})} - \sqrt{\frac{k}{m}} \right),
$$

the last inequality following from classical estimates of $\mathbb{E} \left( \sum_{i=1}^m (g_i^*)^2 \right)^{1/2}$ (see for instance [Gl]).

- If $k \leq ce^{-q}n$, we choose $m = Ne^{-q}$. Since $N \geq n/2$, we get

$$
\mathbb{E} \sup_{|x|=1} \|G_\omega(x)\| \leq \frac{c \sqrt{k}}{C \sqrt{q} n^{1/q}}
$$

and we conclude like in 1..

- If $ce^{-q}n \leq k \leq cn$, we choose $m = k$. We have then

$$
\mathbb{E} \sup_{|x|=1} \|G_\omega(x)\| \leq \frac{ck^{1/s}}{C \ln(1 + n/k)}
$$

and as before, we get (iii).

- If $cn \leq k \leq N$, then by the definition of the $(C, s)$-estimate, one has $d(U, \ell^k_2) \leq N^{1/s}$; thus every $k$-dimensional subspace $F^k$ of $U$ satisfies

$$
d(F^k, \ell^k_2) \leq N^{1/s} \leq n^{1/s} \leq \left( \frac{k}{c} \right)^{1/s}.
$$
Remark

Using inequality (1), it is easy to prove that

$$\mathbb{E} \| \sum_{j=1}^{N} g_j u_j \| \geq c C \sqrt{q} n^{1/q}.$$ 

Indeed, by (1), for all $m \in \{1, \ldots, N\}$, we have

$$\mathbb{E} \| \sum_{j=1}^{N} g_j u_j \| \geq \frac{C}{m^{1/s}} \mathbb{E} \left( \sum_{i=1}^{m} (q_i^*)^2 \right)^{1/2} \geq d C m^{1/q} \sqrt{\ln(1 + \frac{N}{m})},$$

and we choose $m = N e^{-q}$ (recall that $N \geq n/2$).

As a corollary, we get more precise estimates in the particular case of $E = \ell_q^n$.

**Corollary 2** For some universal constant $c_i, d_i > 0$, $1 \leq i \leq 3$, for all $n \geq 1$, and all integer $k = 1, \ldots, n$, there exists a $k$-dimensional subspace $F^k$ of $\ell_q^n$ with $q \geq 2$, such that

(i) If $k \leq c_1 q n^{2/q}$, then $d(F^k, \ell_2^k) \leq 3$.

(ii) If $c_1 q n^{2/q} \leq k \leq c_2 q e^{-q} n$, then $d(F^k, \ell_2^k) \leq \frac{d_1 \sqrt{k}}{\sqrt{q} n^{1/q}}$.

(iii) If $c_2 q e^{-q} n \leq k \leq c_3 n$, then $d(F^k, \ell_2^k) \leq \frac{d_2 k^{1/2-1/q}}{\ln(1 + n/k)}$.

(iv) If $c_3 n \leq k \leq n$, then $d(F^k, \ell_2^k) \leq d_3 k^{1/2-1/q}$.

Moreover, the spaces $F^k$ can be choosen randomly with high probability in $\ell_q^n$.

**Proof:**

Let $(e_1, \ldots, e_n)$ be the canonical basis of $\ell_q^n$; then for all $t_1, \ldots, t_n$ and for all $m = 1, \ldots, n$,

$$\left( \sum_{i=1}^{n} |t_i|^q \right)^{1/q} = \left| \sum_{i=1}^{n} t_i e_i \right|_q \geq \sum_{i=1}^{m} t_i^* e_i \geq \left( \sum_{i=1}^{m} (t_i^*)^q \right)^{1/q},$$

using Hölder’s inequality. Since $q \geq 2$, $(e_1, \ldots, e_n)$ satisfies a $(1, s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It is clear from the preceding remark that

$$\mathbb{E} \| \sum_{j=1}^{n} g_j e_j \|_q \sim c \sqrt{q} n^{1/q}.$$
Then we apply Theorem 1 to get random subspaces in the whole space $\ell_q^n$. □

