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Functional CLT for martingale-like nonstationary dependent structures

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In this paper we develop non-stationary martingale techniques for dependent data. We shall stress the non-stationary version of the projective Maxwell-Woodroofe condition, which will be essential for obtaining maximal inequalities and functional central limit theorem for the following examples: nonstationary ρ-mixing sequences, functions of linear processes with non-stationary innovations, locally stationary processes, quenched version of the functional central limit theorem for a stationary sequence, evolutions in random media such as a process sampled by a shifted Markov chain.

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1. Introduction

Historically, one of the most celebrated limit theorems in non-stationary setting is the functional central limit theorem for non-stationary sequences of martingale differences. For more general dependent sequences, one of the basic techniques is to approximate them with martingales by using projection operators. A remarkable early result obtained by using this technique is due to Dobrushin (1956), who studied non-stationary Markov chains. Later, the technique was also used for Markov chains, in Sethuraman and Varadhan (2005) and in Peligrad (2012). In order to treat more general dependent structures, McLeish (1975, 1977) introduced the notion of mixingales, which are martingale-like structures involving conditions imposed to the bounds of the moments of projections of an individual variable on past sigma fields. This method is very fruitful, but still involves a large degree of stationarity and complicated additional assumptions. In general, the theory of non-stationary martingale approximation is much more difficult and it has remained behind the theory of martingale methods for stationary processes. In the stationary setting, the theory of martingale approximations was steadily developed. We mention the well-known results, such as the celebrated results by Gordin (1969),

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Heyde (1974), Maxwell and Woodroofe (2000) and newer results by Peligrad and Utev (2005), Zhao and Woodroofe (2008), Gordin and Peligrad (2011), among many others. Inspired by these ideas and using a direct martingale approach, we derive alternative conditions to the mixingale-type conditions imposed by McLeish. Our projective conditions lead to a non-stationary version of the weak invariance principle under the so-called Maxwell-Woodroofe condition, which is known to be very sharp. Surprisingly, also, is the fact that our approach leads directly to the quenched invariance principle under the Maxwell-Woodroofe condition, which was first obtained by Cuny and Merlevède (2014) with a completely different proof. In addition, our approach is also efficient enough to get the functional version of the central limit theorem for ρ-mixing sequences satisfying the Lindeberg condition established in Utev (1990). For this class, we completely answer an open problem raised by Ibragimov in 1991. Other applications we shall consider are functions of linear processes with nonstationary innovations, locally stationary processes and evolutions in random media, such as a process sampled by a shifted Markov chain.

We begin by treating nonstationary sequences with the near linear behavior of the variance of the partial sums. Then, we discuss the general non-stationary triangular arrays and give the above mentioned applications. The proofs are given in Section 5.

2. Results under the normalization $\sqrt{n}$

Let $(X_k)_{k\geq 1}$ be a sequence of centered real-valued random variables in $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ and set $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$ and $S_0 = 0$. Let $(\mathcal{F}_i)_{i\geq 0}$ be a non-decreasing sequence of σ-algebras such that X_i is $\mathcal{F}_i$-measurable for any $i \geq 1$. The following notation will be often used: $\mathbb{E}_k(X) := \mathbb{E}(X|\mathcal{F}_k)$. In the sequel we denote by $D([0,1])$ the space of functions defined on $[0,1]$, right continuous, with finite left hand limits, which is endowed with uniform topology and by $[x]$ the integer part of x. For any $k \geq 0$ let

$$\delta(k) = \max_{i\geq 0} \|\mathbb{E}(S_{k+i} - S_i|\mathcal{F}_i)\|_2 \,, \tag{2.1}$$

and for any $k, m \geq 0$ let

$$\theta_k^m = m^{-1} \sum_{i=1}^{m-1} \mathbb{E}_k(S_{k+i} - S_k) \,.$$

To get the functional form of the central limit theorem under the normalization $\sqrt{n}$, we shall assume the Lindeberg condition in the form

$$\sup_{n\geq 1} \frac{1}{n} \sum_{j=1}^n \mathbb{E}(X_j^2) \leq C < \infty, \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E}\{X_k^2 I(|X_k| > \varepsilon\sqrt{n})\} = 0 \,, \text{ for any } \varepsilon > 0. \tag{2.2}$$

Our first result is in the spirit of Theorem 2.4 in McLeish (1977) and gives sufficient conditions to ensure that the partial sums behave asymptotically like a martingale. As we shall see, next theorem is a corollary of Theorem 3.1 of the next section which is using a normalization more general than $\sqrt{n}$.

Theorem 2.1. *Assume that the Lindeberg condition (2.2) holds. Suppose in addition that*

$$\sum_{k\geq 0} 2^{-k/2}\delta(2^k) < \infty \tag{2.3}$$

and there exists a constant c^2 such that, for any $t \in [0,1]$ and any $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{[nt]} \big(X_k^2 + 2X_k\theta_k^m\big) - tc^2\Big| > \varepsilon\Big) = 0\,. \tag{2.4}$$

Then $\big\{n^{-1/2}\sum_{k=1}^{[nt]} X_k, t\in[0,1]\big\}$ converges in distribution in $D([0,1])$ to cW where W is a standard Brownian motion.

***Remark* 2.1.** Note that by the subadditivity property of the sequence $(\delta(k))_{k\geq 0}$, condition (2.3) is equivalent to

$$\sum_{k\geq 1} k^{-3/2}\delta(k) < \infty\,. \tag{2.5}$$

Moreover condition (2.3) holds under the stronger assumption:

$$\sum_{k\geq 1} k^{-1/2} \sup_{i\geq 0} \|\mathbb{E}(X_{k+i}|\mathcal{F}_i)\|_2 < \infty\,. \tag{2.6}$$

***Comment* 2.1.** Using the Cramér-Wold device, we infer that Theorem 2.1 can be extended to the multivariate setting as follows. Assume that $(X_k)_{k\geq 1}$ is a sequence of centered $\mathbb{R}^d$-valued random variables in $\mathbb{L}^2(\Omega,\mathcal{A},\mathbb{P})$. Assume that, for any $\lambda\in\mathbb{R}^d$, the sequence of real-valued random variables $(\lambda\cdot X_k)_{k\geq 1}$ satisfies conditions (2.2), (2.3) and (2.4) with $c^2 = \sigma^2(\lambda)$. Then $\big\{n^{-1/2}\sum_{k=1}^{[nt]} X_k, t\in[0,1]\big\}$ converges in distribution in $D([0,1])$ to $\Sigma^{1/2}W$ where W is a standard Brownian motion on $\mathbb{R}^d$ and $\Sigma = (\sigma_{i,j})_{i,j=1}^d$ is a positive definite symmetric matrix whose entries can be defined as follows: $\sigma_{i,j} = \frac{1}{2}\Big\{\sigma^2(e_i+e_j) - \sigma^2(e_i) - \sigma^2(e_j)\Big\}$ where $(e_1,\dots,e_d)$ is the canonical basis of $\mathbb{R}^d$. Note that the Gaussian approximation for non-stationary multiple time series that are functions of an iid sequence has been obtained by Wu and Zhou (2011), but the conditions of their paper and ours have different range of applications. Indeed, their result is restricted to functions of an iid sequence and their dependence condition is stronger than $\sum_{k\geq 1}\sup_{i\geq 0}\|\mathbb{E}(X_{k+i}|\mathcal{F}_i) - \mathbb{E}(X_{k+i}|\mathcal{F}_{i-1})\|_2 < \infty$. This latter condition is known not to be comparable with (2.3) (see for instance Durieu (2009)).

For stationary sequences, as a corollary to Theorem 2.1, we obtain:

Corollary 2.2. *Let $(X_n)_{n\in\mathbb{Z}}$ be an ergodic stationary sequence of centered random variables with finite second moment, which is adapted to a stationary filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$. Assume that*

$$\sum_{k\geq 0} 2^{-k/2}\|\mathbb{E}_0(S_{2^k})\|_2 < \infty\,, \tag{2.7}$$

Then, $\lim_{m\to\infty} m^{-1}\mathbb{E}(S_m^2)=c^2$ *and the conclusion of Theorem 2.1 holds.*

Note that condition (2.7) is equivalent to $\sum_{k\geq 1} k^{-3/2}\|\mathbb{E}_0(S_k)\|_2<\infty$ and known under the name of Maxwell-Woodroofe condition. Under this condition, Maxwell and Woodroofe (2000) obtained a CLT. Later, Peligrad and Utev (2005) have shown that this condition is, in some sense, minimal in order for the sequence $(S_n/\sqrt{n})_{n\geq 1}$ to be stochastically bounded and they proved a maximal inequality and convergence to the Brownian motion. In order to derive this corollary from Theorem 2.1 we use the fact that $\delta(k)=\|\mathbb{E}(S_k|\mathcal{F}_0)\|_2$ and then condition (2.3) reads as condition (2.7). In addition, for $k\geq 0$, we get, by the ergodic theorem,

$$\lim_{n\to\infty}\mathbb{E}\Big|\frac{1}{n}\sum_{k=1}^{[nt]}(X_k^2+2X_k\theta_k^m)-c^2t\Big|=t\big|\mathbb{E}X_0^2+2\mathbb{E}(X_0\theta_0^m)-c^2\big|\,.$$

It remains to take into account that

$$\frac{1}{m}\mathbb{E}(S_m^2)=\mathbb{E}(X_0^2)+\frac{2}{m}\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}\mathbb{E}(X_0X_j)=\mathbb{E}(X_0^2)+2\mathbb{E}(X_0\theta_0^m)\,,$$

proving the corollary since it has been shown in Peligrad and Utev (2005) that, in the stationary setting, condition (2.7) implies that $\lim_{m\to\infty} m^{-1}\mathbb{E}(S_m^2)$ exists.

3. Results for general triangular arrays

Let $\{X_{i,n}, 1\leq i\leq n\}$ be a triangular array of square integrable ($\mathbb{E}(X_{i,n}^2)<\infty$), centered ($\mathbb{E}(X_{i,n})=0$), real-valued random variables adapted to a filtration $(\mathcal{F}_{i,n})_{i\geq 0}$. We write as before $\mathbb{E}_{j,n}(X)=\mathbb{E}(X|\mathcal{F}_{j,n})$ and set

$$S_{k,n}=\sum_{i=1}^{k}X_{i,n}\ \text{ and }\ \theta_{k,n}^m=m^{-1}\sum_{i=1}^{m-1}\mathbb{E}_{k,n}(S_{k+i,n}-S_{k,n})\,.$$

We shall assume that the triangular array satisfies the following Lindeberg condition:

$$\sup_{n\geq 1}\sum_{j=1}^{n}\mathbb{E}(X_{j,n}^2)\leq C<\infty,\ \text{ and }\ \lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}\{X_{k,n}^2 I(|X_{k,n}|>\varepsilon)\}=0\,,\ \text{ for any }\varepsilon>0. \tag{3.1}$$

Moreover, for a non-negative integer u and positive integers ℓ,m, define martingale-type dependence characteristics by

$$A^2(u)=\sup_{n\geq 1}\sum_{k=0}^{n-1}\|\mathbb{E}_{k,n}(S_{k+u,n}-S_{k,n})\|_2^2 \tag{3.2}$$

and

$$B^2(\ell, m) = \sup_{n\geq 1} \sum_{k=0}^{[n/\ell]} \|\bar{S}_{k,n}(\ell, m)\|_2^2 \,, \tag{3.3}$$

where

$$\bar{S}_{k,n}(\ell, m) = \frac{1}{m} \sum_{u=0}^{m-1} \big(\mathbb{E}_{(k-1)\ell+1,n}(S_{(k+1)\ell+u,n} - S_{k\ell+u,n})\big) \,.$$

Our next theorem provides a general functional CLT under the Lindeberg condition for martingale-like nonstationary triangular arrays.

