Supplement to ”Functional CLT for martingale-like nonstationary dependent structures”

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This supplementary file contains a detailed proof of Corollary 4.6 in [1].

1. Proof of Corollary 4.6

In all the proof, we recall that the chain starts from the origin. To soothe the notation, we will often write $\mathbb{P}$ for $\mathbb{P} \phi_0 = 0$ when no confusion is possible.

We first note that $c^2$ is well defined. Indeed, the series $\sum_{m \geq 1} \mathbb{E}(\zeta_0 \zeta_m) \sum_{j=1}^{m} (P^j)_{0,m-j}$ is absolutely convergent. To see this, we note that since $\pi$ is the stationary distribution, we have that for any non-negative integers $m$ and $i$, $\pi_m = \sum_{j \geq 0} \pi_j (P^i)_{j,m}$. Therefore $(P^j)_{0,m-j} \leq \pi_0^{-1} \pi_{m-j}$ which entails that

$$\sum_{m \geq 1} |\mathbb{E}(\zeta_0 \zeta_m)| \sum_{j=1}^{m} (P^j)_{0,m-j} \leq \sum_{m \geq 1} |\mathbb{E}(\zeta_0 \zeta_m)| \sum_{j \geq 1} j p_j = \mathbb{E}(\tau) \sum_{m \geq 1} |\mathbb{E}(\zeta_0 \zeta_m)|.$$ 

So the absolute convergence of the series $\sum_{m \geq 1} \mathbb{E}(\zeta_0 \zeta_m) \sum_{j=1}^{m} (P^j)_{0,m-j}$ follows from the fact that $\sum_{m \geq 1} |\mathbb{E}(\zeta_0 \zeta_m)| < \infty$, which holds under the first part of condition (4.7) of [1]. Indeed, the first part of condition (4.7) of [1] implies that $\sum_{k \geq 0} \mathbb{E}((\zeta_k | G_0) - \mathbb{E}(\zeta_k | G_{-1}))_2 < \infty$ (see Corollary 2 in [2]), which in turn implies $\sum_{m \geq 1} |\mathbb{E}(\zeta_0 \zeta_m)| < \infty$.

The invariance principle will be proved by an application of Theorem 2.1 together with Proposition 3.2 of [1]. In order to apply it we need to introduce a non-decreasing filtration $(\mathcal{F}_k)_{k \geq 0}$ for which the process $(X_k)_{k \geq 1}$ is adapted. Let

$$\mathcal{A} = \sigma(Y_k, k \geq 0)$$

and for any $k \in \mathbb{Z}$, we define

$$\mathcal{F}_k = \sigma(\mathcal{A}; X_j, 1 \leq j \leq k).$$
This filtration encodes all the history of the Markov chain, and all the history up to
time $k$ of the sampled time scenery $\zeta_{Y_1}, \zeta_{Y_2}, \ldots, \zeta_{Y_k}$. Note that by the above definition
we have: $F_j = A_j$ for all $j \leq 0$.

We first notice that by Condition (A1) of [1] and the independence between the time
scenery and the Markov chain, condition (2.2) of [1] is satisfied. It remains to show that
conditions (2.6), (3.6) and (3.7) (with $v_n(t) = [nt]$ and $X_{k,n} = X_k/\sqrt{n}$) of [1] hold. With
this aim, we start by introducing some notations.

Let us first define a sequence of stopping times $\nu_1, \nu_2, \ldots$ as the times of return to the
origin,

$$\nu_1 \overset{\text{def}}{=} \inf\{k > 0 : \phi_k = 0\}, \quad \nu_{n+1} \overset{\text{def}}{=} \inf\{k > \nu_n : \phi_k = 0\}, \quad n \geq 1.$$  

Define also the random times $\tau_1, \tau_2, \ldots$ as the interarrival times to the origin, i.e. the
times between successive returns to the origin,

$$\tau_1 = \nu_1, \quad \tau_{n+1} = \nu_{n+1} - \nu_n, \quad n \geq 1,$$

and the renewal process

$$N_k = \max\{j \geq 0 : \nu_j \leq k\} + 1,$$

which basically counts the number of visits to the origin up to time $k$, including the
renewal at 0. Let us start with the two following facts that will be useful to prove that
conditions (2.6), (3.6) and (3.7) of [1] are satisfied. Their proof will be given later.

