On the weak invariance principle for non-adapted sequences under projective criteria.

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Abstract

In this paper we study the central limit theorem and its weak invariance principle for sums of non-adapted stationary sequences, under different normalizations. Our conditions involve the conditional expectation of the variables with respect to a given σ-algebra, as done in Gordin (1969) and Heyde (1974). These conditions are well adapted to a large variety of examples, including linear processes with dependent innovations or regular functions of linear processes.

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1 Introduction and notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. Let X_0 be a square integrable random variable with mean 0. Define then the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$, and let

$$S_n = X_1 + \cdots + X_n \quad \text{and } \sigma_n = \|S_n\|_2.$$

In this paper, we shall address the central limit question and its invariance principle; namely we want to find a sequence s_n of positive numbers with $s_n \to \infty$, and conditions ensuring that $s_n^{-1} S_n$ converges in distribution to a mixture of normal distributions (CLT), or more precisely that $\{s_n^{-1} S_{[nt]}, t \in [0,1]\}$ converges in distribution in the Skorohod space to a mixture of Wiener distributions (WIP).

We shall provide sufficient conditions involving quantities of the type $\mathbb{E}(X_k|\mathcal{M}_0)$, where $\mathcal{M}_0$ is a sub-σ-algebra of $\mathcal{A}$ satisfying $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$. We do not assume here that X_0 is $\mathcal{M}_0$-measurable, since in many cases the natural filtration $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$ is generated by some auxiliary sequence, typically the innovations $(\varepsilon_i)_{i\in\mathbb{Z}}$ of a linear process $X_k = \sum_{i\in\mathbb{Z}} a_i \varepsilon_{k-i}$.

The first result to mention in this context was obtained by Gordin (1969), for stationary and ergodic sequences. As a consequence of a general result involving martingale approximations, he proved that the CLT holds with $s_n = \sqrt{n}$ under the conditions

$$\sum_{k\geq 1} \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2 < \infty \quad \text{and} \quad \sum_{k\geq 1} \|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2 < \infty. \tag{1.1}$$

Following Gordin's approach, Heyde obtained the two following results for stationary and ergodic sequences. For regular sequences (i.e. $\mathbb{E}(X_0|\mathcal{M}_{-n}) \to 0$ and $\mathbb{E}(X_0|\mathcal{M}_n) \to X_0$), he proved in 1974 that $S_n/\sqrt{n}$ converges to $\mathcal{N}(0, \sigma^2)$ under the conditions

(1.2)
$$\sum_{k\in\mathbb{Z}} (\mathbb{E}(X_k|\mathcal{M}_0) - \mathbb{E}(X_k|\mathcal{M}_{-1})) \text{ converges in } \mathbb{L}^2 \text{ to } m, \text{ and } \lim_{n\to\infty} \frac{\|S_n\|_2}{\sqrt{n}} = \|m\|_2 = \sigma\,,$$

which is close to optimality in the case where $s_n = \sqrt{n}$ (see our proposition 2). Next, Heyde proved in 1975 that the WIP holds for $s_n = \sqrt{n}$ provided the two series

$$\sum_{k\geq 1} \mathbb{E}(X_k|\mathcal{M}_0) \quad \text{and} \quad \sum_{k\geq 1} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)) \quad \text{converge in } \mathbb{L}^2, \tag{1.3}$$

which clearly improves on (1.1). Notice that (1.3) is a necessary and sufficient condition in order to get the representation, $X_0 = m + g - g \circ T^{-1}$, where $(m \circ T^i)_{i\in\mathbb{Z}}$ is a martingale difference sequence in $\mathbb{L}^2$ and g is in $\mathbb{L}^2$ (see Volný (1993)). Also (1.3) is a sufficient condition for the functional law of the iterated logarithm (see Heyde (1975)).

Following Heyde's approach (1974), our aim is to provide sufficient conditions based on $P_0(X_k) = \mathbb{E}(X_k|\mathcal{M}_0) - \mathbb{E}(X_k|\mathcal{M}_{-1})$, for the CLT (cf. Theorem 1, Section 2) and for the WIP (cf. Theorem 2, Section 3) under general normalizations. For instance, as a consequence of Theorem 1, we obtain that if X_0 is $\mathcal{M}_0$-measurable and $s_n/\sqrt{n}$ is a slowly varying function at infinity, then the CLT holds under the conditions

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o\,(s_n)\,, \text{ and } \frac{\sqrt{n}}{s_n}\sum_{i=0}^{n} P_0(X_i) \to m \text{ in } \mathbb{L}^2.$$

Now, as a consequence of Theorem 2, we obtain that if the sequence is regular and

$$\sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_2 < \infty\,, \tag{1.4}$$

then the WIP holds under the normalization $s_n = \sqrt{n}$. In Proposition 4, we give a counterexample showing that (1.4) cannot be weakened to (1.2) for the WIP to hold with $s_n = \sqrt{n}$.

Of course, such results are well adapted to linear processes with dependent innovations (see Section 4), but they can also be successfully applied to functions of linear processes generated by independent innovations (see Section 5). For instance, we obtain as a consequence of Corollary 6 that if

$$X_k = f\Big(\sum_{i\geq 0}\frac{\varepsilon_{k-i}}{i+1}\Big) - \mathbb{E}\Big(f\Big(\sum_{i\geq 0}\frac{\varepsilon_{k-i}}{i+1}\Big)\Big)\,,$$

where f is Lipschitz with continuous derivative f', and $(\varepsilon_i)_{i\in\mathbb{Z}}$ is iid with mean zero and finite variance, then

$$\Big\{\frac{S_{[nt]}}{\sqrt{n}\log n}, t\in[0,1]\Big\} \text{ converges in distribution to } \|\varepsilon_0\|_2\Big|\mathbb{E}\Big(f'\Big(\sum_{i\geq 0}\frac{\varepsilon_i}{i+1}\Big)\Big)\Big|\,W$$

in the Skohorod space, where W is a standard Brownian motion.

In Section 6, we go back to conditions à la Gordin. More precisely we derive from (1.4) the following improvement of (1.1): the WIP holds with $s_n = \sqrt{n}$ provided that

$$\sum_{k\geq 1}\frac{\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2}{\sqrt{k}} < \infty \quad \text{and} \quad \sum_{k\geq 1}\frac{\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2}{\sqrt{k}} < \infty. \tag{1.5}$$

Most of the results of this paper are new (except Corollary 1). However parts of them were known in the particularly cases where X_0 is $\mathcal{M}_0$-measurable and/or $s_n = \sqrt{n}$. This is the reason why we have made a lot of detailed remarks all along this paper.

1.1 Notations

We have already introduced the map T and the sequence $(X_i)_{i\in\mathbb{Z}}$. We now fix the other notations which we shall use in this paper.

We denote by $\mathcal{I}$ the σ-algebra of all T-invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathcal{I}$ has measure 0 or 1.

We denote by $(D([0,1]), d)$ the space of all functions from $[0,1]$ to $\mathbb{R}$ which have left-hand limits and are continuous from the right, equipped with the Skorohod distance d (see Billingsley (1968), Chapter 3).

For a σ-algebra $\mathcal{M}_0$ satisfying $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$, we define the nondecreasing filtration $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ by $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$. Let $\mathcal{M}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{M}_k$ and $\mathcal{M}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{M}_k$. Let H_i be the space of $\mathcal{M}_i$-measurable and square integrable random variables, and denote by $H_i \ominus H_{i-1}$ the orthogonal of H_{i-1} in H_i. Let P_i be the projection operator from $\mathbb{L}^2$ to $H_i \ominus H_{i-1}$, that is

$$P_i(f) = \mathbb{E}(f|\mathcal{M}_i) - \mathbb{E}(f|\mathcal{M}_{i-1}) \quad \text{for any } f \text{ in } \mathbb{L}^2.$$

Definition 1. We say that the random variable X_0 is regular if $\mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0$ almost surely, and X_0 is $\mathcal{M}_\infty$-measurable.

Definition 2. Following Definition 0.15 in Bradley (2002), a sequence $(h(n))_{n\geq 1}$ of positive numbers is said to be slowly varying in the strong sense if there exists a continuous function $f : (0,\infty) \to (0,\infty)$ such that $f(n) = h(n)$ for all $n \in \mathbb{N}$, and $f(x)$ is slowly varying as x tends to infinity. In what follows, we shall say that $h(n)$ is a svf if the sequence $(h(n))_{n\geq 1}$ is slowly varying in the strong sense.

2 Sufficient conditions for the CLT.

As in the introduction, $(s_n)_{n\geq 1}$ denotes a sequence of positive numbers such that $s_n \to \infty$. In the theorem below, we give a necessary and sufficient condition for the normalized partial sum S_n/s_n to be well approximated by $M_n/\sqrt{n}$, where M_n is a martingale with stationary increments adapted to the filtration $\mathcal{M}_n$.

Theorem 1. *Let m be an element of $H_0 \ominus H_{-1}$. The following conditions are equivalent*

$\mathbf{C_0}(s_n)$**:**
$$\lim_{n\to\infty} \Big\| \frac{S_n}{s_n} - \frac{1}{\sqrt{n}} \sum_{i=1}^n m \circ T^i \Big\|_2 = 0 \ .$$

$$
\mathbf{C_1}(s_n)\text{:}\quad \begin{cases} (a) \quad \|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o\left(s_n\right) \ \textit{ and } \|S_n - \mathbb{E}(S_n|\mathcal{M}_n)\|_2 = o\left(s_n\right), \\ \\ (b) \quad \displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{\ell=1}^{n} \Big\| \frac{\sqrt{n}}{s_n} \sum_{i=1-\ell}^{n-\ell} P_0(X_i) - m \Big\|_2^2 = 0\,. \end{cases}
$$

If one of these conditions holds then $s_n^{-1}S_n$ converges in distribution to $\sqrt{\mathbb{E}(m^2|\mathcal{I})}\,N$, where N is a standard Gaussian random variable independent of $\mathcal{I}$.

Remark 1. Arguing as in the proof of Proposition 1 in Dedecker and Merlevède (2002), we can prove that if $\mathbf{C_0}(s_n)$ holds, then $s_n^{-1}S_n$ satisfies the conditional central limit theorem, that is: for any continuous function φ such that $x \to |(1+x^2)^{-1}\varphi(x)|$ is bounded, and any integer k,

$$
\lim_{n\to\infty} \Big\| \mathbb{E}\left(\varphi(s_n^{-1}S_n)|\mathcal{M}_k\right) - \int \varphi\left(x\sqrt{\mathbb{E}(m^2|\mathcal{I})}\right) g(x)dx \Big\|_1 = 0\,,
$$

where g is the distribution of a standard normal. Recall that this implies the stable convergence of $s_n^{-1}S_n$ in the sense of Rényi (1963).

Remark 2. If X_0 is regular, the following orthogonal decomposition is valid:

$$
X_k = \sum_{i\in\mathbb{Z}} P_i(X_k)\,. \tag{2.1}
$$

It follows that

$$
\mathbb{E}(X_k|\mathcal{M}_0) = \sum_{i\le 0} P_i(X_k) \quad\text{and}\quad X_k - \mathbb{E}(X_k|\mathcal{M}_n) = \sum_{i>n} P_i(X_k)\,. \tag{2.2}
$$

Using the stationarity, we see that $\mathbf{C_1}(s_n)(a)$ is equivalent to

$$
\sum_{i=0}^{\infty} \Big\| \sum_{k=i+1}^{n+i} P_0(X_k) \Big\|_2^2 = o\left(s_n^2\right) \quad\text{and}\quad \sum_{i=n+1}^{\infty} \Big\| \sum_{k=1-i}^{n-i} P_0(X_k) \Big\|_2^2 = o\left(s_n^2\right).
$$

Remark 3. If $\mathbf{C_0}(s_n)$ holds and $\mathbb{E}(m^2) > 0$ then $s_n^{-2}\sigma_n^2$ converges to $\mathbb{E}(m^2)$. Hence $\mathbf{C_0}(\sigma_n)$ holds with $m' = m/\|m\|_2$. It follows that $\mathbf{C_1}(\sigma_n)(a)$ holds, which implies that $\sigma_n/\sqrt{n}$ is a svf (see Theorem 8.13 in Bradley (2002)), and the same is true for $s_n/\sqrt{n}$.

Remark 4. The condition $\mathbf{C_1}(\sigma_n)(a)$ is equivalent to the existence of a sequence m_n in $H_0 \ominus H_{-1}$ such that

$$
\lim_{n\to\infty} \Big\| \frac{S_n}{\sigma_n} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} m_n \circ T^i \Big\|_2 = 0\,.
$$

This has been proved by Wu and Woodroofe (2004) if X_0 is $\mathcal{M}_0$-measurable, and extended to the general case by Volný (2005). Note also that even in the adapted case, the condition $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(\sigma_n)$ alone is not sufficient for the CLT to hold even if $\sigma_n/\sqrt{n} \to 1$ (see Klicnarová and Volný (2006)).

In the following proposition, we give a sufficient condition for $\mathbf{C_1}(s_n)(b)$.

Proposition 1. *The condition* $\mathbf{C_1}(s_n)(b)$ *holds as soon as*

$$\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n} P_0(X_i) \text{ converges to } m \text{ in } \mathbb{L}^2, \text{ and} \tag{2.3}$$
$$\sum_{\ell=1}^{n}\Big\|\sum_{k=\ell}^{n} P_0(X_k)\Big\|_2^2 = o\left(s_n^2\right), \text{ and } \sum_{\ell=1}^{n}\Big\|\sum_{k=\ell}^{n} P_0(X_{-k})\Big\|_2^2 = o\left(s_n^2\right).$$

In particular if X_0 *is* $\mathcal{M}_0$*-measurable and* $s_n/\sqrt{n}$ *is a svf, then* $\mathbf{C_0}(s_n)$ *holds as soon as*

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(s_n), \text{ and } \frac{\sqrt{n}}{s_n}\sum_{i=0}^{n} P_0(X_i) \to m \text{ in } \mathbb{L}^2. \tag{2.4}$$

As a consequence, we obtain the following corollary.

Corollary 1. *Consider the following conditions*

$\mathbf{C_2}$: $\sum_{i\in\mathbb{Z}} P_0(X_i)$ *converges to* m *in* $\mathbb{L}^2$, *and* $\dfrac{\|S_n\|_2}{\sqrt{n}} \to \|m\|_2$,

$\mathbf{C_3}$: X_0 *is regular and* $\sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_2 < +\infty$.

