Moderate deviations for linear processes generated by martingale-like random variables

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Abstract

In this paper we study the moderate deviation principle for linear statistics of the type $S_n = \sum_{i\in\mathbf{Z}} c_{n,i}\xi_i$ where $c_{n,i}$ are real numbers and the variables ξ_i's are in turn stationary martingale differences or dependent sequences satisfying projective criteria. As an application, we obtain the moderate deviation principle and its functional form for sums of a class of linear processes with dependent innovations that might exhibit long memory. A new notion of equivalence of the coefficients allows us to study the difficult case when the variance of S_n behaves slower than n. The main tools are: a new type of martingale approximations and moment and maximal inequalities that are important in themselves.

1 Introduction

Let $(\xi_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of random variables with $\mathbf{E}(\xi_0)^2 < \infty$, $\mathbf{E}(\xi_0) = 0$ and with continuous and bounded spectral density. By ℓ^2 we denote the sequences indexed by $\mathbf{Z}$ that are square summable. Let $(c_{n,i})_{i\in\mathbf{Z},n\geq 1}$ be a double indexed sequence of real numbers such that for all $n \geq 1$, the sequence is in ℓ^2, (i.e. $\sum_{i\in\mathbf{Z}} c_{n,i}^2 < \infty$). For any $n \geq 1$ we consider the following linear statistic

$$S_n = \sum_{i\in\mathbf{Z}} c_{n,i}\xi_i \ . \tag{1}$$

Many random evolutions and also statistical procedures such as nonparametric estimation of non linear regression with fixed design, produce linear statistics of type (1) (see for instance the kernel estimators in Robinson (1997)).

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The aim of this paper is to investigate the moderate deviation principle for S_n, that is an intermediate behavior between CLT and large deviations. This type of behavior holds for many situations for which the large deviation principle fails, such as the class of bounded φ-mixing sequences with polynomial rate (see Bryc and Dembo (1996) and the references therein). It allows for an asymptotic computation of small probabilities on a logarithmic scale with invariant limit.

Linear statistics of type (1) are also useful to study the asymptotic behavior of the moving averages of the form

$$X_k = \sum_{i\in\mathbf{Z}} c_i \xi_{k-i} \text{ where } \sum_{i\in\mathbf{Z}} c_i^2 < \infty\,. \tag{2}$$

We shall refer to such a process as to a linear process with innovations $(\xi_i)_{i\in\mathbf{Z}}$. We address the MDP problem and its functional form for general weights so that the process can exhibit long range dependence when either $\limsup_n \mathrm{Var}(S_n)/n \to \infty$ or $\liminf_n \mathrm{Var}(S_n)/n \to 0$. This problem has not been fully investigated even for i.i.d. observations. Djellout and Guillin (2001) treated the short memory case when $\sum_{i\in\mathbf{Z}} |c_i| < \infty$. The short memory case for stationary innovations centered with $E[\exp(\delta|\xi_0|)]$ for some $\delta > 0$ was investigated by Dong *et al.* (2005). In the same setup Ghosh and Samorodnitsky (2007) prove functional moderate and huge deviation principle under the assumption that the coefficients are absolutely summable with $\sum_{i\in\mathbf{Z}} c_i \neq 0$, or regularly varying, and the innovations are i.i.d.

The class of innovations we consider are martingale differences and a larger class that can be approximated by martingales. The results can be easily applied to linear processes constructed from functions of mixing sequences, contracting Markov chains, expanding maps on the interval and symmetric random walks on the circle. Our weights are as general as possible while the innovations are assumed to have bounded support. We construct an example to show that some of our results are false if only the moment generating function is assumed to be finite as shown in the discussion on Condition (8) (see the comments after Theorem 1). So, some of our results complement those of Dong *et al.* (2005) and Ghosh and Samorodnitsky (2007).

It is noticeable that, in order to solve these problems, we develop new tools, such as estimates of the error of approximation of a linear process with stationary innovations with the corresponding one with martingale difference innovations.

For convenience, we shall use in the rest of the paper the following notations and definitions. We assume from now on that all the strictly stationary sequences $(\xi_i)_{i\in\mathbf{Z}}$ considered in the paper are given by $\xi_i = \xi_0 \circ T^i$ where $T : \Omega \mapsto \Omega$ is a bijective bimeasurable transformation preserving the probability $\mathbf{P}$ on $(\Omega,\, \mathcal{A})$. For a subfield $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{n\geq 0} \mathcal{F}_{-n}$ and $\mathcal{F}_{\infty} = \bigvee_{k\in\mathbf{Z}} \mathcal{F}_k$. $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ will be called a stationary filtration. We will say that a random variable ξ_0 is *regular* if $\mathbf{E}(\xi_0|\mathcal{F}_{-\infty}) = 0$ and ξ_0 is $\mathcal{F}_\infty$-measurable. For any random variable Y, by $\|Y\|_\infty$ we denote the $\mathbf{L}_\infty$-norm, the smallest u such that $P(|Y| > u) = 0$. We consider also in this paper the projection operator P_j defined as

$$P_j(Y) = \mathbf{E}(Y|\mathcal{F}_j) - \mathbf{E}(Y|\mathcal{F}_{j-1})\,. \tag{3}$$

We call two double indexed sequences of real numbers $(b_{n,j})$ and $(u_{n,j})$ (indexed over all integers j and all naturals n) equivalent if

$$\frac{\sum_{j\in\mathbf{Z}}(b_{n,j}-u_{n,j})^2}{\sum_{j\in\mathbf{Z}}(b_{n,j})^2} \to 0 \quad \text{as } n\to\infty\,. \tag{4}$$

If (4) holds, we write $(b_{n,j}) \approx (u_{n,j})$. The notion of equivalent sequences will play a prominent role all over the paper. We shall often use a smoothing sequence to prove our results and weaken the assumptions.

In this paper we shall also use the following notations: $[x]$ denotes the integer part of x, for two positive sequences of numbers the notation $u_n \sim v_n$ means that $\lim_{n\to\infty} u_n/v_n = 1$.

Our paper is organized in the following way: Section 2 concerns the moderate deviation principle for linear statistics of type (1). Section 3 is devoted to the moderate deviation principle and its functional form for the partial sums of some classes of linear processes. Section 4 contains the proofs and also a martingale-type approximation for linear processes (see Lemma 12) which is a key tool when the sequence of innovations is assumed to satisfy certain dependence conditions.

2 Moderate deviations for linear statistics

In this section we investigate sufficient conditions for the linear statistic S_n defined by (1) to satisfy the moderate deviation principle (MDP for short).

To be more precise, we say that the MDP holds for S_n with the speed $a_n \to 0$ and rate function $I(t)$ if for each A Borelian,

$$\begin{aligned} -\inf_{t\in A^o} I(t) &\le \liminf_n a_n \log \mathbf{P}(\sqrt{a_n}S_n \in A) \\ &\le \limsup_n a_n \log \mathbf{P}(\sqrt{a_n}S_n \in A) \le -\inf_{t\in \bar{A}} I(t)\,, \end{aligned} \tag{5}$$

where $\bar{A}$ denotes the closure of A and A^o the interior of A.

For a linear statistic generated by a martingale differences sequence, our first result is the following.

Theorem 1 *Let $(d_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of nondegenerate martingale differences adapted to the stationary filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. Assume $\|d_0\|_\infty < \infty$ and*

$$\|\frac{1}{n}\sum_{i=1}^{n}\mathbf{E}(d_i^2|\mathcal{F}_0) - \mathbf{E}(d_0^2)\|_\infty \to 0 \;\; \textit{as } n\to\infty\,. \tag{6}$$

For any positive integer n, let $\{c_{n,i}, i \in \mathbf{Z}\}$ be a triangular array of numbers satisfying

$$\sum_i c_{n,i}^2 \to 1 \ \ and \ \ \sum_j (c_{n,j} - c_{n,j-1})^2 \to 0 \ \ as\ n \to \infty\ . \tag{7}$$

Let $S_n = \sum_i c_{n,i} d_i$. Assume that $a_n \to 0$ and that there is $(u_{n,j}) \approx (c_{n,j})$ such that

$$\frac{\sup_j |u_{n,j}|}{\sqrt{a_n}} \to 0 \ \ as\ \ n \to \infty\ \ . \tag{8}$$

Then (S_n) satisfies the MDP with speed a_n and rate $I(t) = t^2/(2\mathbf{E}(d_0^2))$.

