# Functional CLT for nonstationary strongly mixing processes

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**Abstract**

This paper deals with the functional central limit theorem for non-stationary dependent sequences of random variables satisfying the Lindeberg condition. The dependence condition which we impose is known under the name of weak strong mixing condition. It is satisfied by a large class of dependent random variables, including functions of strongly mixing or $\alpha$-dependent Markov chains.

*Keywords:* Functional central limit theorem; non-stationary triangular arrays; $\alpha$-mixing arrays.

*MSC:* 60F17, 60G48.

## 1 Introduction

One of the most important limit theorems in probability theory is Donsker's theorem for triangular arrays of independent random variables. This is a functional central limit theorem for sequences of independent random variables satisfying the Lindeberg condition. It is a natural question, rooted in practical applications, to extend this theorem to dependent structures. This problem appears to be very difficult in dependent setting. Regarding the central limit theorem only, there are several remarkable results in the literature. Assuming the Lindeberg condition, a central limit theorem for $\varphi$-mixing sequences was obtained by Utev [19]. In [11], Peligrad obtained a similar result for interlaced mixing sequences, while Rio [15, 16] treated strongly mixing sequences.

In a recent paper Merlevède et al. [9] developed a method, based on martingale approximation, to deal with the functional central limit theorem for dependent structures. The method proved to be useful for obtaining the functional central limit theorem for a class of dependent random variables defined by using as a measure of dependence the maximal coefficient of correlation. The scope of this paper is to further exploit the tools developed in [9] for obtaining the functional central limit theorem for non-stationary strongly mixing sequences. As a matter of fact, we shall use a weak form of strongly mixing coefficients, for including a much larger class of example than the traditional strong mixing condition introduced by Rosenblatt [17]. We include in this paper several applications to linear processes with strongly mixing innovations and functions of strongly mixing or $\alpha$-dependent Markov chains.

Our paper is organized as follows. In Section 2 we state the main result, we comment on its conditions and applications, while in Section 3 we include the proofs.

Throughout the paper we shall denote by $\|X\|_p$ the norm in $\mathbb{L}_p$ of a random variable namely $\|X\|_p^p = \mathbb{E}(|X|^p)$. For two sequences of real numbers $(a_n)$ and $(b_n)$ the notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, whereas the notation $a_n \ll b_n$ means that there exists a positive constant $C$ such that for all $n$, $a_n \leq Cb_n$. The notation $[x]$ is used for the integer part of $x$.

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## 2 Main result

Let $\{X_{i,n}, 1 \le i \le n\}$ be a triangular array of square integrable ($\mathbb{E}(X_{i,n}^2) < \infty$), centered ($\mathbb{E}(X_{i,n}) = 0$), real-valued random variables. We define the partial sums $S_k = S_{k,n} = \sum_{i=1}^{k} X_{i,n}$ and denote by $\sigma_{k,n}^2 = \mathrm{Var}(S_{k,n})$ for $k \le n$. Assume that

$$\sigma_{n,n}^2 = \mathrm{Var}\Big(\sum_{\ell=1}^{n} X_{\ell,n}\Big) = 1 \,, \tag{1}$$

and, for $0 \le t \le 1$, define

$$v_n(t) = \inf\Big\{k;\ 1 \le k \le n\colon\ \sigma_{k,n}^2 \ge t\Big\} \text{ and } W_n(t) = \sum_{i=1}^{v_n(t)} X_{i,n} \,. \tag{2}$$

We shall also assume that the triangular array satisfies the following Lindeberg condition:

$$\sup_{n\ge1} \sum_{j=1}^{n} \mathbb{E}(X_{j,n}^2) \le C < \infty, \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\{X_{k,n}^2 I(|X_{k,n}| > \varepsilon)\} = 0 \,, \text{ for any } \varepsilon > 0. \tag{3}$$

Our aim is to provide sufficient conditions for deriving a functional central limit theorem for the partial sums process $\big\{W_n(t), t \in (0,1]\big\}$. For any integer $i \ge 1$, let $f_{i,n}(t) = \mathbf{1}_{\{X_{i,n} \le t\}} - \mathbb{P}(X_{i,n} \le t)$. For any non-negative integer $k$, set

$$\alpha_{1,n}(k) = \sup_{i\ge0} \max_{i+k\le u} \sup_{t\in\mathbb{R}} \big\|\mathbb{E}\big(f_{u,n}(t)|\mathcal{F}_{i,n}\big)\big\|_1 \,,$$

and

$$\alpha_{2,n}(k) = \sup_{i\ge0} \max_{i+k\le u\le v} \sup_{s,t\in\mathbb{R}} \big\|\mathbb{E}\big(f_{u,n}(t)f_{v,n}(s)|\mathcal{F}_{i,n}\big) - \mathbb{E}\big(f_{u,n}(t)f_{v,n}(s)\big)\big\|_1 \,,$$

where $\mathcal{F}_{i,n} = \sigma(X_{j,n} 1 \le j \le i\}$, $\mathcal{F}_{0,n} = \{\emptyset, \Omega\}$ and in the definitions above we extend the triangular arrays by setting $X_{i,n} = 0$ if $i > n$.

