Rates of convergence for minimal distances in the central limit theorem under projective criteria

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Abstract

In this paper, we give estimates of ideal or minimal distances between the distribution of the normalized partial sum and the limiting Gaussian distribution for stationary martingale difference sequences or stationary sequences satisfying projective criteria. Applications to functions of linear processes and to functions of expanding maps of the interval are given.

1 Introduction and Notations

Let $X_1, X_2, \dots$ be a strictly stationary sequence of real-valued random variables (r.v.) with mean zero and finite variance. Set $S_n = X_1 + X_2 + \cdots + X_n$. By $P_{n^{-1/2}S_n}$ we denote the law of $n^{-1/2}S_n$ and by G_{σ^2} the normal distribution $N(0,\sigma^2)$. In this paper, we shall give quantitative estimates of the approximation of $P_{n^{-1/2}S_n}$ by G_{σ^2} in terms of minimal or ideal metrics.

Let $\mathcal{L}(\mu,\nu)$ be the set of the probability laws on $\mathbb{R}^2$ with marginals μ and ν. Let us consider the following minimal distances (sometimes called Wasserstein distances of order r)

$$W_r(\mu,\nu) = \begin{cases} \inf\Big\{ \displaystyle\int |x-y|^r P(dx,dy) : P \in \mathcal{L}(\mu,\nu)\Big\} & \text{if } 0<r<1 \\ \inf\Big\{ \Big(\displaystyle\int |x-y|^r P(dx,dy)\Big)^{1/r} : P \in \mathcal{L}(\mu,\nu)\Big\} & \text{if } r \geq 1\,. \end{cases}$$

It is well known that for two probability measures μ and ν on $\mathbb{R}$ with respective distributions functions (d.f.) F and G,

$$W_r(\mu,\nu) = \Big(\int_0^1 |F^{-1}(u) - G^{-1}(u)|^r du\Big)^{1/r} \text{ for any } r \geq 1. \tag{1.1}$$

We consider also the following ideal distances of order r (Zolotarev distances of order r). For two probability measures μ and ν, and r a positive real, let

$$\zeta_r(\mu,\nu) = \sup\Big\{ \int f d\mu - \int f d\nu : f \in \Lambda_r \Big\},$$

where Λ_r is defined as follows: denoting by l the natural integer such that $l < r \leq l+1$, Λ_r is the class of real functions f which are l-times continuously differentiable and such that

$$|f^{(l)}(x) - f^{(l)}(y)| \leq |x-y|^{r-l} \text{ for any } (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{1.2}$$

It follows from the Kantorovich-Rubinstein theorem (1958) that for any $0 < r \leq 1$,

$$W_r(\mu,\nu) = \zeta_r(\mu,\nu). \tag{1.3}$$

For probability laws on the real line, Rio (1998) proved that for any $r > 1$,

$$W_r(\mu,\nu) \leq c_r \big(\zeta_r(\mu,\nu)\big)^{1/r}, \tag{1.4}$$

where c_r is a constant depending only on r.

For independent random variables, Ibragimov (1966) established that if $X_1 \in \mathbb{L}^p$ for $p \in]2,3]$, then $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$ (see his Theorem 4.3). Still in the case of independent r.v.'s, Zolotarev (1976) obtained the following upper bound for the ideal distance: if $X_1 \in \mathbb{L}^p$ for $p \in]2,3]$, then $\zeta_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$. From (1.4), the result of Zolotarev entails that, for $p \in]2,3]$, $W_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1/p-1/2})$ (which was obtained by Sakhanenko (1985) for any $p > 2$). From (1.1) and Hölder's inequality, we easily get that for independent random variables in $\mathbb{L}^p$ with $p \in]2,3]$,

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2r}) \quad \text{for any } 1 \leq r \leq p. \tag{1.5}$$

In this paper, we are interested in extensions of (1.5) to sequences of dependent random variables. More precisely, for $X_1 \in \mathbb{L}^p$ and p in $]2,3]$ we shall give $\mathbb{L}^p$-projective criteria under which: for $r \in [p-2, p]$ and $(r,p) \neq (1,3)$,

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2\max(1,r)}). \tag{1.6}$$

As we shall see in Remark 2.4, (1.6) applied to $r = p - 2$ provides the rate of convergence $O(n^{-\frac{p-2}{2(p-1)}})$ in the Berry-Esseen theorem.

When $(r, p) = (1, 3)$, Dedecker and Rio (2008) obtained that $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2})$ for stationary sequences of random variables in $\mathbb{L}^3$ satisfying $\mathbb{L}^1$ projective criteria or weak dependence assumptions (a similar result was obtained by Pène (2005) in the case where the variables are bounded). In this particular case our approach provides a new criterion under which $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$.

Our paper is organized as follows. In Section 2, we give projective conditions for stationary martingales differences sequences to satisfy (1.6) in the case $(r, p) \neq (1, 3)$. To be more precise, let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of martingale differences with respect to some σ-algebras $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ (see Section 1.1 below for the definition of $(\mathcal{F}_i)_{i\in\mathbb{Z}}$). As a consequence of our Theorem 2.1, we obtain that if $(X_i)_{i\in\mathbb{Z}}$ is in $\mathbb{L}^p$ with $p \in]2, 3]$ and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{p/2} < \infty\,, \tag{1.7}$$

then the upper bound (1.6) holds provided that $(r, p) \neq (1, 3)$. In the case $r = 1$ and $p = 3$, we obtain the upper bound $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$.

In Section 3, starting from the coboundary decomposition going back to Gordin (1969), and using the results of Section 2, we obtain $\mathbb{L}^p$-projective criteria ensuring (1.6) (if $(r, p) \neq (1, 3)$). For instance, if $(X_i)_{i\in\mathbb{Z}}$ is a stationary sequence of $\mathbb{L}^p$ random variables adapted to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$, we obtain (1.6) for any $p \in]2, 3[$ and any $r \in [p-2, p]$ provided that (1.7) holds and the series $\mathbb{E}(S_n|\mathcal{F}_0)$ converge in $\mathbb{L}^p$. In the case where $p = 3$, this last condition has to be strengthened. Our approach makes also possible to treat the case of non-adapted sequences.

Section 4 is devoted to applications. In particular, we give sufficient conditions for some functions of Harris recurrent Markov chains and for functions of linear processes to satisfy the bound (1.6) in the case $(r, p) \neq (1, 3)$ and the rate $O(n^{-1/2}\log n)$ when $r = 1$ and $p = 3$. Since projective criteria are verified under weak dependence assumptions, we give an application to functions of ϕ-dependent sequences in the sense of Dedecker and Prieur (2007). These conditions apply to unbounded functions of uniformly expanding maps.

1.1 Preliminary notations

Throughout the paper, Y is a $N(0, 1)$-distributed random variable. We shall also use the following notations. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{F}_k$ and

$\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{F}_k$. We shall denote sometimes by $\mathbb{E}_i$ the conditional expectation with respect to $\mathcal{F}_i$. Let X_0 be a zero mean random variable with finite variance, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$.

2 Stationary sequences of martingale differences.

In this section we give bounds for the ideal distance of order r in the central limit theorem for stationary martingale differences sequences $(X_i)_{i\in\mathbb{Z}}$ under projective conditions.

Notation 2.1. For any $p > 2$, define the envelope norm $\|\,.\,\|_{1,\Phi,p}$ by

$$\|X\|_{1,\Phi,p} = \int_0^1 (1 \vee \Phi^{-1}(1-u/2))^{p-2} Q_X(u)du$$

where Φ denotes the d.f. of the $N(0,1)$ law, and Q_X denotes the quantile function of $|X|$, that is the cadlag inverse of the tail function $x \to \mathbb{P}(|X| > x)$.

Theorem 2.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale differences sequence with respect to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. Let σ denote the standard deviation of X_0. Let $p \in]2,3]$. Assume that $\mathbb{E}|X_0|^p < \infty$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{1,\Phi,p} < \infty\,, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{p/2} < \infty\,. \tag{2.2}$$

Then, for any $r \in [p-2, p]$ with $(r,p) \neq (1,3)$, $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, and for $p = 3$, $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$.

Remark 2.1. Let $a > 1$ and $p > 2$. Applying Hölder's inequality, we see that there exists a positive constant $C(p,a)$ such that $\|X\|_{1,\Phi,p} \le C(p,a)\|X\|_a$. Consequently, if $p \in]2,3]$, the two conditions (2.1) and (2.2) are implied by the condition (1.7) given in the introduction.

Remark 2.2. Under the assumptions of Theorem 2.1, $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-r/2})$ if $r < p-2$. Indeed, let $p' = r+2$. Since $p' < p$, if the conditions (2.1) and (2.2) are satisfied for p, they also hold for p'. Hence Theorem 2.1 applies with p'.

From (1.3) and (1.4), the following result holds for the Wasserstein distances of order r.

Corollary 2.1. *Under the conditions of Theorem 2.1,* $W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2\max(1,r)})$ *for any* r *in* $[p-2,p]$*, provided that* $(r,p) \neq (1,3)$*.*

Remark 2.3. For p in $]2,3]$, $W_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2p})$. This bound was obtained by Sakhanenko (1985) in the independent case. For $p < 3$, we have $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$. This bound was obtained by Ibragimov (1966) in the independent case.

Remark 2.4. Recall that for two real valued random variables X, Y, the Ky Fan metric $\alpha(X,Y)$ is defined by $\alpha(X,Y) = \inf\{\varepsilon > 0 : \mathbb{P}(|X-Y| > \varepsilon) \leq \varepsilon\}$. Let $\Pi(\mu,\nu)$ be the Prokhorov distance between μ and ν. By Theorem 11.3.5 in Dudley (1989) and Markov inequality, one has, for any $r > 0$,

$$\Pi(P_X, P_Y) \leq \alpha(X,Y) \leq (\mathbb{E}(|X-Y|^r))^{1/(r+1)} .$$

Taking the minimum over the random couples (X,Y) with law $\mathcal{L}(\mu,\nu)$, we obtain that, for any $0 < r \leq 1$, $\Pi(\mu,\nu) \leq (W_r(\mu,\nu))^{1/(r+1)}$. Hence, if Π_n is the Prokhorov distance between the law of $n^{-1/2}S_n$ and the normal distribution $N(0,\sigma^2)$,

$$\Pi_n \leq (W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}))^{1/(r+1)} \text{ for any } 0 < r \leq 1 .$$

Taking $r = p-2$, it follows that under the assumptions of Theorem 2.1,

$$\Pi_n = O(n^{-\frac{p-2}{2(p-1)}}) \text{ if } p < 3 \text{ and } \Pi_n = O(n^{-1/4}\sqrt{\log n}) \quad \text{if } p = 3. \tag{2.3}$$

For p in $]2,4]$, under (2.2), we have that $\| \sum_{i=1}^n \mathbb{E}(X_i^2 - \sigma^2|\mathcal{F}_{i-1})\|_{p/2} = O(n^{2/p})$ (apply Theorem 2 in Wu and Zhao (2006)). Applying then the result in Heyde and Brown (1970), we get that if $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale difference sequence in $\mathbb{L}^p$ such that (2.2) is satisfied then

$$\|F_n - \Phi_\sigma\|_\infty = O\big(n^{-\frac{p-2}{2(p+1)}}\big) .$$

where F_n is the distribution function of $n^{-1/2}S_n$ and Φ_σ is the d.f. of G_{σ^2}. Now

$$\|F_n - \Phi_\sigma\|_\infty \leq \big(1 + \sigma^{-1}(2\pi)^{-1/2}\big)\Pi_n .$$

Consequently the bounds obtained in (2.3) improve the one given in Heyde and Brown (1970), provided that (2.1) holds.

Remark 2.5. If $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale difference sequence in $\mathbb{L}^3$ such that $\mathbb{E}(X_0^2) = \sigma^2$ and

$$\sum_{k>0} k^{-1/2}\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{3/2} < \infty, \tag{2.4}$$

then, according to Remark 2.1, the conditions (2.1) and (2.2) hold for $p = 3$. Consequently, if (2.4) holds, then Remark 2.4 gives $\|F_n - \Phi_\sigma\|_\infty = O\big(n^{-1/4}\sqrt{\log n}\big)$. This result has to be compared with Theorem 6 in Jan (2001), which states that $\|F_n - \Phi_\sigma\|_\infty = O(n^{-1/4})$ if $\sum_{k>0} \|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{3/2} < \infty$.

Remark 2.6. Notice that if $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale differences sequence, then the conditions (2.1) and (2.2) are respectively equivalent to

$$\sum_{j\geq 0} 2^{j(p/2-1)}\|2^{-j}\,\mathbb{E}(S_{2^j}^2|\mathcal{F}_0)-\sigma^2\|_{1,\Phi,p}<\infty, \text{ and } \sum_{j\geq 0} 2^{j(1-2/p)}\|2^{-j}\,\mathbb{E}(S_{2^j}^2|\mathcal{F}_0)-\sigma^2\|_{p/2}<\infty\,.$$

To see this, let $A_n = \|\mathbb{E}(S_n^2|\mathcal{F}_0\big) - \mathbb{E}(S_n^2)\|_{1,\Phi,p}$ and $B_n = \|\mathbb{E}(S_n^2|\mathcal{F}_0\big) - \mathbb{E}(S_n^2)\|_{p/2}$. We first show that A_n and B_n are subadditive sequences. Indeed, by the martingale property and the stationarity of the sequence, for all positive i and j

$$\begin{aligned} A_{i+j} &= \|\mathbb{E}(S_i^2+(S_{i+j}-S_i)^2|\mathcal{F}_0\big)-\mathbb{E}(S_i^2+(S_{i+j}-S_i)^2)\|_{1,\Phi,p} \\ &\leq A_i+\|\mathbb{E}\big((S_{i+j}-S_i)^2-\mathbb{E}(S_j^2)\,|\mathcal{F}_0\big)\|_{1,\Phi,p}\,. \end{aligned}$$

Proceeding as in the proof of (4.6), p. 65 in Rio (2000), one can prove that, for any σ-field $\mathcal{A}$ and any integrable random variable X, $\|\mathbb{E}(X|\mathcal{A})\|_{1,\Phi,p}\leq\|X\|_{1,\Phi,p}$. Hence

$$\|\mathbb{E}\big((S_{i+j}-S_i)^2-\mathbb{E}(S_j^2)\,|\mathcal{F}_0\big)\|_{1,\Phi,p}\leq\|\mathbb{E}\big((S_{i+j}-S_i)^2-\mathbb{E}(S_j^2)\,|\mathcal{F}_i\big)\|_{1,\Phi,p}\,.$$

By stationarity, it follows that $A_{i+j}\leq A_i+A_j$. Similarly $B_{i+j}\leq B_i+B_j$. The proof of the equivalences then follows by using the same arguments as in the proof of Lemma 2.7 in Peligrad and Utev (2005).