In some situations, when additional information on either the sequence of martingales or on the filtration is available, we might have $\mathbf{E}(d_1^2|\mathcal{F}_0) = \mathbf{E}(d_0^2)$ almost surely. Our proof reveals that for this situation the second part of Condition (7) is not needed and Condition (6) is trivially satisfied.

To further comment on the conditions of Theorem 1 we mention that there are examples of stationary bounded ergodic martingales that do not satisfy the MDP, so a condition of type (6) is needed (see Comment 8 in Dedecker *et al.* (2008)).

Moreover, we mention that Condition (8) is not enough to ensure the validity of MDP if the variables are not bounded, but instead they are supposed to have finite moment generating function. Indeed, if we take for instance $c_{n,j} = \log n/(n^{1/2} \log j)$ for $j \in [2, n]$ and 0 otherwise, then we can choose $u_{n,j} = n^{-1/2}$ for all $j \in [2, n]$ and 0 otherwise. It is easy to see that this selection of $u_{n,j}$ satisfies (7) and it is such that $(u_{n,j}) \approx (c_{n,j})$. Then, if the variables are i.i.d. centered and bounded, Condition (8) holds as soon as $na_n \to \infty$. However simple computations (see Section 2.1 in Merlevède and Peligrad (2008)) show that if we assume for instance that $\xi_i = \varepsilon_i - 1$ where the $(\varepsilon_i)_{i\in\mathbf{Z}}$ are iid with exponential law with mean 1 ($\mathbf{P}(\varepsilon_0 > x) = e^{-x}$) then a necessary condition for the MDP to hold for $\sum_j \ c_{n,j}\xi_j$ is $a_n n/(\log n)^2 \to \infty$.

By using Theorem 1 and the martingale approximation given in Lemma 12, we can extend Theorem 1 to the case when the linear statistic is generated by a sequence more general than a martingale difference sequence and satisfies projective criteria. Before stating the result, we point out the following known fact (see Peligrad and Utev (2006)):

Remark 2 *Let $(\xi_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of regular random variables and let $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ be a stationary filtration. Then the condition*

$$\sum_{j\in\mathbf{Z}} \|P_0(\xi_j)\|_2 < \infty \tag{9}$$

implies that the sequence $(\xi_i)_{i\in\mathbf{Z}}$ has continuous spectral density $f(x)$,

$$\lim_{n\to\infty} \mathrm{Var}(\xi_1 + ... + \xi_n)/n = s^2 = \| \sum_j P_0(\xi_j)\|_2^2 \ = 2\pi f(0). \tag{10}$$

and S_n introduced in (1) is well defined.

Next theorem contains a limiting result for a larger class of dependent variables.

Theorem 3 *Let $(\xi_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of regular random variables and let $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ be a stationary filtration. Assume that*

$$\sum_{j\in\mathbf{Z}} \|P_0(\xi_j)\|_\infty < \infty\,, \tag{11}$$

s^2 given by (10) is strictly positive and for all $j \geq 0$,

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n}[\mathbf{E}(\xi_i\xi_{i+j}|\mathcal{F}_0) - \mathbf{E}(\xi_0\xi_j)]\|_\infty \ = 0\,. \tag{12}$$

Let $(c_{n,i}, -\infty \leq i \leq \infty)$ be a triangular array of numbers satisfying (7) and $S_n = \sum_i c_{ni}\xi_i$. Assume that $a_n \to 0$ and that there is $(u_{n,j}) \approx (c_{n,j})$ such that (8) is satisfied. Then (S_n) satisfies the MDP with speed a_n and rate $I(t) = t^2/(2s^2)$.

If $s^2 = 0$ then the theorem still holds with the rate function equals to infinity.

It is easy to see that this theorem extends the moderate deviation principle obtained in Dedecker *et al.* (2008) to the case of linear processes (see their Theorem 3). Moreover Condition (12) is slightly weaker than Condition (6) used in Dedecker *et al.* (2008).

Some comments on the conditions imposed on the innovations. The dependence conditions in our theorems are in particular easy to verify for ϕ-mixing sequences.

Recall that if Y is a random variable with values in a Polish space and $\mathcal{M}$ is a σ-field, the ϕ-mixing coefficient between $\mathcal{M}$ and $\sigma(Y)$ is defined by

$$\phi(\mathcal{M}, \sigma(Y)) = \sup_{A\in\mathcal{B}(\mathcal{Y})} \|\mathbf{P}_{Y|\mathcal{M}}(A) - \mathbf{P}_Y(A)\|_\infty\,. \tag{13}$$

For a stationary adapted sequence $(\xi_i)_{i\in\mathbf{Z}}$, let

$$\phi_1(n) = \phi(\mathcal{F}_0, \sigma(\xi_n)) \quad \text{and} \quad \phi_2(n) = \sup_{i>j\geq n} \phi(\mathcal{F}_0, \sigma(\xi_i, \xi_j))\ .$$

It follows from the definition that if $\phi_1(n) \to 0$ then condition (6) is satisfied; if the variables $(\xi_i)_{i\in\mathbf{Z}}$ are bounded and adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$, the condition $\phi_2(n) \to 0$ implies Condition (12); moreover Condition (11) holds as soon as

$$\sum_{k>0} \phi_1(k) < \infty\,. \tag{14}$$

As a consequence all our results hold for innovations satisfying both $\phi_2(n) \to 0$ and Condition (14).

We can also consider functions of mixing sequences as innovations. Let $(\varepsilon_i)_{i\in\mathbf{Z}} = (\varepsilon_0 \circ T^i)_{i\in\mathbf{Z}}$ be a stationary sequence of ϕ-mixing random variables. Starting from the definition (13), we denote by $\phi_\varepsilon(n)$ the coefficient $\phi_\varepsilon(n) = \phi(\sigma(\varepsilon_i, i \le 0), \sigma(\varepsilon_i, i \ge n))$. Let H be a function from $A^{\mathbf{Z}}$ to $\mathbf{R}$ satisfying the condition

$$(*): \quad \text{for any } x, y \text{ in } A^{\mathbf{Z}}, \quad |H(x) - H(y)| \le \sum_{i\in\mathbf{Z}} \Delta_i \mathbf{1}_{x_i \ne y_i}, \quad \text{where } \sum_{i\in\mathbf{Z}} \Delta_i < \infty.$$

Define the stationary sequence ξ_k by

$$\xi_k = H((\varepsilon_{k-i})_{i\in\mathbf{Z}}) - \mathbf{E}(H((\varepsilon_{k-i})_{i\in\mathbf{Z}}))\,. \tag{15}$$

Note that ξ_k is bounded in view of Condition $(*)$. If $\sum_{k>0} \phi_\varepsilon(k)$ is finite then (11) and (12) are satisfied and our results hold with $s^2 = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0 \xi_k)$.

We direct to Section 3 of the paper by Dedecker *et al* (2008) for the proof of this result and other examples of classes of dependent sequences for which (11) and (12) holds. The examples include functions of linear processes, contracting Markov chains, expanding maps and symmetric random walks on the circle.

3 Application to linear processes

In this section, we give the corresponding MDP results for partial sums of a linear process and then we state their functional form.

Let $(c_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2 and define X_k by (2). Define the partial sums and the partial sums process by $S_n = \sum_{j=1}^n X_j$ and $W_n(t) = \sum_{i=1}^{[nt]} X_i$.

In general, the covariances of $(X_k)_{k\in\mathbf{Z}}$ might not be summable so that the linear process might exhibit long range dependence, and therefore the variance of S_n may not be linear in n. As a matter of fact, as a consequence of Lemma A (iii) in Peligrad and Utev (2006), it turns out that when the innovations have a continuous spectral density $f(x)$, the variance of S_n is asymptotically proportional to $2\pi f(0) s_n^2$, where

$$s_n^2 = \sum_{j\in\mathbf{Z}} b_{n,j}^2 \quad \text{with } b_{n,j} = c_{1-j} + \ldots + c_{n-j}\,. \tag{16}$$

In particular we point out the following fact:

Remark 4 *According to Corollary 2 in Peligrad and Utev (2006), if ξ_0 is regular, Condition (9) implies that*

$$\mathrm{Var}(S_n)/s_n^2 \to s^2,$$

where s^2 is given by (10).

To give an example of a linear process we mention the fractionally integrated processes since they play an important role in financial econometrics, climatology and so on and they are widely studied. Such processes are defined for $-1/2 < d < 1/2$ by

$$X_k = (1-B)^{-d}\xi_k = \sum_{i\geq 0} c_i \xi_{k-i} \text{ where } c_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i)}\,, \tag{17}$$

where B is the lag operator and $(\xi_i)_{i\in\mathbf{Z}}$ is a strictly stationary sequence. In case when the innovations satisfy Condition (9) and $0 < d < 1/2$, the covariances of $(X_k)_{k\in\mathbf{Z}}$ are not summable, the variance of partial sums is asymptotically proportional to n^{2d+1} and the linear process exhibits long range dependence.