We have the implications $\mathbf{C_3} \Rightarrow \mathbf{C_2} \Rightarrow \mathbf{C_1}(\sqrt{n})$. *Furthermore, if* $\mathbf{C_3}$ *holds then we have* $\mathbb{E}(m^2|\mathcal{I}) = \sum_{k\in\mathbb{Z}} \mathbb{E}(X_0X_k|\mathcal{I})$.

Remark 5. The fact that $\mathbf{C_2}$ implies $\mathbf{C_0}(\sqrt{n})$ is due to Heyde (1974). Note that the convergence of $\sum_{i\in\mathbb{Z}} P_0(X_i)$ alone is not sufficient for the CLT, as shown by Theorem 4 in Volný (1993). However if we assume that the series $\sum_{i\in\mathbb{Z}} P_0(X_i)$ is *unconditionally* convergent, then $\mathbf{C_2}$ holds (see Theorem 5 in Volný (1993)). In particular, the series $\sum_{i\in\mathbb{Z}} P_0(X_i)$ converges unconditionally as soon as $\mathbf{C_3}$ holds (see Theorem 6 in Volný (1993)). In Section 7, we shall give another proof of the implications $\mathbf{C_3} \Rightarrow \mathbf{C_2} \Rightarrow \mathbf{C_1}(\sqrt{n})$, and we shall prove the last assertion of Corollary 1. Note also that $\mathbf{C_2}$ does not imply $\mathbf{C_3}$ as shown by Theorem 8 in Volný (1993).

Remark 6. If X_0 is $\mathcal{M}_0$-measurable, Heyde's condition $\mathbf{C_2}$ is equivalent to (2.4) with $s_n = \sqrt{n}$. For a centered and square integrable function $X_k = f(Y_k)$ of a stationary Markov chain $(Y_k)_{k\geq 0}$ with transition Kernel K and invariant distribution μ, the condition $\mathbf{C_2}$ is equivalent to the two following items:

1. $\lim_{n\to\infty} \sup_{m>0} \Big[\Big\| K^n \sum_{k=0}^{m-1} K^k f \Big\|_{\mu,2}^2 - \Big\| K^{n+1} \sum_{k=0}^{m-1} K^k f \Big\|_{\mu,2}^2 \Big] = 0\,,$

2. $\lim_{n\to\infty} \Big\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} K^k f \Big\|_{\mu,2} = 0,$

where $\|\cdot\|_{\mu,2}$ is the $\mathbb{L}^2(\mu)$-norm (the condition 1. is just the Cauchy criterion for the convergence of $\sum_{k=1}^n P_0(X_i)$ in $\mathbb{L}^2$, and the condition 2. means exactly that $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(\sqrt{n})$). The conditions 1. and 2. are given in Theorem C of Derriennic and Lin (2001) and are due to Gordin and Lifshitz (see the discussion on page 511 in Derriennic and Lin). Note that, under ergodicity and a condition equivalent to 1., Woodroofe (1992) proved that $n^{-1/2}(S_n - \mathbb{E}(S_n|\mathcal{M}_0))$ is asymptotically normal.

The following proposition shows that the condition $\mathbf{C_3}$ is close to optimality (a proof can be found in Dedecker (1998), Annexe A, Section A.3).

Proposition 2. *Let $\Omega = [0,1]^{\mathbb{Z}}$, $\mathcal{A} = \mathcal{B}^{\mathbb{Z}}$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$, and $\mathbb{P} = \lambda^{\otimes \mathbb{Z}}$, where λ is the Lebesgue measure on $[0,1]$. Let T be the shift from Ω to Ω defined by $(T(\omega))_i = \omega_{i+1}$. For any sequence $(v_i)_{i\geq 0}$ of positive numbers such that $\sum_{i\geq 0} i v_i^2 < \infty$ and $\sum_{i\geq 0} v_i = \infty$, there exists a strictly stationary sequence $(X_i = X_0 \circ T^i)_{i\in\mathbb{Z}}$ of square integrable and centered random variables such that, taking $\mathcal{M}_i = \sigma(X_k, k \leq i)$,*

1. *$\|P_0(X_i)\|_2 \leq v_i$,*

2. *$\|S_n\|_2^2 = n$,*

3. *for any k, ℓ and any $i \neq j$, the variables $P_i(X_k)$ and $P_j(X_\ell)$ are independent,*

but $n^{-1/2} S_n$ does not converge in distribution.

3 Sufficients conditions for the WIP.

The first result of this section is a criterion for the uniform integrability of $s_n^{-2} \max_{1\leq k\leq n} S_k^2$.

Proposition 3. *We say that the condition* $\mathbf{C_4}(s_n)$ *holds if*

$$\mathbf{C_4}(s_n):\begin{cases}(a) & \Big\|\sup_{1\le k\le n}|\mathbb{E}(S_k|\mathcal{M}_0)|\Big\|_2=o\,(s_n)\,,\ \Big\|\sup_{1\le k\le n}|S_k-\mathbb{E}(S_k|\mathcal{M}_n)|\Big\|_2=o\,(s_n)\,,\\ (b) & \text{for some positive sequence } (u_i)_{i\in\mathbb{Z}} \text{ such that } \frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}u_i \text{ is bounded,}\\ & \lim_{A\to\infty}\limsup_{n\to\infty}\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}\mathbb{E}\left(\frac{P_0^2(X_i)}{u_i}\mathbb{I}_{P_0^2(X_i)>Au_i^2}\right)=0.\end{cases}$$

If $\mathbf{C_4}(s_n)$ *holds, then*

$$\text{the sequence } \frac{\max_{1\le k\le n}S_k^2}{s_n^2} \text{ is uniformly integrable.} \tag{3.1}$$

Remark 7. A sufficient condition for $\mathbf{C_4}(s_n)(a)$ is that

$$\sum_{k=1}^{n}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2=o(s_n)\quad\text{and}\quad\sum_{k=1}^{n}\|X_k-\mathbb{E}(X_k|\mathcal{M}_n)\|_2=o(s_n)\,. \tag{3.2}$$

Note that (3.2) implies that

$$X_0 \text{ is regular, and}\quad n\sqrt{\sum_{|k|\ge n}\|P_0(X_k)\|_2^2}=o(s_n)\,. \tag{3.3}$$

Now if $s_n=\sqrt{n}h(n)$ with $h(n)$ a svf, then (3.2) and (3.3) are equivalent. The proof of this equivalence will be done in Section 7.

Theorem 2. *Assume that* $s_{[nt]}/s_n$ *is bounded for any* $t\in[0,1]$. *If* $\mathbf{C_1}(s_n)(b)$ *holds and* $\mathbf{C_4}(s_n)$ *holds, then* $\{s_n^{-1}S_{[nt]},t\in[0,1]\}$ *converges in distribution in* $(D([0,1]),d)$ *to* $\sqrt{\mathbb{E}(m^2|\mathcal{I})}W$, *where* W *is a standard Brownian motion independent of* $\mathcal{I}$.

Remark 8. Again, if $\mathbf{C_1}(s_n)(b)$ holds and $\mathbf{C_4}(s_n)$ holds, then $W_n=\{s_n^{-1}S_{[nt]},t\in[0,1]\}$ satisfies the conditional WIP, that is: for any continuous function φ from $(D([0,1]),d)$ to $\mathbb{R}$ such that $x\to|(1+\|x\|_\infty^2)^{-1}\varphi(x)|$ is bounded, and any integer k,

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi(W_n)|\mathcal{M}_k\right)-\int\varphi\left(x\sqrt{\mathbb{E}(m^2|\mathcal{I})}\right)\mathbb{P}_W(dx)\right\|_1=0\,,$$

where $\mathbb{P}_W$ is the distribution of a standard Wiener Process. Again, this implies the stable convergence of the processes W_n.

Remark 9. In the condition $\mathbf{C_4}(s_n)$, the fact that $\sqrt{n}s_n^{-1}\sum_{i=-n}^{n} u_i$ is bounded ensures that $\liminf_{n\to\infty} n^{-1}s_n^2 > 0$. This excludes the general class of examples discussed in Herrndorf (1983) for which the normalizing sequence satisfies $\liminf_{n\to\infty} n^{-1}s_n^2 = 0$, the central limit theorem holds, but the invariance principle fails.

As a consequence of Theorem 2, we obtain the following corollary.

Corollary 2. *If the condition* $\mathbf{C_3}$ *holds, then* $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ *converges in distribution in* $(D([0,1]), d)$ *to* $\sqrt{\eta}W$*, where* W *is a standard Brownian motion independent of* $\mathcal{I}$*, and* $\eta = \sum_{k\in\mathbb{Z}} \mathbb{E}(X_0 X_k|\mathcal{I})$.

Remark 10. Let us recall a result due to Hannan (1979): if

1. X_0 is $\mathcal{M}_0$-measurable and $\mathbf{C_3}$ holds,
2. $\mathbb{P}$ is weak mixing (which implies that $\mathbb{P}$ is ergodic), that is

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\mathbb{P}(A \cap T^{-k}B) - \mathbb{P}(A)\mathbb{P}(B)| = 0 \quad \text{for any } A, B \text{ in } \mathcal{A},$$

3. $\liminf_{n\to\infty} \sigma_n/\sqrt{n} > 0$,

then $\{\sigma_n^{-1}S_{[nt]}, t \in [0,1]\}$ converges in distribution in $(D([0,1]), d)$ to W, where W is a standard Brownian motion. In fact, if $\mathbf{C_3}$ holds then $n^{-1}\sigma_n^2$ converges to $\sum_{k\in\mathbb{Z}} \mathbb{E}(X_0X_k)$, so that the last condition reduces to $\sum_{k\in\mathbb{Z}} \mathbb{E}(X_0X_k) > 0$. Applying Corollary 2, we see that the condition 2. of Hannan can be replaced by the weaker one $\mathbb{E}(X_0X_k|\mathcal{I}) = \mathbb{E}(X_0X_k)$ almost surely, for any $k \in \mathbb{Z}$. Finally, note that, if X_0 is $\mathcal{M}_0$-measurable, Corollary 2 is due to Dedecker and Merlevède (2003, Corollary 3).

By comparing the corollaries 1 and 2, one can ask if the WIP holds under the Heyde's condition $\mathbf{C_2}$. The following proposition gives a negative answer to this question.

Proposition 4. *There exists* $X_0 \in \mathbb{L}^2$ *measurable with respect to a* σ*-algebra* $\mathcal{M}_0$*, and a bijective and bimeasurable transformation* T *preserving the probability* $\mathbb{P}$ *such that* X_0 *is regular,* $\mathbb{P}$ *is ergodic and the condition* $\mathbf{C_2}$ *is satisfied, but the WIP does not hold for* $s_n = \sqrt{n}$.

4 Applications to linear processes with dependent innovations

Let $X_0 = \sum_{i\in\mathbb{Z}} a_i\, \varepsilon_0 \circ T^{-i}$ with $(a_i)_{i\in\mathbb{Z}}$ belonging to ℓ^1. The following result shows that if $(\varepsilon_0 \circ T^i)_{i\in\mathbb{Z}}$ satisfies $\mathbf{C_3}$, then $(X_i)_{i\in\mathbb{Z}}$ satisfies $\mathbf{C_3}$ also.

Corollary 3. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^1. Let ε_0 be a regular random variable in $\mathbb{L}^2$ and let $\varepsilon_k = \varepsilon_0 \circ T^k$. Define then $X_0 = \sum_{i\in\mathbb{Z}} a_i \varepsilon_{-i}$. If*

$$\sum_{i\in\mathbb{Z}} \|P_0(\varepsilon_i)\|_2 < \infty, \tag{4.1}$$

then $\mathbf{C}_3$ holds and $\{n^{-1/2} S_{[nt]}, t \in [0,1]\}$ converges in distribution in $(D([0,1]), d)$ to $\sqrt{\eta} W$, where W is a standard Brownian motion independent of $\mathcal{I}$, and

$$\eta = \sum_{k\in\mathbb{Z}} \mathbb{E}(X_0 X_k|\mathcal{I}) = \Big(\sum_{i\in\mathbb{Z}} a_i\Big)^2 \sum_{k\in\mathbb{Z}} \mathbb{E}(\varepsilon_0 \varepsilon_k|\mathcal{I})\,.$$

Remark 11. In Theorem 5 of Dedecker and Merlevède (2003), a similar result was given, but for causal linear processes and causal innovations only, that is

$$X_0 = \sum_{i\geq 0} a_i \varepsilon_{-i},\ \varepsilon_0 \text{ is regular and } \mathcal{M}_0\text{-measurable, and } \sum_{i\geq 0} \|P_0(\varepsilon_i)\|_2 < \infty.$$

Now, if $(a_i)_{i\in\mathbb{Z}}$ does not belong to ℓ^1, Theorem 2 can still be successfully applied. For instance, if the innovations are square integrable martingale differences, we obtain the following result.

Corollary 4. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^2. Let ε_0 be a random variable in $H_0 \ominus H_{-1}$ and let $\varepsilon_k = \varepsilon_0 \circ T^k$. Define then $X_0 = \sum_{i\in\mathbb{Z}} a_i \varepsilon_{-i}$. Let $s_n = \sqrt{n}|a_{-n} + \cdots + a_n|$. If the two following conditions hold,*

(1) $\displaystyle\limsup_{n\to\infty} \frac{\sum_{i=-n}^{n} |a_i|}{\left|\sum_{i=-n}^{n} a_i\right|} < \infty,$

(2) *either* $\displaystyle\sum_{k=1}^{n} \sqrt{\sum_{|i|\geq k} a_i^2} = o(s_n),$ *or* $\sum_{i\in\mathbb{Z}} |a_i| < \infty,$

then $\{s_n^{-1} S_{[nt]}, t \in [0,1]\}$ converges in distribution in $(D([0,1]), d)$ to $\sqrt{\mathbb{E}(\varepsilon_0^2|\mathcal{I})} W$, where W is a standard Brownian motion independent of $\mathcal{I}$.