**Remarks:**

1. As it is proved in [C-P], the result of Corollary 2 is optimal up to absolute constant. We include here a short proof of this optimality:
Let $T : \ell_2^k \to \ell_q^n$ a linear operator such that for all $x \in \ell_2^k$,

$$|x|_2 \leq |Tx|_q \leq d |x|_2.$$  

Now we write

$$1 = \int_{S^{k-1}} |x|_2 d\sigma_{k-1}(x)$$

$$\leq \int_{S^{k-1}} |Tx|_q d\sigma_{k-1}(x) = \int_{S^{k-1}} \left( \sum_{i=1}^n |\langle x, T^*(e_i) \rangle|^q \right)^{1/q} d\sigma_{k-1}(x)$$

$$= \frac{1}{d_k} \mathbb{E} \left( \sum_{i=1}^n |\langle G, T^*(e_i) \rangle|^q \right)^{1/q},$$

where $G$ is a gaussian vector of $\mathbb{R}^k$. Since $\langle G, T^*(e_i) \rangle$ is $\mathcal{N}(0, |T^*(e_i)|_2^2)$ and by Hölder inequality, we get

$$\mathbb{E} \left( \sum_{i=1}^n |\langle G, T^*(e_i) \rangle|^q \right)^{1/q} \leq \left( \sum_{i=1}^n \mathbb{E} |\langle G, T^*(e_i) \rangle|^q \right)^{1/q} \leq n^{1/q} \gamma(q) \sup_{1 \leq i \leq n} |T^*(e_i)|_2,$$

where $\gamma(q)$ is the moment of order $q$ of a gaussian $\mathcal{N}(0,1)$-variable. Moreover $|T^*(e_i)|_2 \leq \|T^*\| |e_i|_{q'} = d$, so that we get a universal constant $c > 0$ such that,

$$\sqrt{k} \leq c d n^{1/q} \sqrt{q}.$$

2. A constructive proof of a single subspace of $\ell_q^n$ satisfying the desired conclusion is given in [G-J2].

3. In fact by [L], the inequality $d(F_k^k, \ell_2^k) \leq k^{1/2 - 1/q}$ is always true.

### 3 The case of operator spaces

Let $\tau$ be a 1-symmetric norm on $\mathbb{R}^n$. It is well known that for $m \geq n$, one defines a norm $\| \cdot \|_\tau$ on the $mn$-dimensional vector space $\mathcal{M}_{m \times n}(\mathbb{R})$ of all $[m \times n]$-matrices with real entries by setting

$$\|M\|_\tau = \tau(s_1(M), \ldots, s_n(M))$$

for all $M \in \mathcal{M}_{m \times n}(\mathbb{R})$. 

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where the \( s_i(M), 1 \leq i \leq n, \) are the eigenvalues of \( \sqrt{M^*M} \). If for some \( q \geq 1 \)

\[
\tau(x) = \tau(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} = |x|^q
\]

we get the so called Schatten class \( S_q(m \times n) \) with the norm

\[
\|T\|_q = \left( \sum_{i=1}^{n} |s_i(T)|^q \right)^{\frac{1}{q}}.
\]

**Theorem 3** Let \( \tau \) be a 1-symmetric norm on \( \mathbb{R}^n \) and \( \| \cdot \|_\tau \) the norm on \( \mathcal{M}_{m \times n}(\mathbb{R}) \) associated to \( \tau \). Let \( d_\tau \) be the Banach-Mazur distance between \( (\mathbb{R}^n, \tau) \) and \( \ell_2^n \). Then for some universal constant \( c > 0 \), and for every integer \( k \), \( 1 \leq k \leq nm \), there exists a \( k \)-dimensional subspace \( F_k \) of \( (\mathcal{M}_{m \times n}(\mathbb{R}), \| \cdot \|_\tau) \) such that

\[
(i) \text{ If } k \leq \frac{1}{3}(\mathbb{E}\|G\|_\tau)^2, \text{ then } d(F_k, \ell_2^k) \leq 3.
\]

\[
(ii) \text{ If } \frac{1}{3}(\mathbb{E}\|G\|_\tau)^2 \leq k \leq nm, \text{ then } d(F_k, \ell_2^k) \leq 1 + c d_\tau \sqrt{\frac{k}{nm}}.
\]