Our first result gives the MDP for a linear process whose innovations satisfy projective criteria.

Theorem 5 *Let $(\xi_i)_{i\in\mathbf{Z}}$ and $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ be as in Theorem 3. Construct X_k as in (2) with $(c_i)_{i\in\mathbf{Z}}$ in ℓ^2 and assume* $\mathrm{Var}(S_n) \to \infty$, $a_n \to 0$ *and there is $(u_{n,j}) \approx (b_{n,j})$ satisfying*

$$\frac{\max_{j\in\mathbf{Z}} |u_{n,j}|}{\sqrt{a_n}s_n} \to 0, \text{ as } n \to \infty\ . \tag{18}$$

Then, $\{s_n^{-1}S_n\}$ satisfies the MDP with speed a_n and rate $I(t) = t^2/(2s^2)$.

Notice that, by the proof of Corollary 2.1 in Peligrad and Utev (1997), we have

$$\frac{\max_{j\in\mathbf{Z}} |b_{n,j}|}{s_n} \to 0 \text{ as } n \to \infty\,.$$

Therefore in all the situations there are sequences $a_n \to 0$ that satisfy Condition (18).

Theorem 5 appears to be new even for the martingale case when we have the following corollary.

Corollary 6 *Let $(d_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of nondegenerate martingale differences adapted the stationary filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. Assume $\|d_0\|_\infty < \infty$ and that (6) holds. Then, under Condition (18) the conclusion of Theorem 5 holds.*

We turn now to functional moderate deviations. We say that the process $\{s_n^{-1}W_n, n > 0\}$ satisfies the Moderate Deviation Principle (MDP) in $D[0,1]$ (functions on $[0,1]$ with left-hand limits and continuous from the right) with speed $a_n \to 0$ and good rate function $I(.)$, if the level sets $\{x, I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$, and for all Borel sets $\Gamma \in \mathcal{B}$

$$\begin{aligned} -\inf_{t\in\Gamma^0} I(t) &\leq \liminf_n a_n \log P(\sqrt{a_n}s_n^{-1}W_n \in \Gamma) \\ &\leq \limsup_n a_n \log P(\sqrt{a_n}s_n^{-1}W_n \in \Gamma) \leq -\inf_{t\in\bar{\Gamma}} I(t)\ . \end{aligned} \tag{19}$$

Next theorem provides the functional form of Theorem 5. We shall require that s_n^2 is regularly varying with exponent $\beta \in]0,2]$ and the sequence of numbers (a_n) satisfies the following mild regularity assumption:

(A) There is a positive real number p such that $(n^p a_n)$ is nondecreasing.

Theorem 7 *Assume that all the conditions of Theorem 5 are satisfied and in addition that there exists $\beta \in]0,2]$ such that for any $t \in]0,1]$,*

$$\frac{s_{[nt]}^2}{s_n^2} \to t^\beta \quad \text{as } n \to \infty \,. \tag{20}$$

Assume that (A) holds. Then, the process $\{s_n^{-1} W_n(t), t \in [0,1]\}$ satisfies the MDP in $D[0,1]$ with speed a_n and the rate function $I_{s^2}^{(\beta)}(\cdot)$, inherited from the fractional Brownian motion with Hurst index $\frac{\beta}{2}$ (see Appendix).

There are situations when additional information on the sequence of constants is available, so we mention the following observation:

Remark 8 *The fact that a_n is assumed to satisfy the regularity condition (A) is not needed if instead of (18) we request that for any $t \in]0,1]$*

$$\frac{\max_{j \in \mathbf{Z}} |u_{[nt],j}|}{\sqrt{a_n} s_n} \to 0 \quad \text{as } n \to \infty \,. \tag{21}$$

If we impose some degrees of regularity such as

$$\max_{j \in \mathbf{Z}} |b_{n,j}| = O(s_n/\sqrt{n}) \,, \tag{22}$$

then, Condition (18) and also its stronger form (21) hold under the simple condition $na_n \to \infty$. Notice that if the linear coefficients $(c_i)_{i \in \mathbf{Z}}$ satisfy the assumption 2.3 in Ghosh and Samorodnitsky (2007), then they satisfy (22). It follows that Theorem 7 extends the point (iii) of Theorem 2.4 in Ghosh and Samorodnitsky (2007) to the case where the innovations are not necessarily independent but, as a counterpart, assumed to be bounded.

For the sake of applications, by using the notion of equivalent sequences, we shall further point out several important sequences of constants that give MDP under regularity conditions easy to verify.

The first corollary treats the case $\beta = 1$ in Condition (20), under a recent condition introduced by Wu and Woodroofe (2004).

Corollary 9 *Let $(\xi_i)_{i \in \mathbf{Z}}$ and $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ be as in Theorem 3. Let $(c_i)_{i \in \mathbf{Z}}$ be a sequence of real numbers in ℓ^2 such that $c_i = 0$ for $i < 0$. Let $b_j = c_0 + \ldots + c_j$. Define $(X_k)_{k \geq 1}$ and $W_n(t)$ as above and assume that*

$$\sum_{k=0}^{n-1} b_k^2 \to \infty, \quad \text{as } n \to \infty, \tag{23}$$

and that

$$\sum_{j=0}^{\infty}(b_{n+j}-b_j)^2 = o\,\Big(\sum_{k=0}^{n-1} b_k^2\Big)\,. \tag{24}$$

Assume that $a_n \to 0$ and $na_n \to \infty$. Then the process $\{s_n^{-1}W_n(t), t \in [0,1]\}$ satisfies the MDP in $D[0,1]$ with speed a_n and the good rate function $I_{s^2}(\cdot)$, inherited from the Brownian motion (see Appendix).

It is interesting to notice that the following condition, given in Hall in Heyde (1980) page 146:

$$\sum_{n\geq 1}\Big(\sum_{j\geq n} c_j\Big)^2 < \infty\ ,$$

implies Condition (24). Hence Condition (24) allows for the following possibility: $\sum_{i=0}^{n}|c_i|$ diverges but $\sum_{i=0}^{n} c_i$ converges. For instance if for $n \geq 1$, $c_n = (-1)^n u_n$ for some sequence $(u_n)_{n\geq 1}$ of positive coefficients decreasing to zero, such that $\sum_{n\geq 1} u_n = \infty$, then Corollary 11 applies as soon as $\mathrm{Var}(S_n) \to \infty$ and $\sum_{n\geq 1} u_n^2 < \infty$, which is a minimal condition.

Another interesting situation covered by Corollary (9) is when $c_k = \ell(k)/k$ for $k \geq 1$, where $\ell(n)$ is a slowly varying function at infinite (this means $\ell(xn)/\ell(n) \to 1$ as $n \to \infty$, for all $x > 0$) and with $\sum_{k=1}^{\infty}|c_k| = \infty$ (for exact computations see Example 1 and Lemma 4 in Wu and Min (2005)).

Second corollary applies to many other regular situations when $\beta \in]0,2]$ in Condition (20).

Corollary 10 *Let $(\xi_i)_{i\in\mathbf{Z}}$ and $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ be as in Theorem 3. Let $(c_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2 such that $c_i = 0$ for $i < 0$ and satisfying either*

$$n\sum_{k\geq n} c_k^2 = O(\frac{s_n^2}{n})\,,\ \text{as } n \to \infty \tag{25}$$

or

$$\Big|\sum_{k=0}^{n} c_k\Big|\ \text{ is decreasing to } 0. \tag{26}$$

Assume in addition that (20) holds for a $\beta \in]0,2]$. Then, for all sequences $a_n \to 0$ such that $na_n \to \infty$, the process $\{s_n^{-1}W_n(t), t \in [0,1]\}$ satisfies the functional MDP of Theorem 7.

This corollary is useful for a variety of applications. In all the examples given below both conditions (20) and (25) are satisfied. Therefore the conclusion of Theorem 7 holds for suitable innovations.

Example 1. For instance if we consider that the linear process X_k is defined by (17) with $0 < d < 1/2$ and the innovations satisfy the assumptions of Theorem 3, then Corollary 10 applies with $\beta = 2d + 1$ since $c_k \sim \kappa_d k^{d-1}$ ($\kappa_d > 0$).