3 Rates of convergence for stationary sequences

In this section, we give estimates for the ideal distances of order r for stationary sequences which are not necessarily adapted to $\mathcal{F}_i$.

Theorem 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^p$ with $p\in]2,3[$, and let $\sigma_n^2=n^{-1}\mathbb{E}(S_n^2)$. Assume that*

$$\sum_{n>0}\mathbb{E}(X_n|\mathcal{F}_0) \text{ and } \sum_{n>0}(X_{-n}-\mathbb{E}(X_{-n}|\mathcal{F}_0)) \text{ converge in } \mathbb{L}^p\,, \tag{3.1}$$

and

$$\sum_{n\geq 1} n^{-2+p/2}\|\,n^{-1}\,\mathbb{E}(S_n^2|\mathcal{F}_0)-\sigma_n^2\|_{p/2}<\infty\,. \tag{3.2}$$

Then the series $\sum_{k\in\mathbb{Z}}\mathrm{Cov}(X_0,X_k)$ converges to some nonnegative σ^2, and

1. $\zeta_r(P_{n^{-1/2}S_n},G_{\sigma^2})=O(n^{1-p/2})$ *for* $r\in[p-2,2]$,
2. $\zeta_r(P_{n^{-1/2}S_n},G_{\sigma_n^2})=O(n^{1-p/2})$ *for* $r\in]2,p]$.

Remark 3.1. According to the bound (5.40), we infer that, under the assumptions of Theorem 3.1, the condition (3.2) is equivalent to

$$\sum_{n\geq 1} n^{-2+p/2} \| \, n^{-1}\, \mathbb{E}(S_n^2|\mathcal{F}_0) - \sigma^2 \|_{p/2} < \infty \,. \tag{3.3}$$

The same remark applies to the next theorem with $p = 3$.

Remark 3.2. The result of item 1 is valid with σ_n instead of σ. On the contrary, the result of item 2 is no longer true if σ_n is replaced by σ, because for $r \in]2, 3]$, a necessary condition for $\zeta_r(\mu, \nu)$ to be finite is that the two first moments of ν and μ are equal. Note that under the assumptions of Theorem 3.1, both $W_r(P_{n^{-1/2}S_n}, G_{\sigma^2})$ and $W_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2})$ are of the order of $n^{-(p-2)/2\max(1,r)}$. Indeed, in the case where $r \in]2, p]$, one has that

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) \leq W_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) + W_r(G_{\sigma_n^2}, G_{\sigma^2}) \,,$$

and the second term is of order $|\sigma - \sigma_n| = O(n^{-1/2})$.

In the case where $p = 3$, the condition (3.1) has to be strengthened.

Theorem 3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^3$, and let $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Assume that (3.1) holds for $p = 3$ and that*

$$\sum_{n\geq 1} \frac{1}{n} \Big\| \sum_{k\geq n} \mathbb{E}(X_k|\mathcal{F}_0) \Big\|_3 < \infty \quad \textit{and} \quad \sum_{n\geq 1} \frac{1}{n} \Big\| \sum_{k\geq n} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0)) \Big\|_3 < \infty \,. \tag{3.4}$$

Assume in addition that

$$\sum_{n\geq 1} n^{-1/2} \| \, n^{-1}\, \mathbb{E}(S_n^2|\mathcal{F}_0) - \sigma_n^2 \|_{3/2} < \infty \,. \tag{3.5}$$

Then the series $\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ converges to some nonnegative σ^2 and

1. $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2} \log n)$,
2. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2})$ *for* $r \in]1, 2]$,
3. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{-1/2})$ *for* $r \in]2, 3]$.

4 Applications

4.1 Martingale differences sequences and functions of Markov chains

Recall that the strong mixing coefficient of Rosenblatt (1956) between two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ is defined by $\alpha(\mathcal{A},\mathcal{B}) = \sup\{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \,:\, (A,B) \in \mathcal{A}\times\mathcal{B}\}$. For a strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$, let $\mathcal{F}_i = \sigma(X_k, k \le i)$. Define the mixing coefficients $\alpha_1(n)$ of the sequence $(X_i)_{i\in\mathbb{Z}}$ by

$$\alpha_1(n) = \alpha(\mathcal{F}_0, \sigma(X_n)) .$$

For the sake of brevity, let $Q = Q_{X_0}$ (see Notation 2.1 for the definition). According to the results of Section 2, the following proposition holds.

Proposition 4.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale difference sequence in $\mathbb{L}^p$ with $p \in]2,3[$. Assume moreover that the series*

$$\sum_{k\ge1} \frac{1}{k^{2-p/2}} \int_0^{\alpha_1(k)} (1 \vee \log(1/u))^{(p-2)/2} Q^2(u)du \quad \textit{and} \quad \sum_{k\ge1} \frac{1}{k^{2/p}} \Big(\int_0^{\alpha_1(k)} Q^p(u)du \Big)^{2/p} \tag{4.1}$$

are convergent.Then the conclusions of Theorem 2.1 hold.

Remark 4.1. From Theorem 2.1(b) in Dedecker and Rio (2008), a sufficient condition to get $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$ is

$$\sum_{k\ge0} \int_0^{\alpha_1(n)} Q^3(u)du < \infty .$$

This condition is always strictly stronger than the condition (4.1) when $p = 3$.

We now give an example. Consider the homogeneous Markov chain $(Y_i)_{i\in\mathbb{Z}}$ with state space $\mathbb{Z}$ described at page 320 in Davydov (1973). The transition probabilities are given by $p_{n,n+1} = p_{-n,-n-1} = a_n$ for $n \ge 0$, $p_{n,0} = p_{-n,0} = 1 - a_n$ for $n > 0$, $p_{0,0} = 0$, $a_0 = 1/2$ and $1/2 \le a_n < 1$ for $n \ge 1$. This chain is irreducible and aperiodic. It is Harris positively recurrent as soon as $\sum_{n\ge2} \Pi_{k=1}^{n-1} a_k < \infty$. In that case the stationary chain is strongly mixing in the sense of Rosenblatt (1956).

Denote by K the Markov kernel of the chain $(Y_i)_{i\in\mathbb{Z}}$. The functions f such that $K(f) = 0$ almost everywhere are obtained by linear combinations of the two functions f_1 and f_2 given by $f_1(1) = 1$, $f_1(-1) = -1$ and $f_1(n) = f_1(-n) = 0$ if $n \ne 1$, and $f_2(0) = 1$, $f_2(1) = f_2(-1) = 0$ and $f_2(n+1) = f_2(-n-1) = 1 - a_n^{-1}$ if $n > 0$. Hence the functions f such that $K(f) = 0$ are bounded.

If $(X_i)_{i\in\mathbb{Z}}$ is defined by $X_i = f(Y_i)$, with $K(f) = 0$, then Proposition 4.1 applies if

$$\alpha_1(n) = O(n^{1-p/2}(\log n)^{-p/2-\epsilon}) \text{ for some } \epsilon > 0, \tag{4.2}$$

which holds as soon as $P_0(\tau = n) = O(n^{-1-p/2}(\log n)^{-p/2-\epsilon})$, where P_0 is the probability of the chain starting from 0, and $\tau = \inf\{n > 0, X_n = 0\}$. Now $P_0(\tau = n) = (1 - a_n)\Pi_{i=1}^{n-1} a_i$ for $n \geq 2$. Consequently, if

$$a_i = 1 - \frac{p}{2i}\Big(1 + \frac{1+\epsilon}{\log i}\Big) \text{ for } i \text{ large enough},$$

the condition (4.2) is satisfied and the conclusion of Theorem 2.1 holds.

Remark 4.2. If f is bounded and $K(f) \neq 0$, the central limit theorem may fail to hold for $S_n = \sum_{i=1}^n (f(Y_i) - \mathbb{E}(f(Y_i)))$. We refer to the Example 2, page 321, given by Davydov (1973), where S_n properly normalized converges to a stable law with exponent strictly less than 2.

Proof of Proposition 4.1. Let $B^p(\mathcal{F}_0)$ be the set of $\mathcal{F}_0$-measurable random variables such that $\|Z\|_p \leq 1$. We first notice that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{p/2} = \sup_{Z\in B^{p/(p-2)}(\mathcal{F}_0)} \mathrm{Cov}(Z, X_k^2).$$

Applying Rio's covariance inequality (1993), we get that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{p/2} \leq 2\Big(\int_0^{\alpha_1(k)} Q^p(u)du\Big)^{2/p},$$

which shows that the convergence of the second series in (4.1) implies (2.2). Now, from Fréchet (1957), we have that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} = \sup\big\{\mathbb{E}((1\vee|Z|^{p-2})|\,\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2|\,), Z\ \mathcal{F}_0\text{-measurable}, Z\sim\mathcal{N}(0,1)\big\}.$$

Hence, setting $\varepsilon_k = \mathrm{sign}(\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2)$,

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} = \sup\big\{\mathrm{Cov}(\varepsilon_k(1\vee|Z|^{p-2}), X_k^2), Z\ \mathcal{F}_0\text{-measurable}, Z\sim\mathcal{N}(0,1)\big\}.$$

Applying again Rio's covariance inequality (1993), we get that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} \leq C\Big(\int_0^{\alpha_1(k)} (1\vee\log(u^{-1}))^{(p-2)/2} Q^2(u)du\Big),$$

which shows that the convergence of the first series in (4.1) implies (2.1).

4.2 Linear processes and functions of linear processes

In what follows we say that the series $\sum_{i\in\mathbb{Z}} a_i$ converges if the two series $\sum_{i\geq 0} a_i$ and $\sum_{i<0} a_i$ converge.

Theorem 4.1. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^2 such that $\sum_{i\in\mathbb{Z}} a_i$ converges to some real A. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a stationary sequence of martingale differences in $\mathbb{L}^p$ for $p \in]2,3]$. Let $X_k = \sum_{j\in\mathbb{Z}} a_j \varepsilon_{k-j}$, and $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Let $b_0 = a_0 - A$ and $b_j = a_j$ for $j \neq 0$. Let $A_n = \sum_{j\in\mathbb{Z}}(\sum_{k=1}^n b_{k-j})^2$. If $A_n = o(n)$, then σ_n^2 converges to $\sigma^2 = A^2\mathbb{E}(\varepsilon_0^2)$. If moreover*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{1}{n}\Big(\sum_{j=1}^{n} \varepsilon_j\Big)^2 \Big| \mathcal{F}_0\Big) - \mathbb{E}(\varepsilon_0^2) \Big\|_{p/2} < \infty\,, \tag{4.3}$$

then we have

1. *If $A_n = O(1)$, then $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log(n))$, for $p = 3$,*
2. *If $A_n = O(n^{(r+2-p)/r})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, for $r \in [p-2, 1]$ and $p \neq 3$,*
3. *If $A_n = O(n^{3-p})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, for $r \in]1, 2]$,*
4. *If $A_n = O(n^{3-p})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$, for $r \in]2, p]$.*

Remark 4.3. If the condition given by Heyde (1975) holds, that is

$$\sum_{n=1}^{\infty}\Big(\sum_{k\geq n} a_k\Big)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\Big(\sum_{k\leq -n} a_k\Big)^2 < \infty\,, \tag{4.4}$$

then $A_n = O(1)$, so that it satisfies all the conditions of items 1-4.

Remark 4.4. Under the additional assumption $\sum_{i\in\mathbb{Z}} |a_i| < \infty$, one has the bound

$$A_n \leq 4B_n, \quad \text{where} \quad B_n = \sum_{k=1}^{n}\Big(\Big(\sum_{j\geq k}|a_j|\Big)^2 + \Big(\sum_{j\leq -k}|a_j|\Big)^2\Big)\,. \tag{4.5}$$

Proof of Theorem 4.1. We start with the following decomposition:

$$S_n = A\sum_{j=1}^{n}\varepsilon_j + \sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{n} b_{k-j}\Big)\varepsilon_j\,. \tag{4.6}$$

Let $R_n = \sum_{j=-\infty}^{\infty}(\sum_{k=1}^n b_{k-j})\varepsilon_j$. Since $\|R_n\|_2^2 = A_n\|\varepsilon_0\|_2^2$ and since $|\sigma_n - \sigma| \leq n^{-1/2}\|R_n\|_2$, the fact that $A_n = o(n)$ implies that σ_n converges to σ. We now give an upper bound for $\|R_n\|_p$. From Burkholder's inequality, there exists a constant C such that

$$\|R_n\|_p \leq C\Big\{\Big\|\sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{n} b_{k-j}\Big)^2\varepsilon_j^2\Big\|_{p/2}\Big\}^{1/2} \leq C\|\varepsilon_0\|_p\sqrt{A_n}. \tag{4.7}$$

According to Remark 2.1, since (4.3) holds, the two conditions (2.1) and (2.2) of Theorem 2.1 are satisfied by the martingale $M_n = A\sum_{k=1}^n \varepsilon_k$. To conclude the proof, we use Lemma 5.2 given in Section 5.2, with the upper bound (4.7). □

Proof of Remarks 4.3 and 4.4. To prove Remark 4.3, note first that

$$A_n = \sum_{j=1}^n \Big(\sum_{l=-\infty}^{-j} a_l + \sum_{l=n+1-j}^{\infty} a_l\Big)^2 + \sum_{i=1}^{\infty}\Big(\sum_{l=i}^{n+i-1} a_l\Big)^2 + \sum_{i=1}^{\infty}\Big(\sum_{l=-i-n+1}^{-i} a_l\Big)^2 .$$

It follows easily that $A_n = O(1)$ under (4.4). To prove the bound (4.5), note first that

$$A_n \le 3B_n + \sum_{i=n+1}^{\infty}\Big(\sum_{l=i}^{n+i-1} |a_l|\Big)^2 + \sum_{i=n+1}^{\infty}\Big(\sum_{l=-i-n+1}^{-i} |a_l|\Big)^2 .$$

Let $T_i = \sum_{l=i}^{\infty} |a_l|$ and $Q_i = \sum_{l=-\infty}^{-i} |a_l|$. We have that

$$\begin{aligned}\sum_{i=n+1}^{\infty}\Big(\sum_{l=i}^{n+i-1} |a_l|\Big)^2 &\le T_{n+1}\sum_{i=n+1}^{\infty}(T_i - T_{n+i}) \le nT_{n+1}^2\\ \sum_{i=n+1}^{\infty}\Big(\sum_{l=-i-n+1}^{-i} |a_l|\Big)^2 &\le Q_{n+1}\sum_{i=n+1}^{\infty}(Q_i - Q_{n+i}) \le nQ_{n+1}^2 .\end{aligned}$$

Since $n(T_{n+1}^2 + Q_{n+1}^2) \le B_n$, (4.5) follows. □

In the next result, we shall focus on functions of real-valued linear processes

$$X_k = h\Big(\sum_{i\in\mathbb{Z}} a_i\varepsilon_{k-i}\Big) - \mathbb{E}\Big(h\Big(\sum_{i\in\mathbb{Z}} a_i\varepsilon_{k-i}\Big)\Big), \qquad (4.8)$$

where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables. Denote by $w_h(., M)$ the modulus of continuity of the function h on the interval $[-M, M]$, that is

$$w_h(t, M) = \sup\{|h(x) - h(y)|, |x-y| \le t, |x| \le M, |y| \le M\} .$$

Theorem 4.2. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^2 and $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables in $\mathbb{L}^2$. Let X_k be defined as in (4.8) and $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Assume that h is γ-Hölder on any compact set, with $w_h(t, M) \le Ct^\gamma M^\alpha$, for some $C > 0$, $\gamma \in]0, 1]$ and $\alpha \ge 0$. If for some $p \in]2, 3]$,*

$$\mathbb{E}(|\varepsilon_0|^{2\vee(\alpha+\gamma)p}) < \infty \quad \textit{and} \quad \sum_{i\ge 1} i^{p/2-1}\Big(\sum_{|j|\ge i} a_j^2\Big)^{\gamma/2} < \infty, \qquad (4.9)$$

then the series $\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ converges to some nonnegative σ^2, and

1. $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$, *for* $p = 3$,

2. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$ *for* $r \in [p-2, 2]$ *and* $(r,p) \neq (1,3)$,

3. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$ *for* $r \in]2, p]$.