Theorem 3.1. *Assume that the Lindeberg condition (3.1) holds and that*

$$\lim_{j\to\infty} 2^{-j/2} A(2^j) = 0 \text{ and } \liminf_{j\to\infty} \sum_{\ell\geq j} B(2^\ell, 2^j) = 0 \,. \tag{3.4}$$

Moreover, assume in addition that there exist a sequence of non-decreasing and right-continuous functions $v_n(\cdot) : [0,1] \to \{0, 1, 2, \dots, n\}$ and a non-negative Lebesgue integrable function $\sigma^2(\cdot)$ on $[0,1]$, such that, for any $t \in (0,1]$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{k=1}^{v_n(t)} \big(X_{k,n}^2 + 2X_{k,n}\theta_{k,n}^m\big) - \int_0^t \sigma^2(u)du\Big| > \varepsilon\Big) = 0 \,. \tag{3.5}$$

Then $\big\{ \sum_{k=1}^{v_n(t)} X_{k,n}, t \in [0,1]\big\}$ converges in distribution in $D([0,1])$ to $\big\{ \int_0^t \sigma(u)dW(u), t \in [0,1]\big\}$ where W is a standard Brownian motion.

The following proposition is useful for verifying condition (3.5).

Proposition 3.2. *Assume that the Lindeberg condition (3.1) holds. Assume in addition that for any non-negative integer ℓ,*

$$\lim_{b\to\infty} \limsup_{n\to\infty} \sum_{k=b+1}^{n} \|\mathbb{E}_{k-b,n}(X_{k,n}X_{k+\ell,n}) - \mathbb{E}_{0,n}(X_{k,n}X_{k+\ell,n})\|_1 = 0 \tag{3.6}$$

and, for any $t \in [0,1]$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{k=1}^{v_n(t)} \big(\mathbb{E}_{0,n}(X_{k,n}^2) + 2\mathbb{E}_{0,n}(X_{k,n}\theta_{k,n}^m)\big) - \int_0^t \sigma^2(u)du\Big| > \varepsilon\Big) = 0 \,. \tag{3.7}$$

Then the convergence (3.5) holds.

Let us apply the general Theorem 3.1 to the sequences of random variables when the normalizing sequence is $\sqrt{n}$. For any non-negative integer u and any positive integers ℓ and m, let $a(u)$ and $b(\ell, m)$ be the non-negative quantities defined by

$$a^2(u) = \sup_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \|\mathbb{E}_k(S_{k+u} - S_k)\|_2^2 \quad , \quad b^2(\ell, m) = \sup_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \|\bar{S}_k(\ell, m)\|_2^2 ,$$

$$\text{where} \quad \bar{S}_k(\ell, m) = \frac{1}{m} \sum_{u=0}^{m-1} \big(\mathbb{E}_{(k-1)\ell+1}(S_{(k+1)\ell+u} - S_{k\ell+u})\big) .$$

The conditions are an average version of condition (2.3). They are particularly useful for the analysis of quenched limit theorems. By applying Theorem 3.1 to the triangular array $X_{k,n} = X_k/\sqrt{n}$; $1 \leq k \leq n$ and $v_n(t) = [nt]$ we obtain the following corollary:

Corollary 3.3. *The statement of Theorem 2.1 holds when condition (2.3) is replaced by the following conditions*

$$\lim_{j\to\infty} 2^{-j/2} a(2^j) = 0 \text{ and } \liminf_{j\to\infty} \sum_{\ell \geq j} 2^{-\ell/2} b(2^\ell, 2^j) = 0 . \tag{3.8}$$

By using the definition of $(\delta(k))_{k\geq 1}$ in (2.1), the subadditivity of this sequence and Proposition 2.5 in Peligrad and Utev (2005) we note that condition (2.3) implies that $\lim_{j\to\infty} 2^{-j/2} a(2^j) = 0$. Moreover, condition (2.3) easily implies the second part of condition (3.8). By using this remark we can see that Theorem 2.1 is a consequence of Corollary 3.3. We elected to present the results first for sequences of random variables and then for triangular arrays, for stressing the fact that our results are generalization to nonstationary sequences of the important results in the stationary setting involving condition (2.7). The results are also related to conditions in McLeish (1975, 1977). Our approach uses a suitable martingale approximation whereas McLeish (1975, 1977) proved first tightness of the partial sum process and then he identified the limit by using a suitable characterization of the Wiener process given in Theorem 19.4 in Billingsley (1968).

4. Applications

4.1. ρ-mixing triangular arrays and sequences

For a triangular array $\{X_{i,n}, 1 \leq i \leq n\}$ of square integrable ($\mathbb{E}(X_{i,n}^2) < \infty$), centered ($\mathbb{E}(X_{i,n}) = 0$), real-valued random variables, we denote by $\sigma_{k,n}^2 = \mathrm{Var}\big(\sum_{\ell=1}^k X_{\ell,n}\big)$ for $k \leq n$ and $\sigma_n^2 = \sigma_{n,n}^2$. For $0 \leq t \leq 1$, let

$$v_n(t) = \inf\Big\{k;\, 1 \leq k \leq n \colon \frac{\sigma_{k,n}^2}{\sigma_n^2} \geq t\Big\} \text{ and } W_n(t) = \sigma_n^{-1} \sum_{i=1}^{v_n(t)} X_{i,n} .$$

Define also $S_k = S_{k,n} = \sum_{i=1}^{k} X_{i,n}$. In this section we assume that the triangular array is ρ-mixing in the sense that

$$\rho(k) = \sup_{n\geq 1} \max_{1\leq j\leq n-k} \rho\big(\sigma(X_{i,n}, 1 \leq i \leq j), \sigma(X_{i,n}, j+k \leq i \leq n)\big) \to 0\,,\text{ as } k \to \infty\,,$$

where $\sigma(X_t, t \in A)$ is the σ-field generated by the r.v.'s X_t with indices in A and we recall that the maximal correlation coefficient $\rho(\mathcal{U}, \mathcal{V})$ between two σ-algebras is defined by

$$\rho(\mathcal{U}, \mathcal{V}) = \sup\{|\mathrm{corr}(X, Y)| : X \in \mathbb{L}^2(\mathcal{U}), Y \in \mathbb{L}^2(\mathcal{V})\}\,.$$

Next result gives the functional version of the central limit theorem for ρ-mixing sequences satisfying the Lindeberg condition established in Theorem 4.1 in Utev (1990). It answers an open question raised by Ibragimov in 1991.

Theorem 4.1. *Suppose that*

$$\sup_{n\geq 1} \sigma_n^{-2} \sum_{j=1}^{n} \mathbb{E}(X_{j,n}^2) \leq C < \infty\,, \tag{4.1}$$

and

$$\lim_{n\to\infty} \sigma_n^{-2} \sum_{k=1}^{n} \mathbb{E}\{X_{k,n}^2 I(|X_{k,n}| > \varepsilon\sigma_n)\} = 0\,,\ \textit{ for any } \varepsilon > 0\,. \tag{4.2}$$

Assume in addition that

$$\sum_{k\geq 0} \rho(2^k) < \infty\,. \tag{4.3}$$

Then $\big\{W_n(t), t \in (0,1]\big\}$ *converges in distribution in* $D([0,1])$ *(equipped with the uniform topology) to* W *where* W *is a standard Brownian motion.*

For the $\rho-$mixing sequences we also obtain the following corollary:

Corollary 4.2. *Let* $(X_n)_{n\geq 1}$ *be a sequence of centered random variables in* $\mathbb{L}^2(\mathbb{P})$. *Let* $S_n = \sum_{k=1}^{n} X_k$ *and* $\sigma_n^2 = \mathrm{Var}(S_n)$. *Suppose that conditions* (4.1), (4.2) *and* (4.3) *are satisfied. In addition assume that* $\sigma_n^2 = nh(n)$ *where* h *is a slowly varying function at infinity. Then* $W_n = \big\{\sigma_n^{-1} \sum_{k=1}^{[nt]} X_k, t \in (0,1]\big\}$ *converges in distribution in* $D([0,1])$ *to* W *where* W *is a standard Brownian motion.*

It should be noted that if W_n converges weakly to a standard Brownian motion, then necessarily $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function (i.e. a regularly varying function with exponent 1). This is so since for $t \in [0,1]$ fixed we have $S_{[nt]}/\sigma_n \to^d N(0,t)$ and in addition, taking $t = 1$ we have S_n^2/σ_n^2 is uniformly integrable (by the convergence of moments theorem), implying $\sigma_{[nt]}^2/\sigma_n^2 \to t$. In the stationary case, let us mention that the functional CLT under (4.3) has been obtained by Shao (1989).

***Comment* 4.1.** If in Corollary 4.2 above we assume that $\sigma_n^2 = n^\alpha h(n)$ where $\alpha > 0$ and h is a slowly varying function at infinity, then the proof reveals that, under (4.1), (4.2) and (4.3), $\{\sigma_n^{-1}\sum_{k=1}^{[nt]} X_k, t \in [0,1]\}$ converges in distribution in $D([0,1])$ to $\{G(t), t \in [0,1]\}$ where, for any $t \in [0,1]$, $G(t) = \sqrt{\alpha}\int_0^t u^{(\alpha-1)/2} dW(u)$ with W a standard Brownian motion.

4.2. Functions of linear processes

Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued independent random variables. We shall say that the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ satisfies condition (A) if $(\varepsilon_i^2)_{i\in\mathbb{Z}}$ is an uniformly integrable family and $\sup_{i\in\mathbb{Z}} \|\varepsilon_i\|_2 := \sigma_\varepsilon < \infty$. Let $(a_i)_{i\geq 0}$ be a sequence of reals in ℓ^1. For any integer k, let then $Y_k = \sum_{i\geq 0} a_i \varepsilon_{k-i}$. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ in the class $\mathcal{L}(c)$, meaning that there exists a concave non-decreasing function c from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\lim_{x\to 0} c(x) = 0$ and such that

$$|f(x) - f(y)| \leq c(|x-y|) \text{ for any } (x,y) \in \mathbb{R}^2 .$$

We shall also assume that

$$c\Big(K\sum_{i\geq 0}|a_i|\Big) < \infty \text{ for any finite real } K > 0 \text{ and } \sum_{k\geq 1} k^{-1/2} c\Big(2\sigma_\varepsilon \sum_{i\geq k}|a_i|\Big) < \infty , \quad (4.4)$$

and, for any $k \geq 1$, define

$$X_k = f(Y_k) - \mathbb{E}(f(Y_k)) . \quad (4.5)$$

Corollary 4.3. *Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued independent random variables satisfying condition (A). Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ belonging to the class $\mathcal{L}(c)$ and let $(a_i)_{i\geq 0}$ be a sequence of reals in ℓ^1. Assume that condition (4.4) is satisfied and define $(X_k)_{k\geq 1}$ by (4.5). Let $S_n = \sum_{k=1}^n X_k$ and $\sigma_n^2 = \mathrm{Var}(S_n)$. If $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity such that $\liminf_{n\to\infty} h(n) > 0$, then $\{\sigma_n^{-1}\sum_{k=1}^{[nt]} X_k, t \in [0,1]\}$ converges in distribution in $D([0,1])$ to W where W is a standard Brownian motion.*

Note that, if $|a_i| \leq C\rho^i$ for some $C > 0$ and $\rho \in]0,1[$, condition (4.4) holds as soon as:

$$\int_0^1 \frac{c(t)}{t\sqrt{|\log t|}} dt < \infty . \quad (4.6)$$

Note that this condition is satisfied as soon as $c(t) \leq D|\log(t)|^{-\gamma}$ for some $D > 0$ and some $\gamma > 1/2$. In particular, it is satisfied if f is α-Hölder for some $\alpha \in]0,1]$.