Remark 12. Under the assumptions of Corollary 4, $\mathbf{C_0}(s_n)$ holds. Hence, according to Remark 3, σ_n/s_n converges to $\|\varepsilon_0\|_2$. It follows that we can take $s_n = \sigma_n$ in Corollary 4 and consequently $\{\sigma_n^{-1}S_{[nt]}, t \in [0,1]\}$ converges in distribution to $\sqrt{\eta}W$ where $\eta = \mathbb{E}(\varepsilon_0^2|\mathcal{I})/\mathbb{E}(\varepsilon_0^2)$ (in particular, $\eta = 1$ if $\mathbb{P}$ is ergodic). Note that, in Corollary 3 and 4 we have only required that $\mathbb{E}(\varepsilon_0^2) < \infty$. Now, if we assume that $\mathbb{E}(|\varepsilon_0|^{2+\delta}) < \infty$ for some $\delta > 0$, then the conditions of Corollary 4 can be weakened. For instance, for causal linear processes $X_0 = \sum_{i\geq 0} a_i\varepsilon_{-i}$, Wu and Min (2005, Theorem 1), and independently Merlevède and Peligrad (2005, Proposition 1), have proved that $\{\sigma_n^{-1}S_{[nt]}, t \in [0,1]\}$ converges in distribution to $\sqrt{\eta}W$ as soon as

$$\sum_{i=0}^{n-1}\Big(\sum_{k=0}^{i} a_k^2\Big) \to \infty \quad \text{and} \quad \sum_{i\geq 0}\Big(\sum_{k=i+1}^{n+i} a_k\Big)^2 = o\Big(\sum_{i=0}^{n-1}\Big(\sum_{k=0}^{i} a_k^2\Big)\Big). \tag{4.2}$$

The condition (4.2) means exactly that $\sigma_n \to \infty$, and $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(\sigma_n)$. However, (4.2) together with $\mathbb{E}(\varepsilon_0^2) < \infty$ is not sufficient for the WIP (see the discussion in Wu and Min (2005) and the counterexample given in Merlevède and Peligrad (2005, Section 3.2)). To be complete on this question, note that in Wu and Min (2005), the WIP is proved under (4.2) and $\mathbb{E}(|\varepsilon_0|^{2+\delta}) < \infty$ for innovations which are not necessarily in $H_0 \ominus H_{-1}$, but which satisfy both $\varepsilon_i = F(\ldots, \zeta_{i-1}, \zeta_i)$ for some iid sequence $(\zeta_i)_{i\in\mathbb{Z}}$, and $\sum_{k\geq 0}\|P_0(\varepsilon_k)\|_{2+\delta} < \infty$ (in particular, the first condition implies that $\mathbb{P}$ is ergodic, so that the limiting process is a standard Brownian motion).

Remark 13. According to Remark 7, if $s_n = \sqrt{n}h(n)$ where $h(n)$ is a svf, then

$$n\sqrt{\sum_{|i|\geq n} a_i^2} = o(s_n) \tag{4.3}$$

is equivalent to the first part of the condition (2) of Corollary 4.

Remark 14. The condition (1) of Corollary 4 does not allow the following possibility: $\sum_{i=-n}^{n}|a_i|$ diverges but $\sum_{i=-n}^{n} a_i$ converges. For instance if, for $n < 0$, $a_n = 0$, and for $n \geq 1$, $a_n = (-1)^n u_n$ for some sequence $(u_n)_{n\geq 1}$ of positive coefficients decreasing to zero, such that $\sum_{n\geq 1} u_n = \infty$, then Corollary 4 cannot be applied since the condition (1) fails to hold. However, for this selection of $(a_n)_{n\in\mathbb{Z}}$, the condition given by Heyde (1975)

$$\sum_{n=1}^{\infty}\Big(\sum_{|k|\geq n} a_k\Big)^2 < \infty \tag{4.4}$$

is satisfied as soon as $\sum_{n\geq 1} u_n^2 < \infty$, which is a minimal condition.

Remark 15. Notice that the conditions (1) and (2) of Corollary 4 are satisfied for sequences $(a_i)_{i\in\mathbb{Z}}$ such that for $i \neq 0$, $a_i = |i|^{-1}h(|i|)$ where $h(n)$ is a svf (this class of sequences obviously does not satisfy (4.4)). The condition (2) of Corollary 4 excludes sequences (a_i) such that for $i \neq 0$, $a_i = |i|^{-\alpha}$, for $1/2 < \alpha < 1$. However, for iid innovations, we know that for such sequences neither $\{\sigma_n^{-1} S_{[nt]}, t \in [0,1]\}$ nor $\{s_n^{-1} S_{[nt]}, t \in [0,1]\}$ can converge in distribution to a Wiener process, since they both converge to a fractional Brownian motion of index $1-\alpha$ (see Giraitis and Surgailis (1989)). In fact, if $S_{[nt]}/\sigma_n$ converges weakly to the Brownian motion, then necessarily σ_n^2 has the representation $\sigma_n^2 = nh(n)$ with $h(n)$ a svf. This is obviously not the case here since $\sigma_n \sim \|\varepsilon_0\|_2 n^{3/2-\alpha}$.

Remark 16. The condition (2) of Corollary 4 was used by Wang *et al* (2002) to prove the invariance principle for linear processes (see their Theorem 2.1) under the normalization $\tilde{s}_n^2 = a_0^2 + \sum_{j=1}^{n-1} s_j^2/j$. However instead of using in addition the condition (1) of our corollary, they used (for one-sided linear processes such that $a_0 \neq 0$) the following condition:

$$\lim_{n\to\infty}\left(\frac{1}{\tilde{s}_n}\max_{1\le j\le n}\Big|\sum_{k=0}^{j-1} a_k\Big|\right) = 0\,. \tag{4.5}$$

It appears that this condition combined with the condition (2) is not enough to ensure the weak invariance principle, so that their theorem is false. Indeed, Merlevède and Peligrad (2005) have pointed out the following fact (see the construction of their example 1): there exists a one-sided linear process for which $\sum_{i=0}^n a_i$ converges, $a_n \sim 1/(n\log^2(n))$ and such that $\tilde{s}_n^{-1} S_{[nt]}$ cannot satisfy the weak invariance principle. In this counterexample, $\tilde{s}_n \sim s_n = \sqrt{n}/\log(n+1)$ and $\sum_{k=1}^n (\sum_{|i|>k} a_i^2)^{1/2} \le C\sqrt{n}/\log^2 n$. It follows that the condition (2) of Corollary 4 is satisfied, as well as (4.5). However, the condition (1) of Corollary 4 fails to hold since this condition imposes that $\liminf_{n\to\infty} n^{-1}s_n^2 > 0$. As already mentioned in Wu and Min (2005), the wrong argument in the proof of Theorem 2.1 in Wang *et al* (2002) lies on page 134 between the equations (36) and (37) (the weak invariance principle (6) cannot follow from (36) and (37) only; to derive (6) from (37), the equality in (36) needs not only be true for any $t \in [0,1]$, but also for any finite dimensional marginals of the two processes, which is clearly false).

5 Application to functions of Linear processes

An important class of strictly stationary sequences is the class of processes which can be written as functions of iid random variables. In our context this class can be described as follows: let $\Omega = \mathcal{X}^{\mathbb{Z}}$ and $\mathbb{P} = \mu^{\otimes \mathbb{Z}}$, where μ is a probability measure on $\mathcal{X}$. If x is an element of $\mathcal{X}^{\mathbb{Z}}$, let T be the shift defined by $(T(x))_i = x_{i+1}$. Let $\varepsilon_i = \varepsilon_0 \circ T^i$ be the projection from $\mathcal{X}^{\mathbb{Z}}$ to $\mathcal{X}$ defined by $\varepsilon_i(x) = x_i$. The sequence $\varepsilon = (\varepsilon_i)_{i \in \mathbb{Z}}$ is a sequence of iid random variables with marginal distribution μ. In this section, we assume that X_0 is a square integrable random variable, which can be written as

$$X_0 = G(\varepsilon), \quad \text{so that } X_k = X_0 \circ T^k = G(\varepsilon \circ T^k). \tag{5.1}$$

Note that, since $\mathbb{P} = \mu^{\otimes \mathbb{Z}}$, the probability $\mathbb{P}$ is ergodic: for any $A \in \mathcal{I}$, $\mathbb{P}(A) = 0$ or 1. Moreover, X_0 is regular with respect to the σ-algebras

$$\mathcal{M}_i = \sigma(\varepsilon_j, j \leq i). \tag{5.2}$$

For such sequences, the condition $\mathbf{C_3}$ may be written as

$$\sum_{k \in \mathbb{Z}} \Big\| \mathbb{E}(G(\varepsilon \circ T^k) | \mathcal{M}_0) - \mathbb{E}(G(\varepsilon \circ T^k) | \mathcal{M}_{-1}) \Big\|_2 < \infty \,. \tag{5.3}$$

In this section, we shall focus on functions of real-valued linear processes

$$X_k = G(\varepsilon \circ T^k) = f\Big(\sum_{i \in \mathbb{Z}} a_i \varepsilon_{k-i} \Big) - \mathbb{E}\Big(f\Big(\sum_{i \in \mathbb{Z}} a_i \varepsilon_{k-i} \Big) \Big) \,, \tag{5.4}$$

and we shall give sufficient conditions for the weak invariance principle in terms of the regularity of the function f. As usual, we define the modulus of continuity of f on the interval $[-M, M]$ by

$$w_{\infty,f}(h, M) = \sup_{|t| \leq h, |x| \leq M, |x+t| \leq M} |f(x+t) - f(x)| \,.$$

Corollary 5. *Let $\mathcal{X} = \mathbb{R}$, $(a_i)_{i \in \mathbb{Z}}$ be a sequence of real numbers in ℓ^1, and assume that $\sum_{i \in \mathbb{Z}} a_i \varepsilon_i$ is defined almost surely. Let X_k and $\mathcal{M}_k$ be defined as in (5.4) and (5.2) respectively. Let $(\varepsilon'_i)_{i \in \mathbb{Z}}$ be an independent copy of $(\varepsilon_i)_{i \in \mathbb{Z}}$, and let*

$$M_k = \max \Big\{ \Big| \sum_{i \in \mathbb{Z}} a_i \varepsilon'_i \Big|, \Big| a_k \varepsilon_0 + \sum_{i \neq k} a_i \varepsilon'_i \Big| \Big\} \,.$$

If the following condition holds

$$\sum_{k\in\mathbb{Z}} \Big\| w_{\infty,f}\big(|a_k||\varepsilon_0|, M_k\big) \wedge \|X_0\|_\infty \Big\|_2 < \infty\,, \tag{5.5}$$

then $\mathbf{C_3}$ *holds. In particular,*

1. *if* f *is* γ*-Hölder on any compact set, with* $w_{\infty,f}(h,M) \le Ch^\gamma M^\alpha$ *for some* $C>0$, $\gamma \in]0,1]$, *and* $\alpha \ge 0$, *then* (5.5) *holds as soon as* $\sum |a_k|^\gamma < \infty$ *and* $\mathbb{E}(|\varepsilon_0|^{2(\alpha+\gamma)}) < \infty$.

2. *if* $\|\varepsilon_0\|_\infty = c < \infty$, *then* (5.5) *holds as soon as*

$$\sum_{k\in\mathbb{Z}} w_{\infty,f}\big(c|a_k|, \|M_0\|_\infty\big) < \infty\,.$$

Now, for functions of causal linear processes, that is

$$X_k = G(\varepsilon\circ T^k) = f\Big(\sum_{i\ge 0} a_i\varepsilon_{k-i}\Big) - \mathbb{E}\Big(f\Big(\sum_{i\ge 0} a_i\varepsilon_{k-i}\Big)\Big)\,, \tag{5.6}$$

we can apply Theorem 2 to the case where $\sum_{i\ge 0}|a_i| = \infty$.

Corollary 6. *Let* $\mathcal{X} = \mathbb{R}$ *and assume that* $\mathbb{E}(\varepsilon_0) = 0$ *and that* $\|\varepsilon_0\|_2$ *is finite. Let* $(a_k)_{k\ge 0} \in \ell^2$, *be such that* $\sum_{k\ge 0}|a_k| = \infty$, *and let* $s_n = \sqrt{n}|a_0 + \cdots + a_n|$. *Assume that*

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^n |a_i|}{\Big|\sum_{i=0}^n a_i\Big|} < \infty, \quad \text{and} \quad \sum_{k=1}^n \sqrt{\sum_{i\ge k} a_i^2} = o(s_n). \tag{5.7}$$

Let X_k, $\mathcal{M}_k$ *be defined as in (5.6) and (5.2) respectively. If* f *is Lipschitz and* f' *is continuous, then the process* $\{s_n^{-1}S_{[nt]}, t\in[0,1]\}$ *converges in distribution in the space* $D([0,1],d)$ *to* $\sqrt{\eta}W$, *where* W *is a standard Brownian motion, and*

$$\sqrt{\eta} = \|\varepsilon_0\|_2 \Big|\mathbb{E}\Big(f'\Big(\sum_{i\ge 0} a_i\varepsilon_i\Big)\Big)\Big|. \tag{5.8}$$

Considering (5.8), we see that the normalization $s_n = \sqrt{n}|a_0 + \cdots + a_n|$ may be too large in all the cases where

$$\mathbb{E}\Big(f'\Big(\sum_{i\ge 0} a_i\varepsilon_i\Big)\Big) = 0\,. \tag{5.9}$$

Notice that (5.9) arises in many situations such as: ε_0 is symmetric and f is even. In the following corollary, we give sufficient conditions for the condition $\mathbf{C_3}$ when (5.9) holds and $\sum_{k\ge 0}|a_k|$ is not necessarily finite.

Corollary 7. *Let $\mathcal{X} = \mathbb{R}$ and assume that $\mathbb{E}(\varepsilon_0) = 0$ and that $\|\varepsilon_0\|_4$ is finite. Let $(a_k)_{k\geq 0} \in \ell^2$, be such that*

$$\sum_{k\geq 0} |a_k| \sqrt{\sum_{i=k+1}^{\infty} a_i^2} < \infty\,. \tag{5.10}$$

Let X_k, $\mathcal{M}_k$ be defined as in (5.6) and (5.2) respectively. If f is differentiable, f' is Lipschitz and (5.9) holds, then $\mathbf{C_3}$ holds.