Proof of Theorem 4.2. Theorem 4.2 is a consequence of the following proposition:

Proposition 4.2. *Let* $(a_i)_{i\in\mathbb{Z}}$, $(\varepsilon_i)_{i\in\mathbb{Z}}$ *and* $(X_i)_{i\in\mathbb{Z}}$ *be as in Theorem 4.2. Let* $(\varepsilon'_i)_{i\in\mathbb{Z}}$ *be an independent copy of* $(\varepsilon_i)_{i\in\mathbb{Z}}$. *Let* $V_0 = \sum_{i\in\mathbb{Z}} a_i\varepsilon_{-i}$ *and*

$$M_{1,i} = |V_0| \vee \Big|\sum_{j<i} a_j\varepsilon_{-j} + \sum_{j\geq i} a_j\varepsilon'_{-j}\Big| \quad \text{and} \quad M_{2,i} = |V_0| \vee \Big|\sum_{j<i} a_j\varepsilon'_{-j} + \sum_{j\geq i} a_j\varepsilon_{-j}\Big|.$$

If for some $p \in]2, 3]$,

$$\sum_{i\geq 1} i^{p/2-1}\Big\|w_h\Big(\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p < \infty \quad \text{and} \quad \sum_{i\geq 1} i^{p/2-1}\Big\|w_h\Big(\Big|\sum_{j<-i} a_j\varepsilon_{-j}\Big|, M_{2,-i}\Big)\Big\|_p < \infty, \tag{4.10}$$

then the conclusions of Theorem 4.2 hold.

To prove Theorem 4.2, it remains to check (4.10). We only check the first condition. Since $w_h(t, M) \leq Ct^\gamma M^\alpha$ and the random variables ε_i are iid, we have

$$\Big\|w_h\Big(\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p \;\leq\; C\Big\|\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|^\gamma |V_0|^\alpha\Big\|_p + C\Big\|\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|^\gamma\Big\|_p \||V_0|^\alpha\|_p\,,$$

so that

$$\Big\|w_h\Big(\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p \\ \leq C\Big(2^\alpha\Big\|\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|^{\alpha+\gamma}\Big\|_p + \Big\|\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|^\gamma\Big\|_p\Big(\||V_0|^\alpha\|_p + 2^\alpha\Big\|\Big|\sum_{j<i} a_j\varepsilon_{-j}\Big|^\alpha\Big\|_p\Big)\Big).$$

From Burkholder's inequality, for any $\beta > 0$,

$$\Big\|\Big|\sum_{j\geq i} a_j\varepsilon_{-j}\Big|^\beta\Big\|_p = \Big\|\sum_{j\geq i} a_j\varepsilon_{-j}\Big\|_{\beta p}^\beta \leq K\Big(\sum_{j\geq i} a_j^2\Big)^{\beta/2}\|\varepsilon_0\|_{2\vee\beta p}^\beta\,.$$

Applying this inequality with $\beta = \gamma$ or $\beta = \alpha + \gamma$, we infer that the first part of (4.10) holds under (4.9). The second part can be handled in the same way. □

Proof of Proposition 4.2. Let $\mathcal{F}_i = \sigma(\varepsilon_k, k \le i)$. We shall first prove that the condition (3.2) of Theorem 3.1 holds. We write

$$\begin{aligned}\|\mathbb{E}(S_n^2|\mathcal{F}_0) \;-\; \mathbb{E}(S_n^2)\|_{p/2} &\le 2\sum_{i=1}^{n}\sum_{k=0}^{n-i}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}\\ &\le \;4\sum_{i=1}^{n}\sum_{k=i}^{n}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2} + 2\sum_{i=1}^{n}\sum_{k=1}^{i}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}\,.\end{aligned}$$

We first control the second term. Let ε' be an independent copy of ε, and denote by $\mathbb{E}_\varepsilon(\cdot)$ the conditional expectation with respect to ε. Define

$$Y_i = \sum_{j<i} a_j\varepsilon_{i-j}\ ,\ Y_i' = \sum_{j<i} a_j\varepsilon_{i-j}'\ , Z_i = \sum_{j\ge i} a_j\varepsilon_{i-j}\ , \text{and}\quad Z_i' = \sum_{j\ge i} a_j\varepsilon_{i-j}'\,.$$

Taking $\mathcal{F}_\ell = \sigma(\varepsilon_i, i \le \ell)$, and setting $h_0 = h - \mathbb{E}(h(\sum_{i\in\mathbb{Z}} a_i\varepsilon_i))$, we have

$$\begin{aligned}&\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}\\ &\qquad = \Big\|\mathbb{E}_\varepsilon\Big(h_0(Y_i'+Z_i)h_0(Y_{k+i}'+Z_{k+i})\Big) - \mathbb{E}_\varepsilon\Big(h_0(Y_i'+Z_i')h_0(Y_{k+i}'+Z_{k+i}')\Big)\Big\|_{p/2}\,.\end{aligned}$$

Applying first the triangle inequality, and next Hölder's inequality, we get that

$$\begin{aligned}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} &\le \|h_0(Y_{k+i}'+Z_{k+i})\|_p\|h_0(Y_i'+Z_i) - h_0(Y_i'+Z_i')\|_p\\ &+ \|h_0(Y_i'+Z_i')\|_p\|h_0(Y_{k+i}'+Z_{k+i}) - h_0(Y_{k+i}'+Z_{k+i}')\|_p\,.\end{aligned}$$

Let $m_{1,i} = |Y_i'+Z_i| \vee |Y_i'+Z_i'|$. Since $w_{h_0}(t,M) = w_h(t,M)$, it follows that

$$\begin{aligned}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} &\le \|h_0(Y_{k+i}'+Z_{k+i})\|_p\Big\|w_h\Big(\Big|\sum_{j\ge i} a_j(\varepsilon_{i-j}-\varepsilon_{i-j}')\Big|, m_{1,i}\Big)\Big\|_p\\ &+ \|h_0(Y_i'+Z_i')\|_p\Big\|w_h\Big(\Big|\sum_{j\ge k+i} a_j(\varepsilon_{k+i-j}-\varepsilon_{k+i-j}')\Big|, m_{1,k+i}\Big)\Big\|_p\,.\end{aligned}$$

By subadditivity, we obtain that

$$\Big\|w_h\Big(\Big|\sum_{j\ge i} a_j(\varepsilon_{i-j}-\varepsilon_{i-j}')\Big|, m_{1,i}\Big)\Big\|_p \le \Big\|w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{i-j}\Big|, m_{1,i}\Big)\Big\|_p + \Big\|w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{i-j}'\Big|, m_{1,i}\Big)\Big\|_p\,.$$

Since the three couples $(\sum_{j\ge i} a_j\varepsilon_{i-j}, m_{1,i})$, $(\sum_{j\ge i} a_j\varepsilon_{i-j}', m_{1,i})$ and $(\sum_{j\ge i} a_j\varepsilon_{-j}, M_{1,i})$ are identically distributed, it follows that

$$\Big\|w_h\Big(\Big|\sum_{j\ge i} a_j(\varepsilon_{i-j}-\varepsilon_{i-j}')\Big|, m_{1,i}\Big)\Big\|_p \le 2\Big\|w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p\,.$$

In the same way

$$\Big\|w_h\Big(\Big|\sum_{j\geq k+i}a_j(\varepsilon_{k+i-j}-\varepsilon'_{k+i-j})\Big|,m_{1,k+i}\Big)\Big\|_p\leq 2\Big\|w_h\Big(\Big|\sum_{j\geq k+i}a_j\varepsilon_{-j}\Big|,M_{1,k+i}\Big)\Big\|_p\,.$$

Consequently

$$\sum_{n\geq 1}\frac{1}{n^{3-p/2}}\sum_{i=1}^{n}\sum_{k=1}^{i}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)-\mathbb{E}(X_iX_{k+i})\|_{p/2}<\infty$$

provided that the first condition in (4.10) holds.

We turn now to the control of $\sum_{i=1}^n\sum_{k=i}^n\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2}$. We first write that

$$\begin{aligned}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2} &\leq \|\mathbb{E}\big((X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]}))X_{k+i}|\mathcal{F}_0\big)\|_{p/2}+\|\mathbb{E}\big(\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})X_{k+i}|\mathcal{F}_0\big)\|_{p/2}\\ &\leq \|X_0\|_p\|X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\|_p+\|X_0\|_p\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p\,.\end{aligned}$$

Let $b(k)=k-[k/2]$. Since $\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p=\|\mathbb{E}(X_{b(k)}|\mathcal{F}_0)\|_p$, we have that

$$\begin{aligned}&\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p\\ &\quad=\Big\|\mathbb{E}_\varepsilon\Big(h\Big(\sum_{j<b(k)}a_j\varepsilon'_{b(k)-j}+\sum_{j\geq b(k)}a_j\varepsilon_{b(k)-j}\Big)-h\Big(\sum_{j<b(k)}a_j\varepsilon'_{b(k)-j}+\sum_{j\geq b(k)}a_j\varepsilon'_{b(k)-j}\Big)\Big)\Big\|_p\,.\end{aligned}$$

Using the same arguments as before, we get that

$$\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p=\|\mathbb{E}(X_{b(k)}|\mathcal{F}_0)\|_p\leq 2\Big\|w_h\Big(\Big|\sum_{j\geq b(k)}a_j\varepsilon_{-j}\Big|,M_{1,b(k)}\Big)\Big\|_p\,. \tag{4.11}$$

In the same way,

$$\begin{aligned}&\Big\|X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\Big\|_p\\ &\quad=\Big\|\mathbb{E}_\varepsilon\Big(h\Big(\sum_{j<-[k/2]}a_j\varepsilon_{i-j}+\sum_{j\geq-[k/2]}a_j\varepsilon_{i-j}\Big)-h\Big(\sum_{j<-[k/2]}a_j\varepsilon'_{i-j}+\sum_{j\geq-[k/2]}a_j\varepsilon_{i-j}\Big)\Big)\Big\|_p\,.\end{aligned}$$

Let

$$m_{2,i,k}=\Big|\sum_{j\in\mathbb{Z}}a_i\varepsilon_{i-j}\Big|\vee\Big|\sum_{j<-[k/2]}a_j\varepsilon'_{i-j}+\sum_{j\geq-[k/2]}a_j\varepsilon_{i-j}\Big|\,.$$

Using again the subbadditivity of $t\to w_h(t,M)$, and the fact that $(\sum_{j<-[k/2]}a_j\varepsilon_{i-j},m_{2,i,k})$, $(\sum_{j<-[k/2]}a_j\varepsilon'_{i-j},m_{2,i,k})$ and $(\sum_{j<-[k/2]}a_j\varepsilon_{-j},M_{2,-[k/2]})$ are identically distributed, we obtain that

$$\Big\|X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\Big\|_p=\Big\|X_{-[k/2]}-\mathbb{E}(X_{-[k/2]}|\mathcal{F}_0)\Big\|_p\leq 2\Big\|w_h\Big(\Big|\sum_{j<-[k/2]}a_j\varepsilon_{-j}\Big|,M_{2,-[k/2]}\Big)\Big\|_p\,. \tag{4.12}$$

Consequently

$$\sum_{n\geq 1}\frac{1}{n^{3-p/2}}\sum_{i=1}^{n}\sum_{k=i}^{n}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2}<\infty$$

provided that (4.10) holds. This completes the proof of (3.2).

Using the bounds (4.11) and (4.12) (taking $b(k)=n$ in (4.11) and $[k/2]=n$ in (4.12)), we see that the condition (3.1) of Theorem 3.1 (and also the condition (3.4) of Theorem 3.2 in the case $p=3$) holds under (4.10). $\square$

4.3 Functions of ϕ-dependent sequences

In order to include examples of dynamical systems satisfying some correlations inequalities, we introduce a weak version of the uniform mixing coefficients (see Dedecker and Prieur (2007)).

Definition 4.1. For any random variable $Y=(Y_1,\cdots,Y_k)$ with values in $\mathbb{R}^k$ define the function $g_{x,j}(t)=\mathbb{1}_{t\leq x}-\mathbb{P}(Y_j\leq x)$. For any σ-algebra $\mathcal{F}$, let

$$\phi(\mathcal{F},Y)=\sup_{(x_1,\dots,x_k)\in\mathbb{R}^k}\Big\|\mathbb{E}\Big(\prod_{j=1}^{k}g_{x_j,j}(Y_j)\Big|\mathcal{F}\Big)-\mathbb{E}\Big(\prod_{j=1}^{k}g_{x_j,j}(Y_j)\Big)\Big\|_\infty .$$

For a sequence $\mathbf{Y}=(Y_i)_{i\in\mathbb{Z}}$, where $Y_i=Y_0\circ T^i$ and Y_0 is a $\mathcal{F}_0$-measurable and real-valued r.v., let

$$\phi_{k,\mathbf{Y}}(n)=\max_{1\leq l\leq k}\sup_{i_l>\ldots>i_1\geq n}\phi(\mathcal{F}_0,(Y_{i_1},\ldots,Y_{i_l})).$$

Definition 4.2. For any $p\geq 1$, let $\mathcal{C}(p,M,P_X)$ be the closed convex envelop of the set of functions f which are monotonic on some open interval of $\mathbb{R}$ and null elsewhere, and such that $\mathbb{E}(|f(X)|^p)<M$.