4.3. Application to locally stationary processes

In this section, we are interested by the limiting behavior of the partial sum process $\{n^{-1/2}\sum_{k=1}^{[nt]} X_{k,n}, t \in [0,1]\}$ when $(X_{k,n}, 1 \leq k \leq n)$ is a locally stationary process as considered by Vogt (2012), so in the sense that $X_{k,n}$ can be locally approximated by a stationary process $\tilde{X}_k(u)$ in some neighborhood of u, that is for those k where $|(k/n) - u|$ is small. More precisely, we shall assume Assumptions (S_0) and (S_1) below, which are close to Assumption 2.1 (S_1) in Dahlhaus et al. (2018). Assumption (D) is a weak dependence assumption, which cannot be compared to Assumption 2.3 (M_1) in Dahlhaus et al. (2018). Therefore, even if Corollary 4.4 below is in the spirit of Theorem 2.9 in Dahlhaus et al. (2018), it has a different range of applications.

Assumption 4.1. *Let $(X_{k,n}, 1 \le k \le n)$ be a triangular array of stochastic processes such that $\mathbb{E}(X_{k,n}) = 0$. For each $u \in [0,1]$, let $\tilde{X}_k(u)$ be a stationary and ergodic process such that the following conditions hold.*

(S_0) $\max_{1\le j\le n} n^{-1/2}\Big|\sum_{k=1}^{j} X_{k,n} - \sum_{k=1}^{j}\tilde{X}_k(k/n)\Big| \to^{\mathbb{P}} 0.$
(S_1) $\sup_{u\in[0,1]}\|\tilde{X}_k(u)\|_2 < \infty$ *and* $\lim_{\varepsilon\to 0}\sup_{|u-v|\le\varepsilon}\|\tilde{X}_k(u) - \tilde{X}_k(v)\|_2 = 0.$
(D) *There exists a stationary non-decreasing filtration $(\mathcal{F}_k)_{k\ge 0}$ such that, for each $u \in [0,1]$, $\tilde{X}_k(u)$ is adapted to $\mathcal{F}_k$ and the following condition holds: $\sum_{k\ge 0} 2^{-k/2}\tilde{\delta}(2^k) < \infty$, where $\tilde{\delta}(k) = \sup_{u\in[0,1]}\|\mathbb{E}(\tilde{S}_k(u)|\mathcal{F}_0)\|_2$ and $\tilde{S}_k(u) = \sum_{i=1}^{k}\tilde{X}_i(u)$.*

As a consequence of Theorem 3.1, we obtain

Corollary 4.4. *Assume that Assumption 4.1 holds. Then there exists a Lebesgue integrable function $\sigma^2(\cdot)$ on $[0,1]$ such that, for any $u \in [0,1]$, $\lim_{m\to\infty} m^{-1}\mathbb{E}(\tilde{S}_m(u))^2 = \sigma^2(u)$ and the sequence of processes $\big\{n^{-1/2}\sum_{k=1}^{[nt]} X_{k,n}, t \in [0,1]\big\}$ converges in distribution in $D([0,1])$ to $\big\{\int_0^t \sigma(u)dW(u), t \in [0,1]\big\}$ where W is a standard Brownian motion.*

Note that compared to Theorem 2.9 in Dahlhaus et al. (2018), we do not need to assume that $\|\sup_{u\in[0,1]}|\tilde{X}_k(u)|\|_2 < \infty$ nor that $\tilde{X}_k(u)$ takes the form $H(u, \eta_k)$ with H a measurable function and $\eta_k = (\varepsilon_j, j \le k)$ where $(\varepsilon_j)_{j\in\mathbb{Z}}$ is a sequence of iid real-valued random variables. Moreover, let us consider the following example. For any $u \in [0,1]$, let $Y_k(u) = \sum_{i\ge 0}\alpha(u)^i\varepsilon_{k-i}$ and $\tilde{X}_k(u) = f(Y_k(u)) - \mathbb{E}f(Y_k(u))$ with $\mathbb{E}(\varepsilon_0) = 0$ and $\|\varepsilon_0\|_2 = \sigma_\varepsilon < \infty$, $\alpha(\cdot)$ a Lipschitz continuous function such that $\sup_{u\in[0,1]}|\alpha(u)| = \alpha < 1$ and $f \in \mathcal{L}(c)$ as defined in the beginning of Section 4.2. Define then $X_{k,n} = \tilde{X}_k(k/n) + n^{-3/2}u_n(\varepsilon_k + \cdots + \varepsilon_{k-n})$ with $u_n \to 0$. It follows that (S_0) is satisfied. Moreover, using Lemma 5.1 in Dedecker (2008), one infers that (S_1) is satisfied as well as (D) provided that (4.6) holds. Note that Assumption 2.3 (M_1) in Dahlhaus et al. (2018) requires that $\int_0^1 t^{-1}c(t)dt < \infty$ which is stronger than (4.6). As a counter part, if $f(x) = x$ and $Y_k(u) = \sum_{i\ge 0}\alpha(u,i)\varepsilon_{k-i}$ with $\sup_{i\ge 0}|\alpha(u,i) - \alpha(v,i)| \le C|u-v|$ and $\sup_{u\in[0,1]}|\alpha(u,i)| \le \alpha_i$ with $(\alpha_i)_{i\ge 0} \in \ell^1$, then Assumption 2.3 (M_1) in Dahlhaus et al. (2018) is weaker than (D). Hence (D) and Assumption 2.3 (M_1) in Dahlhaus et al. (2018) have different areas of applications.

4.4. Quenched functional central limit theorems

In this subsection we start with a stationary sequence and address the question of functional CLT when the process is not started from its equilibrium, but it is rather started at a point or from a fixed past trajectory. This process is no longer strictly stationary. This type of result is known under the name of quenched limit theorem. It is convenient to introduce a stationary process by using the dynamical systems language. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. An element A is said to be invariant if $T(A) = A$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathcal{I}$ has measure 0 or 1.

Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ and define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. We assume that there exists a regular version $P_{T|\mathcal{F}_0}$ of T given $\mathcal{F}_0$,

In this subsection, we assume that $\mathbb{P}$ is ergodic and we consider X_0 a $\mathcal{F}_0$-measurable, square integrable and centered random variable. Define then the sequence $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Let $S_n = X_1 + \cdots + X_n$ and $W_n = \{W_n(t), t \in [0, 1]\}$ where $W_n(t) = n^{-1/2}S_{[nt]}$. It is well-known that, by a canonical construction, any stationary sequence can be represented in this way via the translation operator. As we shall see, applying our Corollary 3.3, we derive the following quenched CLT in its functional form under Maxwell and Woodroofe condition (2.7) which, from the subadditivity property of the sequence $(\|\mathbb{E}_0(S_n)\|_2)_{n\geq 0}$, is equivalent to the convergence: $\sum_{k\geq 1} k^{-3/2}\|\mathbb{E}_0(S_k)\|_2 < \infty$. This result was first obtained by Cuny and Merlevède in 2014 (see their Theorem 2.7) with a completely different proof.

Corollary 4.5. *Assume that (2.7) holds. Then there exists a constant c^2 such that $\lim_{n\to\infty} n^{-1/2}\mathbb{E}(S_n^2) = c^2$ and W_n satisfies the following quenched weak invariance principle: on a set of probability one, for any continuous and bounded function f from $(D([0, 1), \|\cdot\|_\infty)$ to $\mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{E}_0(f(W_n)) = \int f(zc)W(dz)\,,$$

where W is the distribution of a standard Wiener process.

The conclusion of this corollary can also be expressed in the following way. Denote by $\mathbb{P}^\omega(A)$ a regular version of conditional probability $\mathbb{P}(A|\mathcal{F}_0)(\omega)$. Then for any ω in a set of probability 1, W_n converges in distribution in $D([0, 1])$ to W under $\mathbb{P}^\omega$.

Since condition (2.7) is verified by a stationary $\rho-$mixing sequence satisfying (4.3) (see for instance Peligrad and Utev (2005)), the quenched functional CLT in Corollary 4.5 holds if (4.3) is satisfied. Note that for a stationary Gaussian process, its spectral density provides an useful tool to bound its associated ρ-mixing coefficients (see for instance Theorem 27.5 in Bradley (2007)).

4.5. Application to a random walk in random time scenery

Consider the partial sums associated with $(X_k)_{k\geq 0}$ which is a sequence of random variables, $\{\zeta_j\}_{j\geq 0}$, called the *random time scenery*, sampled by the process $(Y_k)_{k\geq 0}$, defined as

$$Y_k = k + \phi_k , \quad k \geq 0 ,$$

where $\{\phi_n\}_{n\geq 0}$ is a "renewal"-type Markov chain defined as follows: $\{\phi_k; k \geq 0\}$ is a discrete Markov chain with the state space $\mathbb{Z}^+$ and transition matrix $P = (p_{ij})$ given by $p_{k,k-1} = 1$ for $k \geq 1$ and $p_j = p_{0,j-1} = \mathbb{P}(\tau = j)$, $j = 1, 2, \ldots$, (that is whenever the chain hits 0 it then regenerates with the probability p_j). Therefore the sequence $(X_k)_{k\geq 0}$ is defined by setting

$$X_k = \zeta_{Y_k} .$$

We assume that $\mathbb{E}[\tau] < \infty$ which ensures that $\{\phi_n\}_{n\geq 0}$ has a stationary distribution $\pi = (\pi_i, i \geq 0)$ given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i \ , \ j = 1, 2 \ldots$$

where $\pi_0 = 1/\mathbb{E}(\tau)$. We also assume that $p_j > 0$ for all $j \geq 0$. This last assumption implies the irreducibility of the Markov chain.

In Corollary 4.6 below, we shall make the following assumption on the random time scenery:

Condition (A_1) $\{\zeta_j\}_{j\geq 0}$ *is a strictly stationary sequence of centered random variables in* $\mathbb{L}^2(\mathbb{P})$, *independent of* $(\phi_k)_{k\geq 0}$ *and such that, setting* $\mathcal{G}_i = \sigma(\zeta_k, k \leq i)$,

$$\sum_{k\geq 1} \frac{\|\mathbb{E}(\zeta_k|\mathcal{G}_0)\|_2}{\sqrt{k}} < \infty \ \text{ and } \ \lim_{n\to\infty} \sup_{j\geq i\geq n} \|\mathbb{E}(\zeta_i\zeta_j|\mathcal{G}_0) - \mathbb{E}(\zeta_i\zeta_j)\|_1 = 0 . \quad (4.7)$$

Applying Theorem 2.1 and Proposition 3.2, we can prove the following result concerning the asymptotic behavior of $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ when the chain starts from zero (below $\mathbb{P}_{\phi_0=0}$ is the conditional probability given $\phi_0 = 0$).

Corollary 4.6. *Assume that* $\mathbb{E}(\tau^2) < \infty$ *and that* $\{\zeta_j\}_{j\geq 0}$ *satisfies condition* (A_1). *Let* $S_0 = 0$ *and* $S_k = \sum_{i=1}^{k} X_i$ *for any* $k \geq 1$. *Then, under* $\mathbb{P}_{\phi_0=0}$, $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ *converges in distribution in* $D[0,1]$ *to a Brownian motion with parameter* c^2 *defined by*

$$c^2 = \mathbb{E}(\zeta_0^2)\Big(1 + 2\sum_{i\geq 1} i\pi_i\Big) + 2\sum_{m\geq 1} \mathbb{E}(\zeta_0\zeta_m) \sum_{j=1}^{m} (P^j)_{0,m-j} . \quad (4.8)$$

Note that $\mathbb{E}(\tau^2) < \infty$ is equivalent to $\sum_{i\geq 1} i\,\mathbb{P}(\tau > i) < \infty$ and therefore to $\sum_{i\geq 1} i\pi_i < \infty$.

The proof of the above corollary being long and technical, it is postponed to the supplementary material Merlevède, Peligrad and Utev (2018).