We shall now introduce two conditions that combine the tail distributions of the variables with their associated $\alpha$-dependent coefficients:

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} \sum_{i=m}^{n} \int_0^{\alpha_{1,n}(i)} Q_{k,n}^2(u)du = 0 \tag{4}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} \int_0^{\alpha_{2,n}(m)} Q_{k,n}^2(u)du = 0 \,, \tag{5}$$

where $Q_{k,n}$ is the quantile function of $X_{k,n}$ i.e., the inverse function of $t \mapsto \mathbb{P}(|X_{k,n}| > t)$.

Under the conditions above, the following result holds:

**Theorem 2.1** *Suppose that* (1), (3), (4) *and* (5) *hold. Then* $\big\{W_n(t), t \in (0,1]\big\}$ *converges in distribution in* $D([0,1])$ *(equipped with the uniform topology) to* $W$, *where* $W$ *is a standard Brownian motion.*

## 2.1 Discussions

1. *Discussion on the mixing coefficients.* Note that for any $i \in \mathbb{N}$, $\alpha_{1,n}(i) \leq \alpha_{2,n}(i)$. In addition both these coefficients are decreasing in $i$. So, obviously we can assume instead of conditions (4) and (5) the unique condition

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} \sum_{i=m}^{n} \int_0^{\alpha_{2,n}(i)} Q_{k,n}^2(u)du = 0. \tag{6}$$

Furthermore, these strong mixing-type coefficients are weaker than those which are often used in the literature. Let us recall that the traditional coefficient introduced and studied by Rosenblatt [17] is defined as

$$\bar{\alpha}_n(k) = \sup_i \sup \left|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\right|,$$

where the second supremum is taken over all $A \in \mathcal{F}_{i,n}$ and $B \in \sigma(X_{j,n}, j \geq k+i)$. Equivalently this coefficient can be rewritten as

$$\bar{\alpha}_n(k) = \frac{1}{4} \sup_{i\geq 0} \sup_{Y} \left\|\mathbb{E}\big(Y|\mathcal{F}_{i,n}\big) - \mathbb{E}\big(Y\big)\right\|_1,$$

where the $Y'$s are random variables measurable with respect to $\sigma(X_{j,n}, j \geq k+i)$ and bounded by one (see [3, Th. 4.4]). We can see from this definition that $\alpha_{2,n}(k) \leq 2\bar{\alpha}_n(k)$ and we use not more than two variables in the future and not all the functions but rather only the indicators. This weaker form is important since it will allow us to cover a larger class of stochastic processes than the strong mixing sequences.

2. *Discussion on mixing rates and moments.* We can de-couple the mixing coefficients from the variables. Denote by

$$\alpha_1(i) = \sup_n \alpha_{1,n}(i) \text{ and } a_m(u) = \sum_{i\geq m} \mathbf{1}_{\{u<\alpha_1(i)\}}.$$

Then, we apply the Cauchy-Schwarz inequality and take into account that $Q_{k,n}(U)$ is distributed as $X_{k,n}$ for $U$ uniformly distributed. For some $\delta > 0$ we obtain

$$\sum_{i\geq m} \int_0^{\alpha_{1,n}(i)} Q_{k,n}^2(u)du \leq \Big(\int_0^1 a_m^{(2+\delta)/\delta}(u)du\Big)^{\delta/(2+\delta)} \|X_{k,n}\|_{2+\delta}^2.$$

Now, by computations on page 12 in Rio [16] this inequality shows that condition (4) is satisfied if the following couple of conditions are satisfied:

$$\sup_n \sum_{k=1}^{n} \|X_{k,n}\|_{2+\delta}^2 < \infty \text{ and } \sum_{i\geq 1} i^{2/\delta}\alpha_1(i) < \infty \text{ where } 0 < \delta \leq \infty. \tag{7}$$

Moreover, condition (5) holds provided the first part of (7) is satisfied and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \alpha_{2,n}(m) = 0\,. \tag{8}$$

We have then established the following corollary:

**Corollary 2.2** *Assume that conditions (1), (3), (7) and (8)are satisfied for some $\delta \in (0, \infty]$. Then the conclusion of Theorem 2.1 holds.*

**Remark.** In the corollary above, we can assume instead of the second part of condition (3) that $\max_{1\le k\le n}\|X_{k,n}\|_{2+\delta}\to 0$, as $n\to\infty$.

3. *Discussion on the CLT for $S_n$.* Our result obviously implies that $W_n(1)$ satisfies the CLT. It is important to mention that the CLT also holds for $S_n=\sum_{k=1}^n X_{k,n}$. To see it, according to Theorem 3.1 in Billingsley [1] it is enough to show that

$$\lim_{n\to\infty}\mathbb{E}\Big(\sum_{k=1}^{v_n(1)}X_{k,n}-\sum_{k=1}^{n}X_{k,n}\Big)^2=0. \tag{9}$$

Note first that condition (3) implies that $\max_{1\le k\le n}\|X_{k,n}\|_2\to 0$. Hence, by the definition of $v_n(1)$, we have

$$\lim_{n\to\infty}\mathbb{E}\Big(\sum_{k=1}^{v_n(1)}X_{k,n}\Big)^2=1$$

(see the proof of (5.35) in [9] for more details). Since $\sigma_{n,n}^2=1$, the proof of (9) is then reduced to show that

$$\lim_{n\to\infty}\mathrm{Cov}\Big(\sum_{k=1}^{v_n(1)}X_{k,n},\sum_{j=v_n(1)+1}^{n}X_{k,n}\Big)=0\,.$$

