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Misspecified Filtering Theory applied to Optimal Allocation Problems in Finance

Miguel Martinez Sylvain Rubenthaler

Etienne Tanré

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Misspecified Filtering Theory applied to Optimal Allocation Problems in Finance

Miguel MARTINEZ[1], Sylvain RUBENTHALER[2], Etienne TANRÉ[3]

Abstract:

In this paper, we consider an investor who plays in a market that involves a risky asset whose instantaneous rate of return changes at unknown random times. This return rate is assumed to follow the law of a Compound Poisson Process. We construct optimal mathematical strategies in this context and we show that these strategies are based on a continuous time filter arising in Filtering Theory. We study the case where the model is misspecified and where the observations can only be made at discrete times. We give precise results that show how these misspecifications affect both the behavior of the original filter and the behavior of the trader's wealth.

Keywords:

Filtering, Portfolios, Applications in optimization.

JEL classification:

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[1]Universidad de Valparaíso Departamento de Matemàtica Av. Gran Bretaña 1041 Playa Ancha- Valparaíso 1470 -V Valparaíso, Chile Miguel.Math@gmail.com. Miguel Martinez gratefully acknowledges financial support from the National Centre of Competence in Research "Conceptual Issues in Financial Risk Management" (NCCR–FINRISK),

[2]Laboratoire Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France, rubentha@math.unice.fr

[3]INRIA, Projet OMEGA, 2004 route des Lucioles, BP93, 06902 Sophia-Antipolis, France, Etienne.Tanre@sophia.inria.fr

1 Introduction, description of the model and organization of the paper

1.1 Introduction

The purpose of this study is to address the following problem: consider a non-stationary financial economy. It is impossible to specify and calibrate models which can capture all the sources of instability during a long time interval. In other words, one can only pretend to divide a long investment period into sub-periods such that, in each one of these sub-periods, the market can reasonably be supposed to follow some particular model (e.g., a stochastic differential equation with a fixed volatility function). Because of the market instability, each sub-period is short. Therefore, one can only use small amounts of data during each sub-period to calibrate the model and the calibration errors can be substantial. However, hedging strategies, portfolio management, strategies, etc. highly depend on the underlying model for the market evolution, and also on the values of the parameters involved in the model. One can conclude that, in non-stationary economies, one can use strategies which have been optimally designed under the assumption that the market is perfectly described by a prescribed model, but these strategies may be extremely misleading in practice because the parameters of the model in each sub-period and the random times between each sub-period of the considered non-stationary financial economy are very difficult to know.

Thus, it might be useful to search for optimal strategies when the model leading the non-stationary financial economy is perfectly known and afterwards to compare them with the sub-optimal allocation policies obtained when one uses erroneously calibrated parameters.

Other questions of interest appear when we consider the following fact: even if the underlying non-stationary financial economy market evolves in continuous time, the data concerning its evolution can only be observed and collected at discrete times in practice. In particular, a trader can only reasonably make his decisions and give his orders to buy or sell at discrete times. This is a difficulty that mathematicians have to face since the optimal strategies concerning mathematical financial models are given theoretically in continuous time in general.

Thus, it might be also useful to introduce mathematical tools in order to evaluate the manner in which the time step affects the continuous theoretical optimal strategy. In particular, a natural question is to seek for numerical schemes that approximate the continuous theoretical optimal strategy in an appropriate way, so that it minimizes the effect of time discretization in both ways (the time discretization due to observations/decisions and the time discretization due to the numerical procedure used when we implement the theoretical optimal strategy in a computer).

These questions have been investigated in [3, 2] in a particular case: the authors assume that the prices of the risky asset evolve according to a log-normal model whose return rate performs a rupture occurring at a random time that cannot be directly observed.

The purpose of this paper is to present the mathematical complexity of these problems and to give preliminary results in a slightly more general case than the one studied in [3, 2].

In this paper, we consider the case of an asset whose instantaneous expected rate of return changes at unknown random times. We give results concerning:

- A strategy or allocation procedure which is optimal when the parameters of our mathematical model are perfectly specified and calibrated: the result is given in continuous time. We will see that this optimal allocation procedure can be described in terms of a mathematical object arising in filtering theory: a *continuous time filter* that satisfies a stochastic differential equation.
- Mathematical strategies in misspecified situations:
  - the underlying market is observed only at multiples of some time step;
  - the decisions of the trader can only be taken at discrete times;
  - the parameters of the model are not exactly known, we make some errors of calibration.

In this case we have to deal with a mathematical object that corresponds intuitively to some kind of *misspecified filter*.

Our problem, in which an asset has an instantaneous rate of return changing at unknown random times, is in relation with the rupture detection. One can quote the reference book [1] on the subject. There are some differences between our setting and the framework of [1]:

- We work in continous time and [1] is entirely written for discrete time models.
- We want to detect the changes in the return rate with the objective to maximize our wealth whereas [1] uses other quantities to optimize.
- We suppose the dynamic of the return rate is completly known which is not the case in [1].

1.2 Description of the model and definition of the filter

Consider two assets (a bank account and a risky asset) which are traded continuously.

Prices of the bank account evolve according to:

$$\frac{dS_t^0}{S_t^0} = rdt. \tag{1.1}$$

Prices of the risky asset evolve according to the following equation:

$$\boxed{\frac{dS_t}{S_t} = \mu(t)dt + \sigma dB_t,} \tag{1.2}$$

where $\mu(t) \in \{\mu_1, \mu_2\}$. The law of the time between two jumps of the drift are exponential with parameters λ_1 and λ_2. More precisely, let us consider two independent sequences of random variables:

- $(\xi_{2n+1}, n \geq 1)$, i.i.d. with exponential law of parameter λ_1
- $(\xi_{2n}, n \geq 1)$, i.i.d. with exponential law of parameter λ_2.

The *r.v.* $(\xi_n, n \geq 1)$ are assumed to be independent of the Brownian Motion $(B_t, t \geq 0)$. We now define the time jumps τ_n as:

$$\begin{cases} \tau_0 &= 0, \\ \tau_n &= \displaystyle\sum_{k=1}^{n} \xi_k. \end{cases}$$

Finally, the drift satisfies:

$$\mu(t) = \begin{cases} \mu_1 & \text{if } \tau_{2n} \leq t \leq \tau_{2n+1}, \\ \mu_2 & \text{if } \tau_{2n+1} \leq t \leq \tau_{2n+2}. \end{cases}$$

The solution of the SDE (1.2) is:

$$\boxed{S_t = S_0 \exp\left(\int_0^t \mu(s)ds - \frac{\sigma^2}{2}t + \sigma B_t\right).} \tag{1.3}$$

Notation:

$$X_t = \log\left(\frac{S_t}{S_0}\right).$$

$$X_t = \int_0^t \mu(s)ds - \frac{\sigma^2}{2}t + \sigma B_t. \tag{1.4}$$

Definition 1.1. *For any process Y, we define the filtration $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)_{t\geq 0}$ generated by this process, that is*

$$\mathcal{F}_t^Y = \sigma\left(Y_s\ ;\ 0 \leq s \leq t\right).$$

In particular, the filtration generated by the observations is denoted $\mathbb{F}^X$.

Definition 1.2 (of the filter). *We define the filter $(F_t)_{t\geq 0}$ to be the optional projection of $\mathbb{1}_{\mu(t)=\mu_1}$ on $\mathbb{F}^X$. This means that $(F_t)_{t\geq 0}$ is the unique optional process such that*

$$\mathbb{E}\left[\mathbb{1}_{\mu(\tau)=\mu_1}\mathbb{1}_{\tau<\infty}|\mathcal{F}_\tau^X\right] = F_\tau \mathbb{1}_{\tau<\infty} \quad \text{a.s. } \textit{for every stopping time } \tau.$$

(see Revuz-Yor [14] Theorem (5.6) p.173).

In particular,

$$\boxed{F_t := \mathbb{P}\left(\mu(t) = \mu_1|\mathcal{F}_t^X\right) \quad \text{a.s.} \quad \text{for all } t \in [0,T].}$$

Remark 1.3. The classical definition of the filter is the conditional law $\mathcal{L}(\mu(t)|\mathcal{F}_t^X)$. In our case, the state space has only two elements; our definition is an abuse of notation.