Example 2. Now, if we consider the following selection of $(c_k)_{k\geq 0}$: $c_0 = 1$ and $c_i = (i+1)^{-\alpha} - i^{-\alpha}$ for $i \geq 1$ with $\alpha \in]0, 1/2[$, then Corollary 10 also applies. Indeed for this selection, $s_n^2 \sim \kappa_\alpha n^{1-2\alpha}$ and $\big|\sum_{k=0}^n c_k\big| = (n+1)^{-\alpha}$.

Example 3. For the selection $c_i \sim i^{-\alpha}\ell(i)$ where ℓ is a slowly varying function at infinity and $1/2 < \alpha < 1$ then, $s_n^2 \sim \kappa_\alpha n^{3-2\alpha}\ell^2(n)$ (see for instance Relations (12) in Wang *et al.* (2003)), where κ_α is a positive constant depending on α. It is easy to see that by the properties of slowly varying functions, Condition (25) is verified.

Example 4. Finally, if $c_i \sim i^{-1/2}(\log i)^{-\alpha}$ for $\alpha > 1/2$, then $s_n^2 \sim n^2(\log n)^{1-2\alpha}/(2\alpha - 1)$ (see Relations (12) in Wang *et al.* (2003)). Hence (20) is satisfied with $\beta = 2$ and also (25) follows.

4 Proofs

4.1 An exponential inequality for linear statistics

Next lemma gives an upper bound for the exponential moment of linear statistics of type (1). The proof is omitted since it follows the arguments of Lemma 22 in Dedecker *et al* (2008) which treats the case of the maximum of a finite partial sum. The treatment of the infinite sum is easily justified by Remark 2 and Fatou lemma.

Lemma 11 *Let $(Y_k)_{k\in\mathbf{Z}}$ be a sequence of random variables such that for all k, $\mathbf{E}(Y_k|\mathcal{F}_{-\infty}) = 0$ almost surely and Y_k is $\mathcal{F}_\infty$-measurable. Assume that*

$$\|P_{k-j}(Y_k)\|_\infty \leq p_j \quad \text{and} \quad D := \sum_{j=-\infty}^{\infty} p_j < \infty\,.$$

Let $(c_{m,j})_{j\in\mathbf{Z},m\geq 1}$ be a double indexed sequence of numbers such that for all $m \geq 1$, $\sum_{j\in\mathbf{Z}} c_{m,j}^2 < \infty$. Define $S_m = \sum_{j\in\mathbf{Z}} c_{m,j} Y_j$. Then for all $m \geq 1$ and all $t \in \mathbf{R}$,

$$\mathbf{E}\exp(tS_m) \leq \exp\big(\tfrac{t^2}{2} D^2 \sum_{j\in\mathbf{Z}} c_{m,j}^2\big)\,. \tag{27}$$

4.2 Martingale approximation for linear statistics

For all $j \in \mathbf{Z}$, let $d_j = \sum_{\ell\in\mathbf{Z}} P_j(\xi_\ell)$. Clearly $(d_j)_{j\in\mathbf{Z}}$ is a stationary sequence of martingale differences with respect to the filtration $(\mathcal{F}_j)_{j\in\mathbf{Z}}$.

Lemma 12 *For any positive integer n, let $(c_{n,i}, -\infty \le i \le \infty)$ be a triangular array of numbers satisfying (7). If $\sum_j ||P_0(\xi_j)\|_\infty < \infty$, then for all $\epsilon > 0$, $n \ge N_\epsilon$ and $t \in \mathbf{R}$,*

$$\mathbf{E}(\exp t(\sum_{i=-\infty}^{\infty} c_{n,i}(\xi_i - d_i)) \le c\exp(\epsilon(t^2)) \ \ \textit{for all } t > 0.$$

Proof of Lemma 12. Fix a positive integer m and define

$$\theta_{0,m} = \sum_{k=0}^{2m-2} \sum_{i=k-m+1}^{m-1} P_i(\xi_k) \ \ , \ \ \theta_{j,m} = T^j(\theta_{0,m})$$

Observe that $\theta_{j,m}$ is well defined and

$$\|\theta_{0,m}\|_\infty = \| \sum_{k=0}^{2m-2} \sum_{i=k-m+1}^{m-1} P_i(\xi_k)\|_\infty \le 2m \sum_{i \in \mathbf{Z}} \|P_0(\xi_i)\|_\infty < \infty \ \ .$$

Simple computations lead to the decomposition

$$\sum_{i=-m+1}^{m-1} P_i(\xi_0) - \sum_{\ell=1}^{2m-1} P_m(\xi_\ell) = \theta_{0,m} - \theta_{1,m}$$

implying that

$$\xi_0 - T^m(\sum_k P_0(\xi_k)) = \theta_{0,m} - \theta_{1,m} + \sum_{|i|\ge m} P_i(\xi_0) - T^m(\sum_{|k|\ge m} P_0(\xi_k)) \ .$$

With our notation, ($d_0 = \sum_k P_0(\xi_k)$) we obtain

$$\xi_0 - d_0 = T^m(d_0) - d_0 + \theta_{0,m} - \theta_{1,m} + \sum_{|i|\ge m} P_i(\xi_0) - T^m(\sum_{|k|\ge m} P_0(\xi_k)) \, . \quad (28)$$

We have to estimate the exponential moments of $t\sum_{i=-\infty}^{\infty} c_{n,i}(\xi_i - d_i)$. We shall treat the terms from the error of approximation separately. First notice that

$$R_1 := \sum_{j=-\infty}^{\infty} c_{n,j}(\ T^m(d_j) - d_j) = \sum_{j=-\infty}^{\infty} (c_{n,j-m} - c_{n,j})d_j \, .$$

According to Lemma 11, $\mathbf{E}(\exp(tR_1)) \le \exp(t^2\|d_j\|_\infty^2 \sum_{j=-\infty}^{\infty}(c_{n,j} - c_{n,j-m})^2) \le \exp(\epsilon(t^2))$ for all n sufficiently large, since by (7) we have that $\sum_{j=-\infty}^{\infty}(c_{n,j} - c_{n,j-m})^2 \to 0$ as $n \to \infty$ and also d_j is bounded.

To treat the second difference in the error, notice that

$$R_2 := \sum_{i=-\infty}^{\infty} c_{n,i}(\theta_{i,m} - \theta_{i+1,m}) = \sum_{i=-\infty}^{\infty} (c_{n,i} - c_{n,i-1})\theta_{i,m} \, .$$

By the definition of $\theta_{0,m}$, we have

$$\sum_i \|P_i(\theta_{0,m})\|_\infty \le 4m^2 \sum_i \|P_i(\xi_0)\|_\infty \,.$$

Hence by Lemma 11 together with (7) we easily deduce that $\mathbf{E}\exp(tR_2) \le \exp(\epsilon(t^2))$ for n sufficiently large.

For the term $R_3 := \sum_{i=-\infty}^{\infty} c_{n,i} T^i(\sum_{|j|\ge m} P_j(\xi_0)\ )$ we apply Lemma 11 to obtain

$$\mathbf{E}\exp(tR_3) \le \exp\Big(\frac{t^2}{2}\sum_{i=-\infty}^{\infty} c_{n,i}^2 \big(\sum_{|j|\ge m}\|P_j(\xi_0)\|_\infty\big)^2\Big).$$

Therefore by Condition (11), $\mathbf{E}\exp(tR_3) \le \exp(\epsilon(t^2))$ for all n and m sufficiently large (uniformly in t).