Proposition 4.3. *Let $p\in]2,3]$ and $s\geq p$. Let $X_i=f(Y_i)-\mathbb{E}(f(Y_i))$, where $Y_i=Y_0\circ T^i$ and f belongs to $\mathcal{C}(s,M,P_{Y_0})$. Assume that*

$$\sum_{i\geq 1}i^{(p-4)/2+(s-2)/(s-1)}\phi_{2,\mathbf{Y}}(i)^{(s-2)/s}<\infty\,. \tag{4.13}$$

Then the conclusions of Theorem 4.2 hold.

Remark 4.5. Notice that if $s=p=3$, the condition (4.13) becomes $\sum_{i\geq 1}\phi_{2,\mathbf{Y}}(i)^{1/3}<\infty$, and if $s=\infty$, the condition (4.13) becomes $\sum_{i\geq 1}i^{(p-2)/2}\phi_{2,\mathbf{Y}}(i)<\infty$.

Proof of Proposition 4.3. Let $B^p(\mathcal{F}_0)$ be the set of $\mathcal{F}_0$-measurable random variables such that $\|Z\|_p \leq 1$. We first notice that

$$\|\mathbb{E}(X_k|\mathcal{F}_0)\|_p \leq \|\mathbb{E}(X_k|\mathcal{F}_0)\|_s = \sup_{Z \in B^{s/(s-1)}(\mathcal{F}_0)} \mathrm{Cov}(Z, f(Y_k))\,.$$

Applying Corollary 6.2 with $k = 2$ to the covariance on right hand (take $f_1 = \mathrm{Id}$ and $f_2 = f$), we obtain that

$$\begin{aligned} \|\mathbb{E}(X_k|\mathcal{F}_0)\|_s &\leq \sup_{Z \in B^{s/(s-1)}(\mathcal{F}_0)} 8(\phi(\sigma(Z), Y_k))^{(s-1)/s} \|Z\|_{s/(s-1)} (\phi(\sigma(Y_k), Z))^{1/s} M^{1/s} \\ &\leq 8(\phi_{1,\mathbf{Y}}(k))^{(s-1)/s} M^{1/s}\,, \end{aligned} \tag{4.14}$$

the last inequality being true because $\phi(\sigma(Z), Y_k) \leq \phi_{1,\mathbf{Y}}(k)$ and $\phi(\sigma(Y_k), Z) \leq 1$. It follows that the conditions (3.1) (for $p \in]2,3]$) and (3.4) (for $p = 3$) are satisfied under (4.13). The condition (3.2) follows from the following lemma by taking $b = (4-p)/2$.

Lemma 4.1. *Let X_i be as in Proposition 4.3, and let $b \in]0,1[$.*

$$\textit{If} \quad \sum_{i \geq 1} i^{-b+(s-2)/(s-1)} \phi_{2,\mathbf{Y}}(i)^{(s-2)/s} < \infty, \quad \textit{then} \quad \sum_{n>1} \frac{1}{n^{1+b}} \|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} < \infty\,.$$

Proof of Lemma 4.1. Since,

$$\|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} \leq 2 \sum_{i=1}^{n} \sum_{k=0}^{n-i} \|\mathbb{E}(X_i X_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_i X_{k+i})\|_{p/2},$$

we infer that there exists $C > 0$ such that

$$\sum_{n>1} \frac{1}{n^{1+b}} \|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} \leq C \sum_{i>0} \sum_{k \geq 0} \frac{1}{(i+k)^b} \|\mathbb{E}(X_i X_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_i X_{k+i})\|_{p/2}\,. \tag{4.15}$$

We shall bound up $\|\mathbb{E}(X_i X_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_i X_{k+i})\|_{p/2}$ in two ways. First, using the stationarity and the upper bound (4.14), we have that

$$\|\mathbb{E}(X_i X_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_i X_{k+i})\|_{p/2} \leq 2\|X_0 \mathbb{E}(X_k|\mathcal{F}_0)\|_{p/2} \leq 16 \|X_0\|_p M^{1/s} (\phi_{1,\mathbf{Y}}(k))^{(s-1)/s}\,. \tag{4.16}$$

Next, note that

$$\begin{aligned} \|\mathbb{E}(X_i X_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_i X_{k+i})\|_{p/2} &= \sup_{Z \in B^{s/(s-2)}(\mathcal{F}_0)} \mathrm{Cov}(Z, X_i X_{k+i}) \\ &= \sup_{Z \in B^{s/(s-2)}(\mathcal{F}_0)} \mathbb{E}((Z - \mathbb{E}(Z)) X_i X_{k+i})\,. \end{aligned}$$

Applying Corollary 6.2 with $k = 3$ to the term $\mathbb{E}((Z-\mathbb{E}(Z))X_iX_{k+i})$ (take $f_1 = \mathrm{Id}$, $f_2 = f_3 = f$), we obtain that $\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}$ is smaller than

$$\sup_{Z\in B^{s/(s-2)}(\mathcal{F}_0)} 32(\phi(\sigma(Z), Y_i, Y_{k+i}))^{(s-2)/s}\|Z\|_{s/(s-2)}M^{2/s}(\phi(\sigma(Y_i), Z, Y_{k+i}))^{1/s}(\phi(\sigma(Y_{k+i}), Z, Y_i))^{1/s} .$$

Since $\phi(\sigma(Z), Y_i, Y_{k+i}) \le \phi_{2,\mathbf{Y}}(i)$ and $\phi(\sigma(Y_i), Z, Y_{k+i}) \le 1$, $\phi(\sigma(Y_{k+i}), Z, Y_i) \le 1$, we infer that

$$\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} \le 32(\phi_{2,\mathbf{Y}}(i))^{(s-2)/s}M^{2/s} . \tag{4.17}$$

From (4.15), (4.16) and (4.17), we infer that the conclusion of Lemma 4.1 holds provided that

$$\sum_{i>0}\Big(\sum_{k=1}^{[i^{(s-2)/(s-1)}]}\frac{1}{(i+k)^b}\Big)(\phi_{2,\mathbf{Y}}(i))^{(s-2)/s} + \sum_{k\ge 0}\Big(\sum_{i=1}^{[k^{(s-1)/(s-2)}]}\frac{1}{(i+k)^b}\Big)(\phi_{1,\mathbf{Y}}(k))^{(s-1)/s} < \infty .$$

Here, note that

$$\sum_{k=1}^{[i^{(s-2)/(s-1)}]}\frac{1}{(i+k)^b} \le i^{-b+\frac{s-2}{s-1}} \quad \text{and} \quad \sum_{i=1}^{[k^{(s-1)/(s-2)}]}\frac{1}{(i+k)^b} \le \sum_{m=1}^{[2k^{(s-1)/(s-2)}]}\frac{1}{m^b} \le Dk^{(1-b)\frac{(s-1)}{(s-2)}} ,$$

for some $D > 0$. Since $\phi_{1,\mathbf{Y}}(k) \le \phi_{2,\mathbf{Y}}(k)$, the conclusion of Lemma 4.1 holds provided

$$\sum_{i\ge 1} i^{-b+\frac{s-2}{s-1}}\phi_{2,\mathbf{Y}}(i)^{\frac{s-2}{s}} < \infty \quad \text{and} \quad \sum_{k\ge 1} k^{(1-b)\frac{(s-1)}{(s-2)}}\phi_{2,\mathbf{Y}}(k)^{\frac{s-1}{s}} < \infty .$$

To complete the proof, it remains to prove that the second series converges provided the first one does. If the first series converges, then

$$\lim_{n\to\infty}\sum_{i=n+1}^{2n} i^{-b+\frac{s-2}{s-1}}\phi_{2,\mathbf{Y}}(i)^{\frac{s-2}{s}} = 0 . \tag{4.18}$$

Since $\phi_{2,\mathbf{Y}}(i)$ is non increasing, we infer from (4.18) that $\phi_{2,\mathbf{Y}}(i)^{1/s} = o(i^{-1/(s-1)-(1-b)/(s-2)})$. It follows that $\phi_{2,\mathbf{Y}}(k)^{(s-1)/s} \le C\phi_{2,\mathbf{Y}}(k)^{(s-2)/s}k^{-1/(s-1)-(1-b)/(s-2)}$ for some positive constant C, and the second series converges. □

4.3.1 Application to Expanding maps

Let BV be the class of bounded variation functions from $[0, 1]$ to $\mathbb{R}$. For any $h \in BV$, denote by $\|dh\|$ the variation norm of the measure dh.

Let T be a map from $[0, 1]$ to $[0, 1]$ preserving a probability μ on $[0, 1]$, and let

$$S_n(f) = \sum_{k=1}^{n}(f \circ T^k - \mu(f)) .$$

Define the Perron-Frobenius operator K from $\mathbb{L}^2([0,1],\mu)$ to $\mathbb{L}^2([0,1],\mu)$ *via* the equality

$$\int_0^1 (Kh)(x)f(x)\mu(dx) = \int_0^1 h(x)(f\circ T)(x)\mu(dx)\,. \tag{4.19}$$

A Markov Kernel K is said to be BV-contracting if there exist $C>0$ and $\rho\in[0,1[$ such that

$$\|dK^n(h)\| \leq C\rho^n\|dh\|\,. \tag{4.20}$$

The map T is said to be BV-contracting if its Perron-Frobenius operator is BV-contracting.

Let us present a large class of BV-contracting maps. We shall say that T is uniformly expanding if it belongs to the class $\mathcal{C}$ defined in Broise (1996), Section 2.1 page 11. Recall that if T is uniformly expanding, then there exists a probability measure μ on $[0,1]$, whose density f_μ with respect to the Lebesgue measure is a bounded variation function, and such that μ is invariant by T. Consider now the more restrictive conditions:

(a) T is uniformly expanding.

(b) The invariant measure μ is unique and (T,μ) is mixing in the ergodic-theoretic sense.

(c) $\dfrac{1}{f_\mu}\mathbf{1}_{f_\mu>0}$ is a bounded variation function.

Starting from Proposition 4.11 in Broise (1996), one can prove that if T satisfies the assumptions (a), (b) and (c) above, then it is BV contracting (see for instance Dedecker and Prieur (2007), Section 6.3). Some well known examples of maps satisfying the conditions (a), (b) and (c) are:

1. $T(x)=\beta x-[\beta x]$ for $\beta>1$. These maps are called β-transformations.

2. I is the finite union of disjoint intervals $(I_k)_{1\leq k\leq n}$, and $T(x)=a_kx+b_k$ on I_k, with $|a_k|>1$.

3. $T(x)=a(x^{-1}-1)-[a(x^{-1}-1)]$ for some $a>0$. For $a=1$, this transformation is known as the Gauss map.

Proposition 4.4. *Let $\sigma_n^2=n^{-1}\mathbb{E}(S_n^2(f))$. If T is BV-contracting, and if f belongs to $\mathcal{C}(p,M,\mu)$ with $p\in]2,3]$, then the series $\mu((f-\mu(f))^2)+2\sum_{n>0}\mu(f\circ T^n\cdot(f-\mu(f)))$ converges to some nonnegative σ^2, and*

1. *$\zeta_1(P_{n^{-1/2}S_n(f)},G_{\sigma^2})=O(n^{-1/2}\log n)$, for $p=3$,*

2. *$\zeta_r(P_{n^{-1/2}S_n(f)},G_{\sigma^2})=O(n^{1-p/2})$ for $r\in[p-2,2]$ and $(r,p)\neq(1,3)$,*

3. *$\zeta_r(P_{n^{-1/2}S_n(f)},G_{\sigma_n^2})=O(n^{1-p/2})$ for $r\in]2,p]$.*

Proof of Proposition 4.4. Let $(Y_i)_{i\geq 1}$ be the Markov chain with transition Kernel K and invariant measure μ. Using the equation (4.19) it is easy to see that $(Y_0, \dots, Y_n)$ is distributed as $(T^{n+1}, \dots, T)$. Consequently, to prove Proposition 4.4, it suffices to prove that the sequence $X_i = f(Y_i) - \mu(f)$ satisfies the condition (4.13) of Proposition 4.3.

According to Lemma 1 in Dedecker and Prieur (2007), the coefficients $\phi_{2,\mathbf{Y}}(i)$ of the chain $(Y_i)_{i\geq 0}$ with respect to $\mathcal{F}_i = \sigma(Y_j, j \leq i)$ satisfy $\phi_{2,\mathbf{Y}}(i) \leq C\rho^i$ for some $\rho \in]0,1[$ and some positive constant C. It follows that (4.13) is satisfied for $s = p$.

5 Proofs of the main results

From now on, we denote by C a numerical constant which may vary from line to line.

Notation 5.1. For l integer, q in $]l, l+1]$ and f l-times continuously differentiable, we set

$$|f|_{\Lambda_q} = \sup\{|x-y|^{l-q}|f^{(l)}(x) - f^{(l)}(y)| : (x,y) \in \mathbb{R} \times \mathbb{R}\}.$$

5.1 Proof of Theorem 2.1

We prove Theorem 2.1 in the case $\sigma = 1$. The general case follows by dividing the random variables by σ. Since $\zeta_r(P_{aX}, P_{aY}) = |a|^r \zeta_r(P_X, P_Y)$, it is enough to bound up $\zeta_r(P_{S_n}, G_n)$. We first give an upper bound for $\zeta_{p,N} := \zeta_p(P_{S_{2^N}}, G_{2^N})$.

Proposition 5.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale differences sequence in $\mathbb{L}^p$ for p in $]2,3]$. Let $M_p = \mathbb{E}(|X_0|^p)$. Then for any natural integer N,*

$$2^{-2N/p}\zeta_{p,N}^{2/p} \leq \Big(M_p + \frac{1}{2\sqrt{2}}\sum_{K=0}^{N} 2^{K(p/2-2)}\|Z_K\|_{1,\Phi,p}\Big)^{2/p} + \frac{2}{p}\Delta_N\,, \tag{5.1}$$

where $Z_K = \mathbb{E}(S_{2^K}^2|\mathcal{F}_0) - \mathbb{E}(S_{2^K}^2)$ and $\Delta_N = \sum_{K=0}^{N-1} 2^{-2K/p}\|Z_K\|_{p/2}$.