**Fact 1.1.** Let $k$ and $m$ be in $\mathbb{N}$ and such that $m \leq k$. Then

$$\mathbb{P}(N_{k-m} = N_k) \leq \mathbb{P}(\tau > m).$$

**Fact 1.2.** For any non-negative integer $k$, let

$$b(k) = \|\mathbb{E}(\zeta_k | G_0)\|_2 \quad \text{and} \quad c(k) = \sup_{j \geq 1 \geq 2 \leq k} \|\mathbb{E}(\zeta_i \zeta_j | G_0) - \mathbb{E}(\zeta_i \zeta_j)\|_1.$$  

Then

$$\sup_{k \geq 0} \|\mathbb{E}(X_{k+m} | F_k)\|_2^2 \leq b^2([m/2]) + b^2(0) \mathbb{P}(\tau > [m/2])$$

and

$$\sup_{k, \ell \geq 0} \|\mathbb{E}(X_{k+m}X_{k+m+\ell} | F_k) - \mathbb{E}_A(X_{k+m}X_{k+m+\ell})\|_1 \leq c([m/2]) + c(0) \mathbb{P}(\tau > [m/2]).$$

The second inequality of Fact 1.2 together with the second part of condition (4.7)
of [1] and the fact that $\mathbb{E}(\tau) < \infty$ implies that condition (3.6) of [1] is satisfied. On
another hand, by the first inequality of Fact 1.2 and taking into account the first part
of condition (4.7) of [1], it follows that condition (2.6) will be satisfied provided that
\[ \sum_{m \geq 1} m^{-1/2} \sqrt{\mathbb{P}(\tau > \lceil m/2 \rceil)} < \infty. \]
This last condition obviously holds since \( \mathbb{E}(\tau^2) < \infty \) is equivalent to \( \sum_{i \geq 1} i \mathbb{P}(\tau > i) < \infty. \)

It remains to prove that condition (3.7) of [1] holds. Note that it is satisfied if one can prove that, for any \( \varepsilon > 0 \) and any \( t \in [0, 1], \)
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \frac{1}{m} \sum_{k=1}^{m-1} \left\{ \mathbb{E}_A(X_k^2) + 2 \sum_{\ell=1}^{K} \mathbb{E}_A(X_k X_{k+\ell}) \right\} - tc^2 \geq \varepsilon \right) = 0. \tag{1.2}
\]
Note that for any integer \( \ell \) in \([0, K]\) where \( K \) is a fixed integer, because of the independence between the time scenery and the Markov chain, by stationarity of the random time scenery,
\[
\mathbb{E}_A(X_k X_{k+\ell}) = \sum_{m \geq \ell} 1_{\phi_k = m} \mathbb{E}(\zeta_{k+m}^2) + \sum_{i=1}^{\ell} \sum_{m \geq 0} 1_{\phi_k = \ell - i} 1_{\phi_{k+i} = m} \mathbb{E}(\zeta_{k+i}^2 - i \zeta_{k+i}) + m \mathbb{E} \left( \zeta_{k+i} - i \zeta_{k+i} \right).
\]
It follows that
\[
\left| \frac{1}{n} \sum_{k=1}^{nt} (\mathbb{E}_A(X_k^2) + 2 \sum_{\ell=1}^{K} \mathbb{E}_A(X_k X_{k+\ell})) - ct^2 \right| \leq \sigma^2 \left| \frac{nt}{n} - t \right| + 2 \mathbb{E}(\zeta_0^2) \frac{1}{n} \sum_{k=1}^{nt} \sum_{\ell=1}^{K} 1_{\phi_k = \ell - i} \sum_{i \geq 1} i \pi_i
\]
\[+ 2 \left| \frac{1}{n} \sum_{k=1}^{nt} \sum_{\ell=1}^{\ell} \sum_{i \geq 0} U(k, \ell, i, r) - \sum_{r \geq 1} \sum_{j=1}^{r} (P^j)_{0, j-r} \mathbb{E}(\zeta_0^2) \right| =: \sum_{i \geq 1} \pi_i \]
\[= \sum_{i \geq 1} \pi_i = t \sum_{i \geq 1} i \pi_i \quad \text{a.s.,} \quad \text{as,} \tag{1.3}
\]
where
\[
U(k, \ell, i, r) := 1_{\phi_k = \ell - i} 1_{\phi_{k+i} = \ell} \mathbb{E}(\zeta_0 \zeta_{r+i}). \tag{1.4}
\]
Let us first prove that, for any \( \varepsilon > 0 \) and any \( t \in [0, 1], \)
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \frac{1}{m} \sum_{K=1}^{m-1} |I_{n,K}^{(2)}(t)| \geq \varepsilon \right) = 0. \tag{1.5}
\]
With this aim, note that since the Markov chain is irreducible and positive recurrent, by the law of large numbers for Markov chains, for any \( t \in [0, 1], \)
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{nt} \phi_k = t \sum_{i \geq 1} i \pi_i \quad \text{a.s.,} \tag{1.6}
\]
whatever the initial law is. In addition, by Fact 1.1,