But this easily follows under the conditions of Theorem 2.1 via Rio's covariance inequality [14]. Note that in [15], Rio also proved a CLT for $S_n$ under (1) and the following conditions

$$\limsup_{n\to\infty}\max_{1\le k\le n}\sigma_{k,n}^2<\infty \text{ and } \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1\bar\alpha_n^{-1}(u)Q_{k,n}^2(u)(\bar\alpha_n^{-1}(u)Q_{k,n}(u)\wedge 1)du=0 \tag{10}$$

instead of conditions (3), (4) and (5). Above $\bar\alpha_n^{-1}(u)=\sum_{i=0}^{n-1}\mathbf{1}_{u<\bar\alpha_n(i)}$. Some easy computations show that if (3) and (4) (with $\bar\alpha_n(i)$ replacing $\alpha_{1,n}(i)$) are satisfied then so is the second part of (10). On another hand, even if our conditions and the Rio's ones are very similar in nature, we cannot prove that they are equivalent (even if in all our examples and applications they give the same conditions). However, our conditions lead to the functional central limit theorem, are easier to verify and we specify a weaker version of the strong mixing coefficients.

4. *Forms of stationarity.*
(i) If we assume that $\sigma_n^2=nh(n)$, where $h(n)$ is a slowly varying function such that $\liminf_{n\to\infty}h(n)>0$, and that the sequence is not triangular, (i.e. for all $k$, $X_{k,n}=X_k$) then, we can construct the process $W_n(t)=\sum_{k=1}^{[nt]}X_k/\sigma_n$ and the conclusion of Theorem 2.1 holds for $W_n(t)$ under its assumptions.
(ii) Let us assume that $\sum_{k=1}^n Q_{k,n}$ is decreasing to $Q$, and denote by $\alpha_1(i)=\sup_n\alpha_{1,n}(i)$ and $\alpha_2(i)=\sup_n\alpha_{2,n}(i)$. Then, by monotonicity, conditions (4) and (5) are implied respectively by

$$\sum_{i\ge 0}\int_0^{\alpha_1(i)}Q^2(u)du<\infty \text{ and } \alpha_2(i)\to 0\,. \tag{11}$$

while condition (5) is implied by

$$\alpha_2(i)\to 0. \tag{12}$$

(iii) Let us assume now that there is $Q$, a quantile function, such that $Q_{k,n}\le Q$ and also that $\lim_{n\to\infty}\sigma_n^2/n=\sigma^2>0$. Then the functional CLT holds for $W_n(t)=\sum_{k=1}^{[nt]}X_{k,n}/\sqrt{n}$ under (11). For this case the limiting process will be $|\sigma|W$, where $W$ is a standard Brownian motion.

(iv) If $(X_j)_{j\in Z}$, is a strictly stationary sequence of random variables, our Theorem 2.1 reduces to the invariance principle for $S_{[nt]}/\sqrt{n}$. In this case, as in the point (iii), the conditions on the mixing coefficients reduces to (12) where now $Q$ is the quantile function of $|X_0|$. This fact can be easily seen by a change of time, taken into account that in this case we have $\sigma_n^2/n \to \sigma^2$. The constant $|\sigma|$ in the limiting process could be unfortunately 0. It will be strictly positive if we assume that $\sigma_n^2 \to \infty$ and if we impose instead of the first part of (11) rather the condition $\sum_{i\geq 1} i \int_0^{\alpha_1(i)} Q^2(u)du < \infty$ (See Lemma 1 in Bradley [2]).

5. *Discussion on the minimality of mixing conditions for the CLT.* There are numerous counterexample to the CLT, involving stationary strong mixing sequences, in papers by Davydov [4], Bradley [2], Doukhan et al. [7], Häggström [8] among others. We know that in the stationary case our conditions reduce to the minimal ones. These examples show that we cannot just assume that only the moments of order 2 are finite. Furthermore the mixing rate is minimal in some sense (see [7]).

## 2.2 Examples and Applications

1. *Functions of $\alpha$-dependent Markov chains.* Let $Y_{i,n} = f_{i,n}(X_i)$ where $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$ is a stationary Markov process with Kernel operator $K$ and invariant measure $\nu$ and, for each $i$ and $n$, $f_{i,n}$ is such that $\nu(f_{i,n}) = 0$ and $\nu(f_{i,n}^2) < \infty$. Let $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n Y_{i,n})$ and $X_{i,n} = \sigma_n^{-1} Y_{i,n}$. Note that the weak dependent coefficients $\alpha_1(i)$ of $\mathbf{X}$ can be rewritten as follows: Let $BV_1$ be the class of bounded variation functions $h$ such that $|h|_v \leq 1$ (where $|h|_v$ is the total variation norm of the measure $dh$). Then

$$\alpha_1(i) = \frac{1}{2} \sup_{f\in BV_1} \nu\big(|K^i(f) - \nu(f)|\big)\,.$$

Now, $\alpha_2(i)$ will have the same order of magnitude as $\alpha_1(i)$ if the space $BV_1$ is invariant under the iterates $K^n$ of $K$, uniformly in $n$, i.e., there exists a positive constant $C$ such that, for any function $f$ in $BV_1$ and any $n \geq 1$,