Proof of Proposition 5.1. The proof is done by induction on N. Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of $N(0,1)$-distributed independent random variables, independent of the sequence $(X_i)_{i\in\mathbb{Z}}$. For $m > 0$, let $T_m = Y_1 + Y_2 + \dots + Y_m$. Set $S_0 = T_0 = 0$. For any numerical function f and $m \leq n$, set

$$f_{n-m}(x) = \mathbb{E}(f(x + T_n - T_m)).$$

Then, from the independence of the above sequences,

$$\mathbb{E}(f(S_n) - f(T_n)) = \sum_{m=1}^{n} D_m \quad \text{with} \quad D_m = \mathbb{E}\big(f_{n-m}(S_{m-1} + X_m) - f_{n-m}(S_{m-1} + Y_m)\big). \tag{5.2}$$

For any two-times differentiable function g, the Taylor integral formula at order two writes

$$g(x+h)-g(x)=g'(x)h+\frac{1}{2}h^2g''(x)+h^2\int_0^1(1-t)(g''(x+th)-g''(x))dt. \tag{5.3}$$

Hence, for any q in $]2,3]$,

$$|g(x+h)-g(x)-g'(x)h-\frac{1}{2}h^2g''(x)|\le h^2\int_0^1(1-t)|th|^{q-2}|g|_{\Lambda_q}dt\le\frac{1}{q(q-1)}|h|^q|g|_{\Lambda_q}. \tag{5.4}$$

Let

$$D'_m=\mathbb{E}(f''_{n-m}(S_{m-1})(X_m^2-1))=\mathbb{E}(f''_{n-m}(S_{m-1})(X_m^2-Y_m^2))$$

From (5.4) applied twice with $g=f_{n-m}$, $x=S_{m-1}$ and $h=X_m$ or $h=Y_m$ together with the martingale property,

$$\Big|D_m-\frac{1}{2}D'_m\Big|\le\frac{1}{p(p-1)}\,|f_{n-m}|_{\Lambda_p}\mathbb{E}(|X_m|^p+|Y_m|^p).$$

Now $\mathbb{E}(|Y_m|^p)\le p-1\le(p-1)M_p$. Hence

$$|D_m-(D'_m/2)|\le M_p|f_{n-m}|_{\Lambda_p} \tag{5.5}$$

Assume now that f belongs to Λ_p. Then the smoothed function f_{n-m} belongs to Λ_p also, so that $|f_{n-m}|_{\Lambda_p}\le 1$. Hence, summing on m, we get that

$$\mathbb{E}(f(S_n)-f(T_n))\le nM_p+(D'/2)\quad\text{where } D'=D'_1+D'_2+\cdots+D'_n. \tag{5.6}$$

Suppose now that $n=2^N$. To bound up D', we introduce a dyadic scheme.

Notation 5.2. Set $m_0=m-1$ and write m_0 in basis 2: $m_0=\sum_{i=0}^N b_i2^i$ with $b_i=0$ or $b_i=1$ (note that $b_N=0$). Set $m_L=\sum_{i=L}^N b_i2^i$, so that $m_N=0$. Let $I_{L,k}=]k2^L,(k+1)2^L]\cap\mathbb{N}$ (note that $I_{N,1}=]2^N,2^{N+1}]$), $U_L^{(k)}=\sum_{i\in I_{L,k}}X_i$ and $\tilde{U}_L^{(k)}=\sum_{i\in I_{L,k}}Y_i$. For the sake of brevity, let $U_L^{(0)}=U_L$ and $\tilde{U}_L^{(0)}=\tilde{U}_L$.

Since $m_N=0$, the following elementary identity is valid

$$D'_m=\sum_{L=0}^{N-1}\mathbb{E}\Big((f''_{n-1-m_L}(S_{m_L})-f''_{n-1-m_{L+1}}(S_{m_{L+1}}))(X_m^2-1)\Big).$$

Now $m_L\ne m_{L+1}$ only if $b_L=1$, then in this case $m_L=k2^L$ with k odd. It follows that

$$D'=\sum_{L=0}^{N-1}\sum_{\substack{k\in I_{N-L,0}\\ k\text{ odd}}}\mathbb{E}\Big((f''_{n-1-k2^L}(S_{k2^L})-f''_{n-1-(k-1)2^L}(S_{(k-1)2^L}))\sum_{\{m:m_L=k2^L\}}(X_m^2-\sigma^2)\Big). \tag{5.7}$$

Note that $\{m : m_L = k2^L\} = I_{L,k}$. Now by the martingale property,

$$\mathbb{E}_{k2^L}\Big(\sum_{i\in I_{L,k}}(X_i^2-\sigma^2)\Big) = \mathbb{E}_{k2^L}((U_L^{(k)})^2) - \mathbb{E}((U_L^{(k)})^2) := Z_L^{(k)} .$$

Consequently

$$\begin{aligned} D' &= \sum_{L=0}^{N-1}\sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} \mathbb{E}\Big(\big(f''_{n-1-k2^L}(S_{k2^L}) - f''_{n-1-(k-1)2^L}(S_{(k-1)2^L})\big)Z_L^{(k)}\Big) \\ &= \sum_{L=0}^{N-1}\sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} \mathbb{E}\Big(\big(f''_{n-1-k2^L}(S_{k2^L}) - f''_{n-1-k2^L}(S_{(k-1)2^L}+T_{k2^L}-T_{(k-1)2^L})\big)Z_L^{(k)}\Big) , \end{aligned} \tag{5.8}$$

since $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ are independent. By using (1.2), we get that

$$D' \le \sum_{L=0}^{N-1}\sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} \mathbb{E}(|U_L^{(k-1)} - \tilde U_L^{(k-1)}|^{p-2}|Z_L^{(k)}|) .$$

From the stationarity of $(X_i)_{i\in\mathbb{N}}$ and the above inequality,

$$D' \le \frac{1}{2}\sum_{K=0}^{N-1} 2^{N-K}\mathbb{E}(|U_K - \tilde U_K|^{p-2}|Z_K^{(1)}|). \tag{5.9}$$

Now let V_K be the $N(0,2^K)$-distributed random variable defined from U_K via the quantile transformation, that is

$$V_K = 2^{K/2}\Phi^{-1}(F_K(U_K-0) + \delta_K(F_K(U_K) - F_K(U_K-0)))$$

where F_K denotes the d.f. of U_K, and (δ_K) is a sequence of independent r.v.'s uniformly distributed on $[0,1]$, independent of the underlying random variables. Now, from the subadditivity of $x \to x^{p-2}$, $|U_K - \tilde U_K|^{p-2} \le |U_K - V_K|^{p-2} + |V_K - \tilde U_K|^{p-2}$. Hence

$$\mathbb{E}(|U_K - \tilde U_K|^{p-2}|Z_K^{(1)}|) \le \|U_K - V_K\|_p^{p-2}\|Z_K^{(1)}\|_{p/2} + \mathbb{E}(|V_K - \tilde U_K|^{p-2}|Z_K^{(1)}|) . \tag{5.10}$$

By definition of V_K, the real number $\|U_K - V_K\|_p$ is the so-called Wasserstein distance of order p between the law of $U_K^{(0)}$ and the $N(0,2^K)$ normal law. Therefrom, by Theorem 3.1 of Rio (2007) (which improves the constants given in Theorem 1 of Rio (1998)), we get that, for $p \in]2,3]$,

$$\|U_K - V_K\|_p \le 2(2(p-1)\zeta_{p,K})^{1/p} \le 2(4\zeta_{p,K})^{1/p}. \tag{5.11}$$

Now, since V_K and $\tilde U_K$ are independent, their difference has the $N(0,2^{K+1})$ distribution. Note that if Y is a $N(0,1)$-distributed random variable, $Q_{|Y|^{p-2}}(u) = (\Phi^{-1}(1-u/2))^{p-2}$. Hence, by

Fréchet's inequality (1957) (see also Inequality (1.11b) page 9 in Rio (2000)), and by definition of the norm $\| \, . \, \|_{1,\Phi,p}$,

$$\mathbb{E}(|V_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|) \le 2^{(K+1)(p/2-1)}\|Z_K\|_{1,\Phi,p}. \tag{5.12}$$

From (5.10), (5.11) and (5.12), we get that

$$\mathbb{E}(|U_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|) \le 2^{p-4/p}\zeta_{p,K}^{(p-2)/p}\|Z_K\|_{p/2} + 2^{(K+1)(p/2-1)}\|Z_K\|_{1,\Phi,p} \, . \tag{5.13}$$

Then, from (5.6), (5.9) and (5.13), we get

$$2^{-N}\zeta_{p,N} \le M_p + 2^{p/2-3}\Delta'_N + 2^{p-2-4/p}\sum_{K=0}^{N-1} 2^{-K}\zeta_{p,K}^{(p-2)/p}\|Z_K\|_{p/2},$$

where $\Delta'_N = \sum_{K=0}^{N-1} 2^{K(p/2-2)}\|Z_K\|_{1,\Phi,p}$. Consequently we get the induction inequality

$$2^{-N}\zeta_{p,N} \le M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \sum_{K=0}^{N-1} 2^{-K}\zeta_{p,K}^{(p-2)/p}\|Z_K\|_{p/2} \, . \tag{5.14}$$

We now prove (5.1) by induction on N. First by (5.6) applied with $n = 1$, one has $\zeta_{p,0} \le M_p$, since $D'_1 = f''(0)\mathbb{E}(X_1^2 - 1) = 0$. Assume now that $\zeta_{p,L}$ satisfies (5.1) for any L in $[0, N-1]$. Starting from (5.14), using the induction hypothesis and the fact that $\Delta'_K \le \Delta'_N$, we get that

$$2^{-N}\zeta_{p,N} \le M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \sum_{K=0}^{N-1} 2^{-2K/p}\|Z_K\|_{p/2}\Big(\Big(M_p + \frac{1}{2\sqrt{2}}\Delta'_N\Big)^{2/p} + \frac{2}{p}\Delta_K\Big)^{p/2-1} .$$

Now $2^{-2K/p}\|Z_K\|_{p/2} = \Delta_{K+1} - \Delta_K$. Consequently

$$2^{-N}\zeta_{p,N} \le M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \int_0^{\Delta_N} \Big(\Big(M_p + \frac{1}{2\sqrt{2}}\Delta'_N\Big)^{2/p} + \frac{2}{p}x\Big)^{p/2-1} dx \, ,$$

which implies (5.1) for $\zeta_{p,N}$. □

In order to prove Theorem 2.1, we will also need a smoothing argument. This is the purpose of the lemma below.

Lemma 5.1. *Let S and T be two centered and square integrable random variables with the same variance. For any r in $]0, p]$, $\zeta_r(P_S, P_T) \le 2\zeta_r(P_S * G_1, P_T * G_1) + 4\sqrt{2}$.*

Proof of Lemma 5.1. Throughout the sequel, let Y be a $N(0,1)$-distributed random variable, independent of the σ-field generated by (S, T).

For $r \le 2$, since ζ_r is an ideal metric with respect to the convolution,

$$\zeta_r(P_S, P_T) \le \zeta_r(P_S * G_1, P_T * G_1) + 2\zeta_r(\delta_0, G_1) \le \zeta_r(P_S * G_1, P_T * G_1) + 2\mathbb{E}|Y|^r$$

which implies Lemma 5.1 for $r \le 2$. For $r > 2$, from (5.4), for any f in Λ_r,

$$f(S) - f(S+Y) + f'(S)Y - \frac{1}{2}f''(S)Y^2 \le \frac{1}{r(r-1)}|Y|^r.$$

Taking the expectation and noting that $\mathbb{E}|Y|^r \le r-1$ for r in $]2,3]$, we infer that

$$\mathbb{E}\big(f(S) - f(S+Y) - \frac{1}{2}f''(S)\big) \le \frac{1}{r}.$$

Obviously this inequality still holds for T instead of S and $-f$ instead of f, so that adding the so obtained inequality,

$$\mathbb{E}(f(S) - f(T)) \le \mathbb{E}(f(S+Y) - f(T+Y)) + \frac{1}{2}\mathbb{E}(f''(S) - f''(T)) + 1.$$

Since f'' belongs to Λ_{r-2}, it follows that

$$\zeta_r(P_S, P_T) \le \zeta_r(P_S * G_1, P_T * G_1) + \frac{1}{2}\zeta_{r-2}(P_S, P_T) + 1.$$

Now $r - 2 \le 1$. Hence

$$\zeta_{r-2}(P_S, P_T) = W_{r-2}(P_S, P_T) \le (W_r(P_S, P_T))^{r-2}.$$

Next, by Theorem 3.1 in Rio (2007), $W_r(P_S, P_T) \le (32\zeta_r(P_S, P_T))^{1/r}$. Furthermore

$$(32\zeta_r(P_S, P_T))^{1-2/r} \le \zeta_r(P_S, P_T)$$

as soon as $\zeta_r(P_S, P_T) \ge 2^{(5r/2)-5}$. This condition holds for any r in $]2,3]$ if $\zeta_r(P_S, P_T) \ge 4\sqrt{2}$. Then, from the above inequalities

$$\zeta_r(P_S, P_T) \le \zeta_r(P_S * G_1, P_T * G_1) + \frac{1}{2}\zeta_r(P_S, P_T) + 1,$$

which implies Lemma 5.1. $\square$

We go back to the proof of Theorem 2.1. Let $n \in]2^N, 2^{N+1}]$ and $\ell = n - 2^N$. The main step is then to prove the inequalities below: for $r \ge p-2$ and $(r,p) \ne (1,3)$, for some $\epsilon(N)$ tending to zero as N tends to infinity,

$$\zeta_r(P_{S_n}, G_n) \le c_{r,p}2^{N(r-p)/2}\zeta_p(P_{S_\ell}, G_\ell) + C(2^{N(r+2-p)/2} + 2^{N((r-p)/2+2/p)}\epsilon(N)(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p}) \tag{5.15}$$

and for $r = 1$ and $p = 3$,

$$\zeta_1(P_{S_n}, G_n) \le C(N + 2^{-N}\zeta_3(P_{S_\ell}, G_\ell) + 2^{-N/3}(\zeta_3(P_{S_\ell}, G_\ell))^{1/3}) . \tag{5.16}$$

Assuming that (5.15) and (5.16) hold, we now complete the proof of Theorem 2.1. Let $\zeta^*_{p,N} = \sup_{n\le 2^N} \zeta_p(P_{S_n}, G_n)$, we infer from (5.15) applied to $r = p$ that

$$\zeta^*_{p,N+1} \le \zeta^*_{p,N} + C(2^N + 2^{2N/p}\epsilon(N)(\zeta^*_{p,N})^{(p-2)/p}) .$$

Let N_0 be such that $C\epsilon(N) \le 1/2$ for $N \le N_0$, and let $K \ge 1$ be such that $\zeta^*_{p,N_0} \le K2^{N_0}$. Choosing K large enough such that $K \ge 2C$, we can easily prove by induction that $\zeta^*_{p,N} \le K2^N$ for any $N \ge N_0$. Hence Theorem 2.1 is proved in the case $r = p$. For r in $[p-2, p[$, Theorem 2.1 follows by taking into account the bound $\zeta^*_{p,N} \le K2^N$, valid for any $N \ge N_0$, in the inequalities (5.15) and (5.16).