To deal with the last term we denote by $R_4 = \sum_{i=-\infty}^{\infty} c_{n,i} T^{m+i}(\sum_{|k|\ge m} P_0(\xi_k))$ and apply again Lemma 11 that gives

$$\mathbf{E}\exp(tR_4) \le \exp\Big(\frac{t^2}{2}\sum_{i=-\infty}^{\infty} c_{n,i}^2 \,\|\sum_{|k|\ge m} P_0(\xi_k)\|_\infty^2\Big)\,.$$

We then apply the same argument as for R_3. This shows the desired approximation. $\diamond$

4.3 Proof of Theorem 1

Notice first that by Lemma 11 applied to $S_n - S_n'$ with $S_n = \sum_i c_{n,i} d_i$ and $S_n' = \sum_i u_{n,i} d_i$, we get that for any $t \in \mathbf{R}$,

$$\mathbf{E}\exp\big(t(S_n - S_n')\big) \le \exp\Big(\frac{t^2}{2}\sum_{j\in\mathbf{Z}}(c_{n,j} - u_{n,j})^2 \,(\sum_{k\in\mathbf{Z}}\|P_0(\xi_k)\|_\infty)^2\Big)\,.$$

Hence, the fact that $(u_{n,j}) \approx (c_{n,j})$ entails that for any positive λ,

$$\lim_{n\to\infty} a_n \log \mathbf{E}\exp\Big(\frac{\lambda}{\sqrt{a_n}}\big|S_n - S_n'\big|\Big) = 0\,,$$

and therefore we get that for any $\delta > 0$

$$\limsup_{n\to\infty} a_n \log \mathbf{P}\left(\sqrt{a_n}\big|S_n - S_n'\big| \ge \delta\right) = -\infty\,.$$

Hence by Theorem 4.2.13 in Dembo and Zeitouni (1998) S_n and S_n' satisfy the same moderate deviation principle. So, without restricting the generality we can assume

$$\frac{\sup_j |c_{n,j}|}{\sqrt{a_n}} \to 0 \ \text{ as } \ n \to \infty\ .$$

To prove the result, we just have to apply a MDP for a triangular array of martingale differences sequences as stated in Theorem 3.1 and Lemma 3.1 of

Puhalskii (1994), (see also Djellout, 2002, Proposition 1 and Lemma 2). Since, for all j, by Condition (8), $\|c_{n,j}d_j\|_\infty \le o(\sqrt{a_n})$, we have only to verify that for any $\delta > 0$

$$\limsup_{n\to\infty} a_n \log \mathbf{P}\left(\Big| \sum_j c_{n,j}^2 \mathbf{E}(d_j^2|\mathcal{F}_{j-1}) - \mathbf{E}(d_0^2) \Big| \ge \delta \right) = -\infty \ .$$

Taking into account the first part of Condition (7), it is sufficient to show that for any $\delta > 0$

$$\limsup_{n\to\infty} a_n \log \mathbf{P}\left(\Big| \sum_j c_{n,j}^2 \big(\mathbf{E}(d_j^2|\mathcal{F}_{j-1}) - \mathbf{E}(d_j^2)\big) \Big| \ge \delta \right) = -\infty \ . \tag{29}$$

To verify this condition we use the technique from Peligrad and Utev (2006). To further diminish the dependence we divide the variables in blocks and then we average the coefficients in each block. Let p be a fixed positive integer. For any k in $\mathbf{Z}$, define

$$t_{n,k} = p^{-1} \sum_{j=p(k-1)+1}^{pk} c_{n,j}^2 \text{ and } Z_k = t_{n,k} \sum_{i=p(k-1)+1}^{pk} (\mathbf{E}(d_i^2|\mathcal{F}_{i-1}) - \mathbf{E}(d_i^2)) \ .$$

Then,

$$\begin{aligned} & \sum_j c_{n,j}^2 (\mathbf{E}(d_j^2|\mathcal{F}_{j-1}) - \mathbf{E}(d_j^2)) - \sum_k Z_k \\ = & \sum_k \sum_{i=p(k-1)+1}^{pk} (c_{n,i}^2 - t_{n,k}) \left(\mathbf{E}(d_i^2|\mathcal{F}_{i-1}) - \mathbf{E}(d_i^2)\right) \ . \end{aligned}$$

Since for any pair of indexes $i, \ell \in [p(k-1)+1, pk]$,

$$|c_{n,i}^2 - c_{n,\ell}^2| \le p \sum_{j=p(k-1)+1}^{pk} |c_{n,j}^2 - c_{n,j-1}^2| \,,$$

we easily get

$$\begin{aligned} & |\sum_j c_{n,j}^2 (\mathbf{E}(d_j^2|\mathcal{F}_{j-1}) - \mathbf{E}(d_j^2)) - \sum_k Z_k| \\ & \le 2p^2 \|d_0\|_\infty^2 \sum_j |c_{n,j}^2 - c_{n,j-1}^2| \ . \end{aligned}$$

So, by Condition (7), the relation (29) is reduced to prove that for any $\delta > 0$

$$\limsup_{n\to\infty} a_n \log \mathbf{P}\left(|\sum_k Z_k| \ge \delta \right) = -\infty \ . \tag{30}$$

Now, denoting by $G_k = \sigma(Z_j, j \le pk)$, we have the following martingale decomposition

$$\mathbf{P}\left(|\sum_k Z_k| \ge \delta\right) \le \mathbf{P}\left(\big|\sum_k Z_k - \mathbf{E}(Z_k|G_{k-1})\big| \ge \frac{\delta}{2}\right) + \mathbf{P}\left(\big|\sum_k \mathbf{E}(Z_k|G_{k-1})\big| \ge \frac{\delta}{2}\right) . \qquad (31)$$

Notice that by stationarity and taking into account Condition (6), for a certain $p = p_\delta$ sufficiently large and fixed we have

$$|| \sum_k \mathbf{E}(Z_k|G_{k-1})||_\infty \le \big(\sum_j c_{n,j}^2\big)\ \frac{1}{p}|| \sum_{i=1}^{p} (\mathbf{E}(d_i^2|\mathcal{F}_0) - \mathbf{E}(d_i^2))||_\infty \le \frac{\delta}{4}$$

and so, the second term of (31) vanishes. Applying Azuma inequality to the martingale part, we obtain that

$$\mathbf{P}\left(\big|\sum_k Z_k - \mathbf{E}(Z_k|G_{k-1})\big| \ge \frac{\delta}{2}\right) \le 2\exp\left(\frac{-\delta^2}{32\sum_k \|Z_k\|_\infty^2}\right) .$$

We notice now that (30) holds provided

$$\frac{a_n}{\sum_k \|Z_k\|_\infty^2} \to \infty\ . \qquad (32)$$

It remains to notice that

$$\sum_k \|Z_k\|_\infty^2 \le 4\|d_0\|_\infty^4 \sum_k \Big(\sum_{i=p(k-1)+1}^{pk} c_{ni}^2\Big)^2 \le 4p\|d_0\|_\infty^4 \max_i c_{ni}^2 \sum_j c_{nj}^2\ .$$

This bound combined with (8) and the first part of (7) leads to (32). $\diamond$

4.4 Proof of Theorem 3

Let $\lambda > 0$. By Lemma 12 applied with $t = \lambda/\sqrt{a_n}$, we then obtain that for any $\lambda > 0$,

$$\lim_{n\to\infty} a_n \log \mathbf{E} \exp\Big(\frac{\lambda}{\sqrt{a_n}}|\sum_{i=-\infty}^{\infty} c_{n,i}(\xi_i - d_i)|\Big) = 0\,.$$

This shows that from the point of view of moderate deviations, the behavior of $\sum_{i=-\infty}^{\infty} c_{n,i}\xi_i$ is equivalent to the one of $\sum_{i=-\infty}^{\infty} c_{n,i}d_i$, where d_i are martingale differences with $d_0 = \sum_{k=-\infty}^{\infty} P_0(\xi_k)$. Therefore we just have to apply

Theorem 1 to the martingale part. Hence we have to verify

$$\lim_n \| \frac{1}{n} \mathbf{E}((\sum_{i=1}^n d_i)^2 | \mathcal{F}_0) - s^2 \|_\infty = 0 \,, \tag{33}$$

since by Remark 2, $\mathbf{E}(d_0^2) = s^2$. Now by Condition (11), we have for all $j \in \mathbf{Z}$,

$$\lim_{m\to\infty} \|d_j - \sum_{k=j-m+1}^{j+m+1} P_j(\xi_k)\|_\infty = 0 \,,$$

and we notice that for any $1 \leq m \leq n$

$$\sum_{j=1}^n \Big(\sum_{k=j-m+1}^{j+m-1} P_j(\xi_k) - \sum_{k=j+1}^{j+2m-1} P_{j+m}(\xi_k) \Big) \\ = \sum_{j=1}^m \sum_{k=j-m+1}^{j+m-1} P_j(\xi_k) - \sum_{j=n+1}^{n+m} \sum_{k=j-m+1}^{j+m-1} P_j(\xi_k) \,.$$

Using Condition (11), we then derive that (33) holds if

$$\lim_{m\to\infty} \limsup_{n\to\infty} \| \frac{1}{n} \mathbf{E}((\sum_{i=1}^n d_{i,m})^2 | \mathcal{F}_0) - s^2) \|_\infty = 0 \,, \tag{34}$$

where $d_{j,m} = \sum_{k=j+1}^{j+2m-1} P_{j+m}(\xi_k)$. Noticing that $d_{j,m} = \theta_{j+1,m} - \mathbf{E}(\theta_{j+1,m} | \mathcal{F}_{j+m-1})$ by using the notation of the proof of Lemma 12, we can follow the lines of the end of the proof of Theorem 3 in Dedecker *et al.* (2008) to infer that (34) holds as soon as

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big\| \frac{1}{n} \mathbf{E}\Big(\sum_{i=2m-1}^n \xi_i^2 + 2 \sum_{i=2m-1}^n \sum_{j=i+1}^{(N+i)\wedge n} \xi_i \xi_j | \mathcal{F}_0 \Big) - v_N^2 \Big\|_\infty = 0 \,, \tag{35}$$

where $v_N^2 = \mathbf{E}(\xi_0^2) + 2\mathbf{E}(\xi_0\xi_1) + \cdots + 2\mathbf{E}(\xi_0\xi_{N-1})$. To finish the proof, it suffices to notice that (35) is implied by (12). Hence by using Theorem 1, we get that $\{\sum_{j\in\mathbf{Z}} c_{nj} d_j\}$ satisfies the MDP with speed a_n and rate $I(t) = t^2/(2s^2)$.