$$|K^n(f)|_v \leq C|f|_v\,.$$

There are many Markov chains such that $\alpha_2(n) \to 0$, as $n \to \infty$, but which are not mixing in the sense of Rosenblatt. For instance, let $T_\gamma$ be a GPM map, as defined in [5], that is an expanding map of $[0,1]$ with a neutral fixed point at 0; the behavior of the map around 0 is described by the parameter $\gamma \in (0,1)$. It is well-known that the Markov chain $(X_i)_{i\geq 0}$ associated with $T_\gamma$ is not strong mixing but it is proved in [5] that it is such that $\alpha_2(k) \leq Ck^{1-1/\gamma}$. Moreover, the invariant measure $\nu$ of $(X_i)_{i\geq 0}$ is equivalent to the Lebesgue measure on $[0,1]$ and its density $h$ satisfies $0 < c \leq x^\gamma h(x) \leq C < \infty$.

Then, in what follows, we assume that $(X_i)_{i\geq 0}$ is the Markov chain associated with $T_\gamma$ and that, for any $i$ and $n$ fixed, $f_{i,n}$ is monotonic on some open interval and 0 elsewhere. It follows that the weak dependence coefficients associated with $(X_{i,n})$ are such that $\alpha_{2,n}(k) \leq Ck^{1-1/\gamma}$, where $C$ is a positive constant not depending on $n$. By applying Corollary 2.2, we derive that if the triangular array $(X_{i,n})$ satisfies the Lindeberg condition (3) and if

$$\gamma \in (0,1/2) \text{ and } \sup_{n\geq 1} \frac{1}{\sigma_n^2} \sum_{i=1}^n \Big(\int_0^1 f_{i,n}^{2+\delta}(x) x^{-\gamma} dx\Big)^{2/(2+\delta)} < \infty \text{ for some } \delta > \frac{2\gamma}{1-2\gamma}\,,$$

then the conclusion of Theorem 2.1 is satisfied for the triangular array $(X_{i,n})$ defined above.

2. *Linear statistics.* We shall use our result to establish limit theorems for statistics of the type

$$S_n = \sum_{j=1}^n d_{n,j} X_j, \tag{13}$$

where $d_{n,j}$ are real valued weights and $(X_j)$ is a strictly stationary sequence of centered real-valued r.v.'s in $\mathbb{L}^2$. This model is also useful to analyze linear processes with dependent innovations and regression models. It was studied in Peligrad and Utev [12], Rio [15] and also in Peligrad and Utev [13] where a central limit theorem was obtained by using a stronger form of the mixing coefficients. Our general approach shows that we can weaken the mixing coefficients for this result and in addition we provide a functional central limit theorem.
We shall assume that the sequence of constants satisfy the following two conditions:

$$\sum_{i=1}^{n} d_{n,i}^2 \to c^2 \text{ and } \sum_{i=1}^{n} (d_{n,j} - d_{n,j-1})^2 \to 0 \text{ as } n \to \infty \,, \tag{14}$$

where $c^2 > 0$. Also, we note that the condition

$$\sum_{i \geq 0} \int_0^{\alpha_1(i)} Q^2(u) du < \infty \tag{15}$$

implies condition (2) in [13] (see Corollary 7 there) and therefore the sequence $(X_j)$ has a continuous spectral density, $f(x)$. By first part of (3) in Theorem 1 in the same paper,

$$\lim_{n\to\infty} \frac{\text{Var}(S_n)}{\sum_{i=1}^n d_{n,i}^2} \to 2\pi f(0)\,.$$

Also let us note that, by Lemma 12.12 in [10], we know that (14) implies

$$\max_{1\leq i\leq n} |d_{n,i}| \to 0\,.$$

Therefore the Lindeberg condition is satisfied.
In order to apply our Theorem 2.1 we have only to verify conditions (4) and (5). Since for $U$ a uniform random variable on $[0,1]$ the variable $|d_{n,i}|Q(U)$ is distributed as $|d_{n,i}X_i|$ and so as $Q_{n,i}(U)$ where $Q_{n,i}$ is the quantile function of $|d_{n,i}X_i|$, we can easily see that condition (4) is verified under (14) and (15). Also, we have

$$\sum_{i=1}^{n} \int_0^{\alpha_2(m)} Q_{i,n}^2(u) du \leq \sum_{i=1}^{n} \int_0^{\alpha_2(m)} |d_{n,i}|^2 Q^2(u) du \leq \sum_{i=1}^{n} |d_{n,i}|^2 \int_0^{\alpha_2(m)} Q^2(u) du\,,$$

which leads to (5) if we assume

$$\alpha_2(m) \to 0. \tag{16}$$

Gathering all these arguments, by applying Theorem 2.1 we obtain the following result:

**Theorem 2.3** *Let $S_n = \sum_{j=1}^n d_{n,j} X_j$, where $d_{n,j}$ are real valued weights and $(X_j)$ is a strictly stationary sequence. Assume that (14), (15) and (16) are satisfied. Then $S_n$ converges in distribution to $\sqrt{2\pi f(0)}|c|N$ where $N$ is a standard Gaussian random variable. Let $v_{k,n}^2 = \sum_{i=1}^k d_{n,i}^2$. Define*

$$v_n(t) = \inf\left\{k;\, 1 \leq k \leq n\colon\; v_{k,n}^2 \geq c^2 t\right\} \text{ and } W_n(t) = \sum_{i=1}^{v_n(t)} d_{n,i} X_i\,.$$