Now that the model is settled and that we have introduced the main mathematical tools that will be used in the sequel, we explain briefly the organization of the paper.

1.3 Organization of the paper

The organization of the paper is the following:

In Section 2, we give some results concerning classical filtering theory. We construct the Innovation Process of the filtering theory. We explicit the Kushner-Stratonovich equation satisfied by the filter (see Kurtz-Ocone [8]). We also study some ergodic properties of the continuous time filter.

In Section 3, we give a formula for the optimal allocation policy depending on the choice of the Utility function in the continuous time context (see Karatzas-Shreve [6]). In particular, we show that the optimal strategies depend on the continuous time filter in a crucial manner. We give results concerning the particular case of a Logarithmic Utility function: in this study, we use the ergodic properties of the continuous time filter.

In Section 4, we introduce various types of filters that can be used to approximate the continuous time filter. Our study is divided in three parts:

- The discrete time filters: Euler Scheme and Prediction Filters.
- The construction of the Misspecified Continuous Time Filter, which takes into account the errors concerning a bad specification of the parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ that appear in the description of the model.
- Finally, we construct a Misspecified Prediction Filter that takes into account all sources of errors.

We give results concerning the control of the errors of these filters in comparison with the continuous time filter. This kind of approximations has been investigated in many papers (see for example Kushner [9], Di Masi and Runggaldier [4], Picard [13], Florchinger and Le Gland [5], Körezlioğlu and Runggaldier [7] and the references therein). However, to our knowledge, no result can be applied to our model.

In Section 5, we study how the filters constructed in the previous section affect the strategies and the wealth of the trader. For the seek of consistency, we allow the trader to take his decisions only at discrete times that are multiples of the underlying time-step in which observations are made.

We give results concerning the deviation of the asymptotic behavior (when the horizon time T tends to infinity) of the trader's wealth in the particular case of a Logarithmic Utility function.

2 Classical filtering theory: the innovation process and the continuous time filter

2.1 The Innovation Process

Proposition 2.1. *The following process*

$$\boxed{\bar{B}_t := \frac{1}{\sigma}\left(X_t - \int_0^t \left\{\mu_1 F_s + \mu_2(1-F_s) - \frac{\sigma^2}{2}\right\} ds\right)} \tag{2.1}$$

is an $\mathbb{F}^X$-Brownian motion. It is called the innovation process.

Proof. First of all, note that $\left(\bar{B}_t\right)_{t\geq 0}$ is well defined: more precisely, $(F_t)_{t\geq 0}$ is bounded and is defined as an optional projection. In particular, it is measurable w.r.t the Lebesgue measure and $\int_0^t F_s ds$ is well defined.

From Levy's characterization theorem, it is sufficient to show that $\left(\bar{B}_t\right)_{t\geq 0}$ is a continuous local $\mathbb{F}^X$-martingale with

$$\langle \bar{B}\rangle_t = t, \quad t\geq 0, \quad \text{a.s..}$$

Note that $\bar{B}$ is $\mathbb{F}^X$ adapted (see (2.1)) and continuous because X is. It is also easy to check that

$$\begin{aligned}\langle \bar{B}\rangle_t &= \frac{1}{\sigma^2}\langle X\rangle_t \\ &= \langle B\rangle_t \\ &= t.\end{aligned}$$

Thus, it remains only to prove that $\bar{B}$ is an $\mathbb{F}^X$-martingale.

For notational convenience, we define $\left(\mu^{\text{opt}}(t)\right)_{t\geq 0}$ by

$$\mu^{\text{opt}}(t) := \mu_1 F_t + \mu_2(1-F_t), \quad \forall t\geq 0.$$

In particular, it is easily seen that $\left(\mu^{\text{opt}}(t)\right)_{t\geq 0}$ is the optional projection of $(\mu(t))_{t\geq 0}$ on $\mathbb{F}^X$.

For all $0\leq s\leq t$,

$$\begin{aligned}\mathbb{E}\left[\bar{B}_t - \bar{B}_s|\mathcal{F}_s^X\right] &= \frac{1}{\sigma}\mathbb{E}\left[\int_s^t \left(\mu(u) - \mu^{\text{opt}}(u)\right)du|\mathcal{F}_s^X\right] + \mathbb{E}\left[B_t - B_s|\mathcal{F}_s^X\right] \\ &= \frac{1}{\sigma}\int_s^t \mathbb{E}\left[\mu(u) - \mu^{\text{opt}}(u)|\mathcal{F}_s^X\right]du \\ &\quad + \mathbb{E}\left\{\mathbb{E}\left[B_t - B_s|\mathcal{F}_s^B \vee \mathcal{F}_s^\xi\right]|\mathcal{F}_s^X\right\}\end{aligned}$$

and from the definition of μ^{opt} and the independence of the σ-algebras $\mathcal{F}_s^B$ and $\mathcal{F}_s^\xi$, we see that

$$\begin{aligned}\mathbb{E}\left[\bar{B}_t - \bar{B}_s|\mathcal{F}_s^X\right] &= \frac{1}{\sigma}\int_s^t \mathbb{E}\left\{\left[\mu(u) - \mathbb{E}\left[\mu(u)|\mathcal{F}_u^X\right]|\mathcal{F}_s^X\right]\right\}du \\ &\quad + \mathbb{E}\left\{\mathbb{E}\left[B_t - B_s|\mathcal{F}_s^B\right]|\mathcal{F}_s^X\right\} \\ &= 0.\end{aligned}$$

From all the previous, we can conclude that $\bar{B}$ defines an $\mathbb{F}^X$-Brownian motion. □

Remark 2.2. The process X admits the following decomposition in its own filtration $\mathbb{F}^X$

$$\boxed{dX_t = \left(\mu_1 F_t + \mu_2(1-F_t) - \tfrac{\sigma^2}{2}\right)dt + \sigma d\bar{B}_t.} \tag{2.2}$$

2.2 The continuous time filter

In 1965, Wonham [15] showed that $(F_t)_{t\geq 0}$ satisfies a stochastic differential equation. We precise and detail here this result:

Lemma 2.3. *The filter satisfies the following SDE:*

$$\boxed{dF_t = -\lambda_1 F_t dt + \lambda_2(1-F_t)dt + \frac{\mu_1-\mu_2}{\sigma}F_t(1-F_t)d\bar{B}_t.} \tag{2.3}$$

Proof. The generator $\mathcal{G}$ of the process $(\mu(t)\ ;\ t\geq 0)$ satisfies:

$$\forall\ \ f : \{\mu_1\}\cup\{\mu_2\} \longrightarrow \mathbb{R},$$

$$\begin{cases} \mathcal{G}f(\mu_1) := \lim_{t\to 0+}\frac{1}{t}\left\{f(\mu_1)\left(\mathrm{e}^{-\lambda_1 t}-1\right) - f(\mu_2)\left(\mathrm{e}^{-\lambda_1 t}-1\right)\right\} = \lambda_1\left(f(\mu_2)-f(\mu_1)\right) \\ \mathcal{G}f(\mu_2) = \lambda_2\left(f(\mu_1)-f(\mu_2)\right). \end{cases}$$

Thanks to Kurtz-Ocone [8] (p. 90), we find that the SDE satisfied by the filter is given by (2.3) (with the notation used in [8], $h(x) = \frac{x}{\sigma} - \frac{\sigma}{2}$ and $f(x) = \mathbb{1}_{\mu_1 = x}(x)$). ☐

Lemma 2.4. *There is a unique strong solution $(F_t)_{0\leq t\leq T}$ to equation (2.3). Moreover, $(F_t)_{0\leq t\leq T}$ takes values in $(0,1)$.*

Proof. First of all, we consider the following equation

$$\begin{cases} dF_t^c &= \left(-\lambda_1 F_t^c + \lambda_2\left(1-F_t^c\right)\right)dt + \frac{\mu_1-\mu_2}{\sigma}F_t^c(1-F_t^c)\mathbb{1}_{\{0\leq F_t^c\leq 1\}}d\bar{B}_t \\ F_0^c &= F_0 \in (0,1) \end{cases} \tag{2.4}$$

We introduce the stopping time $\tau = \inf\{t\geq 0, F_t \notin (0,1)\}$.