4.5 Proof of Theorem 5

We first notice that $X_k = \sum_{i=-\infty}^{\infty} c_{k-i}\xi_i$. Hence we can write

$$S_n = \sum_{k=1}^n X_k = \sum_{i=-\infty}^{\infty} (c_{1-i} + ... + c_{n-i})\xi_i = \sum_{i=-\infty}^{\infty} b_{n,i}\xi_i \,.$$

Theorem 5 is then an immediate consequence of Theorem 3 by setting $c_{n,j} = b_{n,j}/s_n$ and applying Lemma A.1 in Peligrad and Utev (2006).

4.6 Proof of Theorem 7

The proof is divided in two steps. First we prove that the finite dimensional laws satisfy the MDP with an appropriate good rate function, and after we prove tightness in the sense of the moderate deviation principle.

Step 1. MDP for the finite dimensional laws. We show first that for m a fixed integer and for any $0 < t_1 < t_2 < \cdots < t_m \leq 1$, the m-tuple

$$\Big(\frac{1}{s_n}\sum_{k=1}^{[nt_1]} X_k, \frac{1}{s_n}\sum_{k=1}^{[nt_2]} X_k, \dots, \frac{1}{s_n}\sum_{k=1}^{[nt_m]} X_k\Big)$$

satisfy the MDP in $\mathbf{R}^m$ with speed a_n and rate function given by

$$I_{t_1,\cdots,t_m}(u_1,\cdots,u_m) = \sup_{\lambda_1,\cdots,\lambda_m \in \mathbf{R}} \Big(\sum_{j=1}^m \lambda_j u_j - \frac{s^2}{2}\sum_{j,k=1}^m \lambda_j\lambda_k R(t_j,t_k)\Big), \quad (36)$$

where $R(s,t) = \frac{1}{2}([t^\beta + s^\beta - |t-s|^\beta]$.

With this aim, we shall apply the Cramér-Wold device. For all integer $1 \leq \ell \leq m$, let $n_\ell = [nt_\ell]$, and for $\lambda_1,\dots,\lambda_m \in \mathbf{R}$, let us consider the decomposition

$$\frac{\sum_{\ell=1}^m \lambda_\ell S_{n_\ell}}{s_n} = \Lambda_{m,\beta}\sum_{j\in\mathbf{Z}} c_{n,j}\xi_j\,, \quad (37)$$

where $\Lambda^2_{m,\beta} = \frac{1}{2}\sum_{\ell,k=1}^m \lambda_\ell\lambda_k\big(t_\ell^\beta + t_k^\beta - |t_k - t_\ell|^\beta\big)$ and $c_{n,j}$ is defined by

$$c_{n,j} = \frac{1}{\Lambda_{m,\beta}}\sum_{\ell=1}^m \frac{\lambda_\ell b_{n_\ell,j}}{s_n}\,. \quad (38)$$

We apply Theorem 3 to $c_{n,j}$ and the ξ_j defined as $\Lambda_{m,\beta}\xi_j$. We have first to calculate the limit over n of the following quantity

$$\sum_{j\in\mathbf{Z}} c^2_{n,j} = \frac{1}{\Lambda^2_{m,\beta}}\frac{\sum_{j\in\mathbf{Z}}\sum_{\ell=1}^m\sum_{k=1}^m \lambda_\ell\lambda_k b_{n_\ell,j}b_{n_k,j}}{s_n^2}\,.$$

For any $1 \leq \ell \leq k \leq m$, by using the fact that for any two real numbers A and B we have $A(A+B) = (A^2 + (A+B)^2 - B^2)/2$, we get that

$$\begin{aligned}\frac{1}{s_n^2}\sum_{j\in\mathbf{Z}} b_{n_\ell,j}b_{n_k,j} &= \frac{1}{2s_n^2}\sum_{j\in\mathbf{Z}}\big(b^2_{n_\ell,j} + b^2_{n_k,j} - \big(b_{n_\ell,j} - b_{n_k,j}\big)^2\big) \\ &= \frac{1}{2s_n^2}\sum_{j\in\mathbf{Z}}\big(b^2_{n_\ell,j} + b^2_{n_k,j} - b^2_{n_k-n_\ell,j}\big)\,.\end{aligned}$$

Hence by using Condition (20), we derive that, for any $1 \leq \ell \leq k \leq m$,

$$\frac{\sum_{j\in\mathbf{Z}} b_{n_\ell,j}b_{n_k,j}}{s_n^2} \to \frac{1}{2}\big(t_\ell^\beta + t_k^\beta - (t_k - t_\ell)^\beta\big)\,. \quad (39)$$

It follows from (39) that

$$\lim_{n\to\infty}\sum_{j\in\mathbf{Z}} c_{nj}^2 = 1\,. \tag{40}$$

Hence the first part of Condition (7) holds. On an other hand, by using Lemma A.1 in Peligrad and Utev (2006), the second part of the condition (7) is satisfied. Now, by our assumption, there is $(u_{n,j}) \approx (b_{n,j})$ such that (18) is satisfied. Hence, by simple computations, the sequence $(v_{n,j})$ defined by

$$v_{n,j} = \frac{1}{\Lambda_{m,\beta}}\sum_{\ell=1}^{m}\frac{\lambda_\ell u_{n_\ell,j}}{s_n}\,,$$

is such that $(v_{n,j}) \approx (c_{n,j})$. In addition,

$$\frac{\max_j |v_{n,j}|}{\sqrt{a_n}} \le \frac{1}{\Lambda_{m,\beta}}\sum_{\ell=1}^{m}|\lambda_\ell|\frac{\max_j |u_{n_\ell,j}|}{\sqrt{a_n}s_n}\,.$$

We just have to notice now that for all $1 \le \ell \le m$,

$$\frac{\max_j |u_{[nt_\ell],j}|}{\sqrt{a_n}s_n} = \frac{\max_j |u_{[nt_\ell],j}|}{\sqrt{a_{[nt_\ell]}}s_{[nt_\ell]}}\,\frac{s_{[nt_\ell]}}{s_n}\,\frac{n^{p/2}\sqrt{a_{[nt_\ell]}}}{n^{p/2}\sqrt{a_n}}\,,$$

which converges to zero by (18) combined with (20) and assumption (A). Hence, applying Theorem 3, the convergence of finite dimensional distributions is proved.

Step2. Tightness. We turn now to prove tightness in the sense of the moderate deviations. With this aim, we want to show that for any $\epsilon > 0$,

$$\limsup_{\delta\to 0}\lim_{n\to\infty} a_n \log \mathbf{P}\big(\frac{\sqrt{a}_n}{s_n}\max_{1\le k\le [n\delta]}|S_k| \ge \epsilon\big) = -\infty\,.$$

This clearly holds if for any $\lambda > 0$,

$$\limsup_{\delta\to 0}\lim_{n\to\infty} a_n \log \mathbf{E}\exp\big(\frac{\lambda}{\sqrt{a}_n s_n}\max_{1\le k\le [n\delta]}|S_k|\big) = 0\,. \tag{41}$$

Applying Lemma 11, we derive that for any $k \ge 1$ and any $t > 0$,

$$\mathbf{E}\exp(t|S_k|) \le 2\exp\Big(t^2 D^2 \sum_i b_{k,i}^2\Big) = 2\exp\Big(t^2 D^2 s_k^2\Big)\,,$$

where $D = \sum_{j\ge 1}\|P_0(\xi_j)\|_\infty$. By taking into account Condition (20) and Remark 17, we can apply Lemma 15 to derive that there exist $n_0 \ge 1$ and constants $A \ge 1$ and $B \ge 1$ depending only on n_0 such that, for all $n \ge n_0$,

$$\mathbf{E}\exp\big(\frac{\lambda}{\sqrt{a}_n s_n}\max_{1\le k\le [n\delta]}|S_k|\big) \le A\exp\Big(\frac{\lambda^2 B D^2}{a_n}\,\frac{s_{[n\delta]}^2}{s_n^2}\Big)\,,$$

Since by (20) $s_{[n\delta]}^2/s_n^2 \to \delta^\beta$, (41) follows by taking $\limsup_n$ followed by $\delta \to 0$.