*Then $W_n(\cdot)$ converges weakly to $\sqrt{2\pi f(0)}|c|W$ where $W$ is the standard Brownian motion.*

We shall apply this result to the model of the nonlinear regression with fixed design. Our goal is to estimate the function $\ell(x)$ such that

$$y(x) = \ell(x) + \xi(x),$$

where $\ell$ is an unknown function and $\xi(x)$ is the noise. If we fix the design points $x_{n,i}$ we get

$$Y_{n,i} = y(x_{n,i}) = \ell(x_{n,i}) + \xi_i(x_{n,i}).$$

According to [18], the nonparametric estimator of $\ell(x)$ is defined to be

$$\hat{\ell}_n(x) = \sum_{i=1}^{n} w_{n,i}(x) Y_{n,i} = \sum_{i=1}^{n} w_{n,i}(x)(\ell(x_{n,i}) + \xi_i(x_{n,i})), \tag{17}$$

where

$$w_{n,i} = K\Big(\frac{x_{n,i}-x}{h}\Big) / \sum_{i=1}^{n} K\Big(\frac{x_{n,i}-x}{h}\Big).$$

We shall apply Theorem 2.3 to find necessary conditions for the convergence of the estimator $\hat{\ell}_n(x)$. To fix the ideas we shall consider the following setting: The kernel $K$ is a density function, continuous with compact support $[0,1]$. The design points will be $x_{n,i} = i/n$ and $(\xi_i(x_{n,1}), ...\xi_i(x_{n,i}))$ is distributed as $(X_1, ..., X_n)$, where $(X_n)_{n\in Z}$ is a stationary sequence of centered sequence of random variables satisfying (15) and (16). We shall find the normal asymptotic limit for

$$V_n(x) = \left(\sum_{i=1}^{n} w_{n,i}^2(x)\right)^{-1/2} \left(\hat{\ell}_n(x) - \mathbb{E}(\hat{\ell}_n(x))\right).$$

According to Theorem 2.3, in order to obtain the central limit theorem we have just to verify the conditions satisfied by the sequence of constants (14). Re-denoting

$$d_{n,i} = w_{n,i}(x)\Big(\sum_{i=1}^{n} w_{n,i}^2(x)\Big)^{-1/2},$$

we have to show that

$$\Big(\sum_{i=1}^{n} w_{n,i}^2(x)\Big)^{-1} \sum_{i=1}^{n} (w_{n,i} - w_{n,i-1})^2 \to 0.$$

Simple computations show that

$$nh \sum_{i=1}^{n} w_{n,i}^2(x) \sim \frac{1}{nh} \sum_{i=1}^{n} K^2\Big(\frac{i/n - x}{h}\Big) \sim \frac{1}{nh} \sum_{j=1}^{nh} K^2\Big(\frac{j}{nh}\Big) \to \int_0^1 K^2(v)dv = c^2.$$

Furthermore

$$nh \sum_{i=1}^{n} (w_{n,i} - w_{n,i-1})^2 \sim \frac{1}{nh} \sum_{i=1}^{n} \left(K\Big(\frac{i/n - x}{h}\Big) - K\Big(\frac{(i-1)/n - x}{h}\Big)\right)^2$$

$$\sim \frac{1}{nh} \sum_{i=1}^{nh} \left(K\Big(\frac{j}{nh}\Big) - K\Big(\frac{j}{nh} - \frac{1}{nh}\Big)\right)^2 \leq \sup_x \left(K(x) - K\Big(x - \frac{1}{nh}\Big)\right)^2 \to 0,$$

since $K$ is uniformly continuous and $nh \to \infty$.
So, we shall unify our computation in the following Theorem.

**Theorem 2.4** *Assume for $x$ fixed that $\hat{\ell}_n(x)$ in defined by (17) and the sequence $(X_j)$ is a stationary sequence satisfying (15), (16). Assume that the kernel $K$ is a density, it is square integrable, has compact support and is continuous. Assume $nh \to \infty$ and $h \to 0$. Then $\sqrt{nh}(\hat{\ell}_n(x) - \mathbb{E}(\hat{\ell}_n(x)))$ converges in distribution to $\sqrt{2\pi f(0)}|c|N$ where $N$ is a standard Gaussian random variable and $c^2$ is the second moment of $K$.*

3. *Functions of a triangular stationary Markov chain.* For any positive integer $n$, $(\xi_{i,n})_{i\geq 0}$ is an homogeneous Markov chain with state space $\mathbb{N}$ and transition probabilities given by

$$\mathbb{P}(\xi_{1,n} = i|\xi_{0,n} = i+1) = 1 \text{ and } \mathbb{P}(\xi_{1,n} = i|\xi_{0,n} = 0) = p_{i+1,n} \text{ for } i \geq 1,$$

where, for $i \geq 2$, $p_{i,n} = c_a/(v_n i^{a+2})$ with $a > 0$, $c_a \sum_{i\geq 2} 1/i^{a+2} = 1/2$, $(v_n)_{n\geq 1}$ a sequence of positive reals and $p_{1,n} = 1 - 1/(2v_n)$. $(\xi_{i,n})_{i\geq 0}$ has a stationary distribution $\pi_n = (\pi_{j,n})_{j\geq 0}$ satisfying

$$\pi_{0,n} = \Big(\sum_{i\geq 1} i p_{i,n}\Big)^{-1} \text{ and } \pi_{j,n} = \pi_{0,n} \sum_{i\geq j+1} p_{i,n} \text{ for } j \geq 1.$$