\[
\frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \mathbb{E}_{\phi_0} \left( 1_{\phi_k \geq \ell} \right) = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \mathbb{P}_{\phi_0} \left( N_{k-1} = N_{k+\ell-1} \right) \leq \sum_{\ell \geq K+1} \mathbb{P}(\tau > \ell).
\]

Hence

\[
\frac{1}{m} \sum_{k=1}^{m-1} \frac{1}{n} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}_{\phi_0} \left( 1_{\phi_k \geq \ell} \right) \leq \frac{1}{m} \sum_{k=1}^{m-1} \sum_{\ell \geq K+1} \mathbb{P}(\tau > \ell) \to 0, \text{ as } m \to \infty. \quad (1.7)
\]

The convergence (1.5) follows from (1.6) and (1.7). To end the proof of (1.2) (and then of condition (3.7) in [1]), it remains to show that, for any \( \varepsilon > 0 \) and any \( t \in [0, 1] \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \frac{1}{m} \sum_{k=1}^{m-1} \sum_{\ell \geq K+1} |T_{n,K}(t)| \geq \varepsilon \right) = 0. \quad (1.8)
\]

Let \( \ell \) be a positive fixed integer, by the ergodic Theorem applied to the irreducible and positive recurrent Markov chain \( Z_k = (\phi_k, \phi_{k+\ell}) \), we have, for \( \ell \geq i \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}_{\phi_k=\ell-i} \mathbb{E}_{\phi_k+\ell=r} = t \pi_{\ell-i}(P^{\ell})_{\ell-i,r} \quad \text{a.s.},
\]

whatever the initial law is. But, for any \( \ell \geq i \), by homogeneity of the Markov chain,

\[
(P^{\ell})_{\ell-i,r} = \mathbb{P}(\phi_{\ell} = r | \phi_0 = \ell - i) = \mathbb{P}(\phi_\ell = r, \phi_{\ell-i} = 0 | \phi_0 = \ell - i) = \mathbb{P}(\phi_{\ell} = r | \phi_{\ell-i} = 0) = (P^{\ell})_{0,r}.
\]

Hence, for any \( K \) and \( L \) fixed,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}_{\phi_0=0} \left( \phi_k = \ell - i, \phi_k + \ell = r \right) \mathbb{E}(\zeta_{r+i}) = \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}(\zeta_{r+i}) \quad \text{a.s.}
\]

whatever the initial law is. On another hand,

\[
\mathbb{E}_{\phi_0=0} \left| \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}(\zeta_{r+i}) \mathbb{E}(\zeta_{r+i}) \right| \leq \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}(\zeta_{r+i}) |\mathbb{P}_{\phi_0=0}(\phi_k = \ell - i, \phi_k + \ell = r)| \leq \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \sum_{\ell \geq K+1} \mathbb{E}(\zeta_{r+i}) |(P^{\ell})_{\ell-i,r}(P^{\ell})_{0,\ell-i}|
\]

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But, since \( \pi \) is the stationary distribution, we have that for any non-negative integers \( m \) and \( i, \pi_m = \sum_{j \geq 0} \pi_j (P^i)_{j,m} \). Therefore \( \pi_0 (P^k)_{0,\ell-i} \leq \pi_{\ell-i} \) which implies that

\[
\mathbb{E}_{\phi_0 = 0} \left| \frac{1}{n} \sum_{k=1}^{[nt]} \sum_{\ell=1}^{K} \sum_{i=1}^{\ell} \sum_{r \geq L+1} 1_{\phi_k = \ell-i, \phi_{k+i} = P_r} \mathbb{E} (\phi_0 \zeta_{r+i}) \right| \leq \pi_0^{-1} \sum_{\ell=1}^{K} \sum_{i=1}^{\ell} \sum_{r \geq L+1} \mathbb{E} (\phi_0 \zeta_{r+i}) |\pi_{\ell-i}|
\]

\[
\leq K \pi_0^{-1} \sum_{r \geq L+2} \mathbb{E} (\phi_0 \zeta_r). \]