End of the proof. According to Theorem 3.2 in Arcones (2003-a), it follows from the steps 1 and 2, that the process $s_n^{-1} W_n$ satisfies the MDP in $D[0,1]$ with speed a_n and the good rate function given by $I_{s^2}^{(\beta)}(z)$. The exact expression of these rate functions can be found in Arcones (2003-b) and are recalled in the appendix (see Section 5.2). $\diamond$

4.7 Proof of Corollary 9

We first start with a technical lemma which is a consequence of the relation (6) in the proof of Theorem 1 in Wu and Woodroofe (2004).

Lemma 13 *Let $(c_i)_{i\geq 0}$ be a sequence of real numbers in ℓ^2. Let $b_j = c_0 + \ldots + c_j$. Under (23) and (24), we have that*

$$\Big(\sum_{k=0}^{n-1} b_k^2\Big)^{-1} \sum_{k=0}^{n-1} \big(b_k - \frac{1}{n}\sum_{j=0}^{n-1} b_j\big)^2 \to 0\,, \tag{42}$$

$$\sum_{k=0}^{n-1} b_k^2 \sim nh(n) \text{ where } h(n) \text{ is a slowly varying function}\,, \tag{43}$$

and

$$|\sum_{j=0}^{n-1} b_j|^2 \sim n \sum_{k=0}^{n-1} b_k^2\,. \tag{44}$$

Under (23) and (24), we have $s_n^2 \sim \sum_{k=0}^{n-1} b_k^2$. Hence setting $u_{n,j} = \frac{1}{n}\sum_{k=0}^{n-1} b_k$ for $j \in [0, n-1]$ and 0 otherwise, the condition (4) is satisfied by taking into account (24) and (42). Now by (44), we get that the condition (21) holds provided $na_n \to \infty$. We finish the proof by applying Theorem 7. $\diamond$

4.8 Proof of Corollary 10

This corollary is based on the following useful lemma that can be also combined with Theorem 5.

Lemma 14 *Let $(c_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2 such that $c_i = 0$ for $i < 0$. Assume that s_n^2 defined in (16) is eventually non-decreasing and for any sequence (p_n), such that $p_n \to \infty$ with $p_n/n \to 0$, we have $s_{p_n}^2/s_n^2 \to 0$. Moreover assume that either (25) or (26) holds. Then, conditions (18) and also (21) are satisfied as soon as $a_n \to 0$ and $na_n \to \infty$.*

The key to this result is also a construction of an equivalent sequence. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of integers decreasing to zero, such that $\varepsilon_1 < 1/2$ and satisfying

$$\frac{1}{\varepsilon_n} = o(\sqrt{na_n})\,.$$

Let $p_n = \varepsilon_n n$. Define a sequence $u_{n,j}$ as follows:

$$u_{n,j} = \begin{cases} b_{n,j} & \text{for } j \in [1, n-p_n] \\ 0 & \text{for } j \in [n-p_n+1, n] \\ b_{n,j} - b_{p_n,j} & \text{for } j \leq 0\,. \end{cases}$$

It follows that

$$\begin{aligned} \sum_{j=-\infty}^{\infty} (b_{n,j} - u_{n,j})^2 &= \sum_{j=n-p_n+1}^{n} (c_0 + \ldots + c_{n-j})^2 + \sum_{j \leq 0} b_{p_n,j}^2 \\ &= \sum_{j=0}^{p_n - 1} (c_0 + \ldots + c_j)^2 + \sum_{j \leq 0} b_{p_n,j}^2 = s_{p_n}^2\,. \end{aligned}$$

Now we have that

$$\begin{aligned} &\max_{j \in \mathbf{Z}} |u_{n,j}| \leq \max_{j \in [1, n-p_n]} |c_0 + \ldots + c_{n-j}| + \max_{j \geq 0} |c_{p_n+1+j} + \ldots + c_{n+j}| \\ &\quad \leq |c_0 + \ldots + c_{p_n-1}| + \max_{j \in [p_n, n-1]} |c_{p_n} + \ldots + c_j| + \max_{j \geq 0} |c_{p_n+1+j} + \ldots + c_{n+j}| \\ &\quad \leq |b_{p_n}| + 2 \max_{j \in [p_n, n-1]} |c_{p_n} + \ldots + c_j| + \max_{j \geq 0} |c_{p_n+1+j} + \ldots + c_{n+j}| \\ &\quad \leq |b_{p_n}| + 3 \max_{j \geq p_n} \max_{j \leq k \leq j+n} |c_j + \ldots + c_k|\,, \end{aligned}$$

where $b_j = c_0 + \ldots + c_j$ for $j \geq 0$. To treat $|b_{p_n}|$, by the definition of s_n^2 and the selection of p_n we write

$$\begin{aligned} b_{p_n}^2 &= \frac{1}{n-p_n} \sum_{j=p_n}^{n-1} b_{p_n}^2 \leq \frac{4}{n} \sum_{j=p_n}^{n} b_j^2 + \frac{2}{n-p_n} \sum_{j=p_n}^{n-1} (b_{p_n} - b_j)^2 \\ &\leq \frac{4 s_n^2}{n} + 2 \max_{p_n+1 \leq j \leq n} (c_{p_n+1} + \ldots + c_j)^2\,. \end{aligned}$$

By Cauchy-Schwarz inequality we easily obtain

$$\max_{j \geq p_n} \max_{j \leq k \leq j+n} (c_j + ... + c_k)^2 \leq (n+1) \sum_{j=p_n}^{\infty} c_j^2$$

and

$$\max_{p_n+1 \leq j \leq n} (c_{p_n+1} + \ldots + c_j)^2 \leq n \sum_{j=p_n}^{\infty} c_j^2\,.$$

Overall

$$\max_{j \in \mathbf{Z}} u_{n,j}^2 \leq \frac{8 s_n^2}{n} + 40 n \sum_{j=p_n}^{\infty} c_j^2\,.$$

Therefore by the fact that s_n^2 is eventually nondecreasing, and by using Condition (25), we obtain for all n sufficiently large

$$\begin{aligned} \frac{\max_{j\in\mathbf{Z}} u_{n,j}^2}{a_n s_n^2} &\leq \frac{8}{a_n n} + K\frac{n\ s_{p_n}^2}{a_n p_n^2 s_n^2} \\ &= o(1) + K\frac{1}{a_n n\ \varepsilon_n^2}\,, \end{aligned}$$

which converges to zero by the selection of $(\varepsilon_n)_{n\geq 1}$. Hence (18) holds. A similar computation taking into account the fact that ε_n is decreasing shows that also Condition (21) is satisfied.

To prove this lemma under condition (26) we have just to notice that

$$\max_{j\in\mathbf{Z}} |u_{n,j}|^2 \leq 9|c_0 + ... + c_{p_n}|^2 = 9\frac{1}{p_n}\sum_{j=0}^{p_n}(\sum_{k=0}^{p_n} c_k)^2$$
$$\leq 9\frac{1}{p_n}\sum_{j=0}^{p_n}(\sum_{k=0}^{j} c_k)^2 \leq 9\ \frac{s_{p_n}^2}{p_n}\,.$$

Consequently,

$$\frac{\max_{j\in\mathbf{Z}} u_{n,j}^2}{a_n s_n^2} \leq C\frac{s_{p_n}^2}{a_n p_n s_n^2} \ \leq \frac{C'}{a_n p_n}\,.$$

The result follows by the selection of p_n .$\diamond$

5 Appendix

5.1 A General maximal inequality.

Denote $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1\leq i\leq n}|S_i|$. The next lemma deals with maximum exponential estimates and is adapted from Theorem 2.1 in Móricz, Serfling and Stout (1982) which cannot be used directly in our context since we only have that s_n^2 is equivalent to a regularly varying function.