Let $Y_{i,n} = I_{\xi_{i,n}=0} - \pi_{0,n}$. Let $b_n^2 = \mathrm{Var}\Big(\sum_{k=1}^n Y_{k,n}\Big)$ and set $X_{i,n} = Y_{i,n}/b_n$. Note that

$$\mathrm{Var}\Big(\sum_{k=1}^{n} Y_{k,n}\Big) = n\pi_{0,n}(1-\pi_{0,n}) + 2\sum_{k=1}^{n-1}(n-k)\pi_{0,n}\Big(\mathbb{P}(\xi_{k,n} = 0|\xi_{0,n} = 0) - \pi_{0,n}\Big).$$

Assume now that $a > 1$. Usual computations imply that there exists $\sigma^2 > 0$ such that $b_n^2 \sim \sigma^2 n v_n^{-1}$. It is easy to see that the first part of condition (3) is satisfied whereas the second part holds provided that $v_n/n \to 0$. Now $\bar\alpha_n(k) \ll \sum_{j\geq k}\sum_{i\geq j+1} p_{i,n} \ll 1/(v_n k^a)$ (see [4, Th. 5] and [3, Chap. 30]). Hence (6) holds by taking into account that $Q_{k,n}(u) \leq 1/b_n$. Then, provided that $a > 1$ and $v_n/n \to 0$, $(X_{k,n})_{k>0}$ satisfies the functional central limit theorem given in Theorem 2.1.

## 3 Proof of Theorem 2.1

Without loss of generality, we assume that $X_{k,n} = 0$ for $k > n$ and $\mathcal{F}_{k,n} = \mathcal{F}_{n,n}$ for $k > n$. Moreover, by abuse of notation, we will often avoid the index $n$. In particular, we shall write $X_k = X_{k,n}$ and $\mathcal{F}_k = \mathcal{F}_{k,n}$, and use the notation

$$\mathbb{E}_j(X) = \mathbb{E}(X|\mathcal{F}_j)\,.$$

For each $n$, let also $S_n = \sum_{k=1}^n X_k$ and $S_0 = 0$.
Let $m$ be a fixed positive integer such that $m < n$. Let us then define

$$\theta_\ell^m = \frac{1}{m}\sum_{i=1}^{m-1}\mathbb{E}_\ell(X_{\ell+1} + \ldots + X_{\ell+i})\,,$$

and

$$Y_\ell^m = \frac{1}{m}\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\,,\; R_k^m = \sum_{\ell=0}^{k-1} Y_\ell^m\,.$$

According to the proof of [9, Lemma 5.3], since (3) is assumed, the theorem will follow provided

$$(H) := \begin{cases} \lim_{m\to\infty}\limsup_{n\geq 1}\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2 = 0\,,\\ \lim_{m\to\infty}\limsup_{n\geq 1}\|\max_{1\leq k\leq n}|R_k^m|\|_2 = 0\,,\\ \lim_{m\to\infty}\limsup_{n\geq 1}\sum_{k=0}^{n-1}\|\theta_k^m\|_2\|Y_k^m\|_2 = 0\,,\end{cases}$$

and that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\Big|\sum_{k=1}^{v_n(t)}\big(X_{k,n}^2 + 2X_{k,n}\theta_{k,n}^m\big) - t\Big| > \varepsilon\Big) = 0\,. \quad (18)$$

Moreover, recall that $\mathcal{F}_{0,n} = \{\Omega, \emptyset\}$ and then $\mathbb{E}_{0,n}(\cdot) = \mathbb{E}(\cdot)$. Hence, according to Proposition 3.2 in [9], under the Lindeberg condition (3), condition (18) is satisfied as soon as

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big| \sum_{k=1}^{v_n(t)} \big(\mathbb{E}(X_{k,n}^2) + 2\mathbb{E}(X_{k,n}\theta_{k,n}^m)\big) - t \Big| = 0 \tag{19}$$

and, for any non-negative integer $\ell$,

$$\lim_{b\to\infty} \limsup_{n\to\infty} \sum_{k=b+1}^{n} \|\mathbb{E}_{k-b,n}(X_{k,n}X_{k+\ell,n}) - \mathbb{E}(X_{k,n}X_{k+\ell,n})\|_1 = 0\,. \tag{20}$$

In the rest of the proof, we show that $(H)$, (19) and (20) are satisfied under the conditions of our theorem. With this aim, we start with the following lemma whose proof is given later.