**Proof:**

Since \( \tau \) is a 1-symmetric norm, we can assume that \( \frac{1}{d_\tau} |x|_2 \leq \tau(x) \leq |x|_2 \). Then for all \( T \in \mathcal{M}_{m \times n}(\mathbb{R}) \) one has

\[
\frac{1}{d_\tau} \|T\|_2 \leq \|T\|_\tau = \tau(s_1(T), \ldots, s_n(T)) \leq \|T\|_2
\]

where \( \|T\|_2 = (\text{tr}(T^*T))^{1/2} \) is the Hilbert-Schmidt norm. For \( 1 \leq p \leq m \) and \( 1 \leq q \leq n \), let \( E_{pq} \) be the canonical basis of \( \mathcal{M}_{m \times n}(\mathbb{R}) \) (with entries \( (E_{pq})_{ij} = \delta_{ip} \delta_{qj} \)). Let \( G_\omega : \ell_2^k \to (\mathcal{M}_{m \times n}(\mathbb{R}), \| \cdot \|_\tau) \) be the Gaussian operator defined by

\[
G_\omega = \sum_{l=1}^{k} \sum_{1 \leq p \leq m \atop 1 \leq q \leq n} g_{lpq}(\omega) \ e_l \otimes E_{pq},
\]

where \( e_1, \ldots, e_k \) is the canonical basis of \( \ell_2^k \) and the \( g_{lpq}, 1 \leq l \leq k, 1 \leq p \leq m, 1 \leq q \leq n, \) are pairwise independent normalized Gaussian variables. By inequalities (2) and (3), we have

\[
\mathbb{E} \sup_{|x|_2 = 1} \|G_\omega(x)\|_\tau \leq \mathbb{E}\|G\|_\tau + a_k \sup \{ \|T\|_\tau \ ; \ T \in \mathcal{M}_{m \times n}(\mathbb{R}), \|T\|_2 = 1 \}
\]

and

\[
\mathbb{E} \inf_{|x|_2 = 1} \|G_\omega(x)\|_\tau \geq \mathbb{E}\|G\|_\tau - a_k \sup \{ \|T\|_\tau \ ; \ T \in \mathcal{M}_{m \times n}(\mathbb{R}), \|T\|_2 = 1 \}
\]
where $G$ is a matrix with pairwise independent normalized real Gaussian entries in $\mathcal{M}_{m\times n}(\mathbb{R})$.

It is clear that $\sup\{\|T\|_r; \ T \in \mathcal{M}_{m\times n}(\mathbb{R}), \|T\|_2 = 1\} = 1$. We distinguish now three cases.

1. If $\mathbb{E}\|G\|_r \geq 2a_k$, we have

$$\mathbb{E}\sup_{|x|_r = 1}\|G_{\omega}(x)\|_r / \mathbb{E}\inf_{|x|_r = 1}\|G_{\omega}(x)\|_r \leq \frac{1 + a_k}{1 - a_k / \mathbb{E}\|G\|_r} \leq 3.$$  

2. If $\mathbb{E}\|G\|_r \leq 2a_k \leq \frac{\sqrt{mn}}{2}$, then by condition (4) and inequality (2) with $G_\omega : \ell^2 \rightarrow (\mathcal{M}_{m\times n}(\mathbb{R}), \|\cdot\|_2)$, we get

$$\mathbb{E}\inf_{|x|_r = 1}\|G_{\omega}(x)\|_r \geq \frac{1}{d_r} \mathbb{E}\inf_{|x|_r = 1}\|G_{\omega}(x)\|_2 \geq \frac{1}{d_r} (\mathbb{E}\|G\|_2 - a_k).$$

It is well known that $\mathbb{E}\|G\|_2 \geq \frac{\sqrt{mn}}{2}$, so that

$$\mathbb{E}\inf_{|x|_r = 1}\|G_{\omega}(x)\|_r \geq \frac{1}{d_r} (\frac{\sqrt{mn}}{2} - a_k).$$