5. Proofs

In all the proofs, we shall use the notation $a_n \ll b_n$ which means that there exists a universal constant C such that, for all $n \geq 1$, $a_n \leq C b_n$.

5.1. Preparatory material

The next result is a version of the functional central limit theorem for triangular arrays of martingale differences essentially due to Aldous (1978) and Gänssler and Häusler (1979) (see also Theorem 3.2 in Helland (1982)).

Theorem 5.1 (Aldous-Gänssler-Häusler)**.** *Let $v_n(\cdot) : [0,1] \to \{0,1,2,\dots,n\}$ be a sequence of integer valued, non-decreasing and right-continuous functions. Assume $(d_{i,n})_{1\leq i\leq n}$ is an array of martingale differences adapted to an array $(\mathcal{F}_{i,n})_{0\leq i\leq n}$ of nested sigma fields. Let $\sigma(\cdot)$ be a non-negative function on $[0,1]$ such that $\sigma^2(\cdot)$ is Lebesgue integrable. Suppose that the following conditions hold:*

$$\max_{1\leq j\leq n} |d_{j,n}| \text{ is uniformly integrable,} \tag{5.1}$$

and, for all $t \in [0,1]$,

$$\sum_{j=1}^{v_n(t)} d_{j,n}^2 \to^{\mathbb{P}} \int_0^t \sigma^2(u)du \quad \text{as } n \to \infty\,. \tag{5.2}$$

Then $\{\sum_{j=1}^{v_n(t)} d_{j,n}, t \in [0,1]\}$ converges in distribution in $D[0,1]$ to $\big\{ \int_0^t \sigma(u)dW(u), t \in [0,1]\big\}$ where W is a standard Brownian motion.

5.1.1. A maximal inequality in the non-stationary setting

The following theorem is an extension of Proposition 2.3 in Peligrad and Utev (2005) to the non-stationary case. The proof follows the lines of the proof of Theorem 3 in Wu and Zhao (2008), but in the non-stationary setting, and is then done by induction. The proof is left to the reader but details can be found in the proof of Theorem 3.2 in Cuny et al. (2017).

Theorem 5.2. *Let $(X_k)_{k\in\mathbb{Z}}$ be a sequence of real-valued random variables in $\mathbb{L}^2$ and adapted to a filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}}$. Let $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$ and $S_n^* = \max_{1\leq k\leq n} |S_k|$. Then, for any $n \geq 1$,*

$$\|S_n^*\|_2 \leq 3\Big(\sum_{j=1}^n \|X_j\|_2^2\Big)^{1/2} + 3\sqrt{2}\Delta_n(X)\,, \tag{5.3}$$

where

$$\Delta_n(X) = \sum_{j=0}^{r-1} \Big(\sum_{k=1}^{2^{r-j}} \|\mathbb{E}(S_{k2^j} - S_{(k-1)2^j} | \mathcal{F}_{(k-2)2^j+1})\|_2^2 \Big)^{1/2} ,$$

with r the unique positive integer such that $2^{r-1} \le n < 2^r$.

5.2. Proof of Theorem 3.1

Recall that $X := \{X_{k,n} : k = 1, \dots, n\}$ is a triangular array of real-valued random variables in $\mathbb{L}^2$ adapted to a filtration $(\mathcal{F}_{k,n})_{0 \le k \le n}$. Without loss of generality, we assume that $X_{k,n} = 0$ for $k > n$ and $\mathcal{F}_{k,n} = \mathcal{F}_{n,n}$ for $k > n$. Moreover, by abuse of notation, we will often avoid the index n. In particular, we shall write $X_k = X_{k,n}$ and $\mathcal{F}_k = \mathcal{F}_{k,n}$, and we will use the notations

$$\mathbb{E}_j(X) = \mathbb{E}(X|\mathcal{F}_j) \, , \, \mathbf{P}_j(x) = \mathbb{E}_j(X) - \mathbb{E}_{j-1}(X) \, .$$

For a positive integer n, define the unique positive integer r such that $2^{r-1} \le n < 2^r$. For each n let also $S_n = \sum_{k=1}^n X_k$ and $S_0 = 0$.

Theorem 3.1 will follow from a martingale approximation and an application of Theorem 5.1, for the approximating martingale.

5.2.1. Step 1: A general Lemma.

Let us first introduce some notations. Let m be a fixed positive integer such that $m < n$. Let us then define

$$\theta_\ell^m = \frac{1}{m} \sum_{i=1}^{m-1} \mathbb{E}_\ell(X_{\ell+1} + \ldots + X_{\ell+i}) \, , \, D_\ell^m = \frac{1}{m} \sum_{i=0}^{m-1} \mathbf{P}_\ell(S_{\ell+i}) = \frac{1}{m} \sum_{i=0}^{m-1} \mathbf{P}_\ell(S_{\ell+i} - S_{\ell-1}) \, , \tag{5.4}$$

and

$$Y_\ell^m = \frac{1}{m} \mathbb{E}_\ell(S_{\ell+m} - S_\ell) \, , \, R_k^m = \sum_{\ell=0}^{k-1} Y_\ell^m \, . \tag{5.5}$$

Then, $D^m = (D_k^m)_{k=1}^n$ is a (triangular) array of martingale differences adapted to the filtration $(\mathcal{F}_k)_{0 \le k \le n}$ and the following decomposition is valid:

$$X_\ell = D_\ell^m + \theta_{\ell-1}^m - \theta_\ell^m + Y_{\ell-1}^m \, . \tag{5.6}$$

Also, for any positive integer m and k, we have

$$S_k = M_k^m + \theta_0^m - \theta_k^m + R_k^m \, . \tag{5.7}$$

As an intermediate step in proving Theorem 3.1 we shall prove a lemma under a set of assumptions which will be verified later. The next assumption (H) aims to guarantee that, in a certain sense, S_k can be approximated by $M_k^{m'}$ (for m' a subsequence of m) and it is then used to verify the conditions of Theorem 5.1.

There exists an increasing subsequence of integers $(m_j)_{j\geq 1}$ with $m_j \to \infty$ as $j \to \infty$ such that

$$(H) := \begin{cases} \lim_{j\to\infty} \sup_{n\geq 1} \sum_{\ell=0}^{n-1} \|Y_\ell^{m_j}\|_2^2 = 0\,, \\ \lim_{j\to\infty} \sup_{n\geq 1} \Delta_n(Y^{m_j}) = 0\,, \\ \lim_{j\to\infty} \sup_{n\geq 1} \sum_{k=0}^{n-1} \|\theta_k^{m_j}\|_2 \|Y_k^{m_j}\|_2 = 0\,, \end{cases}$$

where

$$\Delta_n(Y^m) := \sum_{r=0}^{d} \Big(\sum_{k=1}^{2^{d-r}} \|\mathbb{E}(R_{k2^r}^m - R_{(k-1)2^r}^m | \mathcal{F}_{(k-2)2^r})\|_2^2 \Big)^{1/2}.$$

We are now in the position to state our general lemma.

Lemma 5.3. *Assume that the Lindeberg condition (3.1) holds and that condition (H) is satisfied. Assume in addition that there exist a sequence of non-decreasing and right-continuous functions $v_n(\cdot) : [0,1] \to \{1,2,\dots,n\}$ and a non-negative Lebesgue integrable function $\sigma^2(\cdot)$ on $[0,1]$ such that (3.5) holds. Then $\big\{ \sum_{k=1}^{v_n(t)} X_k, t\in[0,1]\big\}$ converges in distribution in $D([0,1])$ to $\big\{ \int_0^t \sigma(u)dW(u), t\in[0,1]\big\}$ where W is a standard Brownian motion.*

Proof. To soothe the notations, we will often write m instead of m_j. To prove the lemma, let us first analyze the negligibility in some sense of the variables θ_k^m and R_k^m. Notice that from the definition (5.4)

$$\max_{0\leq k\leq n} |\theta_k^m|^2 \leq m^2 \max_{0\leq j\leq n} \mathbb{E}_j\big(\max_{1\leq k\leq n} |X_k|^2\big)\,.$$

By applying the Doob's maximal inequality and next truncation, we derive

$$\mathbb{E}\big[\max_{0\leq j\leq n} \mathbb{E}_j\big(\max_{1\leq k\leq n} |X_k|^2\big)\big] \leq 4\mathbb{E}\Big(\mathbb{E}_n\big(\max_{1\leq k\leq n} |X_k|^2\big)\Big) \leq 4\varepsilon^2 + 4\sum_{k=1}^{n} \mathbb{E}\big(X_k^2 I(|X_k|>\varepsilon)\big)\,.$$

Combining it with the previous estimate, taking into account the Lindeberg condition (3.1) and letting n tend to infinity and then $\varepsilon\to 0$ we obtain for each m, that

$$\mathbb{E}\big(\max_{0\leq k\leq n} |\theta_k^m|^2\big) \to 0 \text{ as } n\to\infty\,. \tag{5.8}$$

Note that, proceeding similarly, we also have that, for each m,

$$\mathbb{E}\big(\max_{1\leq k\leq n} |D_k^m|^2\big) \to 0 \text{ as } n\to\infty\,. \tag{5.9}$$

Now, by applying Theorem 5.2 to the array $(Y_k^m)_{k\in\mathbb{Z}}$, we have

$$\big\| \max_{1\leq k\leq n} |R_k^m| \big\|_2 = \big\| \max_{1\leq k\leq n} \big| \sum_{\ell=0}^{k-1} Y_\ell^m \big| \big\|_2 \leq 3\Big(\sum_{k=0}^{n-1} \|Y_k^m\|_2^2 \Big)^{1/2} + 3\sqrt{2}\Delta_n(Y^m)\,.$$

Taking $m = m_j$, by assumption (H) the terms in the r.h.s tend to 0, uniformly in n by letting $j \to \infty$. Hence, we derive the bound

$$\sup_{n\geq 1} \big\| \max_{1\leq k\leq n} |R_k^{m_j}| \big\|_2 \to 0 \text{ as } j \to \infty\,. \tag{5.10}$$

By the relations (5.8) and (5.10) we have the following martingale approximation

$$\limsup_n \Big\| \max_{1\leq k\leq n} \Big| \sum_{i=1}^k X_{i,n} - \sum_{\ell=1}^k D_\ell^{m_j} \Big| \Big\|_2 \to 0 \text{ as } j \to \infty\,.$$

This limit clearly implies

$$\limsup_n \Big\| \sup_{t\in[0,1]} \Big| \sum_{i=1}^{v_n(t)} X_{i,n} - \sum_{\ell=1}^{v_n(t)} D_\ell^{m_j} \Big| \Big\|_2 \to 0 \text{ as } j \to \infty\,, \tag{5.11}$$

and also for j_0 fixed,

$$\sup_{n\geq 1} \mathbb{E}(S_n^2) \leq \sup_{n\geq 1} \sum_{\ell=1}^n \big\| D_\ell^{m_{j_0}} \big\|_2 + \varepsilon_{j_0}\,,$$

where ε_{j_0} is a finite positive constant. Now, by definition (5.4),

$$\|D_k^{m_{j_0}}\|_2 \leq \frac{1}{m_{j_0}} \sum_{i=0}^{m_{j_0}-1} \|\mathbf{P}_k(S_{k+i} - S_k)\|_2 \leq \frac{1}{m_{j_0}} \sum_{i=0}^{m_{j_0}-1} \|\mathbb{E}_k(S_{k+i} - S_k)\|_2\,.$$

Hence, since $X_k = X_{k,n} = 0$, $k > n$,

$$\sum_{k=1}^n \|D_k^{m_{j_0}}\|_2^2 \leq m_{j_0}^2 \sum_{k=1}^n \|X_k\|_2^2\,. \tag{5.12}$$