Remark 17. Let $a_i = i^{-1}$ for $i > 0$. Then the condition (5.7) holds, and Corollary 6 applies. Now, if in addition (5.9) holds, then Corollary 7 applies. Note also that (5.10) holds as soon as $\sum_{i>0} \sqrt{i} a_i^2$ is finite. In particular, for $f(x) = x^2$, we obtain the weak invariance principle as soon as $\mathbb{E}(\varepsilon_0) = 0$, $\mathbb{E}(\varepsilon_0^4) < \infty$ and $\sum_{i>0} \sqrt{i} a_i^2 < \infty$.

In all the results above, no assumption was made on the law of ε_0, except moment assumptions. Now, if we assume that ε_0 has a density bounded by C, then, for the sequences defined by (5.6), the regularity assumption on f in the condition (5.5) may be weakened by considering the L_p-modulus of continuity. As usual, we define the L_p-modulus of continuity of f by

$$w_{p,f}(h) = \sup_{|t|\leq h} \Big(\int |f(x+t) - f(x)|^p dx \Big)^{1/p}\,.$$

Corollary 8. *Let $(a_i)_{i\geq 0}$ be a sequence of real numbers in ℓ^1, and assume that $\sum_{i\geq 0} a_i \varepsilon_i$ is defined almost surely. Let X_k and $\mathcal{M}_k$ be defined as in (5.6) and (5.2) respectively. Assume that ε_0 has a density bounded by C. If there exists $p \in [1, \infty]$ such that*

$$\sum_{k\geq 0} \|w_{p,f}(|a_k||\varepsilon_0|)\|_2 < \infty, \tag{5.11}$$

then $\mathbf{C_3}$ holds.

Remark 18. In particular (5.11) holds for any function f of bounded variation as soon as there exists $p \in [1, \infty[$ such that $\sum_{k\geq 0} |a_k|^{1/p} < \infty$ and $\mathbb{E}(|\varepsilon_0|^{2/p}) < \infty$.

6 Other types of dependence

We have seen that conditions based on the sequence $(P_0(X_i))_{i\in\mathbb{Z}}$ can be verified for certain functions $X_0 = G((\varepsilon_i)_{i\in\mathbb{Z}})$ of stationary processes. However, in many situations (for instance when we know some property of a Markov kernel), we rather have informations on the decrease of $\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2$ and of $\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2$. In the following proposition, we give sufficient conditions, based on such quantities, for $\mathbf{C_3}$ to hold.

Proposition 5. *Consider the two conditions*

$\mathbf{C_5}$: *There exist two sequences $(a_k)_{k>0}$ and $(b_k)_{k>0}$ of positive numbers such that*

$$\sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} a_k\Big)^{-1} < \infty, \quad \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} b_k\Big)^{-1} < \infty, \tag{6.1}$$

and

$$\sum_{k\geq 1} a_k \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 < \infty, \quad \sum_{k\geq 1} b_k \|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2^2 < \infty.$$

$\mathbf{C_6}$: $\displaystyle\sum_{k\geq 1}\frac{\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2}{\sqrt{k}} < \infty$ *and* $\displaystyle\sum_{k\geq 1}\frac{\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2}{\sqrt{k}} < \infty.$

We have the implications $\mathbf{C_6} \Rightarrow \mathbf{C_5} \Rightarrow \mathbf{C_3}$.

Remark 19. The condition $\mathbf{C_5}$ is a mixingale type condition, in the sense of McLeish (1975). In the case where X_0 is $\mathcal{M}_0$-mesurable, the fact that $\mathbf{C_5}$ implies the WIP with the normalization $s_n = \sqrt{n}$ has been established in Proposition 2 of Dedecker and Merlevède (2002). In the same context, Peligrad and Utev (2005) have proved that the WIP holds under the normalization $s_n = \sqrt{n}$ provided that

$$\sum_{n>0}\frac{\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2}{n^{3/2}} < \infty\,. \tag{6.2}$$

In that case, since $\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2 = 0$, we have the implication $\mathbf{C_6} \Rightarrow$ (6.2).

Let us give a simple application of Proposition 5 to functions of adapted sequences.

Definition 3. Let Y_0 be a $\mathcal{M}_0$-measurable real valued random variable, and let $Y_k = Y_0 \circ T^k$. Let $F_{Y_k|\mathcal{M}_0}$ be the conditional distribution function of Y_k given $\mathcal{M}_0$, and let F be the distribution function of the Y_i's. For any $p \in [1, \infty]$, define the dependence coefficients $\beta_p(i)$ of the sequence $(Y_k)_{k\in\mathbb{Z}}$ by

$$\beta_p(i) = \Big\| \sup_{t\in\mathbb{R}} |F_{Y_i|\mathcal{M}_0}(t) - F(t)| \Big\|_p .$$

For $p = \infty$, we shall use the notation $\phi(i) = \beta_\infty(i)$.

Corollary 9. *Let Y_0 be a $\mathcal{M}_0$-measurable real valued random variable, and assume that*

$$X_0 = (f-g)(Y_0) - \mathbb{E}((f-g)(Y_0))\,, \tag{6.3}$$

where f and g are two non decreasing functions. If both $f(Y_0)$ and $g(Y_0)$ belong to $\mathbb{L}^p$ for some $p \geq 2$, and if the dependence coefficients of the sequence $(Y_k)_{k\in\mathbb{Z}}$ satisfy

$$\sum_{k\geq 1} \frac{\big(\beta_{2(p-1)/(p-2)}(k)\big)^{(p-1)/p}}{\sqrt{k}} < \infty\,, \tag{6.4}$$

then the condition $\mathbf{C_3}$ holds. In particular, for $p=2$ and $p=\infty$, the condition (6.4) becomes respectively

$$\sum_{k\geq 1} \sqrt{\frac{\phi(k)}{k}} < \infty \quad \textit{and} \quad \sum_{k\geq 1} \frac{\beta_2(k)}{\sqrt{k}} < \infty\,.$$

Remark 20. Using the notations of Definition 3, define the dependence coefficients $\alpha(i)$ of the sequence $(Y_k)_{k\in\mathbb{Z}}$ by

$$\alpha(i) = \sup_{t\in\mathbb{R}} \|F_{Y_i|\mathcal{M}_0}(t) - F(t)\|_1\,.$$

From Dedecker and Rio (2000), we know that, if X_0 is $\mathcal{M}_0$-measurable and the sequence $X_0\mathbb{E}(S_n|\mathcal{M}_0)$ converges in $\mathbb{L}^1$, then $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ converges in distribution in $(D([0,1],d))$ to $\sqrt{\eta}W$, where W is a standard Brownian motion independent of $\mathcal{I}$ and $\eta = \sum_{k\in\mathbb{Z}} \mathbb{E}(X_0X_k|\mathcal{I})$. If X_0 is defined by (6.3), we infer from inequality (1.11c) in Rio (2000) that $X_0\mathbb{E}(S_n|\mathcal{M}_0)$ converges in $\mathbb{L}^1$ as soon as

$$\sum_{k\geq 1} \int_0^{\alpha(k)} Q^2(u)du < \infty\,, \tag{6.5}$$

where $Q = Q_f \vee Q_g$, and Q_f is the generalized inverse of $x \to \mathbb{P}(|f(Y_0)| > x)$. Since $\alpha(i) \leq \beta_1(i)$, it follows that if both $f(Y_0)$ and $g(Y_0)$ belong to $\mathbb{L}^p$ for some $p > 2$, then (6.5) holds as soon as

$$\sum_{k\geq 1} k^{2/(p-2)}\beta_1(k) < \infty\,. \tag{6.6}$$

Of course, (6.6) cannot be compared to (6.4), since the coefficients $\beta_1(i)$ are smaller than $\beta_{2(p-1)/(p-2)}(i)$ for any $p \geq 2$. However, if $\beta_1(i)$ is of the same order than $\beta_{2(p-1)/(p-2)}(i)$, then the rate given in (6.4) is better.

Example 1. Linear processes. Assume that X_0 is defined by (6.3), with Y_0 such that $Y_0 = \sum_{i\geq 0} a_i \varepsilon_{-i}$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$ is the iid sequence defined in Section 5. Let $\mathcal{M}_0 = \sigma(\varepsilon_i, i \leq 0)$. If Y_0 has a density bounded by K, then one can prove that (see Dedecker and Prieur (2005), Section 4.1)

$$\beta_2(i) \leq 2\sqrt{2}\|\varepsilon_0\|_2 \sqrt{K\sum_{k\geq i}|a_k|} \quad \text{and} \quad \phi(i) \leq 2K\|\varepsilon_0\|_\infty \sum_{k\geq i}|a_k|\,.$$

This leads us to consider the condition

$$\sum_{k\geq 1}\sqrt{\frac{\sum_{k\geq i}|a_k|}{k}} < \infty\,. \tag{6.7}$$

If (6.7) holds, it follows from Corollary 9 that the condition $\mathbf{C_3}$ is satisfied as soon as

1. $\|\varepsilon_0\|_2 < \infty$ and $f(Y_0)$, $g(Y_0)$ belong to $\mathbb{L}^\infty$. This holds in particular if $X_0 = h(Y_0)$ for some function h of bounded variation. Note that the condition (6.7) is stronger than the condition $\sum_{k\geq 0}|a_k|$ given in Remark 18, but we have not assumed here that ε_0 has a density.

2. $\|\varepsilon_0\|_\infty < \infty$ and $f(Y_0)$, $g(Y_0)$ belong to $\mathbb{L}^2$. Here, the moment assumptions on $f(Y_0)$ and $g(Y_0)$ are sharp, and this result cannot be deduced from any results given in Section 5. This result applies in particular to the well known example where $a_i = 2^{-i-1}$ and ε_0 is a Bernoulli-distributed random variable with parameter $1/2$. In that case, Y_0 is uniformly distributed over $[0,1]$, so that $\mathbf{C_3}$ holds as soon as the increasing functions f, g satisfy $\lambda(f^2) < \infty$ and $\lambda(g^2) < \infty$, for the Lebesgue measure λ over $[0,1]$. Note that, for this particular example, it follows from Lemma 1 in Woodroofe (1992) that the condition $\mathbf{C_3}$ holds for $X_0 = f(Y_0) - \mathbb{E}(f(Y_0))$ if and only if the Fourier coefficients $\hat{f}(k)$ of f are such that

$$\sum_{k=1}^{\infty}\sqrt{\sum_{p=0}^{\infty}|\hat{f}((2p+1)\,2^k)|^2} < \infty\,.$$

Example 2. Uniformly expanding maps. Let τ be a Borel-measurable map from $[0,1]$ to $[0,1]$. If the probability μ is invariant by τ, the sequence $(\tau^i)_{i\geq 0}$ of random variables from $([0,1],\mu)$ to $[0,1]$ is strictly stationary. Define the operator K from $\mathbb{L}^1([0,1],\mu)$ to $\mathbb{L}^1([0,1],\mu)$ *via* the equality

$$\int_0^1 (Kh)(x)k(x)\mu(dx) = \int_0^1 h(x)(k\circ\tau)(x)\mu(dx)$$

where $h \in \mathbb{L}^1([0,1],\mu)$ and $k \in \mathbb{L}^\infty([0,1],\mu)$. It is easy to check that $(\tau, \tau^2, \ldots, \tau^n)$ has the same distribution as $(Y_n, Y_{n-1}, \ldots, Y_1)$ where $(Y_i)_{i\in\mathbb{Z}}$ is a stationary Markov chain with invariant distribution μ and transition kernel K. Hence, we can obtain informations on the distribution of $S_n(h) = h \circ \tau + \cdots + h \circ \tau^i$ by studying that of $h(Y_1) + \cdots + h(Y_n)$. Assume now that τ is uniformly expanding, that is: it satisfies the conditions given in Broise, Section 2.1, page 11, with an unique invariant probability μ which is mixing in the ergodic-theoretic sense (note that under Broise's conditions, μ is absolutely continuous with respect to the Lebesgue measure, with a bounded density). For such maps, Dedecker and Prieur (2005) have proved that the coefficients $\phi(k)$ of the Markov chain $(Y_i)_{i\in\mathbb{Z}}$ satisfy $\phi(k) \le C\rho^k$ for some $C > 0$ and $\rho \in]0,1[$. It follows from Corollary 9, that if $h = (f-g) - \mu(f-g)$ for two non decreasing functions f, g such that $\mu(f^2) < \infty$ and $\mu(g^2) < \infty$, then the process $\{n^{-1/2}S_{[nt]}(h), t \in [0,1]\}$ converges in distribution in the space $(D([0,1]), d)$ to $\sqrt{\eta}W$, where W is a standard Brownian motion and

$$\eta = \mu(h^2) + 2\sum_{k\ge 1}\mu(h \cdot h \circ \tau^k)\,.$$

This result seems to be new, although these dynamical systems have been widely studied. The moment assumptions on f and g are sharp. Usually, the central limit theorem for $n^{-1/2}S_n(h)$ is given for h belonging to some class of bounded functions of $[0,1]$, such as bounded variation functions or γ-Hölder functions for some $\gamma > 0$.