We define the function $\phi(y) = \log\left(\frac{y}{1-y}\right)$; ϕ is a bijection from $(0,1)$ to $\mathbb{R}$ and $\phi^{-1}(y) = \frac{e^y}{1+e^y}$. Applying Itô's formula, we get that

$$\begin{aligned} G_t &:= \phi(F^c_{t\wedge\tau}) \\ &= \phi(F_0) + \int_0^{t\wedge\tau}\frac{1}{(\phi^{-1})'(G_s)}\left(-\lambda_1\phi^{-1}(G_s) + \lambda_2\left(1-\phi^{-1}(G_s)\right)\right)ds \\ &\quad + \frac{\gamma}{2}\int_0^{t\wedge\tau}(\phi^{-1})(G_s)^2(1-\phi^{-1}(G_s))^2\phi''\circ\phi^{-1}(G_s)ds \,+\, \gamma\bar{B}_{t\wedge\tau} \end{aligned}$$

where $\gamma := \frac{\mu_1-\mu_2}{\sigma}$.

Now, making use of the fact that $\phi'(y) = \frac{1}{y(1-y)}$ and $\phi''(y) = \frac{2y-1}{y^2(1-y)^2}$, we find that $(G_t)_{t\geq 0}$ is solution of the following stochastic differential equation:

$$\begin{aligned} G_t &= G_0 \,+ \int_0^{t\wedge\tau}\left(-\frac{\lambda_1}{1-\phi^{-1}(G_s)} + \frac{\lambda_2}{\phi^{-1}(G_s)}\right)ds \\ &\quad + \frac{\gamma^2}{2}\int_0^{t\wedge\tau}\left(2\phi^{-1}(G_s)-1\right)ds + \gamma\bar{B}_{t\wedge\tau} \\ &= G_0 \,+ \int_0^{t\wedge\tau}\left(-\lambda_1(\mathrm{e}^{G_s}+1) + \lambda_2(\mathrm{e}^{-G_s}+1)\right)ds \\ &\quad + \frac{\gamma^2}{2}\int_0^{t\wedge\tau}\frac{\mathrm{e}^{G_s}-1}{\mathrm{e}^{G_s}+1}ds \,+\, \gamma\bar{B}_{t\wedge\tau}. \end{aligned} \tag{2.5}$$

Remark that $\tau = \inf\{t\geq 0\ ;\ |G_t| = +\infty\}$. Using the Feller's test for explosions, we prove have $\tau = +\infty$ *a.s.* and the proof is finished. ☐

2.3 An equality of filtrations

Let us introduce the natural filtrations $\mathbb{F}^{\bar{B}} = \left(\mathcal{F}_t^{\bar{B}}\right)_{t\geq 0}$ and $\mathbb{F}^F = \left(\mathcal{F}_t^F\right)_{t\geq 0}$ generated by the innovation process and the filter respectively, that is

$$\mathcal{F}_t^{\bar{B}} = \sigma\left(\bar{B}_s \ ; 0 \leq s \leq t\right)$$

and

$$\mathcal{F}_t^F = \sigma\left(F_s \ ; 0 \leq s \leq t\right).$$

We have the following result:

Proposition 2.5. *The filtrations $\mathbb{F}^{\bar{B}}$ and $\mathbb{F}^X$ coincide. In particular, the $\mathbb{F}^X$-Brownian motion $\bar{B}$ has the $\mathbb{F}^X$-Previsible Representation Property (PRP in short).*

Proof. Using the equation satisfied by X (2.2), we see that:

$$\mathbb{F}^X \subset \mathbb{F}^{\bar{B}} \vee \mathbb{F}^F.$$

But the filter F satisfies the stochastic differential equation (2.3) and we can easily see that this equation possesses a strong solution adapted to $\mathbb{F}^{\bar{B}}$ whenever $\bar{B}$ is given. This means that $\mathbb{F}^F \subset \mathbb{F}^{\bar{B}}$ and finally $\mathbb{F}^X \subset \mathbb{F}^{\bar{B}}$. Now, since $\bar{B}$ is adapted to $\mathbb{F}^X$ by construction, we have

$$\mathbb{F}^X \equiv \mathbb{F}^{\bar{B}}.$$

Our result follows immediately since every Brownian motion has the PRP with respect to its natural filtration. □

Remark 2.6. This equality of filtration will be crucial in the construction of the optimal strategies (see the Appendix)

2.4 Ergodic properties of the continuous time filter

In this section we state results concerning ergodic properties of the continuous time filter. As we will see in the sequel, theses properties will play a crucial role in Section 3.3 where we study the asymptotic behavior of the wealth in the case of a logarithmic utility function.

2.4.1 Existence of the invariant measure

Proposition 2.7. *The continuous time filter $(F_t)_{t\geq 0}$ possesses a unique invariant measure $\xi(dy)$ over $([0,1]; \mathcal{B}([0,1]))$. Moreover,*

$$\forall \varphi \in \mathcal{L}^1_{b_{loc}}(\xi), \quad \frac{1}{T}\int_0^T \varphi(F_s)ds \xrightarrow[T\to\infty]{a.s.} \int \varphi \, d\xi.$$

Proof. We prove that the diffusion $(G_t)_{t\geq 0}$ solution of (2.5) satisfies Has'minskii's criterion. We can apply [11] (Théorème 1 p. 148) with the Lyapounov function $V(x) = x^2$: it gives us existence and uniqueness of an invariant measure ν for $(G_t)_{t\geq 0}$ and

$$\forall \varphi \in \mathcal{L}^1_{b_{loc}}(\nu), \quad \frac{1}{T}\int_0^T \varphi(G_s)ds \xrightarrow[T\to\infty]{a.s.} \int \varphi \, d\nu.$$

Applying the transformation ϕ^{-1} yields the result of the proposition. □

Remark 2.8. (Speed of convergence towards the invariant measure) If we denote by $\xi^x(t)$ the law of F_t when $F_0 = x$, and using the the results of Haasminsk'ii cited in [12] (Proposition 1 p. 1063), we can show that there exists two strictly positive constants C and c such that

$$||\xi^x(t)(dy) - \xi(dy)||_{TV} \leq C\mathrm{e}^{-ct}$$

where $||.||_{TV}$ stands for the usual total variation norm over signed measures.

2.4.2 Computation of the invariant measure

In order not to deal with possible boundary problems at 0 and 1 that may appear when we write the Fokker-Planck equation directly for $(F_t)_{t\geq 0}$ (taking values in $(0,1)$), we choose to compute the invariant measure for $(G_t)_{t\geq 0}$ instead. We apply then the simple transformation ϕ^{-1}.

Let us introduce $\mathcal{L}$ the operator acting on bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and defined as the classical infinitesimal generator of the process $(G_t)_{t\geq 0}$.

We know that the invariant probability measure of $(G_t)_{t\geq 0}$ is solution to the following Fokker-Planck equation

$$\mathcal{L}^*\rho(dy) = 0 \tag{2.6}$$

where $\mathcal{L}^*$ stands for the adjoint operator of $\mathcal{L}$.

We know that the scale-changed filter $(G_t)_{t\geq 0}$ has an unique invariant measure, and we are going to search for a solution of (2.6) in the restricted class $\mathcal{D}(\mathbb{R})$ of probability measures that possess a twice differentiable density on $\mathbb{R}$.