Since $\mathbb{E}\|G\|_r \leq 2a_k \leq n/2$, we deduce that

$$\mathbb{E}\sup_{|x|_r = 1}\|G_{\omega}(x)\|_r / \mathbb{E}\inf_{|x|_r = 1}\|G_{\omega}(x)\|_r \leq \frac{3a_k}{d_r \frac{\sqrt{mn}}{4}} \leq \frac{12d_r \sqrt{k}}{\sqrt{mn}}.$$  

3. If $a_k \geq \frac{\sqrt{mn}}{4}$, we know from condition (4) that for all subspaces $F^k$ of $(\mathcal{M}_{m\times n}(\mathbb{R}), \tau)$ with $\dim F^k = k$, one has $d(F^k, \ell^2) \leq d_r$. This concludes the proof of the theorem because $a_k \sim \sqrt{k}$. \hfill \Box

As a consequence of the preceding theorem, we get

**Corollary 4** Let $q \geq 2$ and let $S_q(m \times n)$ be the Schatten class. Assume that for some fixed $r > 1$, one has $m = rn$. Then for some universal constant $c > 0$, and for every integer $k$, $1 \leq k \leq nm$, there exists a $k$-dimensional subspace $F^k$ of $S_q(m \times n)$ such that

$$d(F^k, \ell^2) \leq 1 + \frac{c}{\sqrt{r}} n^{-1/q} \sqrt{\frac{k}{n}}.$$  

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Proof:
It is well known that for $q \geq \ln(n)$ the norm on $S_q(m \times n)$ is equivalent up

to universal constant to the norm on $S_\infty(m \times n)$; so we reduce to the case

when $2 \leq q \leq \ln(n)$. We have $\tau(x) = \left( \sum_{i=1}^{n} |x_i|^q \right)^{1/q}$ so that $d_\tau = n^{1/\frac{1}{q}-\frac{1}{2}}$. We

need to compute $\mathbb{E} \|G\|_q$ for a Gaussian matrix. It is well known that

$$a_m - a_n \leq \mathbb{E} \min_{1 \leq i \leq n} s_i(G) \leq \mathbb{E} \sup_{1 \leq i \leq n} s_i(G) \leq a_m + a_n,$$

with

$$a_k = \sqrt{2} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \leq \sqrt{k},$$

(see [H-T] for the more general case of gaussian matrices with operator

entries). Then

$$n^{1/q}(a_m - a_n) \leq \mathbb{E} \|G\|_q \leq n^{1/q}(a_m + a_n),$$

and we apply Theorem 3. \hfill \Box

Remark: Using the same idea as for $\ell^q_2$, we can prove the optimality of this Corollary.

Let $\Theta: \ell^k_2 \rightarrow S_q(m \times n)$ an operator such that for all $x \in \ell^k_2$,

$$|x|_2 \leq \|\Theta x\|_q \leq d |x|_2.$$

Now we write

$$1 = \int_{S^{k-1}} |x|_2 d\sigma_{k-1}(x) \leq \int_{S^{k-1}} \|\Theta x\|_q d\sigma_{k-1}(x) \leq n^{1/q} \int_{S^{k-1}} \|\Theta x\|_\infty d\sigma_{k-1}(x).$$

If $T_i$ denotes the matrix $\Theta(e_i)$ and $G = (g_1, \ldots, g_k)$ is a gaussian vector in

$\mathbb{R}^k$, we have

$$1 \leq \frac{n^{1/q}}{a_k} \mathbb{E} \| \sum_{i=1}^{k} g_i T_i \|_\infty.$$

But

$$\| \sum_{i=1}^{k} g_i T_i \|_\infty = \sup_{|x|_2=1, x \in \mathbb{R}^n} \sup_{|y|_2=1, y \in \mathbb{R}^n} \sum_{i=1}^{k} g_i \langle T_i x, y \rangle.$$

Let $h_1, \ldots, h_n, h'_1, \ldots, h'_n$ be $nm$ independent gaussian variables and define

the two gaussian process:

$$X_{x,y} = \sum_{i=1}^{k} g_i \langle T_i x, y \rangle \quad \text{and} \quad Y_{x,y} = \sqrt{2} \sum_{i=1}^{m} h_i x_i + \sum_{i=1}^{n} h'_i y_i.$$