**Lemma 3.1** *For any integer $\ell \geq 0$ and any positive integer $m$,*

$$\sum_{\ell=0}^{n-1} \|\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\|_2^2 \leq C \sum_{k=1}^{n}\sum_{i=1}^{m} i \int_0^{\alpha_{1,n}(i)} Q_{k,n}^2(u)du\,.$$

Let us verify the first condition in $(H)$. We have

$$\|Y_\ell^m\|_2^2 = \frac{1}{m^2}\|\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\|_2^2\,.$$

Hence, by Lemma 3.1,

$$\begin{aligned} m\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2 = \frac{1}{m}\sum_{\ell=0}^{n-1}\|\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\|_2^2 &\ll \frac{1}{m}\sum_{\ell=1}^{n}\sum_{i=1}^{m} i\int_0^{\alpha_{1,n}(i)} Q_{\ell,n}^2(u)du \\ &\ll \frac{1}{\sqrt{m}}\sum_{\ell=1}^{n}\int_0^1 Q_{\ell,n}^2(u)du + \sum_{\ell=1}^{n}\sum_{i=[m^{1/4}]+1}^{n}\int_0^{\alpha_{1,n}(i)} Q_{\ell,n}^2(u)du\,. \end{aligned}$$

Taking first the supremum over $n$ and then the limit over $m$ and taking into account the first part of condition (3) and condition (4), it follows that

$$\lim_{m\to\infty}\limsup_{n\geq 1} m\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2 = 0\,, \tag{21}$$

which proves in particular the first condition in $(H)$.
Now, by the Dedecker-Rio's maximal inequality [6], we have

$$\|\max_{1\leq k\leq n}|R_k^m|\|_2^2 \ll \sum_{k=0}^{n-1}\|Y_k^m\|_2^2 + \sum_{k=0}^{n-1}\|Y_k^m\mathbb{E}_k(R_n^m - R_k^m)\|_1\,.$$

As previously showed, taking first the supremum over $n$ and then the limit over $m$, the first term in the right-hand side is going to zero. To handle the second term, we note that, by the properties of the conditional expectations and Inequality (4.6) in Rio [16],

$$\|\mathbb{E}_k(X_j)\mathbb{E}_k(X_i)\|_1 \leq \|\mathbb{E}_k(X_j)X_i\|_1 \ll \int_0^{\alpha_{1,n}(i-k)} Q_{i,n}(u)Q_{j,n}(u)du\,.$$

Therefore

$$\sum_{k=0}^{n-1} \|Y_k^m \mathbb{E}_k(R_n^m - R_k^m)\|_1 \ll \frac{1}{m^2} \sum_{k=0}^{n-1} \sum_{j=k+1}^{k+m} \sum_{\ell=k}^{n-1} \sum_{i=\ell+1}^{\ell+m} \int_0^{\alpha_{1,n}(i-k)} Q_{i,n}(u) Q_{j,n}(u) du\,.$$

Easy computations yield

$$\begin{aligned}
\sum_{k=0}^{n-1} \|Y_k^m \mathbb{E}_k(R_n^m - R_k^m)\|_1 &\ll \frac{1}{m^2} \sum_{k=0}^{n-1} \sum_{j=k+1}^{k+m} \sum_{i=1}^{m-1} i \int_0^{\alpha_{1,n}(i)} Q_{i+k,n}(u) Q_{j,n}(u) du \\
&\quad + \frac{1}{m} \sum_{k=0}^{n-1} \sum_{j=k+1}^{k+m} \sum_{i=k+m}^{n} \int_0^{\alpha_{1,n}(i-k)} Q_{i,n}(u) Q_{j,n}(u) du \\
&:= I_1(m,n) + I_2(m,n)\,.
\end{aligned}$$

Using that $2Q_{i+k,n}(u)Q_{j,n}(u) \le Q^2_{i+k,n}(u) + Q^2_{j,n}(u)$ and the fact that $X_{i,n} = 0$ for $i > n$, we easily derive

$$I_1(m,n) \le \frac{1}{m} \sum_{j=1}^{n} \sum_{i=1}^{m-1} i \int_0^{\alpha_{1,n}(i)} Q^2_{j,n}(u) du\,.$$

As a consequence we have

$$I_1(m,n) \le \frac{1}{\sqrt{m}} \sum_{j=1}^{n} \int_0^1 Q^2_{j,n}(u) du + \sum_{j=1}^{n} \sum_{i=[m^{1/4}]+1}^{n} \int_0^{\alpha_{1,n}(i)} Q^2_{j,n}(u) du\,.$$

Taking first the supremum over $n$ and then the limit over $m$, by (4) and the first part of condition (3), we get

$$\lim_{m\to\infty} \limsup_{n\ge 1} I_1(m,n) = 0\,.$$

On the other hand, using again that $2Q_{i,n}(u)Q_{j,n}(u) \le Q^2_{i,n}(u) + Q^2_{j,n}(u)$ and that $X_{i,n} = 0$ for $i > n$, we infer that

$$\begin{aligned}
I_2(m,n) &\le \frac{1}{2} \sum_{i=m}^{n} \sum_{k=0}^{i-m} \int_0^{\alpha_{1,n}(i-k)} Q^2_{i,n}(u) du + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=m}^{n} \int_0^{\alpha_{1,n}(i)} Q^2_{j,n}(u) du \\
&\le \sum_{j=1}^{n} \sum_{i=m}^{n} \int_0^{\alpha_{1,n}(i)} Q^2_{j,n}(u) du\,.
\end{aligned}$$