Thus, we search for a function g^ρ solution of (2.6) in $\mathcal{D}(\mathbb{R})$. The Fokker-Planck equation (2.6) now reads:

$$\frac{d}{dy}\left[\left(\lambda_1(1+\mathrm{e}^y) - \lambda_2\left(1+e^{-y}\right)\right)g^\rho(y) + \frac{\gamma^2}{2}\frac{1-\mathrm{e}^y}{1+\mathrm{e}^y}g^\rho(y) + \frac{\gamma^2}{2}\frac{d}{dy}g^\rho(y)\right] = 0.$$

We search for a function g^ρ such that there exists a constant C with:

$$\frac{2}{\gamma^2}\left(\lambda_1(1+\mathrm{e}^y) - \lambda_2\left(1+e^{-y}\right)\right)g^\rho(y) + \frac{1-\mathrm{e}^y}{1+\mathrm{e}^y}g^\rho(y) + \frac{d}{dy}g^\rho(y) = C.$$

Note that this equation is of the general form:

$$\alpha(y)g(y) + g'(y) = C,$$

so that its solution is given by:

$$g(y) = \left(g(0) + C\int_0^y \exp\left(\int_0^z \alpha(u)du\right)dz\right)\exp\left(-\int_0^y \alpha(z)dz\right).$$

In our particular case, we get that

$$\begin{aligned} g^\rho(y) \quad = \quad & \left\{g^\rho(0) + C\int_0^y \exp\left(\frac{2}{\gamma}\left\{\lambda_1\left(z+\mathrm{e}^z-1\right) - \lambda_2\left(z-\mathrm{e}^{-z}+1\right)\right\}\right)\right. \\ & \left.\frac{1}{\mathrm{ch}^2(z/2)}dz\right\} \times \exp\left(\frac{2}{\gamma}\left\{\lambda_2\left(y-\mathrm{e}^{-y}+1\right) - \lambda_1\left(y+\mathrm{e}^y-1\right)\right\}\right) \\ & \times\mathrm{ch}^2(y/2). \end{aligned}$$

From this expression, we deduce that:

$$\xi(dy) := \frac{1}{\int_0^1 g^\rho\circ\phi(z)\phi'(z)dz}g^\rho\circ\phi(y)\phi'(y)\mathbb{1}_{[0,1]}(y)dy.$$

3 Optimal Portfolio Allocation Strategy

In this section our aim is to explicit the optimal wealth and strategy of a trader who perfectly knows all the parameters μ_1, μ_2, λ_1, λ_2, and σ. Of course, this situation is idealistic.

3.1 Presentation of the Problem

Let π_t denote the proportion of the trader's wealth invested in the stock S at time t; the remaining proportion $1-\pi_t$ is invested in the bond S^0. For a given non random initial capital $x > 0$, let $W_\cdot^{x,\pi}$ denote the wealth process corresponding to the portfolio $\pi_\cdot$. This wealth process is solution of the following equation:

$$\begin{cases} \dfrac{dW_t^{x,\pi}}{W_t^{x,\pi}} = \pi_t \dfrac{dS_t}{S_t} + (1-\pi_t)\dfrac{dS_t^0}{S_t^0} \quad 0 \leq t \leq T, \\ W_0^{x,\pi} = x. \end{cases}$$

We now define the notion of utility function that we will use in the following.

Definition 3.1. *A* ***utility*** *function U is such that:*

1. $U : \mathbb{R}_+^* \to \mathbb{R}$
2. *U is C^2, strictly concave, increasing*
3. $\lim\limits_{+\infty} U'(x) = 0$
4. $\lim\limits_{0} U'(x) = +\infty$

We denote by I the inverse of U'.

Definition 3.2. *The set of* ***admissible*** *portfolios is defined by:*

$$\mathcal{A}(x) := \left\{\pi_\cdot \;\; \mathbb{F}^X - \textit{progressively measurable process with values in } [0,1] \;\; \textit{s.t. } W_0^{x,\pi} = x\right\}$$

The investor's objective is to maximize his expected utility of wealth at the terminal time T; he has to solve the following constrained optimization problem:

$$\mathcal{P}: \quad V^*(x) := \sup_{(\pi_t)_{0\leq t\leq T}\in\mathcal{A}(x)} \mathbb{E}\left[U\left(W_T^{x,\pi}\right)\right].$$

3.2 Main result

The main result is given in the Appendix: we describe the optimal allocation strategy for general utility functions using a Representation Theorem for $\mathbb{F}^X$-martingales thanks to Proposition 2.5. The result is similar to the optimal allocation strategy for the constrained Merton problem but with a return rate described with the help of the filter.

The result and its proof closely follow the ideas of Karatzas-Shreve [6]: in order to solve the investor's constrained optimization problem, we introduce a family of auxiliary unconstrained optimization problems. Proposition 7.1 of the Appendix explains the link between this family of unconstrained problems and our investor's constrained problem.

We find the optimal allocation strategy for each auxiliary unconstrained optimization problem and we conclude with Proposition 7.8, which exhibits the optimal strategy for our investor's original constrained problem.

3.3 The particular case of a Logarithmic Utility function

Thanks to the filter, we can apply the optimal allocation strategy for the logarithmic utility function. We will show that the optimal allocation strategy π^* is given in this case by

$$\boxed{\pi^*_t = \mathrm{proj}_{[0,1]}\left\{\frac{\mu^{\mathrm{opt}}(t)-r}{\sigma^2}\right\}.}$$

Notation 3.3. *Note that π^* is a function of the continuous filter F. In all the following we will set*

$$\pi^*_t := q^*(F_t), \quad 0 \le t \le T. \tag{3.1}$$

In the particular case of a Logarithmic Utility function it is possible to obtain the asymptotic behavior of the wealth. When we apply Itô's formula to the optimal wealth process, we see that:

$$\begin{aligned}\frac{1}{T}\log\left(W^{*,x}_T\right) &= \frac{1}{T}\log(x) + r + \frac{1}{T}\int_0^T q^*(F_t)\left(\mu^{\mathrm{opt}}(t)-r\right)dt \\ &\quad -\frac{1}{T}\int_0^T \frac{\sigma^2\left(q^*(F_t)\right)^2}{2}dt + \frac{\sigma}{T}\int_0^T q^*(F_t)d\bar{B}_t\end{aligned} \tag{3.2}$$

Since $\mu^{\mathrm{opt}}(t) := \mu_1 F_t + \mu_2(1-F_t)$, we can apply Birkoff's theorem. Because

$$\lim_{T\to+\infty}\frac{B_T}{T} = 0, \quad \text{a.s.},$$

we directly see that

$$\lim_{T\to+\infty}\frac{1}{T}\log\left(W^{*,x}_T\right) = \int_0^1 \left\{r\left(1-q^*(y)\right) + q^*(y)\left(\mu_1 y + \mu_2(1-y)\right)\right\}\xi(dy) - \frac{\sigma^2}{2}, \quad \text{a.s.} \tag{3.3}$$

Remark 3.4. The crucial fact used here is that the optimal strategy π^* does not depend on the horizon T. Our result gives the asymptotic behavior of the wealth process governed by the optimal strategy almost-surely. This means that the optimal strategy $(\pi^*_t)_{t\ge0}$ is asymptotically **almost surely** better than any other strategy $(\pi_t)_{t\ge0}$ as long as

$$\lim_{T\to+\infty}\frac{1}{T}\log\left(W^{p,x}_T\right)$$

is almost surely deterministic. This is true for all **reasonable** strategies.

4 The filters

4.1 A first discrete filter: the Euler scheme

We present here a simple method to estimate the filter F. Thanks to (2.3) and (2.2), we write the dynamics of F as:

$$\begin{aligned} dF_t &= -\lambda_1 F_t dt + \lambda_2(1-F_t)dt \\ &\quad -\frac{\mu_1-\mu_2}{\sigma^2}F_t(1-F_t)\left(\mu_1 F_t + \mu_2(1-F_t) - \frac{\sigma^2}{2}\right)dt \\ &\quad +\frac{\mu_1-\mu_2}{\sigma^2}F_t(1-F_t)dX_t.\end{aligned} \tag{4.1}$$

To simplify the notation, we write this SDE in the following way:

$$dF_t = \phi_1(F_t)dt + \phi_2(F_t)dX_t.$$

A naive approach to estimate F is to use an Euler's scheme:

$$\begin{cases} \bar{F}^e_0 &= F_0, \\ \bar{F}^e_{k+1} - \bar{F}^e_k &= \phi_1(\bar{F}^e_k)\Delta t + \phi_2(\bar{F}^e_k)\Delta X_k. \end{cases}$$

It is classical to prove that:

$$\boxed{\mathbb{E}\left[\sup_{k\Delta t \le t}\left(F_{k\Delta t} - \bar{F}^e_k\right)^2\right] \le C_t \Delta t.} \tag{4.2}$$

Remark 4.1. a. As in Theorem 5.3, we can prove that (4.2) is still available with C independent of t.

b. This procedure does not ensure that $\bar{F}^e_t$ remains in $(0,1)$. In practice, we project this scheme in $[0,1]$.