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By definition of $T$, one has
\[ \| \sum_{i=1}^{k} \alpha_i T_i \|_\infty \leq d \left( \sum_{i=1}^{k} \alpha_i^2 \right)^{1/2}. \]
If we choose $\alpha_i = \langle T_i x, y \rangle$, $1 \leq i \leq k$, we get for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,
\[ \left( \sum_{i=1}^{k} |\langle T_i x, y \rangle|^2 \right)^{1/2} \leq d |x|_2 |y|_2. \]
We conclude that for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, $|x|_2 = 1$ and $|y|_2 = 1$,
\[ \mathbb{E} |X_{x,y} - X_{x',y'}|^2 = \sum_{i=1}^{k} |(\langle T_i x, y - y' \rangle + \langle x - x', T_i^* y' \rangle)|^2 \]
\[ \leq 2 \sum_{i=1}^{k} |\langle T_i x, y - y' \rangle|^2 + |\langle x - x', T_i^* y' \rangle|^2 \]
\[ \leq 2d^2 (|y - y'|_2^2 + |x - x'|_2^2) = \mathbb{E}|Y_{x,y} - Y_{x',y'}|^2. \]
Then by Slepian’s lemma, we obtain
\[ \mathbb{E} \sup_{|x|_2=1} \sup_{|y|_2=1} X_{x,y} \leq \mathbb{E} \sup_{|x|_2=1} \sup_{|y|_2=1} Y_{x,y} \]
and since $\mathbb{E} \sup_{|x|_2=1} \sup_{|y|_2=1} Y_{x,y} = \sqrt{2} d(a_m + a_n)$, we get a universal constant $c > 0$ such that
\[ \sqrt{k} \leq c d (\sqrt{r} + 1) n^{1/2+1/q}. \]
If a subspace of $S_q(m \times n)$ with dimension $k$ is at distance $d \leq \frac{c}{\sqrt{r}} n^{-1/q} \sqrt{\frac{k'}{n}}$ then $k \leq ck'$ and it proves the optimality of corollary 4.

4 Volume ratios with respect to quotients of subspaces of $L_q$

In this section we introduce volume ratios of random $k$—dimensional subspaces $F$ of an $n$—dimensional normed space $X$ with respect to the class of all $k$—dimensional subspaces of quotients of $\ell_q$, $2 \leq q \leq \infty$. This volume ratio yields among other things, in the case $q = 2$, a lower bound for the distance $d(F, \ell_2^k)$ for random subspaces $F$ of $X$.

Let us consider the following concept of volume ratios introduced in [G-J1, G-J2]. Given a $n$—dimensional Banach space $X = (\mathbb{R}^n, ||.||)$ with unit ball
$B_X$, and a finite or infinite dimensional Banach space $Z$ with unit ball $B_Z$, we define the volume ratios

$$\nu_r (X, Z) := \inf \left\{ \left( \frac{\text{vol}(B_X)}{\text{vol}(T(B_Z))} \right)^{1/n} ; T(B_Z) \subset B_X \right\},$$

$$\nu_r (X, S(Z)) := \inf \left\{ \left( \frac{\text{vol}(B_X)}{\text{vol}(T(B_F))} \right)^{1/n} ; F \subset Z, \dim F = n, T(B_F) \subset B_X \right\},$$

$$\nu_r (X, S_p) := \nu_r (X, S(\ell_p)),$$

and

$$\nu_r (X, SQ(\ell_p)) := \inf_{Q \text{ quotient of } \ell_p} \nu_r (X, S(Q)).$$

As in [G-J2] the $n$-th volume number of an operator $T : X \to Y$ is defined by

$$v_n(T) = \sup \left\{ \left( \frac{\text{vol}(T(B_F))}{\text{vol}(B_F)} \right)^{1/n} ; E \subset X, T(E) \subset F \subset Y, \dim E = \dim F = n \right\}$$

We shall also need the definition of the $p$–nuclear norm of an operator $T : X \to Y$ between two finite dimensional Banach spaces, which is defined by

$$\nu_p(T) = \inf \{ \| A_N \| \| \sigma_N \| \| B_N \| ; T = B_N \sigma_N A_N, N \geq 1 \}$$

where $A_N : X \to \ell^N$, $\sigma_N : \ell^N \to \ell^N$ is a diagonal operator, $B_N : \ell^N \to Y$.