Therefore, by the first part of (3.1),

$$\sup_{n\geq 1} \mathbb{E}(S_n^2) \leq C_{j_0} < \infty\,. \tag{5.13}$$

From (5.8), (5.9), (5.10), (5.11) and (3.5), we can deduce that we can find a sequence of positive integers $\ell(n)$ such that $\ell(n) \to \infty$ and setting $m'_n = m_{\ell(n)}$,

$$\lim_{n\to\infty} \big\| \max_{0\leq k\leq n} |\theta_k^{m'_n}| \big\|_2 = 0\,,\ \lim_{n\to\infty} \big\| \max_{1\leq k\leq n} |D_k^{m'_n}| \big\|_2 = 0\,, \tag{5.14}$$

$$\lim_{n\to\infty} \big\| \max_{1\leq k\leq n} |R_k^{m'_n}| \big\|_2 = 0\,, \tag{5.15}$$

$$\lim_{n\to\infty} \Big\| \sup_{t\in[0,1]} \Big| \sum_{i=1}^{v_n(t)} X_{i,n} - \sum_{\ell=1}^{v_n(t)} D_\ell^{m'_n} \Big| \Big\|_2 = 0\,, \tag{5.16}$$

and, for any $t \in [0,1]$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{k=1}^{v_n(t)} \big(X_k^2 + 2X_k \theta_k^{m'_n} \big) - \int_0^t \sigma^2(u)du \Big| > \varepsilon \Big) = 0 \,. \tag{5.17}$$

In addition, by condition (H), on the same subsequence (m'_n) we also have

$$(H') := \begin{cases} \lim_{n\to\infty} \sum_{\ell=0}^{n-1} \|Y_\ell^{m'_n}\|_2^2 = 0 \,, \\ \lim_{n\to\infty} \Delta_n(Y^{m'_n}) = 0 \,, \\ \lim_{n\to\infty} \sum_{k=0}^{n-1} \|\theta_k^{m'_n}\|_2 \|Y_k^{m'_n}\|_2 = 0 \,. \end{cases}$$

By (5.16), it suffices to show that $\big\{ \sum_{\ell=1}^{v_n(t)} D_\ell^{m'_n}, t \in [0,1] \big\}$ converges in distribution in $D([0,1])$ to cW. We shall verify now that the triangular array of martingale differences $(D_\ell^{m'_n})_{1\le \ell \le n}$ satisfies the conditions of Theorem 5.1. The condition (5.1) follows from the second part of (5.14). In order to verify condition (5.2) we proceed in the following way. We start from the identity (5.6) written as $(m = m'_n)$

$$X_\ell + \theta_\ell^m = D_\ell^m + \theta_{\ell-1}^m + Y_{\ell-1}^m \,.$$

Therefore

$$X_\ell^2 + 2X_\ell \theta_\ell^m + (\theta_\ell^m)^2 = (D_\ell^m)^2 + (\theta_{\ell-1}^m)^2 + (Y_{\ell-1}^m)^2 + 2\theta_{\ell-1}^m Y_{\ell-1}^m + 2D_\ell^m(\theta_{\ell-1}^m + Y_{\ell-1}^m) \,.$$

We sum over ℓ and get

$$\sum\nolimits_{\ell=1}^{v_n(t)} (X_\ell^2 + 2X_\ell\theta_\ell^m) + (\theta_{v_n(t)}^m)^2 = \sum\nolimits_{\ell=1}^{v_n(t)} (D_\ell^m)^2 + (\theta_0^m)^2 + \sum\nolimits_{\ell=1}^{v_n(t)} 2D_\ell^m(\theta_{\ell-1}^m + Y_{\ell-1}^m) + R'(v_n(t)) \,,$$

where

$$R'(v_n(t)) = \sum_{\ell=0}^{v_n(t)-1} (Y_\ell^{m'_n})^2 + 2 \sum_{k=0}^{v_n(t)-1} \theta_k^{m'_n} Y_k^{m'_n} \,.$$

By the Cauchy-Schwarz inequality and condition (H') we have that

$$\mathbb{E}(\sup_{0\le t\le 1} |R'(v_n(t))|) \le \sum_{\ell=0}^{n-1} \|Y_\ell^{m'_n}\|_2^2 + 2\sum_{k=0}^{n-1} \|\theta_k^{m'_n}\|_2 \|Y_k^{m'_n}\|_2 \to 0 \text{ as } n\to\infty \,.$$

Furthermore, by using the first part of (5.14) we also have

$$\mathbb{E} \sup_{0\le t\le 1} |(\theta_{v_n(t)}^m)^2 - (\theta_0^m)^2| \to 0 \text{ as } n \to \infty \,.$$

Now, by gathering the above considerations and by also using (5.17), we shall have

$$\sum\nolimits_{\ell=1}^{v_n(t)} (D_\ell^{m'_n})^2 \to \int_0^t \sigma^2(u)du \text{ in probability as } n\to\infty \,,$$

if we prove that $\sum_{\ell=1}^{v_n(t)} 2D_\ell^{m'_n}(\theta_{\ell-1}^{m'_n} + Y_{\ell-1}^{m'_n}) \to 0$ in probability. Because $(\theta_{\ell-1}^{m'_n} + Y_{\ell-1}^{m'_n})$ is a previsible (i.e. $\mathcal{F}_{\ell-1,n}$-measurable) random variable, the result follows again from (5.14) and (H'), by using the following fact, which is Theorem 2.11 in Hall and Heyde (1980):

***Fact* 5.1.** Let $(Z_i)_{i=1}^n$ be real-valued martingale differences adapted to a non-increasing filtration $(\mathcal{F}_i)_{0\le i\le n}$ and let $(A_k)_{k=1}^n$ be real-valued random variables such that A_k is $\mathcal{F}_{k-1}$-measurable. Then, there exists a positive constant c such that

$$\mathbb{E}\max_{1\le k\le n}\Big|\sum_{i=1}^k A_i Z_i\Big| \le c\big\{\mathbb{E}\max_{1\le k\le n}|A_k|^2\big\}^{1/2}\Big\{\sum_{i=1}^n \mathbb{E}(Z_i^2)\Big\}^{1/2}.$$

together with the following remark: by (5.16) and (5.13),

$$\limsup_n \sum_{\ell=1}^{v_n(t)} \|D_\ell^{m'_n}\|_2^2 = \limsup_n \Big\|\sum_{\ell=1}^{v_n(t)} D_\ell^{m'_n}\Big\|_2^2$$
$$\le \limsup_{n\to\infty}\sup_{t\in[0,1]}\Big\|\sum_{i=1}^{v_n(t)} X_{i,n} - \sum_{i=1}^{v_n(t)} D_\ell^{m'_n}\Big\|_2^2 + \limsup_{n\to\infty}\Big\|\sum_{i=1}^{v_n(t)} X_{i,n}\Big\|_2^2 \le C_{j_0}\,.$$

This ends the proof of the lemma. ⋄

5.2.2. Step 2: end of the proof of Theorem 3.1.

We are going to prove that Theorem 3.1 follows from an application of Lemma 5.3. With this aim we start by noticing the following fact: if the second part of (3.4) holds then there exists an increasing subsequence of integers $(m(j))_{j\ge1}$ with $m(j)\to\infty$ as $j\to\infty$ and such that

$$\lim_{j\to\infty}\sum_{\ell\ge m(j)} 2^{-\ell/2}B(2^\ell, 2^{m(j)}) = 0\,. \tag{5.18}$$

Hence, to show that condition (H) of Lemma 5.3 holds, we shall prove that its three assumptions are satisfied with $m_j = 2^{m(j)}$. So, in what follows $m_j = 2^{m(j)}$ where $m(j)$ is an increasing subsequence of integers tending to infinity and such that (5.18) holds. As before, we will sometimes write m instead of m_j.

Verifying first condition in (H). We first notice that, by the definition (3.2) and first part of condition (3.4)

$$\sup_{n\ge1}\sum_{k=0}^{n-1}\|Y_k^{m_j}\|_2^2 = m_j^{-2}\sup_{n\ge1}\sum_{k=0}^{n-1}\|\mathbb{E}_k(S_{k+m_j}-S_k)\|_2^2 = m_j^{-2}A^2(m_j)\ \to 0 \text{ as } j\to\infty\,, \tag{5.19}$$

which proves the first condition in (H).

Verifying second condition in (H). This needs more considerations. It is convenient to use the decomposition

$$\sum_{r=0}^{d}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2}=\sum_{r=0}^{b}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2}$$
$$+\sum_{r=b+1}^{d}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2},$$

where b is the unique positive integer such that $2^b\le m<2^{b+1}$. To estimate the first sum in the right-hand side, notice that, by the properties of the conditional expectation, we have

$$\begin{aligned}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2 &\le \sum_{\ell=0}^{2^r-1}\|\mathbb{E}(Y^m_{\ell+(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2\\ &\le \frac{1}{m}\sum_{\ell=0}^{2^r-1}\|\mathbb{E}(S_{\ell+(k-1)2^r+m}-S_{\ell+(k-1)2^r}|\mathcal{F}_{(k-2)2^r+\ell})\|_2\\ &\le \frac{1}{m}\sum_{\ell=(k-1)2^r}^{k2^r-1}\|\mathbb{E}(S_{\ell+m}-S_\ell|\mathcal{F}_{\ell-2^r})\|_2\,. \end{aligned}\tag{5.20}$$

Therefore, by definition (3.2),

$$\begin{aligned}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2} &\le \frac{2^{r/2}}{m}\Big(\sum_{k=1}^{2^{d-r}}\sum_{\ell=(k-1)2^r}^{k2^r-1}\|\mathbb{E}(S_{\ell+m}-S_\ell|\mathcal{F}_{\ell-2^r})\|_2^2\Big)^{1/2}\\ &\le \frac{2^{r/2}}{m}\Big(\sum_{\ell=0}^{2^d-1}\|\mathbb{E}(S_{\ell+m}-S_\ell|\mathcal{F}_{\ell-2^r})\|_2^2\Big)^{1/2}\le\frac{2^{r/2}}{m}A(m)\,,\end{aligned}$$

giving

$$\sum_{r=0}^{b}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2}\le\frac{2}{\sqrt{2}-1}\frac{A(m)}{\sqrt{m}}\,.$$

To estimate the second sum we also apply the properties of the conditional expectation and write this time

$$\begin{aligned}&\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2\\ &\qquad\le\Big\|\frac{1}{m}\sum_{u=0}^{m-1}\mathbb{E}(S_{k2^r+u}-S_{(k-1)2^r+u}|\mathcal{F}_{(k-2)2^r+1})\Big\|_2:=\|\bar{S}_{k-1}(2^r,m)\|_2\,.\end{aligned}\tag{5.21}$$

Hence, by definition (3.3)

$$\sum_{r=b+1}^{d}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2}\le\sum_{r=b+1}^{d}B(2^r,m)\,.$$

So, overall,

$$\sum_{r=0}^{d}\Big(\sum_{k=1}^{2^{d-r}}\|\mathbb{E}(R^m_{k2^r}-R^m_{(k-1)2^r}|\mathcal{F}_{(k-2)2^r})\|_2^2\Big)^{1/2}\le\frac{2}{\sqrt{2}-1}\frac{A(m)}{\sqrt{m}}+\sqrt{2}\sum_{r=b+1}^{d}B(2^r,m)\,. \tag{5.22}$$

This gives

$$\sup_{n\ge 1}\Delta_n(Y^{m_j})\le\frac{2}{\sqrt{2}-1}2^{-m(j)/2}A(2^{m(j)})+\sqrt{2}\sum_{r\ge m(j)}B(2^r,2^{m(j)})$$

which, together with condition (3.4), prove the second condition in (H).