In the following, we describe another approximation filter (of higher order) based on the filtering theory.

4.2 The prediction filter

We set $\Delta t > 0$.

Notation 4.2. $\Delta X_k = X_{(k+1)\Delta t} - X_{k\Delta t}$.

4.2.1 Construction

Let us now describe in details the evolution of a classical discrete approximation $(\bar{F}_k)_{k\ge 0}$ of $(F_{k\Delta t})_{k\ge 0}$. We set $\bar{F}_0 = F_0$ and for all k, $\bar{G}_k = 1 - \bar{F}_k$.

First step: selection

$$\begin{pmatrix} \bar{F}_k \\ \bar{G}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{F}'_k = \bar{F}_k \dfrac{1}{\sigma\sqrt{2\pi\Delta t}} \exp - \dfrac{\left(\Delta X_k - (\mu_1 - \frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t} \\ \bar{G}'_k = \bar{G}_k \dfrac{1}{\sigma\sqrt{2\pi\Delta t}} \exp - \dfrac{\left(\Delta X_k - (\mu_2 - \frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t} \end{pmatrix}$$

Second step: normalization

$$\begin{pmatrix} \bar{F}'_k \\ \bar{G}'_k \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{F}''_k = \dfrac{\bar{F}'_k}{\bar{F}'_k + \bar{G}'_k} \\ \bar{G}''_k = \dfrac{\bar{G}'_k}{\bar{F}'_k + \bar{G}'_k} \end{pmatrix}$$

Third step: evolution

$$\begin{pmatrix} \bar{F}''_k \\ \bar{G}''_k = 1 - \bar{F}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{F}_{k+1} = \bar{F}''_k e^{-\lambda_1\Delta t} + \bar{G}''_k(1 - e^{-\lambda_2\Delta t}) \\ \bar{G}_{k+1} = \bar{G}''_k e^{-\lambda_2\Delta t} + \bar{F}''_k(1 - e^{-\lambda_1\Delta t}) \end{pmatrix}$$

We will prove that the construction is linked with a slightly different model involving classical conditional expectations and that we describe below.

We take $(\bar{\mu}_k)_{k\ge 0}$ a Markov chain taking values in $\{\mu_1, \mu_2\}$ such that:

$$\begin{aligned} \mathcal{L}(\bar{\mu}_0) &= \mathcal{L}(\mu_0) \\ \mathbb{P}(\bar{\mu}_{k+1} = \mu_1 | \bar{\mu}_k = \mu_1) &= Q(\mu_1, \mu_1) := e^{-\lambda_1\Delta t} \\ \mathbb{P}(\bar{\mu}_{k+1} = \mu_2 | \bar{\mu}_k = \mu_1) &= Q(\mu_1, \mu_2) := 1 - Q(\mu_1, \mu_1) \\ \mathbb{P}(\bar{\mu}_{k+1} = \mu_2 | \bar{\mu}_k = \mu_2) &= Q(\mu_2, \mu_2) := e^{-\lambda_2\Delta t} \\ \mathbb{P}(\bar{\mu}_{k+1} = \mu_1 | \bar{\mu}_k = \mu_2) &= Q(\mu_2, \mu_1) := 1 - Q(\mu_2, \mu_2) \end{aligned}$$

and $(\bar{X}_k)_{k\geq 0}$ is a Markov chain such as

$$\begin{aligned}\bar{X}_0 &= 0,\\ \bar{X}_{k+1} &= \bar{X}_k + \left(\bar{\mu}_k - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}U_k\end{aligned}$$

where $(U_k)_{k\geq 0}$ are i.i.d. variables of law $\mathcal{N}(0,1)$. $(\bar{\mu}, \bar{X})$ may be viewed as an approximation of (μ, X) where $\bar{\mu}$ is only allowed to jump at the discretization times $k\Delta t$ (with probabilities near the probabilities that $\mu(t)$ may jump between $k\Delta t$ and $(k+1)\Delta t$). We set

$$\begin{aligned}g(y,u) &= \frac{1}{\sigma\sqrt{2\pi\Delta t}}\exp\left(-\frac{\left(y-(u-\frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t}\right), \qquad (4.3)\\ \Delta\bar{X}_k &= \bar{X}_{k+1} - \bar{X}_k.\end{aligned}$$

The law $\mathcal{L}(\bar{\mu}_n|\bar{X}_0,\ldots,\bar{X}_n)$ is called the prediction filter.

Lemma 4.3. *For any function $f:\{\mu_1,\mu_2\}\to\mathbb{R}$, for all $n\geq 0$,*

$$\mathbb{E}(f(\bar{\mu}_n)|\bar{X}_0,\ldots,\bar{X}_n) = \qquad (4.4)$$
$$\frac{\displaystyle\sum_{i_0,\ldots,i_n\in\{1,2\}} f(\mu_{i_n})\mathbb{P}(\bar{\mu}_0=\mu_{i_0})\prod_{k=0}^{n-1} g\left(\Delta\bar{X}_k,\mu_{i_k}\right)Q(\mu_{i_k},\mu_{i_{k+1}})}{\displaystyle\sum_{i_0,\ldots,i_n\in\{1,2\}} \mathbb{P}(\bar{\mu}_0=\mu_{i_0})\prod_{k=0}^{n-1} g\left(\Delta\bar{X}_k,\mu_{i_k}\right)Q(\mu_{i_k},\mu_{i_{k+1}})}.$$

If we set

$$\hat{F}_n = \mathbb{P}(\bar{\mu}_n=\mu_1|\bar{X}_0\ldots,\bar{X}_n) \qquad (4.5)$$
$$\hat{F}'_n = \frac{g(\Delta\bar{X}_n,\mu_1)\hat{F}_n}{g(\Delta\bar{X}_n,\mu_1)\hat{F}_n + (1-\hat{F}_n)g(\Delta\bar{X}_n,\mu_2)}$$

we then have:

$$\hat{F}_{n+1} = F'_n Q(\mu_1,\mu_1) + (1-F'_n)Q(\mu_2,\mu_1). \qquad (4.6)$$

Proof. To obtain (4.4), we remark that the density $\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_n},x_1,\cdots,x_n)$ of the random vector $(\bar{\mu}_0, \cdots, \bar{\mu}_n, \bar{X}_1, \cdots, \bar{X}_n)$ with respect to $\delta^{\otimes n+1}\otimes\lambda^{\otimes n}$ is equal to

$$\mathbb{P}(\bar{\mu}_0=\mu_{i_0})g(x_1,\mu_{i_0})Q(\mu_{i_0},\mu_{i_1})\prod_{k=1}^{n-1}Q(\mu_{i_k},\mu_{i_{k+1}})g(x_{k+1}-x_k,\mu_{i_k})$$

where λ is the Lebesgue measure on $\mathbb{R}$ and δ is equal to the counting measure on $\{\mu_1,\mu_2\}$ (i.e. $\delta=\delta_{\mu_1}+\delta_{\mu_2}$).