We now prove (5.15) and (5.16). We will bound up $\zeta^*_{p,N}$ by induction on N. For $n \in]2^N, 2^{N+1}]$ and $\ell = n - 2^N$, we notice that

$$\zeta_r(P_{S_n}, G_n) \le \zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) + \zeta_r(P_{S_\ell} * G_{2^N}, G_\ell * G_{2^N}) .$$

Let ϕ_t be the density of the law $N(0, t^2)$. With the same notation as in the proof of Proposition 5.1, we have

$$\zeta_r(P_{S_\ell} * G_{2^N}, G_\ell * G_{2^N}) = \sup_{f\in\Lambda_r} \mathbb{E}(f_{2^N}(S_\ell) - f_{2^N}(T_\ell)) \le |f * \phi_{2^{N/2}}|_{\Lambda_p}\zeta_p(P_{S_\ell}, G_\ell) .$$

Applying Lemma 6.1, we infer that

$$\zeta_r(P_{S_n}, G_n) \le \zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) + c_{r,p}2^{N(r-p)/2}\zeta_p(P_{S_\ell}, G_\ell) . \tag{5.17}$$

On the other hand, setting $\tilde S_\ell = X_{1-\ell} + \cdots + X_0$, we have that S_n is distributed as $\tilde S_\ell + S_{2^N}$ and, $S_\ell + T_{2^N}$ as $\tilde S_\ell + T_{2^N}$. Let Y be a $N(0, 1)$-distributed random variable independent of $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$. Using Lemma 5.1, we then derive that

$$\zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) \le 4\sqrt{2} + 2 \sup_{f\in\Lambda_r} \mathbb{E}(f(\tilde S_\ell + S_{2^N} + Y) - f(\tilde S_\ell + T_{2^N} + Y)) . \tag{5.18}$$

Let $D'_m = \mathbb{E}(f''_{2^N-m+1}(\tilde S_\ell + S_{m-1})(X_m^2 - 1))$. We follow the proof of Proposition 5.1. From the Taylor expansion (5.3) applied twice with $g = f_{2^N-m+1}$, $x = \tilde S_\ell + S_{m-1}$ and $h = X_m$ or $h = Y_m$ together with the martingale property, we get that

$$\begin{aligned}\mathbb{E}(f(\tilde S_\ell + S_{2^N} + Y) \; &- \; f(\tilde S_\ell + T_{2^N} + Y)) \\ &= \sum_{m=1}^{2^N} \mathbb{E}(f_{2^N-m+1}(\tilde S_\ell + S_{m-1} + X_m) - f_{2^N-m+1}(\tilde S_\ell + S_{m-1} + Y_m)) \\ &= (D'_1 + \cdots + D'_{2^N})/2 + R_1 + \cdots + R_{2^N} ,\end{aligned} \tag{5.19}$$

where, as in (5.5),

$$R_m \le M_p |f_{2^N-m+1}|_{\Lambda_p} \,. \tag{5.20}$$

In the case $r = p - 2$, we will need the more precise upper bound

$$R_m \le \mathbb{E}\Big(X_m^2\big(\|f''_{2^N-m+1}\|_\infty \wedge \frac{1}{6}\|f^{(3)}_{2^N-m+1}\|_\infty |X_m|\big)\Big) + \frac{1}{6}\|f^{(3)}_{2^N-m+1}\|_\infty \mathbb{E}(|Y_m|^3)\,, \tag{5.21}$$

which is derived from the Taylor formula at orders two and three. From (5.20) and Lemma 6.1, we have that

$$R := R_1 + \cdots + R_{2^N} = O(2^{N(r-p+2)/2}) \quad \text{if } r > p-2, \text{ and} \quad R = O(N) \quad \text{if } (r,p) = (1,3)\,. \tag{5.22}$$

It remains to consider the case $r = p - 2$ and $r < 1$. Applying Lemma 6.1, we get that for $i \ge 2$,

$$\|f^{(i)}_{2^N-m+1}\|_\infty \le c_{r,i}(2^N - m + 1)^{(r-i)/2}\,. \tag{5.23}$$

It follows that

$$\begin{aligned} \sum_{m=1}^{2^N} \mathbb{E}\Big(X_m^2\big(\|f''_{2^N-m+1}\|_\infty \wedge \|f^{(3)}_{2^N-m+1}\|_\infty |X_m|\big)\Big) &\le C\sum_{m=1}^{\infty} \frac{1}{m^{1-r/2}} \mathbb{E}\left(X_0^2\Big(1 \wedge \frac{|X_0|}{\sqrt{m}}\Big)\right) \\ &\le C\mathbb{E}\Big(\sum_{m=1}^{[X_0^2]} \frac{X_0^2}{m^{1-r/2}} + \sum_{m=[X_0^2]+1}^{\infty} \frac{|X_0|^3}{m^{(3-r)/2}}\Big)\,. \end{aligned}$$

Consequently for $r = p - 2$ and $r < 1$,

$$R_1 + \cdots + R_{2^N} \le C(M_p + \mathbb{E}(|Y|^3))\,. \tag{5.24}$$

We now bound up $D'_1 + \cdots + D'_{2^N}$. Using the dyadic scheme as in the proof of Proposition 5.1, we get that

$$\begin{aligned} D'_m &= \sum_{L=0}^{N-1} \mathbb{E}\Big(\big(f''_{2^N-m_L}(\tilde S_\ell + S_{m_L}) - f''_{2^N-m_{L+1}}(\tilde S_\ell + S_{m_{L+1}})\big)(X_m^2 - 1)\Big) + \mathbb{E}(f''_{2^N}(\tilde S_\ell)(X_m^2 - 1)) \\ &:= D''_m + \mathbb{E}(f''_{2^N}(\tilde S_\ell)(X_m^2 - 1))\,. \end{aligned}$$

Notice first that

$$\sum_{m=1}^{2^N} \mathbb{E}(f''_{2^N}(\tilde S_\ell)(X_m^2 - 1)) = \mathbb{E}((f''_{2^N}(\tilde S_\ell) - f''_{2^N}(T_\ell))Z_N^{(0)})\,. \tag{5.25}$$

Since f belongs to Λ_r (*i.e.* $|f|_{\Lambda_r} \le 1$), we infer from Lemma 6.1 that $|f_i|_{\Lambda_p} \le Ci^{(r-p)/2}$ which means exactly that

$$|f''_i(x) - f''_i(y)| \le Ci^{(r-p)/2}|x-y|^{p-2}\,. \tag{5.26}$$

Starting from (5.25) and using (5.26) (with $i = 2^N$), it follows that

$$\sum_{m=1}^{2^N} \mathbb{E}(f''_{2^N}(\tilde{S}_\ell)(X_m^2 - 1)) \le C 2^{N(r-p)/2} \mathbb{E}(|\tilde{S}_\ell - T_\ell|^{p-2} |Z_N^{(0)}|) .$$

Proceeding as to get (5.13) (that is, using similar upper bounds as in (5.10), (5.11) and (5.12)), we obtain that

$$\mathbb{E}(|\tilde{S}_\ell - T_\ell|^{p-2}|Z_N^{(0)}|) \le 2^{p-4/p}(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p} \|Z_N^{(0)}\|_{p/2} + (2\ell)^{p/2-1} \|Z_N^{(0)}\|_{1,\Phi,p} .$$

Using Remark 2.6, (2.1) and (2.2) entail that $\|Z_N^{(0)}\|_{p/2} = o(2^{2N/p})$ and $\|Z_N^{(0)}\|_{1,\Phi,p} = o(2^{N(2-p/2)})$. Hence, for some $\epsilon(N)$ tending to 0 as N tends to infinity, one has

$$\sum_{m=1}^{2^N} D'_m \le \sum_{m=1}^{2^N} D''_m + C(\epsilon(N) 2^{N((r-p)/2+2/p)}(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p} + 2^{N(r+2-p)/2}) . \tag{5.27}$$

Next, proceeding as in the proof of (5.8), we get that

$$\sum_{m=1}^{2^N} D''_m \le \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} \mathbb{E}\Big(\big(f''_{2^N - k2^L}(\tilde{S}_\ell + S_{k2^L}) - f''_{2^N-k2^L}(\tilde{S}_\ell + S_{(k-1)2^L} + T_{k2^L} - T_{(k-1)2^L}) \big) Z_L^{(k)} \Big) . \tag{5.28}$$

Let $r > p-2$ or $(r,p) = (1,3)$. Using (5.26) (with $i = 2^N - k2^L$), (5.28), and the stationarity of $(X_i)_{i\in\mathbb{N}}$, we infer that

$$\sum_{m=1}^{2^N} D''_m \;\le\; C \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} (2^N - k2^L)^{(r-p)/2} \mathbb{E}\big(|U_L - \tilde{U}_L|^{p-2} \big|Z_L^{(1)}\big|\big) .$$

It follows that

$$\sum_{m=1}^{2^N} D''_m \;\le\; C 2^{N(r+2-p)/2} \sum_{L=0}^{N} 2^{-L} \mathbb{E}\big(\big|U_L - \tilde{U}_L\big|^{p-2} \big|Z_L^{(1)}\big|\big) \quad \text{if } r > p-2, \tag{5.29}$$

$$\sum_{m=1}^{2^N} D''_m \;\le\; CN \sum_{L=0}^{N} 2^{-L} \mathbb{E}\big(\big|U_L - \tilde{U}_L\big| \big|Z_L^{(1)}\big|\big) \quad \text{if } r = 1 \text{ and } p = 3. \tag{5.30}$$

In the case $r = p-2$ and $r < 1$, we have

$$\sum_{m=1}^{2^N} D''_m \;\le\; C \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} \mathbb{E}\Big(\big(\|f''_{2^N-k2^L}\|_\infty \wedge \|f'''_{2^N-k2^L}\|_\infty \big|U_L - \tilde{U}_L\big| \big) \big|Z_L^{(1)}\big| \Big) .$$

Applying (5.23) to $i=2$ and $i=3$, we obtain

$$\sum_{m=1}^{2^N} D''_m \le C \sum_{L=0}^{N} 2^{(r-2)L/2} \mathbb{E}\Big(\big|Z_L^{(1)}\big| \sum_{k=1}^{2^{N-L}} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big)\Big) ,$$

Proceeding as to get (5.24), we have that

$$\sum_{k=1}^{2^{N-L}} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big) \le \sum_{k=1}^{\infty} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big) \le C 2^{-Lr/2} \big|U_L - \tilde{U}_L\big|^r .$$

It follows that

$$\sum_{m=1}^{2^N} D''_m \le C \sum_{L=0}^{N} 2^{-L} \mathbb{E}\Big(\big|U_L - \tilde{U}_L\big|^r \big|Z_L^{(1)}\big| \Big) \quad \text{if } r = p-2 \text{ and } r < 1. \tag{5.31}$$

Now by Remark 2.6, (2.1) and (2.2) are respectively equivalent to

$$\sum_{K \ge 0} 2^{K(p/2-2)} \|Z_K\|_{1,\Phi,p} < \infty , \quad \text{and} \quad \sum_{K \ge 0} 2^{-2K/p} \|Z_K\|_{p/2} < \infty .$$

Next, by Proposition 5.1, $\zeta_{p,K} = O(2^K)$ under (2.1) and (2.2). Therefrom, taking into account the inequality (5.13), we derive that under (2.1) and (2.2),

$$2^{-L} \mathbb{E}\Big(\big|U_L - \tilde{U}_L\big|^{p-2} \big|Z_L^{(1)}\big| \Big) \quad \le \quad C 2^{-2L/p} \|Z_L\|_{p/2} + C 2^{L(p/2-2)} \|Z_L\|_{1,\Phi,p} . \tag{5.32}$$

Consequently, combining (5.32) with the upper bounds (5.29), (5.30) and (5.31), we obtain that

$$\sum_{m=1}^{2^N} D''_m = \begin{cases} O(2^{N(r+2-p)/2}) & \text{if } r \ge p-2 \text{ and } (r,p) \ne (1,3) \\ O(N) & \text{if } r = 1 \text{ and } p = 3. \end{cases} \tag{5.33}$$

From (5.17), (5.18), (5.19), (5.22), (5.24), (5.27) and (5.33), we obtain (5.15) and (5.16).

5.2 Proof of Theorem 3.1

By (3.1), we get that (see Volný (1993))

$$X_0 = D_0 + Z_0 - Z_0 \circ T, \tag{5.34}$$

where

$$Z_0 = \sum_{k=0}^{\infty} \mathbb{E}(X_k|\mathcal{F}_{-1}) - \sum_{k=1}^{\infty} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_{-1})) \quad \text{and} \quad D_0 = \sum_{k \in \mathbb{Z}} \mathbb{E}(X_k|\mathcal{F}_0) - \mathbb{E}(X_k|\mathcal{F}_{-1}) .$$

Note that $Z_0 \in \mathbb{L}^p$, $D_0 \in \mathbb{L}^p$, D_0 is $\mathcal{F}_0$-measurable, and $\mathbb{E}(D_0|\mathcal{F}_{-1}) = 0$. Let $D_i = D_0 \circ T^i$, and $Z_i = Z_0 \circ T^i$. We obtain that

$$S_n = M_n + Z_1 - Z_{n+1} , \tag{5.35}$$

where $M_n = \sum_{j=1}^n D_j$. We first bound up $\mathbb{E}(f(S_n) - f(M_n))$ by using the following lemma

Lemma 5.2. *Let $p \in]2,3]$ and $r \in [p-2,p]$. Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^{2\vee r}$. Assume that $S_n = M_n + R_n$ where $(M_n - M_{n-1})_{n>1}$ is a strictly stationary sequence of martingale differences in $\mathbb{L}^{2\vee r}$, and R_n is such that $\mathbb{E}(R_n) = 0$. Let $n\sigma^2 = \mathbb{E}(M_n^2)$, $n\sigma_n^2 = \mathbb{E}(S_n^2)$ and $\alpha_n = \sigma_n/\sigma$.*

1. *If $r \in [p-2,1]$ and $\mathbb{E}|R_n|^r = O(n^{(r+2-p)/2})$, then $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.*
2. *If $r \in]1,2]$ and $\|R_n\|_r = O(n^{(3-p)/2})$, then $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.*
3. *If $r \in]2,p]$, $\sigma^2 > 0$ and $\|R_n\|_r = O(n^{(3-p)/2})$, then $\zeta_r(P_{S_n}, P_{\alpha_n M_n}) = O(n^{(r+2-p)/2})$.*
4. *If $r \in]2,p]$, $\sigma^2 = 0$ and $\|R_n\|_r = O(n^{(r+2-p)/2r})$, then $\zeta_r(P_{S_n}, G_{n\sigma_n^2}) = O(n^{(r+2-p)/2})$.*

Remark 5.1. All the assumptions on R_n in items 1, 2, 3 and 4 of Lemma 5.2 are satisfied as soon as $\sup_{n>0}\|R_n\|_p < \infty$.

Proof of Lemma 5.2. For $r \in]0,1]$, $\zeta_r(P_{S_n}, P_{M_n}) \le \mathbb{E}(|R_n|^r)$, which implies item 1.