It is worth to notice that we have proved the following maximal inequality for the array $(Y_k^m)_{k\in\mathbb{Z}}$ (the proof follows from an application of inequality (5.3) to the array $(Y_k^m)_{k\in\mathbb{Z}}$ and by taking into account the bounds in (5.19) and (5.22)).

Lemma 5.4. *There exists a positive constant C such that, for every positive integers n and m such that $m\le n$,*

$$\|\max_{1\le j\le n}|\sum_{k=0}^{j-1}Y_k^m|\|_2\le 3\Big(1+\frac{2\sqrt{2}}{\sqrt{2}-1}\Big)\frac{A(m)}{\sqrt{m}}+6\sum_{r=[\log_2(m)]}^{d}2^{-r/2}B(2^r,m)\,,$$

where d is the unique positive integer such that $2^{d-1}\le n<2^d$.

Verifying third condition in (H). For any positive integer i such that $i<m_j$, we write its decomposition in basis 2,

$$i=\sum_{k=0}^{[\log_2(i)]+1}c_k(i)2^k\quad\text{where } c_k(i)\in\{0,1\}\,.$$

Denote by $i_u=\sum_{k=0}^{u}c_k(i)2^k$ (hence $i_{[\log_2(i)]+1}=i$), for $u\ge 0$ and set $i_{-1}=0$. We have

$$\begin{aligned}\|\theta_\ell^m\|_2\le\frac{1}{m}\sum_{i=1}^{m-1}\|\mathbb{E}_\ell(S_{\ell+i}-S_\ell)\|_2&\le\frac{1}{m}\sum_{i=1}^{m-1}\sum_{u=0}^{[\log_2(i)]+1}\|\mathbb{E}_{\ell+i_{u-1}}(S_{\ell+i_u}-S_{\ell+i_{u-1}})\|_2\\&=\frac{1}{m}\sum_{i=1}^{m-1}\sum_{u=0}^{[\log_2(i)]+1}c_u(i)\|\mathbb{E}_{\ell+i_{u-1}}(S_{\ell+i_{u-1}+2^u}-S_{\ell+i_{u-1}})\|_2\,.\end{aligned}$$

Hence, by taking into account definition (3.2),

$$\sum_{\ell=0}^{n-1}\|\theta_\ell^m\|_2\|Y_\ell^m\|_2$$
$$\le \frac{1}{m}\sum_{i=1}^{m-1}\sum_{u=0}^{[\log_2(i)]+1}\Big(\sum_{\ell=0}^{n-1}c_u(i)\|\mathbb{E}_{\ell+i_{u-1}}(S_{\ell+i_{u-1}+2^u}-S_{\ell+i_{u-1}})\|_2^2\Big)^{1/2}\Big(\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2\Big)^{1/2}$$
$$\le \frac{1}{m}\sum_{i=1}^{m-1}\sum_{u=0}^{[\log_2(i)]+1}A(2^u)\Big(\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2\Big)^{1/2}. \quad (5.23)$$

So, by the first part of condition (3.4), there exists a constant C such that

$$\sum_{\ell=0}^{n-1}\|\theta_\ell^m\|_2\|Y_\ell^m\|_2\le\frac{C}{m}\sum_{i=1}^{m-1}\sum_{u=0}^{[\log_2(i)]+1}2^{u/2}\Big(\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2\Big)^{1/2}\le\frac{2C}{\sqrt{2}-1}\sqrt{m}\Big(\sum_{\ell=0}^{n-1}\|Y_\ell^m\|_2^2\Big)^{1/2}.$$

With $m=m_j=2^{m(j)}$ and taking now into account (5.19), it follows that

$$\sum_{\ell=0}^{n-1}\|\theta_\ell^{m_j}\|_2\|Y_\ell^{m_j}\|_2\le\frac{2C}{\sqrt{2}-1}\frac{1}{\sqrt{m_j}}A(m_j)=\frac{2C}{\sqrt{2}-1}2^{-m(j)/2}A(2^{m(j)}),$$

which converges to zero as $j\to\infty$ by the first part of (3.4). This shows that the third condition in (H) is satisfied and ends the proof of the theorem.

5.3. Proof of Proposition 3.2

Once again, to soothe the notation, we will avoid the index n involved in the variables and in the σ-algebras. In particular, we shall write $X_k=X_{k,n}$ and $\mathcal{F}_k=\mathcal{F}_{k,n}$, and we will use the notations $\mathbb{E}_j(X)=\mathbb{E}(X|\mathcal{F}_j)$ and $\mathbf{P}_j(x)=\mathbb{E}_j(X)-\mathbb{E}_{j-1}(X)$. Moreover, without loss of generality, we assume that $X_{k,n}=0$ for $k>n$.

Clearly it is enough to show that, for any $t\in[0,1]$ and any fixed integer $\ell\ge 0$,

$$\lim_{n\to\infty}\Big\|\sum_{k=1}^{v_n(t)}\big(\mathbb{E}_k(X_kX_{k+\ell})-\mathbb{E}_0(X_kX_{k+\ell})\big)\Big\|_1=0. \quad (5.24)$$

With this aim, note that for any positive fixed integer b (less than $v_n(t)$), by the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{b}\|\mathbb{E}_k(X_kX_{k+\ell})-\mathbb{E}_0(X_kX_{k+\ell})\|_1\le 2n^{-1}\sum_{k=1}^{b}\|X_k\|_2\|X_{k+\ell}\|_2\le 2\sum_{k=1}^{b+\ell}\|X_k\|_2^2.$$

Hence, for any $\varepsilon > 0$,

$$\sum_{k=1}^{b} \|\mathbb{E}_k(X_k X_{k+\ell}) - \mathbb{E}_0(X_k X_{k+\ell})\|_1 \le 2\big\{\varepsilon^2 (b+\ell) + \sum_{k=1}^{b+\ell} \mathbb{E}\big(X_k^2 \mathbf{1}_{|X_k|>\varepsilon}\big)\big\} ,$$

which converges to zero as $n \to \infty$ followed by $\varepsilon \to 0$, by taking into account condition (3.1). Now

$$\sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(X_k X_{k+\ell}) - \mathbb{E}_0(X_k X_{k+\ell})\big) = \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(X_k X_{k+\ell}) - \mathbb{E}_{k-b}(X_k X_{k+\ell})\big) + \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_{k-b}(X_k X_{k+\ell}) - \mathbb{E}_0(X_k X_{k+\ell})\big) .$$

Taking into account condition (3.6), we have

$$\lim_{b\to\infty} \limsup_{n\to\infty} \Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_{k-b}(X_k X_{k+\ell}) - \mathbb{E}_0(X_k X_{k+\ell})\big) \Big\|_1 = 0 .$$

We show now that

$$\lim_{b\to\infty} \limsup_{n\to\infty} \Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(X_k X_{k+\ell}) - \mathbb{E}_{k-b}(X_k X_{k+\ell})\big) \Big\|_1 = 0 . \tag{5.25}$$

Together with the convergences proved above, this will show that (5.24) is satisfied.

To prove (5.25), we fix a positive real ε and write

$$\Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(X_k X_{k+\ell}) - \mathbb{E}_{k-b}(X_k X_{k+\ell})\big) \Big\|_1 \le \Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(Y'_{k,\ell}) - \mathbb{E}_{k-b}(Y'_{k,\ell})\big) \Big\|_1 + \Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}_k(Y''_{k,\ell}) - \mathbb{E}_{k-b}(Y''_{k,\ell})\big) \Big\|_1$$

where

$$Y'_{k,\ell} = X_k X_{k+\ell} \mathbf{1}_{|X_k X_{k+\ell}| \le \varepsilon^2} \quad \text{and} \quad Y''_{k,\ell} = X_k X_{k+\ell} \mathbf{1}_{|X_k X_{k+\ell}| > \varepsilon^2} .$$

Note now that the following inequalities are valid: for any reals a and b and any positive real M,

$$|ab| \mathbf{1}_{\{|ab|>M\}} \le 2^{-1} \Big(|a^2 + b^2| \mathbf{1}_{\{|a^2+b^2|>2M\}}\Big) \le a^2 \mathbf{1}_{\{a^2>M\}} + b^2 \mathbf{1}_{\{b^2>M\}} .$$

Hence,

$$\Big\| \sum_{k=b+1}^{v_n(t)} \big(\mathbb{E}(Y''_{k,\ell}|\mathcal{F}_k) - \mathbb{E}(Y''_{k,\ell}|\mathcal{F}_{k-b})\big) \Big\|_1 \le 4 \sum_{k=1}^{n} \mathbb{E}(X_k^2 \mathbf{1}_{|X_k|>\varepsilon}) ,$$

which together with condition (3.1) imply that

$$\lim_{n\to\infty}\Big\|\sum_{k=b+1}^{v_n(t)}\big(\mathbb{E}_k(Y''_{k,\ell})-\mathbb{E}_{k-b}(Y''_{k,\ell})\big)\Big\|_1=0\,. \tag{5.26}$$

On another hand,

$$\sum_{k=b+1}^{v_n(t)}\big(\mathbb{E}_k(Y'_{k,\ell})-\mathbb{E}_{k-b}(Y'_{k,\ell})\big)=\sum_{k=b+1}^{v_n(t)}\sum_{j=0}^{b-1}\mathbf{P}_{k-j}(Y'_{k,\ell})=\sum_{j=0}^{b-1}\sum_{k=b+1}^{v_n(t)}\mathbf{P}_{k-j}(Y'_{k,\ell})\,,$$

where we recall $\mathbf{P}_j(\cdot)=\mathbb{E}(\cdot|\mathcal{F}_j)-\mathbb{E}(\cdot|\mathcal{F}_{j-1})$. Since $(\mathbf{P}_{k-j}(Y'_{k,\ell}))_{k\geq 1}$ is a sequence of martingale differences,

$$\Big\|\sum_{k=b+1}^{v_n(t)}\big(\mathbb{E}_k(Y'_{k,\ell})-\mathbb{E}_{k-b}(Y'_{k,\ell})\big)\Big\|_1\leq\sum_{j=0}^{b-1}\Big\|\sum_{k=b+1}^{v_n(t)}\mathbf{P}_{k-j}(Y'_{k,\ell})\Big\|_2\leq\sum_{j=0}^{b-1}\Big(\sum_{k=b+1}^{v_n(t)}\|\mathbf{P}_{k-j}(Y'_{k,\ell})\|_2^2\Big)^{1/2}.$$

By the Cauchy–Schwarz inequality,

$$\|\mathbf{P}_{k-j}(Y'_{k,\ell})\|_2^2\leq\|\mathbb{E}_{k-j}(Y'_{k,\ell})\|_2^2\leq\varepsilon^2\|X_kX_{k+\ell}\|_1\leq 2^{-1}\varepsilon^2(\|X_k\|_2^2+\|X_{k+\ell}\|_2^2)\,.$$

Therefore

$$\Big\|\sum_{k=b+1}^{v_n(t)}\big(\mathbb{E}_k(Y'_{k,\ell})-\mathbb{E}_{k-b}(Y'_{k,\ell})\big)\Big\|_1\leq b\,\varepsilon\sup_{n\geq1}\Big(\sum_{k=b+1}^{n}\|X_k\|_2^2\Big)^{1/2},$$

which converges to zero by taking into account condition (3.1) and by letting ε going to 0. This last convergence together with (5.26) entail (5.25) and then (5.24). This ends the proof of the proposition.

5.4. Proof of Theorem 4.1

We apply Theorem 3.1 to the triangular array $\{\sigma_n^{-1}X_{k,n},1\leq k\leq n\}_{n\geq1}$ and the σ-algebras $\mathcal{F}_{k,n}=\sigma(X_{i,n},1\leq i\leq k)$ for $k\geq1$ and $\mathcal{F}_{k,n}=\{\emptyset,\Omega\}$ for $k\leq 0$. For convenience, we can set $X_{k,n}=0$ for $k>n$. Again, to soothe the notations, we will omit the index n involved in the variables and in the σ-algebras.