Let $f(x)=\delta_{\mu_1}(x)$:

$$\begin{aligned}\hat{F}_{n+1} &= \mathbb{E}(f(\bar{\mu}_{n+1})|\bar{X}_0,\ldots,\bar{X}_{n+1})\\ &= \frac{\displaystyle\sum_{i_0,\ldots,i_{n+1}} f(\mu_{i_{n+1}})\mathcal{P}_{n+1}(\mu_{i_0},\ldots,\mu_{i_{n+1}},\bar{X}_1,\ldots,\bar{X}_{n+1})}{\displaystyle\sum_{i_0,\ldots,i_{n+1}}\mathcal{P}_{n+1}(\mu_{i_0},\cdots,\mu_{i_{n+1}},\bar{X}_1,\ldots,\bar{X}_{n+1})}\\ &=: \frac{\Upsilon_{n+1}}{\Xi_{n+1}}.\end{aligned}$$

We have:

$$
\begin{aligned}
\Upsilon_{n+1} &= \sum_{i_0,\ldots,i_n} g(\Delta\bar{X}_n,\mu_{i_n})\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_n},\bar{X}_1,\ldots,\bar{X}_n)Q(\mu_{i_n},\mu_1)\\
&= g(\Delta\bar{X}_n,\mu_1)Q(\mu_1,\mu_1)\sum_{i_0,\ldots,i_{n-1}}\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_{n-1}},\mu_1,\bar{X}_1,\ldots,\bar{X}_n)\\
&\quad + g(\Delta\bar{X}_n,\mu_2)Q(\mu_2,\mu_1)\sum_{i_0,\ldots,i_{n-1}}\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_{n-1}},\mu_2,\bar{X}_1,\ldots,\bar{X}_n)\\
&= \left\{g(\Delta\bar{X}_n,\mu_1)Q(\mu_1,\mu_1)\hat{F}_n + g(\Delta\bar{X}_n,\mu_2)Q(\mu_2,\mu_1)(1-\hat{F}_n)\right\}\Xi_n.
\end{aligned}
$$

In the same way:

$$
\begin{aligned}
\Xi_{n+1} &= \sum_{i_0,\cdots,i_n} g(\Delta\bar{X}_n,\mu_{i_n})Q(\mu_{i_n},\mu_1)\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_n},\bar{X}_1,\ldots,\bar{X}_n)\\
&\quad + \sum_{i_0,\cdots,i_n} g(\Delta\bar{X}_n,\mu_{i_n})Q(\mu_{i_n},\mu_2)\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_n},\bar{X}_1,\ldots,\bar{X}_n)\\
&= \sum_{i_0,\cdots,i_n} g(\Delta\bar{X}_n,\mu_{i_n})\mathcal{P}_n(\mu_{i_0},\cdots,\mu_{i_n},\bar{X}_1,\ldots,\bar{X}_n)\\
&= g(\Delta\bar{X}_n,\mu_1)\sum_{i_0,\cdots,i_n}\mathcal{P}_n(\mu_{i_0},\cdots,\mu_1,\bar{X}_1,\ldots,\bar{X}_n)\\
&\quad + g(\Delta\bar{X}_n,\mu_2)\sum_{i_0,\cdots,i_n}\mathcal{P}_n(\mu_{i_0},\cdots,\mu_2,\bar{X}_1,\ldots,\bar{X}_n)\\
&= g(\Delta\bar{X}_n,\mu_1)\Upsilon_n + g(\Delta\bar{X}_n,\mu_2)(\Xi_n-\Upsilon_n),
\end{aligned}
$$

and the result follows immediately. □

We denote by Θ^n the function such that

$$\hat{F}_n = \Theta^n\left(\bar{X}_0,\cdots,\bar{X}_n\right). \tag{4.7}$$

We set $\bar{F}_n = \Theta^n\left(X_0,\ldots,X_n\right)$.

Remark 4.4. Notice that $\bar{F}_n$ is constructed exactly in the same way as $\hat{F}_n$ but in which we injected observations $X_0,\ldots,X_n$ (instead of $\bar{X}_0,\ldots,\bar{X}_n$). The process $\bar{F}_n$ cannot be easily seen as a conditional probability. Nevertheless, we will prove later that this discrete filter approximates $F_{n\Delta t}$.

4.2.2 Convergence of the prediction filter

In the following, we will write C in the place of some constant depending on the parameters $\mu_1,\mu_2,\sigma,\lambda_1,\lambda_2$. This constant may change from line to line. We suppose also that $\Delta t\le 1$. We set, for all integer $k\ge 0$ $\bar{\bar{F}}_{k\Delta t} = \bar{F}_k$ and $\eta(t) = \Delta t\left[\frac{t}{\Delta t}\right]$.

Notations 4.5. From now on, we will denote by $\mathcal{R}$ the set of sequences of random variables $(R_k)_{k\ge 0}$ such that:

$$\sup_k \mathbb{E}(|R_k|^2) \le C\Delta t^4. \tag{4.8}$$

We have the following properties:

(P1) If $(R_k)_{k\ge0}\in\mathcal{R}$ and $(R'_k)_{k\ge0}\in\mathcal{R}$ then $(R_k+R'_k)_{k\ge0}\in\mathcal{R}$.

(P2) If $(R_k)_{k\geq 0}\in\mathcal{R}$ and $(R'_k)_{k\geq 0}$ is such that $\sup_k |R'_k|\leq C$ for all ω then $(R_kR'_k)_{k\geq 0}\in\mathcal{R}$.

By (2.2), we have :

$$\Delta X_k = \int_{k\Delta t}^{(k+1)\Delta t} (\mu^{\text{opt}}(s) - \frac{\sigma^2}{2})ds + \sigma\Delta\bar{B}_k \tag{4.9}$$

where

$$\Delta\bar{B}_k = \bar{B}_{(k+1)\Delta t} - \bar{B}_{k\Delta t}\ . \tag{4.10}$$

From (4.9) we can easily notice that the following sequences are in $\mathcal{R}$: $(\Delta t^2)_{k\geq 0}$, $(\Delta t\Delta X_k^2)_{k\geq 0}$, $(\Delta t^2\Delta X_k)_{k\geq 0}$, $(\Delta t^3)_{k\geq 0}$, $(\Delta X_k^4)_{k\geq 0}$.

In the following, we will write $(R_k)_{k\geq 0}$ for a sequence in $\mathcal{R}$. This sequence may change from line to line.

Let us now present a technical lemma:

Lemma 4.6. *We have the following decomposition:*

$$\Delta X_k^2 = \sigma^2\Delta\bar{B}_k^2 + D_k + R_k$$

where (D_k) is a sequence of random variables such that

$$\begin{cases} D_k \text{ is } \mathcal{F}^X_{t_{k+1}} \text{ measurable;} \\ \mathbb{E}(D_k|\mathcal{F}^X_{t_k}) = 0; \\ \mathbb{E}(D_k^2)\leq C\Delta t^3. \end{cases} \tag{4.11}$$

Proof.

$$\begin{aligned}
\Delta X_k^2 &= 2\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{s}\left(\mu_1 F_u + \mu_2(1-F_u) - \frac{\sigma^2}{2}\right)du\right)\left(\mu_1 F_s + \mu_2(1-F_s) - \frac{\sigma^2}{2}\right)ds \\
&+ 2\sigma\int_{t_k}^{t_{k+1}}\left(\bar{B}_s - \bar{B}_{t_k}\right)\left(\mu_1 F_s + \mu_2(1-F_s) - \frac{\sigma^2}{2}\right)ds \\
&+ \sigma^2\left(\bar{B}_{t_{k+1}} - \bar{B}_{t_k}\right)^2 + 2\sigma\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\left(\mu_1 F_u + \mu_2(1-F_u) - \frac{\sigma^2}{2}\right)dud\bar{B}_s
\end{aligned}$$

Obviously, the first term is in $\mathcal{R}$.