7 Proofs

Proof of Theorem 1. We first show that $\mathbf{C_0}(s_n)$ implies $\mathbf{C_1}(s_n)$. To this aim, define $M_n = \sum_{i=1}^n m \circ T^i$, and notice that $\mathbb{E}(M_n|\mathcal{M}_0) = 0$. Then

$$\Big\|\mathbb{E}\Big(\frac{S_n}{s_n}\Big|\mathcal{M}_0\Big)\Big\|_2 = \Big\|\mathbb{E}\Big(\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big|\mathcal{M}_0\Big)\Big\|_2 \le \Big\|\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2$$

which proves that $\mathbf{C_0}(s_n)$ implies the first part of $\mathbf{C}_1(s_n)(a)$. Notice now that $\mathbb{E}(M_n|\mathcal{M}_n) = M_n$. It follows that

$$\begin{aligned}\frac{\|S_n - \mathbb{E}(S_n|\mathcal{M}_n)\|_2}{s_n} &= \Big\|\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}} - \mathbb{E}\Big(\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big|\mathcal{M}_n\Big)\Big\|_2 \\ &\le 2\Big\|\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2,\end{aligned}$$

which proves that $\mathbf{C_0}(s_n)$ implies the second part of $\mathbf{C_1}(s_n)(a)$. Noticing now that the following decomposition holds

$$S_n = S_n - \mathbb{E}(S_n|\mathcal{M}_n) + \mathbb{E}(S_n|\mathcal{M}_0) + \mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0)\,, \tag{7.1}$$

we write that

$$\Big\|\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2 = \Big\|\frac{S_n - \mathbb{E}(S_n|\mathcal{M}_n)}{s_n} + \frac{\mathbb{E}(S_n|\mathcal{M}_0)}{s_n} + \frac{\mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0)}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2 .$$

Next by orthogonality, we derive that

$$\Big\|\frac{S_n}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2^2 = \Big\|\frac{S_n - \mathbb{E}(S_n|\mathcal{M}_n)}{s_n}\Big\|_2^2 + \Big\|\frac{\mathbb{E}(S_n|\mathcal{M}_0)}{s_n}\Big\|_2^2 + \Big\|\frac{\mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0)}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2^2 . \tag{7.2}$$

Consequently, if $\mathbf{C_0}(s_n)$ holds,

$$\lim_{n\to\infty}\Big\|\frac{\mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0)}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2^2 = 0\,.$$

Since $\mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0) = \sum_{i=1}^n \sum_{k=1}^n P_i(X_k)$, we have, by orthogonality and stationarity,

$$\begin{aligned}
\Big\|\frac{\mathbb{E}(S_n|\mathcal{M}_n) - \mathbb{E}(S_n|\mathcal{M}_0)}{s_n} - \frac{M_n}{\sqrt{n}}\Big\|_2^2 &= \Big\|\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(\frac{\sqrt{n}\sum_{k=1}^n P_i(X_k)}{s_n} - m\circ T^i\Big)\Big\|_2^2 \\
&= \frac{1}{n}\sum_{i=1}^n \Big\|\frac{\sqrt{n}\sum_{k=1}^n P_i(X_k)}{s_n} - m\circ T^i\Big\|_2^2 \\
&= \frac{1}{n}\sum_{i=1}^n \Big\|\frac{\sqrt{n}\sum_{k=1}^n P_0(X_{k-i})}{s_n} - m\Big\|_2^2 ,
\end{aligned} \tag{7.3}$$

which ends the proof of $\mathbf{C_0}(s_n) \Rightarrow \mathbf{C_1}(s_n)$. The fact now that $\mathbf{C_1}(s_n) \Rightarrow \mathbf{C_0}(s_n)$ follows directly from (7.2) and (7.3).

Proof of Proposition 1. The fact that (2.3) implies $\mathbf{C_1}(s_n)(b)$ is straightforward. Now, if X_0 is $\mathcal{M}_0$-measurable, then $\mathbf{C_1}(s_n)(a)$ reduces to $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(s_n)$. In the same way, (2.3) reduces to

$$\frac{\sqrt{n}}{s_n}\sum_{i=0}^n P_0(X_i) \to m \text{ in } \mathbb{L}^2, \text{ and } \sum_{\ell=1}^n \Big\|\sum_{k=\ell}^n P_0(X_k)\Big\|_2^2 = o\left(s_n^2\right) . \tag{7.4}$$

Let $s_n = \sqrt{n}h(n)$. Using the decomposition

$$\frac{\sqrt{n}}{s_n}\sum_{i=\ell}^n P_0(X_i) = \frac{\sqrt{n}}{s_n}\sum_{i=0}^n P_0(X_i) - \frac{\sqrt{\ell-1}}{s_{\ell-1}}\sum_{i=0}^{\ell-1} P_0(X_i) + \frac{\sqrt{\ell-1}}{s_{\ell-1}}\sum_{i=0}^{\ell-1} P_0(X_i)\Big(1 - \frac{h(\ell-1)}{h(n)}\Big) ,$$

we see that the first part of (7.4) implies the second part of (7.4) provided that

$$\frac{1}{n}\sum_{\ell=1}^{n}\Big(1-\frac{h(\ell-1)}{h(n)}\Big)^2 \quad \text{converges to } 0. \tag{7.5}$$

Now (7.5) is true as soon as $h(n)$ is a svf. To see this, note that $h^2(n)$ is a svf also, and that, for any svf sequence $g(n)$,

$$\frac{1}{ng(n)}\sum_{\ell=1}^{n} g(\ell-1) \quad \text{converges to } 1.$$

Proof of Corollary 1. Clearly, if the first part of $\mathbf{C_2}$ holds, then (2.3) holds with $s_n=\sqrt{n}$ and $m=\sum_{i\in\mathbb{Z}}P_0(X_i)$, and consequently $\mathbf{C_1}(\sqrt{n})(b)$ holds. Now, from the decomposition (7.1), we obtain that

$$\frac{\|S_n\|_2^2}{n}=\frac{1}{n}\|S_n-\mathbb{E}(S_n|\mathcal{M}_n)\|_2^2+\frac{1}{n}\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2^2+\frac{1}{n}\|\mathbb{E}(S_n|\mathcal{M}_n)-\mathbb{E}(S_n|\mathcal{M}_0)\|_2^2\,. \tag{7.6}$$

By assumption $n^{-1}\|S_n\|_2^2$ converges to $\|m\|_2^2$. Since $\mathbf{C_1}(\sqrt{n})(b)$ holds, it follows from (7.3) that $n^{-1}\|\mathbb{E}(S_n|\mathcal{M}_n)-\mathbb{E}(S_n|\mathcal{M}_0)\|_2^2$ converges to $\|m\|_2^2$. Consequently, we infer from (7.6) that $\mathbf{C_1}(\sqrt{n})(a)$ holds also, so that $\mathbf{C_2}\Rightarrow\mathbf{C_1}(\sqrt{n})$.

Clearly, if $\mathbf{C_3}$ holds, then the first part of $\mathbf{C_2}$ does. Next, we shall prove that

$$\frac{1}{n}\mathbb{E}(S_n^2|\mathcal{I})\to\sum_{k\in\mathbb{Z}}\mathbb{E}(X_0X_k|\mathcal{I}) \text{ a.s., and } \mathbb{E}(m^2|\mathcal{I})=\sum_{k\in\mathbb{Z}}\mathbb{E}(X_0X_k|\mathcal{I})\ a.s., \tag{7.7}$$

which clearly implies the second part of $\mathbf{C_2}$, so that $\mathbf{C_3}\Rightarrow\mathbf{C_2}$. To prove (7.7), note that, since X_0 is regular, the decomposition (2.1) is valid. Then

$$\mathbb{E}(X_0X_k|\mathcal{I})=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})\,.$$

Here we need the following lemma whose proof will be done at the end of this paragraph.

Lemma 1. *If $i\neq j$, then $\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})=0$ almost surely.*

By Lemma 1 and stationarity,

$$\mathbb{E}(X_0X_k|\mathcal{I})=\sum_{i\in\mathbb{Z}}\mathbb{E}(P_0(X_i)P_0(X_{k+i})|\mathcal{I}).$$

Hence

$$\|\mathbb{E}(X_0X_k|\mathcal{I})\|_1 \le \sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_2\|P_0(X_{i+k})\|_2\,,$$

so that

$$\sum_{k\in\mathbb{Z}} \|\mathbb{E}(X_0X_k|\mathcal{I})\|_1 \le \Big(\sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_2\Big)^2,$$

which is finite under $\mathbf{C_3}$. It follows that, almost surely, the series $\sum_{k\in\mathbb{Z}}\mathbb{E}(X_0X_k|\mathcal{I})$ converges absolutely and that

$$\sum_{k\in\mathbb{Z}} \mathbb{E}(X_0X_k|\mathcal{I}) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \mathbb{E}(P_0(X_i)P_0(X_j)|\mathcal{I})\ a.s.. \tag{7.8}$$

Consequently

$$\frac{1}{n}\mathbb{E}(S_n^2|\mathcal{I}) = \sum_{k=-n}^{n}\Big(1-\frac{|k|}{n}\Big)\mathbb{E}(X_0X_k|\mathcal{I})$$

converges almost surely to $\sum_{k\in\mathbb{Z}}\mathbb{E}(X_0X_k|\mathcal{I})$ and the first part of (7.7) is proved. Now

$$\mathbb{E}(m^2|\mathcal{I}) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \mathbb{E}(P_0(X_i)P_0(X_j)|\mathcal{I})\ a.s.,$$

and the second part of (7.7) follows from (7.8).

Now we turn to the proof of Lemma 1. By the $\mathbb{L}^1$-ergodic theorem, we get that for $j \ge i$ and every integer N,

$$\Big(\frac{1}{n}\sum_{\ell=1}^{n} P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell})\Big)\circ T^{-n-j-N} \to \mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I}) \text{ in } \mathbb{L}^1.$$

Since for $\ell \le n$, $j \ge i$ and every integer N, the variables $P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell})\circ T^{-n-j-N}$ are $\mathcal{M}_{-N}$-measurable, it follows that

$$\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I}) = \mathbb{E}\{\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})|\mathcal{M}_{-N}\} \text{ almost surely.}$$

Letting N tend to infinity, we obtain that

$$\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I}) = \mathbb{E}\{\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})|\mathcal{M}_{-\infty}\} \text{ almost surely.} \tag{7.9}$$

Conditioning with respect to $\mathcal{M}_{-\infty}$, we infer from (7.9) that

$$\Big\|\frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}(P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell})|\mathcal{M}_{-\infty}) - \mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})\Big\|_1 \le \Big\|\frac{1}{n}\sum_{\ell=1}^{n} P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell}) - \mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I})\Big\|_1.$$

Consequently, applying the $\mathbb{L}^1$-ergodic theorem on the right hand term, we get that

$$\frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}(P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell})|\mathcal{M}_{-\infty}) \to \mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I}) \text{ in } \mathbb{L}^1.$$

Now by orthogonality if $j > i$, $\mathbb{E}(P_{i+\ell}(X_\ell)P_{j+\ell}(X_{k+\ell})|\mathcal{M}_{-\infty}) = 0$ almost surely, so that $\mathbb{E}(P_i(X_0)P_j(X_k)|\mathcal{I}) = 0$ almost surely.

Proof of Proposition 3. We first consider the following decomposition: for every $1 \le k \le n$,

$$S_k = S_k - \mathbb{E}(S_k|\mathcal{M}_n) + \mathbb{E}(S_k|\mathcal{M}_0) + \mathbb{E}(S_k|\mathcal{M}_n) - \mathbb{E}(S_k|\mathcal{M}_0)\,. \tag{7.10}$$

Then, due to the condition $\mathbf{C_4}(s_n)(a)$, $\{s_n^{-2}\max_{1\le k\le n} S_k^2\}$ will be uniformly integrable as soon as

$$\lim_{\lambda\to\infty}\limsup_{n\to\infty} s_n^{-2}\mathbb{E}((\widetilde{S}_n^+ - \lambda s_n)_+^2) = 0 \quad\text{and}\quad \lim_{\lambda\to\infty}\limsup_{n\to\infty} s_n^{-2}\mathbb{E}((\widetilde{S}_n^- - \lambda s_n)_+^2) = 0\,, \tag{7.11}$$

with $\widetilde{S}_k = \mathbb{E}(S_k|\mathcal{M}_n) - \mathbb{E}(S_k|\mathcal{M}_0)$, $\widetilde{S}_n^+ = \max(0, \widetilde{S}_1, ..., \widetilde{S}_n)$ and $\widetilde{S}_n^- = \max(0, -\widetilde{S}_1, ..., -\widetilde{S}_n)$. We shall only prove the first part of (7.11), the second part being similar. First, note that

$$\widetilde{S}_k = \sum_{j=1}^{k}\sum_{i=j-n}^{j-1} P_{j-i}(X_j) = \sum_{i=1-n}^{k-1}\sum_{j=1\vee(i+1)}^{k\wedge(n+i)} P_{j-i}(X_j)\,.$$

For any positive integer i, let $(Y_{i,k,n})_{k\ge1}$ be the martingale

$$Y_{i,\,k,n} = \sum_{j=1\vee(i+1)}^{k\wedge(n+i)} P_{j-i}(X_j) \quad \text{and define} \quad Y_{i,j,n}^+ = \max\{0, Y_{i,1,n}, \dots, Y_{i,j,n}\}\,.$$

With these notations, $\widetilde{S}_k = \sum_{i=1-n}^{k-1} Y_{i,\,k,n}$ and therefore setting $b_{i,n} = u_i\big(\sum_{\ell=-n}^{n} u_\ell\big)^{-1}$, we have for all $k \le n$,

$$(\widetilde{S}_k - \lambda s_n)_+ \le \sum_{i=1-n}^{k-1} (Y_{i,\,k,n} - \lambda b_{i,n} s_n)_+\,.$$

Next applying Hölder's inequality, and taking the maximum on k on both sides, we get

$$(\widetilde{S}_n^+ - \lambda s_n)_+^2 \le \Big(\sum_{\ell=1-n}^{n-1} u_\ell\Big)\Big(\sum_{i=1-n}^{n-1}\frac{1}{u_i}(Y_{i,n,n}^+ - \lambda b_{i,n}s_n)_+^2\Big)$$

Taking the expectation and applying Proposition 1(a) of Dedecker and Rio (2000) to the martingale $(Y_{i,k,n})_{k\geq 1}$, we get that

$$s_n^{-2}\mathbb{E}\left((\widetilde{S}_n^{+}-\lambda s_n)_+^2\right)\leq 4s_n^{-2}\Big(\sum_{\ell=1-n}^{n-1}u_\ell\Big)\Big(\sum_{i=1-n}^{n-1}\frac{1}{u_i}\Big(\sum_{j=1\vee(i+1)}^{n\wedge(n+i)}\mathbb{E}(P_{j-i}^2(X_j)\mathbb{1}_{\Gamma(i,j,\lambda b_{i,n}s_n)})\Big)\Big)\,,$$

where $\Gamma(i,j,\lambda b_{i,n}s_n)=\{Y_{i,j,n}^{+}>\lambda b_{i,n}s_n\}$. Since $\{u_i\}_{i\in\mathbb{Z}}$ is such that $\dfrac{\sqrt{n}}{s_n}\displaystyle\sum_{i=-n}^{n}u_i$ is bounded, the first part of (7.11) will hold if we can prove that

$$\lim_{\lambda\to\infty}\limsup_{n\to\infty}\frac{1}{\sqrt{n}s_n}\sum_{i=1-n}^{n-1}\frac{1}{u_i}\Big(\sum_{j=1}^{n}\mathbb{E}(P_{j-i}^2(X_j)\mathbb{1}_{\Gamma(i,n,\lambda b_{i,n}s_n)})\Big)=0\,. \tag{7.12}$$