As a matter of fact, we shall first prove that under the conditions of Theorem 4.1, the following reinforced version of condition (3.4) is satisfied:

$$\lim_{m\to\infty}m^{-1/2}A(m)=0\text{ and }\lim_{m\to\infty}\sum_{\ell\geq[\log_2(m)]}B(2^\ell,m)=0\,. \tag{5.27}$$

In order to check the conditions below, we shall apply the following inequality, derived in Theorem 1.1 in Utev (1991). More exactly, under (4.3) there exists a finite positive constant κ such that for any positive integers $a < b$,

$$\|S_b - S_a\|_2^2 \leq \kappa \sum_{i=a+1}^{b} \|X_i\|_2^2 \,. \tag{5.28}$$

The first characteristic $A^2(m)$ defined by (3.2) is then estimated as follows. Write first the following decomposition:

$$\sum_{k=0}^{n-1} \|\mathbb{E}_k(S_{k+m} - S_k)\|_2^2 \leq 2\sum_{k=0}^{n-1} \|\mathbb{E}_k(S_{k+m} - S_{k+[\sqrt{m}]})\|_2^2 + 2\sum_{k=0}^{n-1} \|\mathbb{E}_k(S_{k+[\sqrt{m}]} - S_k)\|_2^2. \tag{5.29}$$

Note now that for any integer k and any positive integers a, b with $a < b$,

$$\begin{aligned}\|\mathbb{E}_k(S_{k+b} - S_{k+a})\|_2^2 &= \operatorname{cov}\big(\mathbb{E}_k(S_{k+b} - S_{k+a}), S_{k+b} - S_{k+a}\big) \\ &\leq \rho(a)\|\mathbb{E}_k(S_{k+b} - S_{k+a})\|_2\|S_{k+b} - S_{k+a}\|_2 \,.\end{aligned}$$

Hence

$$\|\mathbb{E}_k(S_{k+b} - S_{k+a})\|_2 \leq \rho(a)\|S_{k+b} - S_{k+a}\|_2 \,,$$

which combined with (5.28) implies, under (4.3), that there exists a finite positive constant κ such that that

$$\|\mathbb{E}_k(S_{k+b} - S_{k+a})\|_2^2 \leq \kappa\rho^2(a) \sum_{i=k+a+1}^{k+b} \|X_i\|_2^2 \,. \tag{5.30}$$

Therefore, starting from (5.29) and taking into account (5.28) and (5.30), we get, under (4.3) and (4.1), that

$$\begin{aligned}\sigma_n^{-2} \sum_{k=0}^{n-1} \|\mathbb{E}_k(S_{k+m} - S_k)\|_2^2 &\leq 2\kappa\sigma_n^{-2} \sum_{k=0}^{n-1} \rho^2([\sqrt{m}]) \sum_{i=k+[\sqrt{m}]+1}^{k+m} \|X_i\|_2^2 + 2\kappa\sigma_n^{-2} \sum_{k=0}^{n-1} \sum_{i=k+1}^{k+[\sqrt{m}]} \|X_i\|_2^2 \\ &\leq 2\kappa C\big\{m\rho^2([\sqrt{m}]) + \sqrt{m}\big\} \,.\end{aligned}$$

Hence

$$m^{-1}A^2(m) \leq 2\kappa C\big\{\rho^2([\sqrt{m}]) + m^{-1/2}\big\} \,, \tag{5.31}$$

which tends to zero as $m \to \infty$. This proves the first part of assumption (5.27).

Next, observe that by (5.30), under (4.3),

$$\Big\|\frac{1}{m}\sum_{u=0}^{m-1} \mathbb{E}(S_{k2^r+u} - S_{(k-1)2^r+u}|\mathcal{F}_{(k-2)2^r+1})\Big\|_2^2 \leq \frac{\kappa}{m}\sum_{u=0}^{m-1} \rho^2(2^r+u-1) \sum_{i=(k-1)2^r+u+1}^{k2^r+u} \|X_i\|_2^2 \,.$$

Thus, by taking into account (4.1), we derive that

$$B^2(2^r,m)=\sup_{n\geq 1}\sigma_n^{-2}\sum_{k=1}^{[n/2^r]+1}\Big\|\frac{1}{m}\sum_{u=0}^{m-1}\mathbb{E}(S_{k2^r+u}-S_{(k-1)2^r+u}|\mathcal{F}_{(k-2)2^r+1})\Big\|_2^2$$
$$\leq C\frac{\kappa}{m}\sum_{u=0}^{m-1}\rho^2(2^r+u-1)\leq C\kappa\rho^2(2^r-1)\,,$$

where the last inequality comes from the fact that ρ is non-increasing. Taking into account (4.3), this shows that the second part of (5.27) is satisfied.

Now, we apply Proposition 3.2 to verify the last condition (3.5). To do it we need to verify its assumptions (3.6) and (3.7) by recalling that $\mathcal{F}_{0,n}=\{\emptyset,\Omega\}$ and then $\mathbb{E}_0(\cdot)=\mathbb{E}(\cdot)$.

First, we notice that, by the definition of the ρ-mixing coefficients and the condition (4.1), for any non-negative integer ℓ,

$$\sigma_n^{-2}\sum_{k=b+1}^{n}\|\mathbb{E}_{k-b}(X_kX_{k+\ell})-\mathbb{E}(X_kX_{k+\ell})\|_1\leq\rho(b)\sigma_n^{-2}\sum_{k=b+1}^{n}\|X_kX_{k+\ell}-\mathbb{E}(X_kX_{k+\ell})\|_2$$
$$\leq\rho(b)\sigma_n^{-2}\sum_{k=b+1}^{n}\|X_k\|_2\|X_{k+\ell}\|_2\leq\rho(b)\sigma_n^{-2}\Big(\sum_{k=1}^{n+\ell}\|X_k\|_2^2\Big)\leq\rho(b)C\to 0\text{ as }b\to\infty,$$

which proves the first assumption (3.6).

To end the proof of the theorem, it remains to prove that (3.7) holds. Note that since we have proved that condition (5.27) is satisfied, a careful analysis of the proof of Lemma 5.3 reveals that, setting $D_\ell^m=m^{-1}\sum_{i=0}^{m-1}\mathbf{P}_\ell(S_{\ell+i})$ and $\theta_\ell^m=m^{-1}\sum_{i=1}^{m-1}\mathbb{E}_\ell(X_{\ell+1}+\ldots+X_{\ell+i})$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sigma_n^{-2}\sup_{t\in[0,1]}\Big|\sum_{\ell=1}^{v_n(t)}\Big(\mathbb{E}(X_\ell^2+2X_\ell\theta_\ell^m)-\mathbb{E}(D_\ell^m)^2\Big)\Big|=0\,,\tag{5.32}$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sigma_n^{-2}\Big\|\sup_{t\in[0,1]}\Big|\sum_{i=1}^{v_n(t)}X_{i,n}-\sum_{\ell=1}^{v_n(t)}D_\ell^m\Big|\Big\|_2^2=0\,.\tag{5.33}$$

Taking into account (5.32), to prove that (3.7) holds, we then need to show that, for any $t\in[0,1]$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\big|\sigma_n^{-2}\sum_{\ell=1}^{v_n(t)}\mathbb{E}(D_\ell^m)^2-t\big|=0\,.\tag{5.34}$$

But, since $\sum_{\ell=1}^{v_n(t)}\mathbb{E}(D_\ell^m)^2=\|\sum_{\ell=1}^{v_n(t)}D_\ell^m\|_2^2$, by taking into account (5.33), the convergence (5.34) follows if one can prove that, for any $t\in[0,1]$,

$$\sigma_n^{-2}\mathbb{E}\big(S_{v_n(t)}^2\big)\to t\,,\ \text{ as }n\to\infty\,.\tag{5.35}$$

With this aim, we note that since $\left\| \sum_{k=1}^{v_n(t)} X_{k,n} \right\|_2 \le \left\| \sum_{k=1}^{v_n(t)-1} X_{k,n} \right\|_2 + \|X_{v_n(t),n}\|_2$, by definition of $v_n(t)$, we have $\sqrt{t} \le \sigma_n^{-1} \| \sum_{k=1}^{v_n(t)} X_{k,n} \|_2 \le \sqrt{t} + \sigma_n^{-1} \|X_{v_n(t),n}\|_2$. This implies (5.35) by noticing that the Lindeberg condition (4.2) implies that $\lim_{n\to\infty} \frac{\|X_{v_n(t),n}\|_2}{\sigma_n} = 0$. The proof of Theorem 4.1 is complete.

5.5. Proof of Corollary 4.2

By taking $v_n(t) = [nt]$, we need to ensure that (5.35) holds, which is straightforward since we assume that $\sigma_n^2 = nh(n)$ where h is a slowly varying function at infinity.

5.6. Proof of Corollary 4.3

Since c is non-decreasing and concave, by Lemma 5.1 in Dedecker (2008), we note first that, for any $k \ge 1$,

$$\|X_k\|_2 \le 2\|f(Y_k) - f(0)\|_2 \le 2\|c(|Y_k|)\|_2 \le 2c(\|Y_k\|_2)\,.$$

Therefore by (4.4)

$$\sup_{k\ge 1} \|X_k\|_2 \le 2c\Big(\sigma_\varepsilon \sum_{i\ge 0} |a_i|\Big) < \infty\,.$$

This proves the first part of (2.2). Now, to prove the second part of (2.2), it suffices to show that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\big(|f(Y_k) - f(0)|^2 \mathbf{1}_{\{|f(Y_k)-f(0)|>\varepsilon\sqrt{n})\}}\big) = 0\,. \tag{5.36}$$

With this aim, we set $C = \sum_{i\ge 0} |a_i|$ and let K be a positive integer. We denote by $\varepsilon_i'' = \varepsilon_i \mathbf{1}_{|\varepsilon_i|>K}$. Using the fact that for positive reals a, b and ε, $(a+b)^2\mathbf{1}_{\{a+b>2\varepsilon\}} \le 4a^2\mathbf{1}_{\{a>\varepsilon\}} + 4b^2\mathbf{1}_{\{b>\varepsilon\}}$, we infer that, for any $\varepsilon > 0$,

$$\mathbb{E}\big(|f(Y_k) - f(0)|^2 \mathbf{1}_{\{|f(Y_k)-f(0)|>2\varepsilon\sqrt{n})\}}\big) \le 4\Big\|c\Big(\sum_{i\ge 0} |a_i\varepsilon_{k-i}''|\Big)\Big\|_2^2 + 4c^2(KC)\mathbf{1}_{\{c(KC)>\varepsilon\sqrt{n})\}}\,.$$

The last term in the right-hand side converges to zero as $n \to \infty$. Next, since c is non-decreasing and concave, Lemma 5.1 in Dedecker (2008) gives

$$\frac{1}{n}\sum_{k=1}^{n} \Big\|c\Big(\sum_{i\ge 0} |a_i\varepsilon_{k-i}''|\Big)\Big\|_2^2 \le \frac{1}{n}\sum_{k=1}^{n} c^2\Big(\sum_{i\ge 0} |a_i|\|\varepsilon_{k-i}''\|_2\Big) \le c^2\Big(\sup_{k\in\mathbb{Z}} \|\varepsilon_k''\|_2 \sum_{i\ge 0} |a_i|\Big)\,.$$

But, since $(\varepsilon_i^2)_{i\in\mathbb{Z}}$ is an uniformly integrable family, $\limsup_{K\to\infty} \sup_{k\in\mathbb{Z}} \|\varepsilon_k''\|_2 = 0$. Together with the fact that $\lim_{x\to 0} c(x) = 0$, this proves that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \Big\|c\Big(\sum_{i\ge 0} |a_i\varepsilon_{k-i}''|\Big)\Big\|_2^2 = 0\,,$$

ending the proof of (5.36) and then of (2.2).