Hence, by condition (4), it follows that

$$\lim_{m\to\infty} \limsup_{n\ge 1} I_2(m,n) = 0\,.$$

So, overall, the second part of condition $(H)$ holds.
We show now that the third part of condition $(H)$ is satisfied. We have

$$\begin{aligned}
A(m,n) := \sum_{\ell=0}^{n-1} \|\theta_\ell^m\|_2 \|Y_\ell^m\|_2 &\le \frac{1}{m^2} \sum_{i=1}^{m-1} \sum_{\ell=0}^{n-1} \|\mathbb{E}_\ell(S_{\ell+i} - S_\ell)\|_2 \|\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\|_2 \\
&\le \frac{1}{m^2} \sum_{i=1}^{m-1} \Big( \sum_{\ell=0}^{n-1} \|\mathbb{E}_\ell(S_{\ell+i} - S_\ell)\|_2^2 \Big)^{1/2} \Big( \sum_{\ell=0}^{n-1} \|\mathbb{E}_\ell(S_{\ell+m} - S_\ell)\|_2^2 \Big)^{1/2}\,.
\end{aligned}$$

Hence, by Lemma 3.1,

$$A(m,n) \ll \frac{1}{m^2}\sum_{i=1}^{m-1}\Big(\sum_{\ell=1}^{n}\sum_{k=1}^{i} k\int_0^{\alpha_{1,n}(k)} Q^2_{\ell,n}(u)du\Big)^{1/2}\Big(\sum_{\ell=1}^{n}\sum_{k=1}^{m} k\int_0^{\alpha_{1,n}(k)} Q^2_{\ell,n}(u)du\Big)^{1/2}$$
$$\ll \frac{1}{m}\sum_{\ell=1}^{n}\sum_{k=1}^{m} k\int_0^{\alpha_{1,n}(k)} Q^2_{\ell,n}(u)du\,,$$

Therefore, as previously showed, by condition (4), the third part of condition $(H)$ holds.
It remains to prove that (19) and (20) are satisfied. The proof of (19) follows by using the same arguments as those developed at the end of the proof of Theorem 4.1 in [9] and, in particular, that the Lindeberg's condition (3) and the definition of $v_n(t)$ imply

$$\mathbb{E}(S^2_{v_n(t),n}) \to t\,, \text{ as } n\to\infty.$$

We prove now that (20) is satisfied. According to the computations made on page 204 in [10] to handle their term $\|B_{k,0}\|_1$, we have, for any non-negative $\ell$ and any $k \geq b+1$,

$$\|\mathbb{E}_{k-b,n}(X_{k,n}X_{k+\ell,n}) - \mathbb{E}(X_{k,n}X_{k+\ell,n})\|_1 \leq 2^4\int_0^{\alpha_{2,n}(b)} Q_{k,n}(u)Q_{k+\ell,n}(u)du\,.$$

Since $X_{i,n} = 0$ for $i > n$, it follows that

$$\sum_{k=b+1}^{n}\|\mathbb{E}_{k-b,n}(X_{k,n}X_{k+\ell,n}) - \mathbb{E}(X_{k,n}X_{k+\ell,n})\|_1 \leq 2^5\sum_{k=1}^{n}\int_0^{\alpha_{2,n}(b)} Q^2_{k,n}(u)du\,,$$

which proves (20) by taking into account (5).
To end the proof of the theorem, it remains to prove Lemma 3.1
**Proof of Lemma 3.1**. By Rio's covariance inequality [14], we have

$$\|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)\|_2^2 = \mathrm{Cov}\big(\mathbb{E}_\ell(S_{\ell+m}-S_\ell), S_{\ell+m}-S_\ell\big)$$
$$\ll \int_0^1 \sum_{j=\ell+1}^{\ell+m}\mathbf{1}_{u<\alpha_{1,n}(j-\ell)}Q_{j,n}(u)Q_{|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)|}(u)du$$
$$\ll \|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)\|_2\Big(\int_0^1\Big(\sum_{j=\ell+1}^{\ell+m}\mathbf{1}_{u<\alpha_{1,n}(j-\ell)}Q_{j,n}(u)\Big)^2du\Big)^{1/2},$$

Hence

$$\|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)\|_2^2 \ll 2\int_0^1\sum_{j=1}^{m}\sum_{k=j}^{m}\mathbf{1}_{u<\alpha_{1,n}(k)}Q_{j+\ell,n}(u)Q_{k+\ell,n}(u)du\,.$$

Using that $2Q_{j+\ell,n}(u)Q_{k+\ell,n}(u) \leq Q^2_{j+\ell,n}(u) + Q^2_{k+\ell,n}(u)$, we get

$$\|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)\|_2^2 \ll \int_0^1\sum_{k=1}^{m}\mathbf{1}_{u<\alpha_{1,n}(k)}\sum_{j=1}^{k}Q^2_{j+\ell,n}(u)du + \int_0^1\sum_{k=1}^{m}k\mathbf{1}_{u<\alpha_{1,n}(k)}Q^2_{k+\ell,n}(u)du\,.$$

Since $X_{i,n} = 0$ for $i > n$, it follows that

$$\sum_{\ell=0}^{n-1}\|\mathbb{E}_\ell(S_{\ell+m}-S_\ell)\|_2^2 \ll \sum_{\ell=1}^{n}\sum_{i=1}^{m} i\int_0^{\alpha_{1,n}(i)} Q^2_{\ell,n}(u)du\,,$$

which ends the proof of the lemma. □
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