With this aim, notice that for any positive A, we have

$$\mathbb{E}(P_{j-i}^2(X_j)\mathbb{1}_{\Gamma(i,n,\lambda b_{i,n}s_n)})\leq\mathbb{E}(P_{j-i}^2(X_j)\mathbb{1}_{P_{j-i}^2(X_j)>Au_i^2})+Au_i^2\mathbb{P}(\Gamma(i,n,\lambda b_{i,n}s_n))\,.$$

Using this inequality and the stationarity, we get that for any positive A

$$\begin{aligned}\frac{1}{\sqrt{n}s_n}\sum_{i=1-n}^{n-1}\frac{1}{u_i}\Big(\sum_{j=1}^{n}\mathbb{E}(P_{j-i}^2(X_j)\mathbb{1}_{\Gamma(i,n,\lambda b_{i,n}s_n)})\Big)\quad&\leq\quad\frac{\sqrt{n}}{s_n}\sum_{i=1-n}^{n-1}\frac{1}{u_i}\Big(\mathbb{E}(P_0^2(X_i)\mathbb{1}_{P_0^2(X_i)>Au_i^2}\Big)\\&\quad+\frac{A\sqrt{n}}{s_n}\sum_{i=1-n}^{n-1}u_i\mathbb{P}(\Gamma(i,n,\lambda b_{i,n}s_n))\end{aligned}$$

The condition $\mathbf{C_4}(s_n)(b)$ ensures that the first term in the right hand side converges to zero by first letting n tend to infinity and after A. Now to treat the second one, we use Doob's inequality followed by stationarity which leads to

$$\mathbb{P}(\Gamma(i,n,\lambda b_{i,n}s_n))\leq\frac{4}{\lambda^2b_{i,n}^2s_n^2}\sum_{j=1\vee(i+1)}^{n}\mathbb{E}(P_{j-i}^2(X_j))\leq\frac{4n}{\lambda^2b_{i,n}^2s_n^2}\mathbb{E}(P_0^2(X_i))\,.$$

By taking into account the choice of $b_{i,n}$, it follows that

$$\frac{A\sqrt{n}}{s_n}\sum_{i=-n}^{n}u_i\mathbb{P}(\Gamma(i,n,\lambda b_{i,n}s_n))\leq 4\frac{A}{\lambda^2}\Big(\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}\frac{\mathbb{E}(P_0^2(X_i))}{u_i}\Big)\Big(\frac{\sqrt{n}}{s_n}\sum_{\ell=-n}^{n}u_\ell\Big)^2\,. \tag{7.13}$$

Now notice that

$$\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}\frac{\mathbb{E}(P_0^2(X_i))}{u_i}\leq\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}\mathbb{E}\left(\frac{P_0^2(X_i)}{u_i}\mathbb{1}_{P_0^2(X_i)>Au_i^2}\right)+A\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}u_i\,.$$

Then by taking into account the selection of $\{u_i\}_{i\in\mathbb{Z}}$ and the condition $\mathbf{C_4}(s_n)(b)$, it follows that

$$\sup_{n\geq 1}\frac{\sqrt{n}}{s_n}\sum_{i=-n}^{n}\frac{\mathbb{E}(P_0^2(X_i))}{u_i}<+\infty\,. \tag{7.14}$$

Starting from (7.13) and using (7.14) together with the selection of $\{u_i\}_{i\in\mathbb{Z}}$, it follows that for any positive A,

$$\limsup_{\lambda\to\infty}\limsup_{n\to\infty}\frac{A\sqrt{n}}{s_n}\sum_{i=-n}^{n}u_i\mathbb{P}(\Gamma(i,n,\lambda b_{i,n}s_n))=0\,,$$

which ends the proof of (7.12).

Proof of Remark 7. Notice first that (3.2) is equivalent to

$$\sum_{k=1}^{n}\sqrt{\sum_{|i|\geq k}\|P_0(X_i)\|_2^2}=o(s_n)\,. \tag{7.15}$$

Then (3.2) implies (3.3). Now notice that, for $2^{r-1}\leq n<2^r$, we have that

$$\sum_{k=1}^{n}\sqrt{\sum_{|i|\geq k}\|P_0(X_i)\|_2^2}\leq\sum_{k=0}^{r-1}\sum_{\ell=2^k}^{2^{k+1}-1}\sqrt{\sum_{|i|\geq \ell}\|P_0(X_i)\|_2^2}\leq\sum_{k=0}^{r-1}2^k\sqrt{\sum_{|i|\geq 2^k}\|P_0(X_i)\|_2^2}\,.$$

Hence, using (3.3), we derive that

$$\sum_{k=1}^{n}\sqrt{\sum_{|i|\geq k}\|P_0(X_i)\|_2^2}\leq\sqrt{\sum_{|i|\geq 0}\|P_0(X_i)\|_2^2}\sum_{k=0}^{N}2^k+\epsilon_N s_{2^{r-1}}\sum_{k=N+1}^{r-1}\frac{s_{2^k}}{s_{2^{r-1}}}\,,$$

where ϵ_N is such that $\lim_{N\to\infty}\epsilon_N=0$. Now if $s_n=\sqrt{n}h(n)$ with $h(n)$ a svf, we infer from the properties of the slowly varying functions that $\sum_{k=N+1}^{r-1}s_{2^k}s_{2^{r-1}}^{-1}<C$, where C is a constant not depending on N nor r. Then, by first letting $r\to\infty$ and next $N\to\infty$, it follows easily that

$$\sum_{k=1}^{n}\sqrt{\sum_{|i|\geq k}\|P_0(X_i)\|_2^2}=o(s_{2^{r-1}})\,.$$

Now, since $2^{r-1}\leq n<2^r$, it follows that if $s_n=\sqrt{n}h(n)$ where $h(n)$ is a svf, then $s_{2^{r-1}}=O(s_n)$. This completes the proof.

Proof of Theorem 2. Since $\mathbf{C_4}(s_n)$ holds, the sequence $s_n^{-2}\max_{1\leq k\leq n}S_k^2$ is uniformly integrable, and the process $\{s_n^{-1}S_{[nt]},t\in[0,1]\}$ is tight (apply (8.17) in Billingsley (1968) and Markov inequality). It remains to prove that for any $0 \le t_1 < t_2 < \cdots < t_k \le 1$, the k-tuple

$$\Big(\frac{1}{s_n}\sum_{k=1}^{[nt_1]} X_k, \frac{1}{s_n}\sum_{k=1}^{[nt_2]} X_k, \dots, \frac{1}{s_n}\sum_{k=1}^{[nt_k]} X_k\Big)$$

converges in distribution to $\sqrt{\mathbb{E}(m^2|\mathcal{I})}\,(W(t_1), W(t_2), \dots, W(t_k))$. Clearly, this will follow from the invariance principle for stationary martingale difference sequences, provided that, for any $t \in [0,1]$,

$$\lim_{n\to\infty} \Big\| \frac{S_{[nt]}}{s_n} - \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} m \circ T^k \Big\|_2 = 0. \tag{7.16}$$

To prove (7.16), note that $\mathbf{C_4}(s_n)(a)$ together with $\mathbf{C_1}(s_n)(b)$ implies that $\mathbf{C_0}(s_n)$ holds (see Theorem 1). Two cases arise: either $\mathbb{E}(m^2) > 0$, and then $s_n/\sqrt{n}$ is a svf (cf. Remark 3), so that (7.16) is equivalent to $\mathbf{C_0}(s_n)$; or $m = 0$ almost surely, and (7.16) follows from $\mathbf{C_0}(s_n)$ and the fact that $s_{[nt]}/s_n$ is bounded.

Proof of Corollary 2. From Corollary 1, we know that $\mathbf{C_3}$ implies $\mathbf{C_1}(\sqrt{n})$. From Theorem 2, it remains to prove that $\mathbf{C_3}$ implies $\mathbf{C_4}(\sqrt{n})$. The fact that $\mathbf{C_3}$ implies $\mathbf{C_4}(\sqrt{n})(b)$ is clear, by taking $u_i = \|P_0(X_i)\|_2$. To prove that $\mathbf{C_3}$ implies $\mathbf{C_4}(\sqrt{n})(a)$, note first that, since X_0 is regular,

$$\mathbb{E}(S_k|\mathcal{M}_0) = \sum_{i=1}^{\infty}\sum_{j=1}^{k\wedge i} P_{j-i}(X_j) \text{ and } S_k - \mathbb{E}(S_k|\mathcal{M}_n) = \sum_{i=-\infty}^{k-n-1}\ \sum_{j=1\vee(i+n+1)}^{k} P_{j-i}(X_j)\,.$$

Let

$$Y_{i,k} = \sum_{j=1}^{k\wedge i} P_{j-i}(X_j) \text{ and } Z_{i,k,n} = \sum_{j=1\vee(i+n+1)}^{k} P_{j-i}(X_j)\,.$$

Obviously

$$\sup_{1\le k\le n} |\mathbb{E}(S_k|\mathcal{M}_0)| \le \sum_{i=1}^{\infty} \sup_{1\le k\le n} |Y_{i,k}| \text{ and } \sup_{1\le k\le n} |S_k - \mathbb{E}(S_k|\mathcal{M}_n)| \le \sum_{i=-\infty}^{-1} \sup_{1\le k\le n} |Z_{i,k,n}|$$

Next, taking $u_i = \|P_0(X_i)\|_2$ and denoting $C = \sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_2$, we obtain that

$$\begin{aligned}\sup_{1\le k\le n} |\mathbb{E}(S_k|\mathcal{M}_0)|^2 &\le C\sum_{i=1}^{\infty} \frac{1}{u_i} \sup_{1\le k\le n} |Y_{i,k}|^2\,, \quad \text{and}\\ \sup_{1\le k\le n} |S_k - \mathbb{E}(S_k|\mathcal{M}_n)|^2 &\le C\sum_{i=-\infty}^{-1} \frac{1}{u_i} \sup_{1\le k\le n} |Z_{i,k,n}|^2\,.\end{aligned}$$

Applying Doob's maximal inequality to the martingales $(Y_{i,k})_{k\geq 1}$ and $(Z_{i,k,n})_{k\geq 1}$, we infer that

$$\Big\| \sup_{1\leq k\leq n} |\mathbb{E}(S_k|\mathcal{M}_0)| \Big\|_2^2 \leq C\sum_{i=1}^{\infty}(i\wedge n)\|P_0(X_i)\|_2\,, \tag{7.17}$$

and

$$\Big\| \sup_{1\leq k\leq n} |S_k-\mathbb{E}(S_k|\mathcal{M}_n)| \Big\|_2^2 \leq C\sum_{i=-\infty}^{-1}(n\wedge (-i))\|P_0(X_i)\|_2\,. \tag{7.18}$$

From (7.17) and (7.18), we easily infer that, under $\mathbf{C_3}$,

$$\lim_{n\to\infty}\frac{1}{n}\Big\| \sup_{1\leq k\leq n} |\mathbb{E}(S_k|\mathcal{M}_0)| \Big\|_2^2 = 0 \text{ and } \lim_{n\to\infty}\frac{1}{n}\Big\| \sup_{1\leq k\leq n} |S_k-\mathbb{E}(S_k|\mathcal{M}_n)| \Big\|_2^2 = 0\,,$$

which is exactly $\mathbf{C_4}(\sqrt{n})(a)$.

Proof of Proposition 4. Notice that it is sufficient to find a centered random variable X_0, a transformation T, and a sigma-algebra $\mathcal{M}_0$ such that $(1/n)\|\sum_{i=1}^n X_i\|_2^2 \to 0$, $\sum_{i=0}^{\infty} P_0(X_i)$ converges in $\mathbb{L}^2$ to a constant zero, but the tightness condition in the Donsker invariance principle is not satisfied.

Let $(\Omega,\mathcal{A},\mathbb{P}) = \big([0,1],\mathcal{B}([0,1]),\lambda\big)^{\mathbb{Z}}$, where λ is the Lebesgue measure, and let T be the left shift i.e. $(T(\omega))_i = \omega_{i+1}$ for all $i\in\mathbb{Z}$. For all $i\in\mathbb{Z}$, let $\pi_i:\Omega\to[0,1]$ be the projections such that $\pi_i(\omega)=\omega_i$, and let $\mathcal{M}_k=\sigma(\pi_i, i\leq k)$. For $k\geq 1$ and $1\leq j\leq k$, let $\bar{A}_{k,j}$ be independent subsets of $[0,1]$ such that $\lambda(\bar{A}_{k,j}) = 1/(k4^k)$, and let $A_{k,j}=\pi_0^{-1}(\bar{A}_{k,j})$. Notice that for all $k\geq 1$, $1\leq j\leq k$ and $i\in\mathbb{Z}$, the sets $A_{k,j}\circ T^i$ are independent hence the random variables $\mathbb{1}_{A_{k,j}}\circ T^i$, $i\in\mathbb{Z}$ are mutually independent. We define

$$\tilde{e}_{k,j} = 2^{k-j}\mathbb{1}_{A_{k,j}} \text{ and } e_{k,j}=\tilde{e}_{k,j}-\mathbb{E}(\tilde{e}_{k,j})\,,$$

$$n_j = \sum_{i=1}^{j-1} 2^i,\ m_k=\sum_{\ell=1}^{k-1} n_{\ell+1},$$

$$f_{k,j} = \Big(\sum_{i=0}^{2^j-1} e_{k,j}\circ T^{-i} - \sum_{i=2^j}^{2^{j+1}-1} e_{k,j}\circ T^{-i}\Big)\circ T^{-2(m_k+n_j)},\ f_k=\sum_{j=1}^k f_{k,j}$$

$$\text{and finally}\quad X_0=\sum_{k\geq 1} f_{2^k}\,.$$

Note that $\sum_{k\geq 1}\sum_{j=1}^{2^k}\|f_{2^k,j}\|_2 < +\infty$, so that X_0 is well defined in $\mathbb{L}^2$.