Let us consider now the following choice of $(\mathcal{F}_i)_{i\geq 0}$: $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$, for $i \geq 1$. If one can prove that conditions (2.6) and (3.6) are satisfied and also that, for any $t \in [0,1]$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{\sigma_n^2}\Big| \sum_{k=1}^{[nt]} \Big\{ \mathbb{E}(X_k^2) + 2\mathbb{E}(X_k \theta_k^m) \Big\} - t \Big| = 0 \, , \tag{5.37}$$

then the corollary follows by applying Theorem 2.1 and by taking into account Proposition 3.2.

To prove that (2.6) holds, we set $\mathbb{E}_\varepsilon$ the expectation with respect to $\varepsilon := (\varepsilon_i)_{i\in\mathbb{Z}}$ and note that since $\mathcal{F}_i \subset \mathcal{F}_{\varepsilon,i}$ where $\mathcal{F}_{\varepsilon,i} = \sigma(\varepsilon_k, k \leq i)$, for any $i \geq 0$,

$$\|\mathbb{E}(X_{k+i}|\mathcal{F}_i)\|_2 \leq \|\mathbb{E}(X_{k+i}|\mathcal{F}_{\varepsilon,i})\|_2 \, . \tag{5.38}$$

For any $i \geq 0$,

$$\begin{aligned} &\big|\mathbb{E}(X_{k+i}|\mathcal{F}_{\varepsilon,i})\big| \\ &= \Big|\mathbb{E}_\varepsilon\Big(f\Big(\sum_{\ell=0}^{k-1} a_\ell \varepsilon'_{k+i-\ell} + \sum_{\ell\geq k} a_\ell \varepsilon_{k+i-\ell}\Big)\Big) - \mathbb{E}_\varepsilon\Big(f\Big(\sum_{\ell=0}^{k-1} a_\ell \varepsilon'_{k+i-\ell} + \sum_{\ell\geq k} a_\ell \varepsilon'_{k+i-\ell}\Big)\Big)\Big| \, , \end{aligned}$$

where $(\varepsilon'_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Hence, by Lemma 5.1 in Dedecker (2008),

$$\|\mathbb{E}(X_{k+i}|\mathcal{F}_i)\|_2 \leq \Big\| c\Big(\sum_{\ell\geq k} |a_\ell||\varepsilon_{k+i-\ell} - \varepsilon'_{k+i-\ell}|\Big)\Big\|_2 \leq c\Big(2\sigma_\varepsilon \sum_{\ell\geq k} |a_\ell|\Big) \, ,$$

proving that (2.6) holds under (4.4). We prove now that (3.6) is satisfied. With this aim we recall that $\mathcal{F}_0$ is the trivial σ-field, and we first write that for any non-negative integer k, j, and n,

$$\begin{aligned} &\mathbb{E}_k(X_{k+n}X_{j+n}) - \mathbb{E}(X_{k+n}X_{j+n}) = \mathbb{E}_k(f(Y_{k+n})f(Y_{j+n})) - \mathbb{E}(f(Y_{k+n})f(Y_{j+n})) \\ &\quad - \mathbb{E}(f(Y_{k+n}))\mathbb{E}_k(f(Y_{j+n}) - \mathbb{E}(f(Y_{j+n}))) - \mathbb{E}(f(Y_{j+n}))\mathbb{E}_k(f(Y_{k+n}) - \mathbb{E}(f(Y_{k+n}))) \, . \end{aligned} \tag{5.39}$$

Since $\lim_{x\to 0} c(x) = 0$ and the first part of (4.4) is assumed, by using coupling arguments as before and Lemma 5.1 in Dedecker (2008), we infer that

$$\lim_{n\to\infty} \sup_{j\geq k\geq 0} \|\mathbb{E}_k(f(Y_{k+n})f(Y_{j+n}) - \mathbb{E}(f(Y_{k+n})f(Y_{j+n}))\|_1 = 0 \, , \tag{5.40}$$

and

$$\lim_{n\to\infty} \sup_{j\geq k\geq 0} \|\mathbb{E}(f(Y_{j+n}))\mathbb{E}_k(f(Y_{k+n}) - \mathbb{E}(f(Y_{k+n})))\|_1 = 0 \, . \tag{5.41}$$

Starting from (5.39) and taking into account (5.40) and (5.41), the convergence (3.6) follows since we have assumed that $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity such that $\liminf_{n\to\infty} h(n) > 0$.

We turn now to the proof of (5.37). With this aim, note first that since condition (2.6) is satisfied and $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity such that $\liminf_{n\to\infty} h(n) > 0$, condition (5.27) holds. Now as quoted in the proof of Theorem 4.1, if the Lindeberg condition (3.1) and condition (5.27) are both satisfied, then to prove (5.37) it is enough to show that (5.35) holds (here with $v_n(t) = [nt]$). This comes obviously from the fact that we assumed that $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity. This ends the proof of the corollary.

5.7. Proof of Corollary 4.4

The fact that, under (D), $\lim_{m\to\infty} m^{-1}\mathbb{E}(\tilde S_m(u))^2 = \sigma^2(u)$ has been proved in Peligrad and Utev (2005). Note now that, by (S_0), it suffices to prove the functional CLT for the process $\big\{n^{-1/2}\sum_{k=1}^{[nt]} \tilde X_k(k/n), t \in [0,1]\big\}$. With this aim, we shall apply Theorem 3.1 with $X_{k,n} = n^{-1/2}\tilde X_k(k/n)$. Note first that condition (3.4) clearly holds under (D). The first part of the Lindeberg condition (3.1) holds since $\sup_{u\in[0,1]} \|\tilde X_0(u)\|_2 < \infty$. For the second part we note that, for any $A > 0$,

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} \mathbb{E}\{\tilde X_0^2(k/n) I(|\tilde X_0^2(k/n)| > A)\} = \int_0^1 \mathbb{E}\{\tilde X_0^2(u) I(|\tilde X_0^2(u)| > A)\}du\,,$$

which converges to zero as $A\to\infty$ by the dominated convergence theorem. It remains to prove that (3.5) is satisfied. Using (S_1) and proceeding as in the proof of Theorem 2.7 in Dahlhaus et al. (2018), one can easily prove that

$$\frac{1}{n}\sum_{k=1}^{[nt]}\Big\{\tilde X_k^2(k/n) + \frac{2}{m}\sum_{i=1}^{m}\tilde X_k(k/n)\mathbb{E}_k\Big(\sum_{\ell=k+1}^{k+i}\tilde X_\ell(\ell/n)\Big)\Big\} \to_{n\to\infty} \int_0^t \frac{1}{m}\mathbb{E}(\tilde S_m(u))^2 du\,.$$

Now, taking into account assumption (D) and the fact that $\sup_{u\in[0,1]}\|\tilde X_k(u)\|_2 < \infty$, Theorem 5.2 entails that $\frac{1}{m}\sup_{u\in[0,1]}\mathbb{E}(\tilde S_m(u))^2 \le K$. Hence, by the dominated convergence theorem, $\int_0^t \frac{1}{m}\mathbb{E}(\tilde S_m(u))^2 du \to_{m\to\infty} \int_0^t \sigma^2(u)du$. This completes the proof of (3.5) and then of the corollary.

5.8. Proof of Corollary 4.5

For any integrable random variable f from Ω to $\mathbb{R}$ we write $K(f) = P_{T|\mathcal{F}_0}(f)$. Since $\mathbb{P}$ is T-invariant, for any integer k, a regular version $P_{T|\mathcal{F}_k}$ of T given $\mathcal{F}_k$ is then obtained *via* $P_{T|\mathcal{F}_k}(f) = K(f \circ T^{-k}) \circ T^k$. With these notations, for any positive integer ℓ, $\mathbb{E}(f \circ T^\ell | \mathcal{F}_0) = K^\ell(f)$. We denote

$$\mathcal{M}_{2^r}(|f|) = \sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} K^{k2^r}(|f|) \, .$$

Applying Corollary 3.3, Corollary 4.5 follows if one can prove that, with probability one,

$$\sup_{n \geq 1} n^{-1} \sum_{j=1}^{n} \mathbb{E}_0(X_j^2) \leq C < \infty \, , \tag{5.42}$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_0\{X_k^2 I(|X_k| > \varepsilon\sqrt{n})\} = 0 \, , \text{ for any } \varepsilon > 0 \, , \tag{5.43}$$

there exists a constant c^2 such that, for any $t \in [0,1]$ and any $\varepsilon > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}_0\Big(\Big| \frac{1}{n} \sum_{k=1}^{[nt]} \big(X_k^2 + \frac{2}{m} X_k \sum_{i=1}^{m-1} \mathbb{E}_k(S_{k+i} - S_k) \big) - tc^2 \Big| > \varepsilon \Big) = 0 \, , \tag{5.44}$$

$$\sum_{\ell \geq 0} 2^{-\ell/2} \mathcal{M}_1^{1/2}(|\mathbb{E}_0(S_{2^\ell})|^2) < \infty \, , \tag{5.45}$$

and

$$\liminf_{j \to \infty} \sum_{\ell \geq j} 2^{-\ell/2} \mathcal{M}_{2^\ell}^{1/2}\big(\big| 2^{-j} \sum_{u=0}^{2^j-1} \mathbb{E}_{-2^\ell+1}(S_{2^\ell} \circ T^u) \big|^2 \big) = 0 \, . \tag{5.46}$$

To prove (5.42) and (5.43), it suffices to apply, for instance, Lemma 7.1 in Dedecker et al. (2014). To show (5.45) and (5.46), let introduce the weak $\mathbb{L}^2$-spaces: $\mathbb{L}^{2,w} := \{f \in \mathbb{L}^1 \, : \, \sup_{\lambda > 0} \lambda^2 \mathbb{P}\{|f| \geq \lambda\} < \infty\}$. Recall that, when $p > 1$, there exists a norm $\| \cdot \|_{2,w}$ on $\mathbb{L}^{2,w}$ that makes $\mathbb{L}^{2,w}$ a Banach space and which is equivalent to the "pseudo"-norm $(\sup_{\lambda > 0} \lambda^2 \mathbb{P}\{|f| \geq \lambda\})^{1/2}$. Moreover, by the Dunford–Schwartz (or Hopf) ergodic theorem (see Krengel (1985), Lemma 6.1, page 51, and Corollary 3.8, page 131), there exists $C > 0$ and such that for every $f \in \mathbb{L}^2$ and any non-negative integer ℓ,

$$\|(\mathcal{M}_{2^\ell}(|f|^2))^{1/2}\|_{2,w} \leq C\|f\|_2 \, . \tag{5.47}$$

With the help of (5.47), it is then easy to see that (5.45) and (5.46) are satisfied under (2.7).

It remains to prove that (5.44) is satisfied. Since, under (2.7), $\lim_{m \to \infty} m^{-1/2} \mathbb{E}(S_m^2) = c^2$, by the ergodic theorem and the proof of Corollary 2.2,

$$\lim_{m \to \infty} \lim_{n \to \infty} \Big| \frac{1}{n} \sum_{k=1}^{[nt]} \big(X_k^2 + \frac{2}{m} X_k \sum_{i=1}^{m-1} \mathbb{E}_k(S_{k+i} - S_k) \big) - tc^2 \Big| = 0 \, , \text{ almost surely.}$$

This proves (5.44) by taking into account the properties of the conditional expectation (see, e.g., Theorem 34.3, item (v) in Billingsley (1995)). The proof of the corollary is complete.

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Supplementary Material

Supplement A: Supplement to "Functional CLT for martingale-like nonstationary dependent structures"
(doi: COMPLETED BY THE TYPESETTER; .pdf). The supplementary file Merlevède, Peligrad and Utev (2018) contains a detailed proof of Corollary 4.6.

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