Furthermore, the sum of the second and the last terms is equal to:

$$\begin{aligned}
A &= 2\sigma\Delta\bar{B}_k\int_{t_k}^{t_{k+1}}\left[\mu_1 F_u + \mu_2(1-F_u) - \frac{\sigma^2}{2}\right]du \\
&= 2\sigma\Delta\bar{B}_k\Delta t\left[\mu_1 F_{t_k} + \mu_2(1-F_{t_k}) - \frac{\sigma^2}{2}\right] \\
&+ 2\sigma\Delta\bar{B}_k(\mu_1-\mu_2)\int_{t_k}^{t_{k+1}}(F_s - F_{t_k})ds \\
&= D_k + R_k.
\end{aligned}$$

We verify that $(D_k)_{k\geq 0}$ satisfies (4.11). □

Proposition 4.7. *For all $N\in\mathbb{N}$, there exists a constant $C_{N\Delta t}$ (depending continuously of $N\Delta t$ and the parameters of the problem) such that for any X_0, F_0:*

$$\boxed{\sup_{0\leq k\leq N}\mathbb{E}\left[\left(F_{k\Delta t} - \bar{F}_k\right)^2\right]\leq C_{N\Delta t}\Delta t^2.} \tag{4.12}$$

There exists a continuous extension $(\bar{\bar{F}}_t)_{t\geq 0}$ of $\bar{\bar{F}}_0, \bar{\bar{F}}_{\Delta t}, \bar{\bar{F}}_{2\Delta t}, \ldots$ defined below in (4.14) and for all t_0:

$$\boxed{\mathbb{E}\left[\sup_{0\leq t\leq t_0} (\bar{\bar{F}}_t - F_t)^2\right] \leq C_{t_0}\Delta t^2}$$

where C_{t_0} depends only on t_0 and the parameters of the problem.

Proof.

First step

We will show in this step that the sequence $(\bar{F}_k)_{k\geq 0}$ is the Euler-Milstein scheme associated to $(F_t)_{t\geq 0}$ up to negligible terms. We have:

$$\begin{aligned}
\bar{F}_k'' &= \frac{\bar{F}_k}{\bar{F}_k + (1-\bar{F}_k)\exp\dfrac{-\left(\Delta X_k - (\mu_2 - \frac{\sigma^2}{2})\Delta t\right)^2 + \left(\Delta X_k - (\mu_1 - \frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t}} \\
&= \frac{\bar{F}_k}{\bar{F}_k + (1-\bar{F}_k)\underbrace{\exp\dfrac{(\mu_2-\mu_1)(2\Delta X_k + (\sigma^2 - \mu_1 - \mu_2)\Delta t)}{2\sigma^2}}_{=:A}}.
\end{aligned}$$

Let us set:

$$\begin{aligned}
\alpha &:= \left(\frac{\sigma^2 - \mu_1 - \mu_2}{2\sigma^2}\right)(\mu_2 - \mu_1), \\
\beta &:= \frac{\mu_2 - \mu_1}{\sigma^2}\ .
\end{aligned}$$

We have:

$$\begin{aligned}
\bar{F}_k'' &= \frac{\bar{F}_k}{1-(1-A)(1-\bar{F}_k)} \\
&= \bar{F}_k(1 + (1-A)(1-\bar{F}_k) + (1-A)^2(1-\bar{F}_k)^2 \qquad (4.13)\\
&\qquad + (1-A)^3(1-\bar{F}_k)^3 + R_k^{(1)}) \\
\text{with} \qquad R_k^{(1)} &= \int_0^{(1-A)(1-\bar{F}_k)} 4((1-A)(1-\bar{F}_k) - s)^3 \frac{1}{(1-s)^5} ds\ .
\end{aligned}$$

We will prove that $(R_k^{(1)})_k \in \mathcal{R}$.

First, we have

$$\left|R_k^{(1)}\right| \leq \begin{cases} 4|1-A|^4|1-F_k|^4 & \text{if } 1-A<0 \\ \dfrac{4|1-A|^4|1-F_k|^4}{(1-(1-A)(1-\bar{F}_k))^5} & \text{if } 0\leq 1-A\leq 1. \end{cases}$$

And thus,

$$\left|R_k^{(1)}\right| \leq 4|1-A|^4 \sup\left(1, A^{-5}\right).$$

$$A = \exp\left(\beta\Delta X_k + \alpha\Delta t\right) =: e^U.$$

Thanks to the classical inequality

$$\left|e^U - 1\right| \leq |U| e^{|U|},$$

the following inequality is satisfied:

$$|A-1| \leq C(|\Delta X_k| + \Delta t) e^{C(|\Delta X_k| + \Delta t)}.$$

$$\begin{aligned}
\mathbb{E}\left(|R_k^{(1)}|^2\right) &\leq \mathbb{E}\left[C|1-A|^8 \sup\left(1, A^{-10}\right)\right] \\
&\leq \mathbb{E}\left[C(|\Delta X_k| + \Delta t)^8 e^{C(|\Delta X_k| + \Delta t)}\right] \\
&\leq \mathbb{E}\left[C(|\Delta \bar{B}_k| + \Delta t)^8 e^{C(|\Delta \bar{B}_k| + \Delta t)}\right] \\
&\leq C(\Delta t)^4 \text{ for } \Delta t \text{ small enough .}
\end{aligned}$$

We now develop A:

$$\begin{aligned}
A &= 1 + \alpha \Delta t + \beta \Delta X_k + \frac{1}{2}(\alpha \Delta t + \beta \Delta X_k)^2 + \frac{1}{3!}(\alpha \Delta t + \beta \Delta X_k)^3 + R_k^{(2)} \\
\text{with} \qquad & R_k^{(2)} = \int_0^{\alpha \Delta t + \beta \Delta X_k} \frac{(\alpha \Delta t + \beta \Delta X_k - s)^3}{3!} e^s ds.
\end{aligned}$$

We prove also that $(R_k^{(2)})_k \in \mathcal{R}$.

$$\begin{aligned}
\mathbb{E}(|R_k^{(2)}|^2) &\leq \mathbb{E}\left\{C(\Delta t + |\Delta X_k|)^8 e^{C(\Delta t + |\Delta X_k|)}\right\} \\
&\leq C(\Delta t)^4 \ .
\end{aligned}$$

And so, we can write:

$$\begin{aligned}
A - 1 &= \alpha \Delta t + \beta \Delta X_k + \frac{\beta^2}{2} \Delta X_k^2 + \alpha \beta \Delta t \Delta X_k + \frac{\beta^3}{6} \Delta X_k^3 + R_k. \\
(A-1)^2 &= 2\alpha\beta \Delta t \, \Delta X_k + \beta^2 \left(\Delta X_k\right)^2 + \beta^3 \left(\Delta X_k\right)^3 + R_k \\
(A-1)^3 &= \beta^3 \left(\Delta X_k\right)^3 + R_k.
\end{aligned}$$

Getting back to (4.13), we have:

$$\begin{aligned}
\bar{F}_k'' = \; & \bar{F}_k - \bar{F}_k(1-\bar{F}_k)\left(\alpha \Delta t + \beta \Delta X_k + \frac{\beta^2}{2} \Delta X_k^2 + \alpha \beta \Delta t \Delta X_k + \frac{\beta^3}{6} \Delta X_k^3\right) \\
& + \bar{F}_k(1-\bar{F}_k)^2 (\beta^2 \Delta X_k^2 + 2\alpha\beta \Delta t \Delta X_k + \beta^3 \Delta X_k^3) \\
& - \bar{F}_k(1-\bar{F}_k)^3 \beta^3 \Delta X_k^3 + R_k.
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{k+1} &= \bar{F}_k'' e^{-\lambda_1 \Delta t} + (1 - \bar{F}_k'')(1 - e^{-\lambda_2 \Delta t}) \\
&= \bar{F}_k''(1 - \lambda_1 \Delta t - \lambda_2 \Delta t) + \lambda_2 \Delta t + R_k \\
&= \bar{F}_k - \bar{F}_k(\lambda_1 + \lambda_2)\Delta t + \lambda_2 \Delta t \\
&\quad - \bar{F}_k(1-\bar{F}_k)[\alpha \Delta t + \beta \Delta X_k + \frac{\beta^2}{2} \Delta X_k^2 + \alpha\beta \Delta t \Delta X_k \\
&\qquad + \frac{\beta^3}{6} \Delta X_k^3 - \beta(\lambda_1 + \lambda_2)\Delta t \Delta X_k] \\
&\quad + \bar{F}_k(1-\bar{F}_k)^2 (\beta^2 \Delta X_k^2 + 2\alpha\beta \Delta t \Delta X_k + \beta^3 \Delta X_k^3) \\
&\quad - \bar{F}_k(1-\bar{F}_k)^3 \beta^3 \Delta X_k^3 + R_k.
\end{aligned}$$