**Theorem 5** Let $X = (\mathbb{R}^n, \| . \| )$ be a $n$–dimensional normed space, \{ $b_i, b_i^*$\}$_{i=1}^n$ be a biorthogonal basis for $X$ and $J = \sum_{j=1}^k e_j^* \otimes b_j : \mathbb{R}^n \to X$. For all $u \in O_n$, define $u_k : \mathbb{R}^k \to \mathbb{R}^n$ by $u_k(e_j) = u(e_j)$ for all $1 \leq j \leq k$ and $A_u$ by $A_u = J \circ u_k : \ell^2_k \to X$.

Then for some universal constant $c > 0$ and for all $2 \leq q \leq \infty$ the $k$–dimensional random subspace $F_u = A_u(\ell^k_2) \subset X$ satisfies

$$\mathbb{E}_u \nu_r (F_u, SQ(\ell_q)) \geq \frac{c \sqrt{k}}{\left( \sqrt{q} + \sqrt[k]{\sqrt{q}} \right)^{\frac{1}{q}}} \max_{1 \leq i \leq n} \| b_i^* \| \mathbb{E}_u \left\| \sum_{i=1}^n g_i b_i \right\|$$

where $\mathbb{E}_u$ denotes the expectation with respect to the Haar measure on $O_n$. 

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Proof: 
For \( u \in O_n \), we define also \( B_u : X \rightarrow \ell_2^k \) by \( B_u = u_k^* J^{-1} u_k \) where \( u_k^* : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is the adjoint of \( u_k \). Clearly \( B_u A_u = i d_{\ell_2^k} \).

**Claim:** Let \( q' \) be the conjugate of \( q \), i.e. \( \frac{1}{q} + \frac{1}{q'} = 1 \), then

\[
\mathbb{E}_u \nu_{q'}(B_u : X \rightarrow \ell_2^k) \leq c \sqrt{n} \left( \sqrt{q} + \frac{\sqrt{k}}{n^{1/q}} \right) \max_{1 \leq j \leq n} \| b_j^* \|. \tag{5}
\]

To show this proceed as in the definition of the \( q \)-nuclear ideal norm to factor \( B_u|_{X \rightarrow \ell_2^k} = u_k^* |_{\ell_q^m \rightarrow \ell_2^k} I J^{-1} \) where \( I : \ell_\infty^n \rightarrow \ell_\infty^m = \sum_{i=1}^n e_i \otimes e_i \) is the identity map on \( \mathbb{R}^n \), and \( J^{-1} = \sum_{i=1}^n b_i^* \otimes e_i : X \rightarrow \ell_\infty^n \). Then clearly

\[
\nu_{q'}(B_u|_{X \rightarrow \ell_2^k}) \leq \| J^{-1} \| \| I \| \| u_k^* |_{\ell_q^m \rightarrow \ell_2^k} \| = \max_{1 \leq i \leq n} \| b_i^* \| n^{1/q' - 1/2} \| u_k^* |_{\ell_q^m \rightarrow \ell_2^k} \|.
\]

Let \( G = \sum_{i,j} g_{i,j} e_i \otimes e_j \) denote the Gaussian operator which maps \( \ell_q^m \) to \( \ell_2^k \); we have by [B-G]

\[
\mathbb{E}_u \| u_k^* |_{\ell_q^m \rightarrow \ell_2^k} \| \leq \frac{c_0}{\sqrt{n}} \mathbb{E} \| G : \ell_q^m \rightarrow \ell_2^k \| \leq \frac{c_1}{\sqrt{n}} (c n^{1/q} + \sqrt{k})
\]

hence

\[
\mathbb{E}_u \nu_{q'}(B_u|_{X \rightarrow \ell_2^k}) \leq c_0 n^{1/2} (c \sqrt{q} + n^{-1/4} \sqrt{k}) \max_{1 \leq i \leq n} \| b_i^* \|
\]

and (5) is proved.