But, as already mentioned, \( \sum_{m \geq 0} |\mathbb{E} (\phi_0 \zeta_m)| < \infty \) under the first part of condition (4.7) of [1]. Hence, for any positive integer \( K \) fixed,

\[
\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{E}_{\phi_0 = 0} \left| \frac{1}{n} \sum_{k=1}^{[nt]} \sum_{\ell=1}^{K} \sum_{i=1}^{\ell} \sum_{r \geq L+1} 1_{\phi_k = \ell-i, \phi_{k+i} = P_r} \mathbb{E} (\phi_0 \zeta_{r+i}) \right| = 0. \tag{1.10}
\]

Next, we show that, for any positive integer \( K \) fixed,

\[
\lim_{L \to \infty} \sum_{\ell=1}^{K} \sum_{i=1}^{\ell} \pi_{\ell-i} \sum_{r=L+1}^{\infty} (P^i)_{0,r} \mathbb{E} (\phi_0 \zeta_{r+i}) = 0 \tag{1.11}
\]

and

\[
\lim_{K \to \infty} \sum_{\ell=K+1}^{\infty} \sum_{i=1}^{\ell} \pi_{\ell-i} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E} (\phi_0 \zeta_{r+i}) = 0. \tag{1.12}
\]

Noting that

\[
\sum_{\ell=1}^{\infty} \sum_{i=1}^{\ell} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E} (\phi_0 \zeta_{r+i}) = \sum_{i=1}^{\infty} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E} (\phi_0 \zeta_{r+i}) = \sum_{i=1}^{\infty} \mathbb{E} (\phi_0 \zeta_r) \sum_{j=1}^{r} \sum_{i=0}^{r-j} (P^i)_{0,r-j}
\]

and considering (1.9), (1.10), (1.11) and (1.12), we get (1.8) and then the convergence (1.2).

Hence, to end the proof of (1.2), it remains to show (1.11) and (1.12). We start by proving (1.11). We observe that

\[
\sum_{\ell=1}^{K} \sum_{i=1}^{\ell} \sum_{r=L+1}^{\infty} (P^i)_{0,r} |\mathbb{E} (\phi_0 \zeta_{r+i})| \leq \sum_{i=1}^{\infty} \sum_{\ell=1}^{K} \pi_{\ell-i} \sum_{r=L+1+i}^{\infty} \mathbb{E} (\phi_0 \zeta_r) \leq K \sum_{r=L+2}^{\infty} |\mathbb{E} (\phi_0 \zeta_r)|,
\]

which converges to zero as \( L \to \infty \) since, as mentioned before, \( \sum_{r \geq 0} |\mathbb{E} (\phi_0 \zeta_r)| < \infty \) under the first part of condition (4.7) of [1]. This proves (1.11).
To prove (1.12), we first write the following decomposition

$$
\sum_{\ell=K+1}^{\infty} \sum_{i=1}^{\ell} \pi_{\ell-i} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E}(\zeta_r) \]

$$

$$
\leq \sum_{\ell=K+1}^{\infty} \sum_{i=1}^{\ell} \pi_{\ell-i} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E}(\zeta_r) + \sum_{\ell=K+1}^{\infty} \sum_{i=[\ell/2]+1}^{\ell} \pi_{\ell-i} \sum_{r=0}^{\infty} (P^i)_{0,r} \mathbb{E}(\zeta_r) 

:= I_K + J_K.

$$

Now, since \( \pi \) is the stationary distribution, \((P^i)_{0,r} \leq \pi_0^{-1} \pi_r \) for any positive integer \( i \).