If $f \in \Lambda_r$ with $r \in]1,2]$, from the Taylor integral formula and since $\mathbb{E}(R_n) = 0$, we get

$$\mathbb{E}(f(S_n) - f(M_n)) = \mathbb{E}\Big(R_n\Big(f'(M_n) - f'(0) + \int_0^1 (f'(M_n + t(R_n)) - f'(M_n))dt\Big)\Big).$$

Using that $|f'(x) - f'(y)| \le |x-y|^{r-1}$ and applying Hölder's inequality, it follows that

$$\mathbb{E}(f(S_n) - f(M_n)) \le \|R_n\|_r \|f'(M_n) - f'(0)\|_{r/(r-1)} + \|R_n\|_r^r \le \|R_n\|_r \|M_n\|_r^{r-1} + \|R_n\|_r^r\,.$$

Since $\|M_n\|_r \le \|M_n\|_2 = \sqrt{n}\sigma$, we infer that $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.

Now if $f \in \Lambda_r$ with $r \in]2,p]$ and if $\sigma > 0$, we define g by

$$g(t) = f(t) - tf'(0) - t^2 f''(0)/2\,.$$

The function g is then also in Λ_r and is such that $g'(0) = g''(0) = 0$. Since $\alpha_n^2\mathbb{E}(M_n^2) = \mathbb{E}(S_n^2)$, we have

$$\mathbb{E}(f(S_n) - f(\alpha_n M_n)) = \mathbb{E}(g(S_n) - g(\alpha_n M_n))\,. \tag{5.36}$$

Now from the Taylor integral formula at order two, setting $\tilde{R}_n = R_n + (1-\alpha_n)M_n$,

$$\begin{aligned}\mathbb{E}(g(S_n) - g(\alpha_n M_n)) = \mathbb{E}(\tilde{R}_n g'(\alpha_n M_n)) + \frac{1}{2}\mathbb{E}((\tilde{R}_n)^2 g''(\alpha_n M_n))\\ + \mathbb{E}\Big((\tilde{R}_n)^2 \int_0^1 (1-t)(g''(\alpha_n M_n + t\tilde{R}_n) - g''(\alpha_n M_n))dt\Big).\end{aligned} \tag{5.37}$$

Note that, since $g'(0) = g''(0) = 0$, one has

$$\mathbb{E}(\tilde{R}_n g'(\alpha_n M_n)) = \mathbb{E}\Big(\tilde{R}_n \alpha_n M_n \int_0^1 (g''(t\alpha_n M_n) - g''(0))dt\Big)$$

Using that $|g''(x) - g''(y)| \le |x-y|^{r-2}$ and applying Hölder's inequality in (5.37), it follows that

$$\begin{aligned}\mathbb{E}(g(S_n) - g(\alpha_n M_n)) &\le \frac{1}{r-1}\mathbb{E}(|\tilde{R}_n||\alpha_n M_n|^{r-1}) + \frac{1}{2}\|\tilde{R}_n\|_r^2\|g''(\alpha_n M_n)\|_{r/(r-2)} + \frac{1}{2}\|\tilde{R}_n\|_r^r \\ &\le \frac{1}{r-1}\alpha_n^{r-1}\|\tilde{R}_n\|_r\|M_n\|_r^{r-1} + \frac{1}{2}\alpha_n^{r-2}\|\tilde{R}_n\|_r^2\|M_n\|_r^{r-2} + \frac{1}{2}\|\tilde{R}_n\|_r^r .\end{aligned}$$

Now $\alpha_n = O(1)$ and $\|\tilde{R}_n\|_r \le \|R_n\|_r + |1-\alpha_n|\|M_n\|_r$. Since $|\|S_n\|_2 - \|M_n\|_2| \le \|R_n\|_2$, we infer that $|1-\alpha_n| = O(n^{(2-p)/2})$. Hence, applying Burkhölder's inequality for martingales, we infer that $\|\tilde{R}_n\|_r = O(n^{(3-p)/2})$, and consequently $\zeta_r(P_{S_n}, P_{\alpha_n M_n}) = O(n^{(r+2-p)/2})$.

If $\sigma^2 = 0$, then $S_n = R_n$. Let Y be a $N(0,1)$ random variable. Using that

$$\mathbb{E}(f(S_n) - f(\sqrt{n}\sigma_n Y)) = \mathbb{E}(g(R_n) - g(\sqrt{n}\sigma_n Y))$$

and applying again Taylor's formula, we obtain that

$$\sup_{f \in \Lambda_r} |\mathbb{E}(f(S_n) - f(\sqrt{n}\sigma_n Y))| \le \frac{1}{r-1}\|\bar{R}_n\|_r\|\sqrt{n}\sigma_n Y\|_r^{r-1} + \frac{1}{2}\|\bar{R}_n\|_r^2\|\sqrt{n}\sigma_n Y\|_r^{r-2} + \frac{1}{2}\|\bar{R}_n\|_r^r ,$$

where $\bar{R}_n = R_n - \sqrt{n}\sigma_n Y$. Since $\sqrt{n}\sigma_n = \|R_n\|_2 \le \|R_n\|_r$ and since $\|R_n\|_r = O(n^{(r+2-p)/2r})$, we infer that $\sqrt{n}\sigma_n = O(n^{(r+2-p)/2r})$ and that $\|\bar{R}_n\|_r = O(n^{(r+2-p)/2r})$. The result follows. □

By (5.35), we can apply Lemma 5.2 with $R_n := Z_1 - Z_{n+1}$. Then for $p-2 \le r \le 2$, the result follows if we prove that under (3.1) and (3.2), M_n satisfies the conclusion of Theorem 2.1. Now if $2 < r \le p$ and $\sigma^2 > 0$, we first notice that

$$\zeta_r(P_{\alpha_n M_n}, G_{n\sigma_n^2}) = \alpha_n^r \zeta_r(P_{M_n}, G_{n\sigma^2}) .$$

Since $\alpha_n = O(1)$, the result will follow by Item 3 of Lemma 5.2, if we prove that under (3.1) and (3.2), M_n satisfies the conclusion of Theorem 2.1. We shall prove that

$$\sum_{n \ge 1} \frac{1}{n^{3-p/2}}\|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(M_n^2)\|_{p/2} < \infty . \tag{5.38}$$

In this way, according to Remark 2.1, both (2.1) and (2.2) will be satisfied. Suppose that we can show that

$$\sum_{n \ge 1} \frac{1}{n^{3-p/2}}\|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{p/2} < \infty , \tag{5.39}$$

then by taking into account the condition (3.2), (5.38) will follow. Indeed, it suffices to notice that (5.39) also entails that

$$\sum_{n \ge 1} \frac{1}{n^{3-p/2}}|\mathbb{E}(S_n^2) - \mathbb{E}(M_n^2)| < \infty , \tag{5.40}$$

and to write that

$$\|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(M_n^2)\|_{p/2} \le \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{p/2} + \|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} + |\mathbb{E}(S_n^2) - \mathbb{E}(M_n^2)|\,.$$

Hence, it remains to prove (5.39). Since $S_n = M_n + Z_1 - Z_{n+1}$, and since $Z_i = Z_0 \circ T^i$ is in $\mathbb{L}^p$, (5.39) will be satisfied provided that

$$\sum_{n\ge1} \frac{1}{n^{3-p/2}} \|S_n(Z_1 - Z_{n+1})\|_{p/2} < \infty\,. \tag{5.41}$$

Notice that

$$\|S_n(Z_1 - Z_{n+1})\|_{p/2} \le \|M_n\|_p \|Z_1 - Z_{n+1}\|_p + \|Z_1 - Z_{n+1}\|_p^2\,.$$

From Burkholder's inequality, $\|M_n\|_p = O(\sqrt{n})$ and from (3.1), $\sup_n \|Z_1 - Z_{n+1}\|_p < \infty$. Consequently (5.41) is satisfied for any p in $]2,3[$.

5.3 Proof of Theorem 3.2

Starting from (5.35) we have that

$$M_n := S_n + R_n + \tilde{R}_n \tag{5.42}$$

in $\mathbb{L}^p$, where

$$R_n = \sum_{k\ge n+1} \mathbb{E}(X_k|\mathcal{F}_n) - \sum_{k\ge1} \mathbb{E}(X_k|\mathcal{F}_0) \text{ and } \tilde{R}_n = \sum_{k\ge0} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0)) - \sum_{k\ge -n} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_n))\,.$$

Arguing as in the proof of Theorem 3.1 the theorem will follow from (3.5), if we prove that

$$\sum_{n\ge1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{3/2} < \infty\,. \tag{5.43}$$

Under (3.1), $\sup_{n\ge1} \|R_n\|_3 < \infty$ and $\sup_{n\ge1} \|\tilde{R}_n\|_3 < \infty$. Hence (5.43) will be verified as soon as

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n(R_n + \tilde{R}_n)|\mathcal{F}_0)\|_{3/2} < \infty\,. \tag{5.44}$$

We first notice that the decomposition (5.42) together with Burkholder's inequality for martingales and the fact that $\sup_n \|R_n\|_3 < \infty$ and $\sup_n \|\tilde{R}_n\|_3 < \infty$, implies that

$$\|S_n\|_3 \le C\sqrt{n}\,. \tag{5.45}$$

Now to prove (5.44), we first notice that

$$\Big\|\mathbb{E}\Big(S_n\sum_{k\geq 1}\mathbb{E}(X_k|\mathcal{F}_0)\Big|\mathcal{F}_0\Big)\Big\|_{3/2}\leq \|\mathbb{E}(S_n|\mathcal{F}_0)\|_3\Big\|\sum_{k\geq 1}\mathbb{E}(X_k|\mathcal{F}_0)\Big\|_3\,, \tag{5.46}$$

which is bounded by using (3.1). Now write

$$\mathbb{E}\Big(S_n\sum_{k\geq n+1}\mathbb{E}(X_k|\mathcal{F}_n)\Big|\mathcal{F}_0\Big)=\mathbb{E}\Big(S_n\sum_{k\geq 2n+1}\mathbb{E}(X_k|\mathcal{F}_n)\Big|\mathcal{F}_0\Big)+\mathbb{E}(S_n\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\,.$$

Clearly

$$\begin{aligned}\Big\|\mathbb{E}\Big(S_n\sum_{k\geq 2n+1}\mathbb{E}(X_k|\mathcal{F}_n)\Big|\mathcal{F}_0\Big)\Big\|_{3/2} &\leq \|S_n\|_3\Big\|\sum_{k\geq 2n+1}\mathbb{E}(X_k|\mathcal{F}_n)\Big\|_3\\ &\leq C\sqrt{n}\Big\|\sum_{k\geq 2n+1}\mathbb{E}(X_k|\mathcal{F}_0)\Big\|_3\,,\end{aligned} \tag{5.47}$$

by using (5.45). Considering the bounds (5.46) and (5.47) and the condition (3.4), in order to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\|\mathbb{E}(S_nR_n|\mathcal{F}_0)\|_{3/2}<\infty\,, \tag{5.48}$$

it is sufficient to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\|\mathbb{E}(S_n\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2}<\infty\,. \tag{5.49}$$

With this aim, take $p_n=[\sqrt{n}]$ and write

$$\begin{aligned}\mathbb{E}(S_n\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0) &= \mathbb{E}((S_n-S_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\\ &\quad+\mathbb{E}(S_{n-p_n}\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0).\end{aligned} \tag{5.50}$$

By stationarity and (5.45), we get that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\|\mathbb{E}((S_n-S_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2}\leq C\sum_{n=1}^{\infty}\frac{\sqrt{p_n}}{n^{3/2}}\|\mathbb{E}(S_n|\mathcal{F}_0)\|_3\,,$$

which is finite by using (3.1) and the fact that $p_n=[\sqrt{n}]$. Hence from (5.50), (5.49) will follow if we prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\|\mathbb{E}(S_{n-p_n}\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2}<\infty\,. \tag{5.51}$$

With this aim we first notice that

$$\begin{aligned}&\|\mathbb{E}((S_{n-p_n}-\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n}))\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2}\\ &\qquad\leq\|S_{n-p_n}-\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\|_3\|\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)\|_3\,,\end{aligned}$$

which is bounded under (3.1). Consequently (5.51) will hold if we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2} < \infty\,. \tag{5.52}$$

We first notice that

$$\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_n)|\mathcal{F}_0) = \mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_{n-p_n})|\mathcal{F}_0)\,,$$

and by stationarity and (5.45),

$$\begin{aligned}\|\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_{n-p_n})|\mathcal{F}_0)\|_{3/2} &\leq \|S_{n-p_n}\|_3\|\mathbb{E}(S_{2n}-S_n|\mathcal{F}_{n-p_n})\|_3 \\ &\leq C\sqrt{n}\|\mathbb{E}(S_{n+p_n}-S_{p_n}|\mathcal{F}_0)\|_3\,.\end{aligned}$$

Hence (5.52) will hold provided that

$$\sum_{n\geq 1} \frac{1}{n}\Big\| \sum_{k\geq[\sqrt{n}]} \mathbb{E}(X_k|\mathcal{F}_0)\Big\|_3 < \infty\,. \tag{5.53}$$

The fact that (5.53) holds under the first part of the condition (3.4) follows from the following elementary lemma applied to $h(x) = \|\sum_{k\geq[x]} \mathbb{E}(X_k|\mathcal{F}_0)\|_3$.