Now recall that if \( T : \ell_2^k \rightarrow X \) and \( \text{rad} \ (T) =: \int_0^1 \| \sum_{i=1}^k r_i(t)T(e_i) \|_X \, dt \), then using the Marcus-Pisier inequality [B-G], [M-P]

\[
\sqrt{n} \mathbb{E} \text{rad} \ (A_u : \ell_2^k \rightarrow X) = \sqrt{n} \mathbb{E} u \int_0^1 \| \sum_{j=1}^k r_j(t)A_u(e_j) \| \, dt \\
\leq c \mathbb{E} \int_0^1 \| \sum_{j=1}^k \sum_{i=1}^n r_j(t)g_{i,j}b_i \| \, dt \\
= c \sqrt{k} \mathbb{E} \| \sum_{i=1}^n g_i b_i \|.
\]

Denote by \( e_k(T) \) the \( k \)-th entropy number of an operator \( T : Y \rightarrow X \), then by [C-P] one has

\[
\mathbb{E}_u \sqrt{k} e_k(A_u) \leq 4 \mathbb{E}_u \sqrt{k} e_k(A_u) \leq 4c_1 \mathbb{E}_u \text{rad} \ (A_u) \leq c \sqrt{k} \mathbb{E} \| \sum_{i=1}^n g_i b_i \|.
\]

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By [G-J2] Lemma 1.3, we have for any \( k = 1, 2, \ldots, 2 \leq q \leq \infty \), and any operator \( T \) from a Banach space \( Z \) to \( \ell_2 \)

\[
\sqrt{k} v_k(T) \leq c_0 \sup_{F \subseteq Z, \dim(F) = k} \varr(F; SQ(\ell_q)).
\]

Applying this to \( B_u|_{F_k} \rightarrow \ell_2^k \) we have

\[
\sqrt{k} v_k(B_u) \leq c_0 v_q(B_u) \varr(F_u; SQ(\ell_q)).
\]

Since \( B_u A_u = id_{\ell_2^k} \), we have \( 1 = v_k(A_u^* A_u) = v_k(A_u) v_k(A_u^*|_{F_k}) \).

Hence we obtain

\[
1 \leq c_0 v_k(A_u) \frac{v_q(B_u)}{\sqrt{k}} \varr(F_u, SQ(\ell_q))
\]

and taking the 3-nd root we get by Hölder inequality

\[
1 \leq c_0 \mathbb{E}_u v_k(A_u) \mathbb{E}_u \left( \frac{v_q(B_u)}{\sqrt{k}} \right) \mathbb{E}_u \varr(F_u, SQ(\ell_q))
\]

\[
\leq \frac{c_0}{\sqrt{n}} \mathbb{E} \left( \sum_{i=1}^n g_i b_i \right) \frac{c_0}{\sqrt{n}} \left( \sqrt{q} + \frac{\sqrt{k}}{n^{1/2}} \right) \max_i \| b_i^* \| \mathbb{E}_u \varr(F_u, SQ(\ell_q)).
\]

This concludes the proof. \( \square \)

Remarks :

1. It was proved in [G-J2] that

\[
\varr(X, SQ(\ell_p)) \leq \varr(X, S(\ell_p)) \leq c_0 \sqrt{p + p'} \varr(X, SQ(\ell_p))
\]

with \( \frac{1}{p} + \frac{1}{p'} = 1 \).

2. Estimates in the case \( q = 2 \) and \( X = \ell_p^n \) or \( X = S_p(m \times n) \) with \( 2 \leq p \leq \ln n \) which give optimal lower bound in expectation for random \( k \)-dimensional subspaces of \( X \) (in correlation with part 2 and 3 of this paper). (I have to rewrite that and also for \( q \geq 2 \) as Yoram said us by mail)

References


O. Guédon, *Gaussian version of a theorem of Milman and Schechtman*, Positivity **1** (1997), 1-5.