Therefore

$$
I_K \leq \pi_0^{-1} \sum_{\ell=K+1}^{\infty} \sum_{i=1}^{[\ell/2]} \sum_{r=0}^{\infty} \pi_{\ell-i} \pi_r \mathbb{E}(\zeta_r) 

\leq \pi_0^{-1} \sum_{i=1}^{\infty} \sum_{\ell=K+1}^{\infty} \pi_{\ell-i} \sum_{r=0}^{\infty} \pi_r \mathbb{E}(\zeta_r) + \pi_0^{-1} \sum_{i=[(K+1)/2]}^{\infty} \sum_{\ell=[(K+1)/2]}^{\infty} \pi_{\ell-i} \sum_{r=0}^{\infty} \pi_r \mathbb{E}(\zeta_r) 

\leq \pi_0^{-1} \sum_{\ell=[(K+1)/2]}^{\infty} \pi_{\ell} \sum_{r=0}^{\infty} \pi_r \mathbb{E}(\zeta_r) + \pi_0^{-1} \sum_{\ell=[(K+1)/2]}^{\infty} \pi_{\ell} \sum_{j=[(K+1)/2]}^{\infty} \mathbb{E}(\zeta_j) 

\leq 2 \pi_0^{-1} \sum_{\ell=[(K+1)/2]}^{\infty} \pi_{\ell} \sum_{j=[(K+1)/2]}^{\infty} \mathbb{E}(\zeta_r) 

= 2 \pi_0^{-1} \sum_{\ell=[(K+1)/2]}^{\infty} \pi_{\ell} \sum_{j=[(K+1)/2]}^{\infty} \mathbb{E}(\zeta_r),

$$

which converges to zero as \( K \to \infty \) using again that \( \sum_{r=0}^{\infty} \mathbb{E}(\zeta_r) < \infty \). On the other hand, since \((P^i)_{0,r} \leq \pi_0^{-1} \pi_r \) for any positive integer \( i \),

$$
J_K \leq \pi_0^{-1} \sum_{i=[(K+1)/2]+1}^{2i} \sum_{\ell=i}^{\infty} \pi_{\ell-i} \sum_{r=0}^{\infty} \pi_r \mathbb{E}(\zeta_r) \leq \pi_0^{-1} \sum_{i=[(K+1)/2]+1}^{\infty} \sum_{r=0}^{\infty} \pi_r \mathbb{E}(\zeta_r) 

\leq \pi_0^{-1} \sum_{j=[(K+1)/2]+1}^{\infty} \mathbb{E}(\zeta_r),

$$

which converges to zero as \( K \to \infty \) using once again that \( \sum_{r=0}^{\infty} \mathbb{E}(\zeta_r) < \infty \). This ends the proof of (1.12) and then of (1.2).

To end the proof of the corollary it remains to prove Facts 1.1 and 1.2.

**Proof of Fact 1.1.** To prove the fact above, we denote by \( \tilde{\nu} \) the last renewal before time
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Proof of Fact 1.2. Note first that $Y_j = j + \phi_j = \nu_r$, for $\nu_{r-1} < j \leq \nu_r$, and by definition of $N_k$, we have that $N_k - 1 = \max\{j \geq 1 : \nu_j \leq k\}$, that is the last time the increasing random walk \{\nu_j : j \geq 0\} is below the level $k$. Thus $N_k$ is the hitting time of the set $(k, \infty)$, or the time of first entry in the interval $(k+1, \infty)$ of the random walk \{\nu_j : j \geq 0\}. Thus $\nu_{N_k-1} < k + 1$ and $\nu_{N_k} \geq k + 1$. This fact with $k$ replaced by $k - 1$ gives

\[ \nu_{N_k-1-1} < k \leq \nu_{N_k-1}. \]

Since $Y_j = \nu_r$ for $\nu_{r-1} < j \leq \nu_r$, we can conclude that $Y_k = \nu_{N_k-1}$, for all $k \geq 1$. Then once again writing $A$ for $\sigma(\phi_j, j \in \mathbb{Z})$, $\mathcal{F}_k = \sigma(A, X_j, 1 \leq j \leq k)$, and $\mathbb{E}_A(\cdot)$ for $\mathbb{E}(\cdot|A)$, we have

\[ \mathbb{E}(X_k | \mathcal{F}_{k-m}) = \mathbb{E}_A(\zeta Y_k \zeta Y_1, \ldots, \zeta Y_{k-m}) = \mathbb{E}_A(\zeta \nu_{N_{k-1}} \zeta \nu_1, \ldots, \zeta \nu_{N_{k-m-1}}). \]