1. We prove that $\frac{1}{n}\Big\|\sum_{i=1}^{n} X_i\Big\|_2^2 \to 0$.

Notice first that each of the functions $f_{k,j}$ is a coboundary:

$$f_{k,j} = g_{k,j} - g_{k,j} \circ T^{-1}$$

where

$$\begin{aligned} g_{k,j} &= \Big(\sum_{i=0}^{2^j-1}\sum_{\ell=0}^{2^j-1} \tilde{e}_{k,j} \circ T^{-(i+\ell)}\Big) \circ T^{-2(m_k+n_j)} \\ &= \Big(\sum_{i=0}^{2^j-1}(i+1)\tilde{e}_{k,j} \circ T^{-i} + \sum_{i=2^j}^{2^{j+1}-1} (2^{j+1}-i-1)\tilde{e}_{k,j} \circ T^{-i}\Big) \circ T^{-2(m_k+n_j)} \,. \end{aligned}$$

We then have that

$$\sum_{\ell=1}^{n} f_{k,j} \circ T^{\ell} = g_{k,j} \circ T^n - g_{k,j} \,. \tag{7.19}$$

Since

$$\big\|g_{k,j} - \mathbb{E}(g_{k,j})\big\|_2^2 \le 2\sum_{i=0}^{2^j-1}(i+1)^2\|e_{k,j}\|_2^2 \le \frac{2^{j+1}}{k} \,.$$

Then, for all $n \ge 1$ (and in particular for $n \ge 2^j$),

$$\Big\|\sum_{\ell=1}^{n} f_{k,j} \circ T^{\ell}\Big\|_2^2 \le \frac{2^{j+3}}{k} \,. \tag{7.20}$$

On an other hand,

$$\begin{aligned} \sum_{\ell=1}^{n} f_{k,j} \circ T^{\ell} &= \Big(\sum_{\ell=1}^{n}\sum_{i=0}^{2^j-1} e_{k,j} \circ T^{\ell-i} - \sum_{\ell=1}^{n}\sum_{i=2^j}^{2^{j+1}-1} e_{k,j} \circ T^{\ell-i}\Big) \circ T^{-2(m_k+n_j)} \\ &= \Big(\sum_{m=2-2^j}^{n} (e_{k,j} \circ T^m)a(n,m,j) - \sum_{m=2-2^{j+1}}^{n-2^j} (e_{k,j} \circ T^m)b(n,m,j)\Big) \circ T^{-2(m_k+n_j)} \,, \end{aligned}$$

where $a(n,m,j) = \sum_{\ell=1}^{n} \mathbb{1}_{m \le \ell \le m+2^j-1}$ and $b(n,m,j) = \sum_{\ell=1}^{n} \mathbb{1}_{m+2^j \le \ell \le m+2^{j+1}-1}$. Then

$$\Big\|\sum_{\ell=1}^{n} f_{k,j} \circ T^{\ell}\Big\|_2 \le \Big\|\sum_{m=2-2^j}^{n} (e_{k,j} \circ T^m)a(n,m,j)\Big\|_2 + \Big\|\sum_{m=2-2^{j+1}}^{n-2^j} (e_{k,j} \circ T^m)b(n,m,j)\Big\|_2 \,.$$

Now using the independence of the random variables $(e_{k,j} \circ T^m)_{m\in\mathbb{Z}}$, we get that

$$\Big\| \sum_{m=2-2^j}^{n} (e_{k,j} \circ T^m) a(n,m,j) \Big\|_2^2 = \sum_{m=2-2^j}^{n} \|e_{k,j}\|_2^2 \big(a(n,m,j)\big)^2 \le n^2(n+2^j)\|e_{k,j}\|_2^2$$

and

$$\Big\| \sum_{m=2-2^{j+1}}^{n-2^j} (e_{k,j} \circ T^m) b(n,m,j) \Big\|_2^2 = \sum_{m=2-2^{j+1}}^{n-2^j} \|e_{k,j}\|_2^2 \big(b(n,m,j)\big)^2 \le n^2(n+2^j)\|e_{k,j}\|_2^2 .$$

It follows that if $n \le 2^j$ then

$$\Big\| \sum_{\ell=1}^{n} f_{k,j} \circ T^\ell \Big\|_2^2 \le 4n^2(n+2^j)\|e_{k,j}\|_2^2 \le \frac{8n^2}{k2^j} . \tag{7.21}$$

Consequently, from (7.20) and (7.21), we get that

- for $n > 2^k$: $\Big\| \sum_{\ell=1}^{n} f_{k,j} \circ T^\ell \Big\|_2 \le 2\sqrt{\frac{2}{k}} 2^{k/2} 2^{(j-k)/2} \le 2\sqrt{n}\sqrt{\frac{2}{k}} 2^{(j-k)/2}$
- for $2 \le 2^{j-1} \le n < 2^j \le 2^k$:

$$\begin{aligned} \Big\| \sum_{\ell=1}^{n} f_{k,m} \circ T^\ell \Big\|_2 &\le 2\sqrt{\frac{2}{k}} 2^{(j-1)/2} 2^{(m-j+1)/2} \le 2\sqrt{n}\sqrt{\frac{2}{k}} 2^{(m-j+1)/2} \text{ for } m \le j-1 \\ \Big\| \sum_{\ell=1}^{n} f_{k,m} \circ T^\ell \Big\|_2 &\le 2\sqrt{n}\sqrt{\frac{2}{k}}\sqrt{n} 2^{-m/2} < 2\sqrt{n}\sqrt{\frac{2}{k}} 2^{(j-m)/2} \text{ for } m \ge j . \end{aligned}$$

From these last considerations, it follows that uniformly in k and in $n \ge 2$, there exists a positive constant C such that

$$\frac{1}{\sqrt{n}} \Big\| \sum_{\ell=1}^{n} f_k \circ T^\ell \Big\|_2 \le C \frac{1}{\sqrt{k}} . \tag{7.22}$$

Using (7.22), we then derive that

$$\frac{1}{\sqrt{n}} \Big\| \sum_{i=1}^{n} X_i \Big\|_2 \le \sum_{k=1}^{N} \frac{1}{\sqrt{n}} \Big\| \sum_{\ell=1}^{n} f_{2^k} \circ T^\ell \Big\|_2 + C \sum_{k=N+1}^{\infty} \frac{1}{2^{k/2}} \tag{7.23}$$

Since each of the functions f_{2^k} is a coboundary, it follows that for all n, $\big\| \sum_{\ell=1}^{n} f_{2^k} \circ T^\ell \big\|_2 \le C(k)$, where $C(k)$ is a constant only depending on k. Then starting from (7.23) and letting n tend to infinity and after N, we get that $(1/n)\big\| \sum_{i=1}^{n} X_i \big\|_2^2 \to 0$.

2. We prove that $\sum_{\ell=0}^{N} P_0(X_\ell)$ *converges in* $\mathbb{L}^2$ *to zero.*

Notice first that for $\ell = 2(m_{2^k} + n_j) + i$ and $0 \le i \le 2^j - 1$, we have that

$$P_{-\ell}(X_0) = e_{2^k,j} \circ T^{-\ell}\,,$$

and that for $\ell = 2(m_{2^k} + n_j) + i$ and $2^j \le i \le 2^{j+1} - 1$, we have

$$P_{-\ell}(X_0) = -e_{2^k,j} \circ T^{-\ell}\,.$$

In addition if $\ell \neq 2(m_{2^k} + n_j) + i$, for $0 \le i \le 2^{j+1} - 1$, $k \ge 1$ and $1 \le j \le 2^k$, then we get $P_{-\ell}(X_0) = 0$. The sequence of $P_0(X_\ell)$ is then a sequence where 2^j terms equal $e_{2^k,j}$ are followed by 2^j terms equal $-e_{2^k,j}$. Then since

$$\mathbb{E}(2^j e_{2^k,j})^2 \le \frac{1}{2^k}\,,$$

the sum $\sum_{\ell=1}^{N} P_0(X_\ell)$ converges in $\mathbb{L}^2$ to zero.

3. We prove that $\Big\{\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i,\, 0 \le t \le 1\Big\}$ *is not tight.*

Notice first that $\sum_{m \ge 1} \sum_{j=1}^{2^m} g_{2^m,j}$ is almost surely finite and that

$$X_0 = \sum_{m \ge 1} \sum_{j=1}^{2^m} g_{2^m,j} - \Big(\sum_{m \ge 1} \sum_{j=1}^{2^m} g_{2^m,j} \Big) \circ T^{-1}\,.$$

Then X_0 is a coboundary and according to Theorem 1 in Volný and Samek (2000), in order for the process $\{n^{-1/2} \sum_{i=1}^{[nt]} X_i,\, 0 \le t \le 1\}$ to be tight, it is necessary that $n^{-1/2} \max_{1 \le \ell \le n} |\sum_{i=1}^{\ell} X_i|$ converges to zero in probability. Using (7.19) and the fact that the functions $g_{k,j}$ are nonnegative, notice first that for all $k \ge 1$,

$$\begin{aligned} \max_{1 \le \ell \le n} |\sum_{i=1}^{\ell} X_i| &\ge \max_{1 \le \ell \le n} \Big(\sum_{j=1}^{2^k} g_{2^k,j} \circ T^\ell + \sum_{m \ge 1, m \neq k} \sum_{j=1}^{2^m} g_{2^m,j} \circ T^\ell \Big) - \sum_{m \ge 1} \sum_{j=1}^{2^m} g_{2^m,j} \\ &\ge \max_{1 \le \ell \le n} \sum_{j=1}^{2^k} g_{2^k,j} \circ T^\ell - \sum_{m \ge 1} \sum_{j=1}^{2^m} g_{2^m,j}\,. \end{aligned}$$

Then, since the functions $g_{k,j}$ are nonnegative, to show that $\{n^{-1/2}\sum_{i=1}^{[nt]} X_i,\, 0 \le t \le 1\}$ is not tight, it suffices to prove that there exists a subsequence $n(k)$ such that

$$\frac{1}{\sqrt{n(k)}} \max_{1\le \ell \le n(k)} \sum_{j=1}^{2^k} g_{2^k,j} \circ T^\ell \text{ does not converge to 0 in probability.} \tag{7.24}$$

Take $n(k) = 4^{2^k}$ and notice that

$$\mathbb{P}\Big(\max_{1\le \ell\le n(k)} \sum_{j=1}^{2^k} g_{2^k,j}\circ T^\ell \ge \frac{\sqrt{n(k)}}{2}\Big) = \mathbb{P}\Big(\bigcup_{\ell=1}^{4^{2^k}} \Big\{ \sum_{j=1}^{2^k} g_{2^k,j}\circ T^\ell \ge \frac{2^{2^k}}{2}\Big\}\Big)\,.$$

Now let B_k be the sets defined by

$$\begin{aligned} B_k \;=\; & \{\omega \in \Omega \text{ such that } \omega \text{ is in only one of the sets } A_{2^k,j}\circ T^{-(2^j-1)-2(m_{2^k}+n_j)+\ell} \\ & \text{for } 1 \le \ell \le 4^{2^k} \text{ and } 1 \le j \le 2^k\} \end{aligned}$$

On B_k, one of the functions $g_{2^k,j}\circ T^\ell$ reaches its maximum which is equal to 2^{2^k} and the others are nonnegative. Then it follows that

$$\mathbb{P}\Big(\max_{1\le \ell\le n(k)} \sum_{j=1}^{2^k} g_{2^k,j}\circ T^\ell \ge \frac{\sqrt{n(k)}}{2}\Big) \ge \mathbb{P}(B_k) > \big(1 - \frac{1}{2^k 4^{2^k}}\big)^{2^k 4^{2^k}} \to 1/e\,,$$

which proves (7.24).

Proof of Corollary 3. Clearly, if ε_0 is regular, then so is X_0. It remains to see that $\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_2$ is finite. Clearly

$$\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_2 \le \Big(\sum_{i\in\mathbb{Z}} |a_i|\Big)\Big(\sum_{k\in\mathbb{Z}} \|P_0(\varepsilon_k)\|_2\Big)\,,$$

and $\mathbf{C_3}$ follows from (4.1). The approximating martingale is given by

$$m = \sum_{k\in\mathbb{Z}} P_0(X_k) = \Big(\sum_{i\in\mathbb{Z}} a_i\Big) \sum_{k\in\mathbb{Z}} P_0(\varepsilon_k)\,.$$

The identification of the variance follows by applying Corollary 1 (for the second equality, note that, by assumption, $\mathbf{C_3}$ holds for $(\varepsilon_i)_{i\in\mathbb{Z}}$).

Proof of Corollary 4. First note that if $(a_i)_{i\in\mathbb{Z}}$ is a sequence of real numbers in ℓ^1, Corollary 4 follows easily from Corollary 3, since, according to the condition (1),

$s_n/\sqrt{n}$ converges to $|\sum_{i\in\mathbb{Z}} a_i| > 0$ (in that case the approximating martingale is $m = \mathrm{sign}(\sum_{i\in\mathbb{Z}} a_i)\varepsilon_0$). We shall now focus on the case where $\sum_{i\in\mathbb{Z}} |a_i| = \infty$. According to Theorem 2, it is enough to prove that $\mathbf{C_1}(s_n)(b)$ and $\mathbf{C_4}(s_n)$ hold (the fact that $s_{[nt]}/s_n$ is bounded follows from the condition (1)). Since the condition (1) holds, we can take $u_i = |a_i|$ in $\mathbf{C_4}(s_n)(b)$. We first prove $\mathbf{C_4}(s_n)$. From Remark 7, $\mathbf{C_4}(s_n)(a)$ follows from (3.2), which is equivalent to (since X_0 is regular)

$$\sum_{k=1}^{n} \sqrt{\sum_{i\geq k} \|P_0(X_i)\|_2^2} = o(s_n) \quad \text{and} \quad \sum_{k=1}^{n} \sqrt{\sum_{i\geq k} \|P_0(X_{-i})\|_2^2} = o(s_n)\,. \tag{7.25}$$

Since $P_0(X_i) = a_i\varepsilon_0$, (7.25) follows from

$$\sum_{k=1}^{n} \sqrt{\sum_{|i|\geq k} a_i^2} = o(s_n)\,. \tag{7.26}$$

Now, with $u_i = |a_i|$, $\mathbf{C_4}(s_n)(b)$ holds as soon as

$$\lim_{A\to\infty} \limsup_{n\to\infty} \frac{\sum_{i=-n}^{n} |a_i|}{|\sum_{i=-n}^{n} a_i|} \mathbb{E}\big(\varepsilon_0^2 \mathbb{1}_{\varepsilon_0^2 > A}\big) = 0\,,$$

which follows from the condition (1). It remains to prove $\mathbf{C_1}(s_n)(b)$. By Corollary 1, it is enough to prove (2.3). Since $\sum_{i\in\mathbb{Z}} |a_i| = \infty$, we infer from the condition (1) that $a_{-n} + \cdots + a_n$ converges to $+\infty$ or to $-\infty$. Hence

$$\frac{\sqrt{n}}{s_n} \sum_{i=-n}^{n} P_0(X_i) = \varepsilon_0 \frac{\sum_{i=-n}^{n} a_i}{|\sum_{i=-n}^{n} a_i|}$$

converges in $\mathbb{L}^2$ to ε_0 or to $-\varepsilon_0$. Now, according to the condition (1), the second condition in (2.3) will hold as soon as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \left(\frac{\sum_{\ell\leq|k|\leq n} |a_k|}{\sum_{|k|\leq n} |a_k|} \right)^2 = 0. \tag{7.27}$$

We shall prove that (7.27) holds without the square, which is clearly sufficient. Applying Hölder's inequality, we have that

$$\frac{1}{n} \sum_{\ell=1}^{n} \frac{\sum_{\ell\leq|k|\leq n} |a_k|}{\sum_{|k|\leq n} |a_k|} \leq \frac{1}{\sqrt{n}\sum_{|k|\leq n} |a_k|} \sum_{\ell=1}^{n} \sqrt{\sum_{\ell\leq|k|\leq n} a_k^2}\,,$$

and the right hand term converges to 0 according to the condition (1) and (7.26). This completes the proof.