Lemma 5.3. *Assume that h is a positive function on $\mathbb{R}^+$ satisfying $h(\sqrt{x+1}) = h(\sqrt{n})$ for any x in $[n-1, n[$. Then $\sum_{n\geq 1} n^{-1}h(\sqrt{n}) < \infty$ if and only if $\sum_{n\geq 1} n^{-1}h(n) < \infty$.*

It remains to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n\tilde{R}_n|\mathcal{F}_0)\|_{3/2} < \infty\,. \tag{5.54}$$

Write

$$\begin{aligned} S_n\tilde{R}_n &= S_n\Big(\sum_{k\geq 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0)) - \sum_{k\geq -n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_n))\Big) \\ &= S_n\Big(\mathbb{E}(S_n|\mathcal{F}_n) - S_n + \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big)\,.\end{aligned}$$

Notice first that

$$\begin{aligned}\|\mathbb{E}\big(S_n(S_n-\mathbb{E}(S_n|\mathcal{F}_n))|\mathcal{F}_0\big)\|_{3/2} &= \|\mathbb{E}\big((S_n-\mathbb{E}(S_n|\mathcal{F}_n))^2|\mathcal{F}_0\big)\|_{3/2} \\ &\leq \|S_n-\mathbb{E}(S_n|\mathcal{F}_n)\|_3^2\,,\end{aligned}$$

which is bounded under (3.1). Now for $p_n = [\sqrt{n}]$, we write

$$\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_0)) = \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})) + \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0)).$$

Note that

$$\Big\|\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3 = \Big\|\sum_{k\ge 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))-\sum_{k\ge 0}(X_{-k}-(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3$$
$$\le \Big\|\sum_{k\ge 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3+\Big\|\sum_{k\ge p_n}(X_{-k}-(\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\,,$$

which is bounded under (3.1). Next, since the random variable $\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))$ is $\mathcal{F}_{p_n}$-measurable, we get

$$\Big\|\mathbb{E}\Big(S_n\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))|\mathcal{F}_0\Big)\Big\|_{3/2}$$
$$\le \Big\|\mathbb{E}\Big(S_{p_n}\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))|\mathcal{F}_0\Big)\Big\|_{3/2}$$
$$+\|\mathbb{E}(S_n-S_{p_n}|\mathcal{F}_{p_n})\|_3\Big\|\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3$$
$$\le \Big(\|S_{p_n}\|_3+\|\mathbb{E}(S_{n-p_n}|\mathcal{F}_0)\|_3\Big)\Big\|\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\le C\sqrt{p_n}\,,$$

by using (3.1) and (5.45). Hence, since $p_n=[\sqrt{n}]$, we get that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\Big\|\mathbb{E}\Big(S_n\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big|\mathcal{F}_0\Big)\Big\|_{3/2}<\infty\,.$$

It remains to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\Big\|\mathbb{E}\Big(S_n\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big|\mathcal{F}_0\Big)\Big\|_{3/2}<\infty\,. \tag{5.55}$$

Note first that

$$\Big\|\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3 = \Big\|\sum_{k\ge 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_n))-\sum_{k\ge 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3$$
$$\le \Big\|\sum_{k\ge n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3+\Big\|\sum_{k\ge p_n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\,.$$

It follows that

$$\Big\|\mathbb{E}\Big(S_n\sum_{k\ge 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))|\mathcal{F}_0\Big)\Big\|_{3/2}$$
$$\le C\sqrt{n}\Big(\Big\|\sum_{k\ge p_n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3+\Big\|\sum_{k\ge n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\Big)\,.$$

by taking into account (5.45). Consequently (5.55) will follow as soon as

$$\sum_{n\geq 1}\frac{1}{n}\Big\|\sum_{k\geq[\sqrt{n}]}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3<\infty\,,$$

which holds under the second part of the condition (3.4), by applying Lemma 5.3 with $h(x) = \|\sum_{k\geq[x]}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\|_3$. This ends the proof of the theorem.

6 Appendix

6.1 A smoothing lemma.

Lemma 6.1. *Let $r>0$ and f be a function such that $|f|_{\Lambda_r}<\infty$ (see Notation 5.1 for the definition of the seminorm $|\cdot|_{\Lambda_r}$). Let ϕ_t be the density of the law $N(0,t^2)$. For any real $p\geq r$ and any positive t, $|f*\phi_t|_{\Lambda_p}\leq c_{r,p}t^{r-p}|f|_{\Lambda_r}$ for some positive constant $c_{r,p}$ depending only on r and p. Furthermore $c_{r,r}=1$.*

Remark 6.1. In the case where p is a positive integer, the result of Lemma 6.1 can be written as $\|f*\phi_t^{(p)}\|_\infty\leq c_{r,p}t^{r-p}|f|_{\Lambda_r}$.

Proof of Lemma 6.1. Let j be the integer such that $j<r\leq j+1$. In the case where p is a positive integer, we have

$$(f*\phi_t)^{(p)}(x)=\int\big(f^{(j)}(u)-f^{(j)}(x)\big)\phi_t^{(p-j)}(x-u)du\qquad\text{since } p-j\geq 1\,.$$

Since $|f^{(j)}(u)-f^{(j)}(x)|\leq|x-u|^{r-j}|f|_{\Lambda_r}$, we obtain that

$$|(f*\phi_t)^{(p)}(x)|\leq|f|_{\Lambda_r}\int|x-u|^{r-j}|\phi_t^{(p-j)}(x-u)|du\leq|f|_{\Lambda_r}\int|u|^{r-j}|\phi_t^{(p-j)}(u)|du\,.$$

Using that $\phi_t^{(p-j)}(x)=t^{-p+j-1}\phi_1^{(p-j)}(x/t)$, we conclude that Lemma 6.1 holds with the constant $c_{r,p}=\int|z|^{r-j}|\phi_1^{p-j}(z)|dz$.

The case $p=r$ is straightforward. In the case where p is such that $j<r<p<j+1$, by definition

$$|f^{(j)}*\phi_t(x)-f^{(j)}*\phi_t(y)|\leq|f|_{\Lambda_r}|x-y|^{r-j}\,.$$

Also, by Lemma 6.1 applied with $p=j+1$,

$$|f^{(j)}*\phi_t(x)-f^{(j)}*\phi_t(y)|\leq|x-y|\|f^{(j+1)}*\phi_t\|_\infty\leq|f|_{\Lambda_r}c_{r,j+1}t^{r-j-1}|x-y|\,.$$

Hence by interpolation,

$$|f^{(j)} * \phi_t(x) - f^{(j)} * \phi_t(y)| \le |f|_{\Lambda_r} t^{r-p} c_{r,j+1}^{(p-r)/(j+1-r)} |x-y|^{p-j}\,.$$

It remains to consider the case where $r \le i < p \le i+1$. By Lemma 6.1 applied successively with $p = i$ and $p = i+1$, we obtain that

$$|f^{(i)} * \phi_t(x)| \le |f|_{\Lambda_r} c_{r,i} t^{r-i} \text{ and } |f^{(i+1)} * \phi_t(x)| \le |f|_{\Lambda_r} c_{r,i+1} t^{r-i-1}\ .$$

Consequently

$$|f^{(i)} * \phi_t(x) - f^{(i)} * \phi_t(y)| \le |f|_{\Lambda_r} t^{r-i}(2c_{r,i} \wedge c_{r,i+1} t^{-1}|x-y|)\,,$$

and by interpolation,

$$|f^{(i)} * \phi_t(x) - f^{(i)} * \phi_t(y)| \le |f|_{\Lambda_r} t^{r-p}(2c_{r,i})^{1-p+i} c_{r,i+1}^{p-i} |x-y|^{p-i}\,.$$

6.2 Covariance inequalities.

In this section, we give an upper bound for the expectation of the product of k centered random variables $\Pi_{i=1}^k (X_i - \mathbb{E}(X_i))$.

Proposition 6.1. *Let $X = (X_1, \cdots, X_k)$ be a random variable with values in $\mathbb{R}^k$. Define the number*

$$\begin{aligned} \phi^{(i)} &= \phi(\sigma(X_i), X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_k) \qquad (6.1) \\ &= \sup_{x \in \mathbb{R}^k} \Big\| \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_j} - \mathbb{P}(X_j > x_j)) | \sigma(X_i)\Big) - \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_j} - \mathbb{P}(X_j > x_j))\Big) \Big\|_\infty. \end{aligned}$$

Let F_i be the distribution function of X_i and Q_i be the quantile function of $|X_i|$ (see Section 4.1 for the definition). Let F_i^{-1} be the generalized inverse of F_i and let $D_i(u) = (F_i^{-1}(1-u) - F_i^{-1}(u))_+$. We have the inequalities

$$\Big|\mathbb{E} \prod_{i=1}^k \Big(X_i - \mathbb{E}(X_i)\Big)\Big| \le \int_0^1 \Big(\prod_{i=1}^k D_i(u/\phi^{(i)}) \Big) du \qquad (6.2)$$

and

$$\Big|\mathbb{E} \prod_{i=1}^k \Big(X_i - \mathbb{E}(X_i)\Big)\Big| \le 2^k \int_0^1 \Big(\prod_{i=1}^k Q_i(u/\phi^{(i)}) \Big) du\,. \qquad (6.3)$$

In addition, for any k-tuple $(p_1, \dots, p_k)$ such that $1/p_1 + \dots + 1/p_k = 1$, we have

$$\Big|\mathbb{E} \prod_{i=1}^k \Big(X_i - \mathbb{E}(X_i)\Big)\Big| \le 2^k \prod_{i=1}^k (\phi^{(i)})^{1/p_i} \|X_i\|_{p_i}\,. \qquad (6.4)$$

Proof of Proposition 6.1. We have that

$$\mathbb{E}\prod_{i=1}^{k}\Big(X_i-\mathbb{E}(X_i)\Big)=\int\mathbb{E}\prod_{i=1}^{k}\Big(\mathbb{1}_{X_i>x_i}-\mathbb{P}(X_i>x_i)\Big)dx_1\dots dx_k\,. \tag{6.5}$$

Now for all i,

$$\begin{aligned}&\mathbb{E}\prod_{i=1}^{k}\Big(\mathbb{1}_{X_i>x_i}-\mathbb{P}(X_i>x_i)\Big)\\ &=\mathbb{E}\left(\mathbb{1}_{X_i>x_i}\Big(\mathbb{E}\Big(\prod_{j=1,j\neq i}^{k}(\mathbb{1}_{X_j>x_j}-\mathbb{P}(X_j>x_j))|\sigma(X_i)\Big)-\mathbb{E}\Big(\prod_{j=1,j\neq i}^{k}(\mathbb{1}_{X_j>x_j}-\mathbb{P}(X_j>x_j))\Big)\Big)\right)\\ &=\mathbb{E}\left(\mathbb{1}_{X_i\leq x_i}\Big(\mathbb{E}\Big(\prod_{j=1,j\neq i}^{k}(\mathbb{1}_{X_j>x_j}-\mathbb{P}(X_j>x_j))|\sigma(X_i)\Big)-\mathbb{E}\Big(\prod_{j=1,j\neq i}^{k}(\mathbb{1}_{X_j>x_j}-\mathbb{P}(X_j>x_j))\Big)\Big)\right).\end{aligned}$$

Consequently, for all i,

$$\mathbb{E}\prod_{i=1}^{k}\Big(\mathbb{1}_{X_i>x_i}-\mathbb{P}(X_i>x_i)\Big)\leq\phi^{(i)}\big(\mathbb{P}(X_i\leq x_i)\wedge\mathbb{P}(X_i>x_i)\big)\,. \tag{6.6}$$

Hence, we obtain from (6.5) and (6.6) that

$$\begin{aligned}\Big|\mathbb{E}\prod_{i=1}^{k}\Big(X_i-\mathbb{E}(X_i)\Big)\Big| &\leq \int_0^1\Big(\prod_{i=1}^{k}\int\mathbb{1}_{u/\phi^{(i)}<\mathbb{P}(X_i>x_i)}\mathbb{1}_{u/\phi^{(i)}\leq\mathbb{P}(X_i\leq x_i)}dx_i\Big)du\\ &\leq \int_0^1\Big(\prod_{i=1}^{k}\int\mathbb{1}_{F_i^{-1}(u/\phi^{(i)})\leq x_i<F_i^{-1}(1-u/\phi^{(i)})}dx_i\Big)du,\end{aligned}$$

and (6.2) follows. Now (6.3) comes from (6.2) and the fact that $D_i(u)\leq 2Q_i(u)$ (see Lemma 6.1 in Dedecker and Rio (2008)). Finally (6.4) follows by applying Hölder's inequality to (6.3). □

Definition 6.1. For a quantile function Q in $\mathbb{L}_1([0,1],\lambda)$, let $\mathcal{F}(Q,P_X)$ be the set of functions f which are nondecreasing on some open interval of $\mathbb{R}$ and null elsewhere and such that $Q_{|f(X)|}\leq Q$. Let $\mathcal{C}(Q,P_X)$ denote the set of convex combinations $\sum_{i=1}^{\infty}\lambda_i f_i$ of functions f_i in $\mathcal{F}(Q,P_X)$ where $\sum_{i=1}^{\infty}|\lambda_i|\leq 1$ (note that the series $\sum_{i=1}^{\infty}\lambda_i f_i(X)$ converges almost surely and in $\mathbb{L}_1(P_X)$).

Corollary 6.1. *Let $X=(X_1,\cdots,X_k)$ be a random variable with values in $\mathbb{R}^k$ and let the $\phi^{(i)}$'s be defined by (6.1). Let $(f_i)_{1\leq i\leq k}$ be k functions from $\mathbb{R}$ to $\mathbb{R}$, such that $f_i\in\mathcal{C}(Q_i,P_{X_i})$. We have the inequality*

$$\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_i(X_i)-\mathbb{E}(f_i(X_i))\Big)\Big|\leq 2^{2k-1}\int_0^1\prod_{i=1}^{k}Q_i\Big(\frac{u}{\phi^{(i)}}\Big)du\,.$$

Proof of Corollary 6.1. Write for all $1 \le i \le k$, $f_i = \sum_{j=1}^{\infty} \lambda_{j,i} f_{j,i}$ where $\sum_{j=1}^{\infty} |\lambda_{j,i}| \le 1$ and $f_{j,i} \in \mathcal{F}(Q_i, P_{X_i})$. Clearly

$$\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| \le \sum_{j_1=1}^{\infty}\cdots\sum_{j_k=1}^{\infty}\Big(\prod_{i=1}^{k}|\lambda_{j_i,i}|\Big)\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_{j_i,i}(X_i) - \mathbb{E}(f_{j_i,i}(X_i))\Big)\Big|$$
$$\le \sup_{j_1\ge 1,\ldots,j_k\ge 1}\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_{j_i,i}(X_i) - \mathbb{E}(f_{j_i,i}(X_i))\Big)\Big|. \qquad (6.7)$$

Since each $f_{j_i,i}$ is nondecreasing on some interval and null elsewhere,

$$\phi(\sigma(f_{j_i,i}(X_i)), f_{j_1,1}(X_1), \ldots, f_{j_{i-1},i-1}(X_{i-1}), f_{j_{i+1},i+1}(X_{i+1}), \ldots, f_{j_k,k}(X_k)) \le 2^{k-1}\phi^{(i)}\,.$$

Applying (6.3) to the right hand side of (6.7), we then derive that

$$\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| \le 2^k\int_0^1\prod_{i=1}^{k}Q_i\Big(\frac{u}{2^{k-1}\phi^{(i)}}\Big)du\,,$$

and the result follows by a change-of-variables. □

Recall that for any $p \ge 1$, the class $\mathcal{C}(p, M, P_X)$ has been introduced in the definition 4.2.

Corollary 6.2. *Let $X = (X_1, \cdots, X_k)$ be a random variable with values in $\mathbb{R}^k$ and let the $\phi^{(i)}$'s be defined by (6.1). Let $(p_1, \ldots, p_k)$ be a k-tuple such that $1/p_1 + \ldots + 1/p_k = 1$ and let $(f_i)_{1\le i\le k}$ be k functions from $\mathbb{R}$ to $\mathbb{R}$, such that $f_i \in \mathcal{C}(p_i, M_i, P_{X_i})$. We have the inequality*

$$\Big|\mathbb{E}\prod_{i=1}^{k}\Big(f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| \le 2^{2k-1}\prod_{i=1}^{k}(\phi^{(i)})^{1/p_i}M_i^{1/p_i}\,.$$

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