Lemma 15 *Assume that $(X_i)_{i\in\mathbf{Z}}$ is a stationary sequence and there is a constant $K\geq 1$, and a positive sequence of numbers $s(i)$ such that for all $t > t_0 \geq 0$ and all $1\leq i\leq n$*

$$\mathbf{E}(\exp(t|S_i|)) \leq K\exp(f(t)s(i))\,. \qquad (45)$$

Assume that the sequence $s(i)$ is such that there exists $Q\in]0,1[$ satisfying

$$s(n) \leq Q\big(s(2n)\wedge s(2n+1)\big) \text{ and } s(n+1)\leq Qs(2n+1) \text{ for all } n\geq n_0\ . \qquad (46)$$

In addition, suppose that $f(t) > 0$ for $t > t_0$ and for each constant $v > 1$,

$$\sup_{t>t_0} f(vt)/f(t) \leq d(v) < \infty \text{ with } \lim_{v\to 1^+} d(v) = 1\,. \qquad (47)$$

Then there exist constants $A \geq 1$ and $B \geq 1$ independent on both t and n such that for all $n \geq 1$ and $t > t_0$

$$\mathbf{E}(\exp(tM_n)) \leq AK \exp(Bf(t)s(n)) . \tag{48}$$

Remark 16 *If $f(t) = t^\gamma$ for $\gamma \geq 0$ then the function satisfies (47) for all $t > 0$ and then the result holds for all $t > 0$.*

Remark 17 *The condition (46) holds for a large class of sequences $s(n)$. For instance they are valid if $s(n)$ is regularly varying. Indeed assume that $s(n)$ satisfies: for any $t \in [0, 1]$,*

$$\frac{s([nt])}{s(n)} \to t^\alpha \text{ with } \alpha > 0 , \tag{49}$$

then, by Karamata representation in Seneta (1976) page 2, we have

$$\lim_{n\to\infty} \frac{s(n+1)}{s(n)} = 1 .$$

Then by applying (49) with $t = 2$ we get

$$\lim_{n\to\infty} \frac{s(n)}{s(2n)} = \lim_{n\to\infty} \frac{s(n)}{s(2n+1)} = \lim_{n\to\infty} \frac{s(n+1)}{s(2n+1)} = \frac{1}{2^\alpha} < 1$$

and (46) holds for $n \geq n_0$ with $1/2^\alpha < Q < 1$.

Proof of Lemma 15. The proof follows the lines of the proof of Theorem 2.1 in Móricz, Serfling and Stout (1982). Choose $q > 1$, such that

$$d(q) \leq Q^{-1} .$$

Define $C = \max_{1\leq k\leq 2n_0} \max_{1\leq i\leq k} s(i)/s(k)$, and so

$$s(i) \leq Cs(k) \text{ for all } 1 \leq i \leq k \leq 2n_0 . \tag{50}$$

Set

$$A = 2^p \text{ and } B = \max\left\{C \max_{1\leq k\leq 2n_0} d(k), d(p)\right\} ,$$

where $1/p + 1/q = 1$. The proof goes by induction. We first show that (48) holds for all $n \in [1, 2n_0]$. The theorem holds trivially for $n = 1$. Now, notice that for all $k \in [2, 2n_0]$,

$$M_k \leq \sum_{i=1}^{k} |S_i| \ .$$

Now since $\exp(x)$ is convex and non–decreasing, we get that for all $t > 0$

$$\mathbf{E}(\exp(tM_k)) \leq \frac{1}{k} \sum_{i=1}^{k} \mathbf{E}\big(\exp(tk|S_i|)\big) \ .$$

Then according to (45), (47) and (50), we get that for all $t > t_0$,

$$\mathbf{E}(\exp(tM_k)) \le \frac{K}{k}\sum_{i=1}^{k}\exp(f(kt)s(i))$$

$$\le \frac{K}{k}\sum_{i=1}^{k}\exp\big(f(t)d(k)s(i)\big) \le K\exp\big(Cf(t)d(k)s(k)\big)\ .$$

Since $C\max_{1\le k\le 2n_0} d(k) \le B$ and $A \ge 1$, the result holds for all $n \in [1, 2n_0]$.

Now take $n > 2n_0$, and assume by the induction hypothesis that the result holds for all integers m satisfying $1 \le m \le n-1$. We will show that the result follows also for $m = n$. Notice that, for any $1 \le m \le n$ and $t > 0$,

$$M_n \le \max(M_{m-1}, |S_m| + \max_{m+1\le j\le n}|S_j - S_m|)$$

giving

$$\mathbf{E}(\exp tM_n) \le \mathbf{E}(\exp tM_{m-1}) + \mathbf{E}\Big(\exp(t|S_m| + \max_{m+1\le j\le n} t|S_j - S_m|)\Big)\ .$$

Hölder inequality and stationarity entail that

$$\mathbf{E}\Big(\exp(t|S_m| + \max_{m+1\le j\le n} t|S_j - S_m|)\Big) \le \|\exp(t|S_m|)\|_p \|\exp(tM_{n-m})\|_q\ .$$

Also

$$\mathbf{E}(\exp(tM_{m-1})) \le \|\exp(tM_{m-1})\|_q\,.$$

Overall

$$\mathbf{E}(\exp tM_n) \le 2\|\exp(t|S_m|)\|_p(\|\exp(tM_{m-1})\|_q \vee \|\exp(tM_{n-m})\|_q)\,.$$

We apply it with $m = k$ in the case where $n = 2k$ and $m = k+1$ in the case where $n = 2k+1$.

For the first case

$$\mathbf{E}(\exp tM_{2k}) \le 2\|\exp(t|S_k|)\|_p * \|\exp(tM_k)\|_q\,.$$

So since $k \ge n_0$, by the induction hypothesis and the properties (45) and (46),

$$\begin{aligned}\mathbf{E}(\exp tM_{2k}) &\le 2||\exp(t|S_k|)||_p * ||\exp(tM_k)||_q\\ &\le 2A^{1/q}K\exp(f(pt)s(k)/p)\exp(Bf(qt)s(k)/q)\\ &\le 2A^{1/q}K\exp\big(BQs(2k)(f(pt)/Bp + f(qt)/q)\big)\,.\end{aligned}$$

Then, by the property of f, we get for any $t \ge t_0$ and any $n > 2n_0$,

$$\mathbf{E}(\exp tM_{2k}) \le 2A^{1/q}K\exp\big(BQf(t)s(2k)(d(p)/Bp + d(q)/q)\big)\,.$$

Since $Q < 1$, $d(q) \leq Q^{-1}$ and $d(p) \leq B$, we get for any $t > t_0$ and any $n > 2n_0$,

$$\mathbf{E}(\exp tM_{2k}) \leq 2A^{1/q} K \exp(Bf(t)s(2k)) ,$$

which is the desired inequality since $A = 2^p$ and $n = 2k$.

For the case $n = 2k + 1$ we have similar computations starting from

$$\mathbf{E}(\exp tM_{2k+1}) \leq 2\| \exp(t|S_{k+1}|)\|_p * \| \exp(tM_k)\|_q$$

and applying the fact that by (46), $s(k+1) \leq Qs(2k+1)$ and $s(k) \leq Qs(2k+1)$. $\diamond$

5.2 Rate Functions

We give here the different form of the rate function $I_{s^2}^{(\beta)}(\cdot)$ for $\beta \in]0, 2]$. When $\beta = 1$ it is inherited from the Brownian motion. If $0 < \beta < 2$ and $\beta \neq 1$, then it is inherited from the fractional Brownian motion with Hurst index $\beta/2$ (see the computations done in Arcones (2003-b)).

1. If $\beta = 1$,

$$I_{s^2}^{(\beta)}(z) = \frac{1}{2s^2} \int_0^1 (z'(u))^2 du$$

if $z(0) = 0$ and z is absolutely continuous, and $I(z) = \infty$ otherwise.

2. If $0 < \beta < 2$ and $\beta \neq 1$,

$$I_{s^2}^{(\beta)}(z) = \inf \Big\{ \frac{\tau_\beta}{2s^2} \int_{-\infty}^{\infty} h^2(x) dx : \tau_\beta \int_{-\infty}^{\infty} h(x)\big(|x-t|^{(\beta-1)/2} - |x|^{(\beta-1)/2}\big) dx = z(t) \text{ for each } 0 \leq t \leq 1 \Big\} ,$$

where $z \in \ell^\infty([0,1])$ and $\tau_\beta = \Big(\int_{-\infty}^{\infty} \big(|x-1|^{(\beta-1)/2} - |x|^{(\beta-1)/2}\big)^2 dx \Big)^{-1}$.

3. If $\beta = 2$,

$$I_{s^2}^{(\beta)}(z) = \frac{A^2}{2s^2}$$

if for some A, $z(t) = At$ for each $t \in [0,1]$, and $I(z) = \infty$ otherwise.

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