Proof of Corollary 5. Since X_0 is regular, we only have to prove that $\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_2$ is finite. Let ε' be an independent copy of ε, and denote by $\mathbb{E}_\varepsilon(\cdot)$ the conditional expectation with respect to ε. Clearly

(7.28)
$$P_0(X_k) = \mathbb{E}_\varepsilon\Big(f\Big(\sum_{i<k} a_i\varepsilon'_{k-i} + a_k\varepsilon_0 + \sum_{i>k} a_i\varepsilon_{k-i}\Big) - f\Big(\sum_{i<k} a_i\varepsilon'_{k-i} + a_k\varepsilon'_0 + \sum_{i>k} a_i\varepsilon_{k-i}\Big)\Big)\,.$$

Since $w_{\infty,f}(t_1+t_2, M) \le w_{\infty,f}(t_1, M) + w_{\infty,f}(t_2, M)$, it follows that

(7.29)
$$|P_0(X_k)| \le \mathbb{E}_\varepsilon\Big(2\|X_0\|_\infty \wedge \Big(w_{\infty,f}\left(|a_k||\varepsilon_0|, |Y_1|\vee|Y_2|\right) + w_{\infty,f}\left(|a_k||\varepsilon'_0|, |Y_1|\vee|Y_2|\right)\Big)\Big)\,,$$

where $Y_1 = \sum_{i\le k} a_i\varepsilon'_{k-i} + \sum_{i>k} a_i\varepsilon_{k-i}$ and $Y_2 = \sum_{i<k} a_i\varepsilon'_{k-i} + \sum_{i\ge k} a_i\varepsilon_{k-i}$. The result follows by noting that $(\varepsilon_0, |Y_1|\vee|Y_2|)$ and $(\varepsilon'_0, |Y_1|\vee|Y_2|)$ are both distributed as (ε_0, M_k), and by taking the $\mathbb{L}^2$-norm in (7.29). Items 1 and 2 are straightforward.

Proof of Corollary 6. Starting from (7.28) (with $a_i = 0$ for $i < 0$), we obtain that

$$P_0(X_k) = a_k\mathbb{E}_\varepsilon\Big((\varepsilon_0 - \varepsilon'_0)\int_0^1 f'\Big(\sum_{i=0}^{k} a_i\varepsilon'_{k-i} + ta_k(\varepsilon_0-\varepsilon'_0) + \sum_{i>k} a_i\varepsilon_{k-i}\Big)dt\Big)\,. \tag{7.30}$$

Since f' is continuous and bounded, and $|a_k|(|\varepsilon_0| + |\varepsilon'_0|) + |\sum_{i>k} a_i\varepsilon_{k-i}|$ converges in probability to 0, it follows that

(7.31)
$$Z_k = \mathbb{E}_\varepsilon\Big((\varepsilon_0 - \varepsilon'_0)\int_0^1 \Big(f'\Big(\sum_{i=0}^{k} a_i\varepsilon'_{k-i} + ta_k(\varepsilon_0-\varepsilon'_0) + \sum_{i>k} a_i\varepsilon_{k-i}\Big) - f'\Big(\sum_{i=0}^{k-1} a_i\varepsilon'_{k-i}\Big)\Big)dt\Big)$$

converges to 0 in $\mathbb{L}^2$. Since $\sum_{i=0}^{k-1} a_i\varepsilon'_{k-i}$ is independent of $(\varepsilon_0 - \varepsilon'_0)$ and converges in distribution to $\sum_{i=0}^{\infty} a_i\varepsilon_i$, it follows that

(7.32)
$$\lim_{k\to\infty} \mathbb{E}_\varepsilon\Big((\varepsilon_0 - \varepsilon'_0)\int_0^1 f'\Big(\sum_{i=0}^{k} a_i\varepsilon'_{k-i} + ta_k(\varepsilon_0-\varepsilon'_0) + \sum_{i>k} a_i\varepsilon_{k-i}\Big)dt\Big) = \varepsilon_0\mathbb{E}\Big(f'\Big(\sum_{i=0}^{\infty} a_i\varepsilon_i\Big)\Big)\,,$$

in $\mathbb{L}^2$. Since $s_n/\sqrt{n}$ tends to infinity and, by the first condition in (5.7),

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n a_k}{\left|\sum_{k=0}^n a_k\right|} = a \quad \text{with } |a| = 1,$$

it follows from (7.30) and (7.32) that

$$\lim_{n\to\infty}\frac{\sqrt{n}}{s_n}\sum_{k=0}^{n}P_0(X_k)=a\varepsilon_0\mathbb{E}\Big(f'\Big(\sum_{i=0}^{\infty}a_i\varepsilon_i\Big)\Big)\quad\text{in }\mathbb{L}^2. \tag{7.33}$$

Now, since (5.7) implies that $s_n/\sqrt{n}$ is a svf (see Remark 12), it follows from Corollary 1 that $\mathbf{C_1}(s_n)(b)$ holds. According to Theorem 2, it remains to prove that $\mathbf{C_4}(s_n)$ holds. Since, from (7.30),

$$|P_0(X_k)|\le\|f'\|_\infty|a_k|(|\varepsilon_0|+\|\varepsilon_0\|_1)\,,$$

the proof may be done as in Theorem 4, by choosing $u_i=|a_i|$. To conclude, note that the limiting variance is given by the variance of the right hand term in (7.33).

Proof of Corollary 7. Starting from (7.30) and using (5.9), we obtain that

$$P_0(X_k)=a_k\Big(Z_k+\varepsilon_0\mathbb{E}\Big(f'\Big(\sum_{i=0}^{k-1}a_i\varepsilon_i\Big)-f'\Big(\sum_{i=0}^{\infty}a_i\varepsilon_i\Big)\Big)\Big)\,,$$

where Z_k is defined in (7.31). Since f' is Lipschitz, we obtain

$$\|P_0(X_k)\|_2\le 2a_k^2\|f''\|_\infty\|(|\varepsilon_0|+|\varepsilon_0'|)^2\|_2+\|f''\|_\infty(\|\varepsilon_0\|_2+\|\varepsilon_0-\varepsilon_0'\|_2)|a_k|\Big\|\sum_{i>k}a_i\varepsilon_i\Big\|_2\,.$$

Since

$$|a_k|\Big\|\sum_{i>k}a_i\varepsilon_i\Big\|_2=\|\varepsilon_0\|_2|a_k|\sqrt{\sum_{i>k}a_i^2}\,,$$

the condition $\mathbf{C_3}$ follows from (5.10).

Proof of Corollary 8. Assume without loss that $a_0\neq 0$. Let $Y_k=\sum_{i=0}^{k-1}a_i\varepsilon_{k-i}$. The density of Y_k is given by $f_{Y_k}=|a_0|^{-1}f_{\varepsilon_0}(\cdot/a_0)\star\cdots\star|a_{k-1}|^{-1}f_{\varepsilon_0}(\cdot/a_{k-1})$ and hence, it is bounded by $C|a_0|^{-1}$. Starting from (7.28) (with $a_i=0$ for $i<0$), we have that

$$P_0(X_k)=\int\int\Big(f(y+a_k\varepsilon_0+\sum_{i>k}a_i\varepsilon_{k-i})-f(y+a_kx+\sum_{i>k}a_i\varepsilon_{k-i})\Big)f_{Y_k}(y)f_{\varepsilon_0}(x)dydx\,.$$

Consequently

$$|P_0(X_k)|\le\int\Big(\int\Big|f(y+a_k\varepsilon_0+\sum_{i>k}a_i\varepsilon_{k-i})-f(y+a_kx+\sum_{i>k}a_i\varepsilon_{k-i})\Big|^pf_{Y_k}(y)dy\Big)^{1/p}f_{\varepsilon_0}(x)dx\,.$$

Now since f_{Y_k} is bounded by $C|a_0|^{-1}$ and $w_{p,f}(|t_1+t_2|)\le w_{p,f}(|t_1|)+w_{p,f}(|t_2|)$, we obtain that

$$|P_0(X_k)|\le(C|a_0|^{-1})^{1/p}\left(w_{p,f}(|a_k||\varepsilon_0|)+\mathbb{E}\left(w_{p,f}(|a_k||\varepsilon_0|)\right)\right)\,. \tag{7.34}$$

The result follows by taking the $\mathbb{L}^2$-norm in (7.34).

Proof of Proposition 5. Note first that $\mathbf{C_5}$ implies that $\mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0$ and that $\mathbb{E}(X_0|\mathcal{M}_{\infty}) = X_0$ almost surely, so that X_0 is regular. Consequently the decomposition (2.2) is valid. It follows that

$$\sum_{k>0} a_k \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 = \sum_{k>0} a_k \sum_{i\le 0} \|P_i(X_k)\|_2^2 = \sum_{i>0} \Big(\sum_{k=1}^{i} a_k\Big) \|P_0(X_i)\|_2^2$$

$$\sum_{k>0} b_k \|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2^2 = \sum_{k>0} b_k \sum_{i>0} \|P_i(X_{-k})\|_2^2 = \sum_{i<-1} \Big(\sum_{k=1}^{-i-1} b_k\Big) \|P_0(X_i)\|_2^2 .$$

Now, by Hölder's inequality in ℓ^2,

$$\sum_{i>0} \|P_0(X_i)\|_2 \le \Big(\sum_{i\ge 1} \Big(\sum_{k=1}^{i} a_k\Big)^{-1}\Big)^{1/2} \Big(\sum_{i>0} \Big(\sum_{k=1}^{i} a_k\Big) \|P_0(X_i)\|_2^2\Big)^{1/2}$$

$$\sum_{i<-1} \|P_0(X_i)\|_2 \le \Big(\sum_{i\ge 1} \Big(\sum_{k=1}^{i} b_k\Big)^{-1}\Big)^{1/2} \Big(\sum_{i<-1} \Big(\sum_{k=1}^{-i-1} b_k\Big) \|P_0(X_i)\|_2^2\Big)^{1/2},$$

which proves that $\mathbf{C_5}$ implies $\mathbf{C_3}$. To prove that $\mathbf{C_6}$ implies $\mathbf{C_5}$, it suffices to prove that, under $\mathbf{C_6}$, the sequences $a_k^{-1} = \sqrt{k}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2$ and $b_k^{-1} = \sqrt{k}\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2$ satisfy (6.1). Since the sequences $\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2$ and $\|X_{-k} - \mathbb{E}(X_{-k}|\mathcal{M}_0)\|_2$ are non increasing, we have that

$$\sum_{k=1}^{i} a_k \ge \frac{\sqrt{i}}{2\|\mathbb{E}(X_{[i/2]}|\mathcal{M}_0)\|_2} \quad \text{and} \quad \sum_{k=1}^{i} b_k \ge \frac{\sqrt{i}}{2\|X_{-[i/2]} - \mathbb{E}(X_{-[i/2]}|\mathcal{M}_0)\|_2} ,$$

and (6.1) follows easily from $\mathbf{C_6}$.

Proof of Corollary 9. Let $S_1(\mathcal{M}_0)$ be the set of all $\mathcal{M}_0$-measurable random variables Z such that $\mathbb{E}(Z^2) = 1$. Clearly, if X_0 is defined by (6.3), then

$$\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2 = \sup_{Z\in S_1(\mathcal{M}_0)} |\mathrm{cov}(Z, (f-g)(Y_k))| \tag{7.35}$$

Let $b(\mathcal{M}_0, Y_k) = \sup_{t\in\mathbb{R}} |F_{Y_k|\mathcal{M}_0}(t) - F(t)|$. Applying Corollary 2.2 in Dedecker (2004), we have, for any conjugate exponents p, q,

$$|\mathrm{cov}(Z, (f-g)(Y_k))| \le 2(\|f(Y_0)\|_p + \|g(Y_0)\|_p)\big\{\mathbb{E}(|Z|^q b(\mathcal{M}_0, Y_k))\big\}^{1/q} .$$

Let $p \geq 2$. Applying Hölder's inequality on the last term of the right hand side, we obtain, for Z such that $\mathbb{E}(Z^2) = 1$,

$$|\mathrm{cov}(Z, (f-g)(Y_k))| \leq 2(\|f(Y_0)\|_p + \|g(Y_0)\|_p)\big(\beta_{2(p-1)/(p-2)}(k)\big)^{(p-1)/p}. \tag{7.36}$$

Combining (7.35) and (7.36), we obtain that

$$\|\mathbb{E}(X_k|\mathcal{M}_0)\|_2 \leq 2(\|f(Y_0)\|_p + \|g(Y_0)\|_p)\big(\beta_{2(p-1)/(p-2)}(k)\big)^{(p-1)/p}.$$

Hence, $\mathbf{C_6}$ follows from (6.4), and Corollary 9 follows from Proposition 5.

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