Now, writing $k$ instead of $k - 1$ to simplify notation, we have

\[ \| \mathbb{E}_A(\zeta \nu_{N_k} | \zeta \nu_1, \ldots, \zeta \nu_{N_{k-m}}) \|^2_2 \leq \| \mathbb{E}_A(\zeta \nu_{N_k} | \mathcal{G}_{\nu_{N_{k-m}}}) \|^2_2 \leq \mathbb{E}[b^2(\nu_{N_k} - \nu_{N_{k-m}})]. \]

Recall that $N_k$ is the first time the positive random walk $(\nu_j)$ exceeds the value $k$. Thus $\nu_{N_k} - \nu_{N_{k-m}} = \nu_{\sigma_k^+} - \nu_{\sigma_{k-m}^+} > k - (\nu_{\sigma_k^+} - (k - m)) - (k - m) = m - (\nu_{\sigma_k^+} - (k - m))$, where $\sigma_k^+ = \inf\{j \geq 0 : \nu_j > k\}$, and thus $\nu_{\sigma_{k-m}^+} - (k - m)$ is the overshoot. A quick calculation gives for $l = [m/2]$:

\[ \mathbb{E}[b^2(\nu_{N_k} - \nu_{N_{k-m}})] = \mathbb{E}[b^2(\nu_{N_k} - \nu_{N_{k-m}}) \mathbb{1}\{N_{k-m} \leq N_{k-l} - 1\}] + \mathbb{E}[b^2(\nu_{N_k} - \nu_{N_{k-m}}) \mathbb{1}\{N_{k-m} = N_{k-l}\}] =: I_1 + I_2, \]

where the first event corresponds to there is at least one renewal between times $k - m$ and $k - l$, while the second event corresponds to no renewals. Then, we have

\[ I_1 = \mathbb{E}[b^2(\nu_{N_k} - \nu_{N_{k-l} - 1} + \nu_{N_{k-l} - 1} - \nu_{N_{k-m}}) \mathbb{1}\{N_{k-m} \leq N_{k-l} - 1\}] \leq b^2([m/2]) \mathbb{P}(N_{k-l} > N_{k-m}) \leq b^2([m/2]). \]
Indeed, by previous considerations, we have that $\nu_{N_k} - \nu_{N_{k-1}} - 1 > l$, while on the event \{\(N_k - m \leq N_{k-1} - 1\)\}, clearly $\nu_{N_k} - \nu_{N_{k-1}} - 1 \geq 0$ and thus

$$\nu_{N_k} - \nu_{N_{k-1}} - 1 > l.$$ 

Therefore, since $b(\cdot)$ is non-increasing,

$$b^2(\nu_{N_k} - \nu_{N_{k-1}} - 1 + \nu_{N_{k-1}} - \nu_{N_{k-1}}) \leq b^2(l).$$

For the second term we have $I_2 \leq b^2(0)P(N_k - m = N_{k-1})$. So, according to Fact 1.1, we get that for $k \geq m$,

$$I_2 \leq b^2(0)P(\tau \geq m - l) \leq b^2(0)P(\tau \geq [m/2]).$$

Taking into account the bounds of $I_1$ and $I_2$, the first inequality of Fact 1.2 follows.

To prove the second one, we proceed similarly. Since $Y_k = \zeta_{N_{k-1}}$ for all $k \geq 0$, we have

$$\mathbb{E}(X_{k+m}X_{k+m+l}|F_k) - \mathbb{E}_A(X_{k+m}X_{k+m+l})$$

$$= \mathbb{E}_A(\zeta_{v_{N_k+m-1}}\zeta_{v_{N_{k+m+l-1}}}|\zeta_{v_1}, \ldots, \zeta_{v_{N_{k-1}}}) - \mathbb{E}_A(\zeta_{v_{N_k+m-1}}\zeta_{v_{N_{k+m+l-1}}}).$$

Therefore

$$\|\mathbb{E}(X_{k+m}X_{k+m+l}|F_k) - \mathbb{E}_A(X_{k+m}X_{k+m+l})\|_1 \leq \mathbb{E}(c(v_{N_k+m-1} - v_{N_{k-1}})),$$

and the second inequality follows from the previous computations with $c(\cdot)$ replacing $b^2(\cdot)$. 

References