By (4.9), we have

$$\Delta t \Delta X_k = \sigma \Delta t \Delta \bar{B}_k + R_k,$$
$$\Delta X_k^3 = \sigma^3 \Delta \bar{B}_k^3 + R_k.$$

Thanks to Lemma 4.6, we have:

$$\Delta X_k^2 = \sigma^2 \Delta \bar{B}_k^2 + D_k + R_k$$

where (D_k) is a sequence of independent random variables satisfying (4.11).

$$\begin{aligned}\bar{F}_{k+1} &= \bar{F}_k - \bar{F}_k(\lambda_1+\lambda_2)\Delta t + \lambda_2 \Delta t + \Delta t \bar{F}_k(1-\bar{F}_k)\left[-\alpha - \frac{\sigma^2\beta^2}{2} + \sigma^2\beta^2(1-\bar{F}_k)\right] \\ &\quad -\beta \Delta X_k \bar{F}_k(1-\bar{F}_k) + (\Delta \bar{B}_k^2 - \sigma^2 \Delta t)\bar{F}_k(1-\bar{F}_k)\left[-\frac{\beta^2}{2} + \beta^2(1-\bar{F}_k)\right] + \tilde{D}_k + A_k \Delta \bar{B}_k^3 + R_k\end{aligned}$$

where $(\tilde{D}_k)_{k\geq 0}$ is a sequence satisfying also (4.11) and $(A_k)_{k\geq 0}$ is some bounded sequence such that A_k is $\mathcal{F}^X_{k\Delta t}$-measurable. And so:

$$\begin{aligned}\bar{F}_{k+1} &= \bar{F}_k - \lambda_1 \bar{F}_k \Delta t + \lambda_2 \Delta t (1-\bar{F}_k) \\ &\quad +\Delta t \bar{F}_k(1-\bar{F}_k)\frac{(\mu_2-\mu_1)}{\sigma^2}\left[\mu_1 \bar{F}_k + \mu_2(1-\bar{F}_k) - \frac{\sigma^2}{2}\right] \\ &\quad +\frac{(\mu_1-\mu_2)}{\sigma^2}\bar{F}_k(1-\bar{F}_k)\Delta X_{k+1} \\ &\quad +\left(\Delta \bar{B}_k^2 - \Delta t\right)\bar{F}_k(1-\bar{F}_k)\frac{(\mu_2-\mu_1)^2}{2\sigma^2}\left(1-2\bar{F}_k\right) \\ &\quad +\tilde{D}_k + A_k \Delta \bar{B}_k^3 + R_k \ .\end{aligned}$$

Second step

In this step, we reduce the problem to the Euler-Milstein method. We set $\forall x \in \mathbb{R}$:

$$\begin{aligned} u(x) &:= \mu_1 x + \mu_2(1-x) - \frac{\sigma^2}{2}; \\ v(x) &:= -\lambda_1 x + \lambda_2(1-x) + x(1-x)\frac{\mu_2-\mu_1}{\sigma^2}\left(\mu_1 x + \mu_2(1-x) - \frac{\sigma^2}{2}\right); \\ w(x) &:= \frac{\mu_1-\mu_2}{\sigma^2}x(1-x). \end{aligned}$$

We extend $\bar{\bar{F}}$ for all $t \geq 0$ (using Itô's formula):

$$\begin{aligned}
\bar{\bar{F}}_t &= \bar{\bar{F}}_0 + \int_0^t v(\bar{\bar{F}}_{\eta(s)})ds + \int_0^t w(\bar{\bar{F}}_{\eta(s)})dX_s \qquad (4.14)\\
&\quad + \int_0^t \sigma^2 (ww')(\bar{\bar{F}}_{\eta(s)})(\bar{B}_s - \bar{B}_{\eta(s)})d\bar{B}_s \\
&\quad + \sum_{0\leq k\leq [t/\Delta t]} \left(\tilde{D}_k + A_k \Delta \bar{B}_k^3 + R_k\right) \\
&= \bar{\bar{F}}_0 + \int_0^t v(\bar{\bar{F}}_{\eta(s)}) + w(\bar{\bar{F}}_{\eta(s)})u(F_s)ds + \int_0^t \sigma w(\bar{\bar{F}}_{\eta(s)})d\bar{B}_s \\
&\quad + \int_0^t \sigma^2 (ww')(\bar{\bar{F}}_{\eta(s)})(\bar{B}_s - \bar{B}_{\eta(s)})d\bar{B}_s \\
&\quad + \sum_{0\leq k\leq [t/\Delta t]} \left(\tilde{D}_k + A_k \Delta \bar{B}_k^3 + R_k\right)
\end{aligned}$$

where we have used (2.2) in the second equality.

For all t, we define

$$\bar{\bar{F}}'_t := F_0 + \int_0^t v(\bar{\bar{F}}'_{\eta(s)}) + w(\bar{\bar{F}}'_{\eta(s)})u(F_s)ds + \int_0^t \sigma w(\bar{\bar{F}}'_{\eta(s)})d\bar{B}_s + \int_0^t \sigma^2(ww')(\bar{\bar{F}}'_{\eta(s)})(\bar{B}_s - \bar{B}_{\eta(s)})d\bar{B}_s \ .$$

We have for all t_0:

$$\begin{aligned}
\sup_{0\leq t\leq t_0} (\bar{\bar{F}}_t - \bar{\bar{F}}'_t)^2 &\leq Ct_0 \int_0^{t_0} (v(\bar{\bar{F}}_{\eta(s)}) - v(\bar{\bar{F}}'_{\eta(s)}) + u(F_s)(w(\bar{\bar{F}}_{\eta(s)}) - w(\bar{\bar{F}}'_{\eta(s)})))^2 ds \qquad (4.15)\\
&\quad + C \sup_{0\leq t\leq t_0} \left(\int_0^t (w(\bar{\bar{F}}_{\eta(s)}) - w(\bar{\bar{F}}'_{\eta(s)}))\sigma d\bar{B}_s\right)^2 \\
&\quad + C \sup_{0\leq t\leq t_0} \left(\int_0^t ((ww')(\bar{\bar{F}}_{\eta(s)}) - (ww')(\bar{\bar{F}}'_{\eta(s)}))\sigma^2(\bar{B}_s - \bar{B}_{\eta(s)})d\bar{B}_s\right)^2 \\
&\quad + C \sup_{0\leq t\leq t_0} \left(\sum_{0\leq k\leq \lfloor t/\Delta t\rfloor} \tilde{D}_k + A_k\Delta\bar{B}_k^3 + R_k\right)^2 .
\end{aligned}$$

We have:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq t_0}\left(\sum_{0\leq k\leq \lfloor t/\Delta t\rfloor} \tilde{D}_k\right)^2\right) &\leq C\mathbb{E}\left(\left(\sum_{0\leq k\leq \lfloor t_0/\Delta t\rfloor} \tilde{D}_k\right)^2\right) \\
&= C\mathbb{E}\left(\sum_{0\leq k,q\leq \lfloor t_0/\Delta t\rfloor} \tilde{D}_k\tilde{D}_q\right).
\end{aligned}$$

Notice that we have for $k > q$ (recall the properties of $\tilde{D}_k$),

$$\begin{aligned}
\mathbb{E}(\tilde{D}_k\tilde{D}_q) &= \mathbb{E}(\mathbb{E}(\tilde{D}_k\tilde{D}_q|\mathcal{F}^X_{k\Delta t})) \\
&= \mathbb{E}(\tilde{D}_q\mathbb{E}(\tilde{D}_k|\mathcal{F}^X_{k\Delta t})) = 0.
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq t_0}\left(\sum_{0\leq k\leq \lfloor t/\Delta t\rfloor} \tilde{D}_k\right)^2\right) &\leq C\mathbb{E}\left(\sum_{0\leq k\leq \lfloor t_0/\Delta t\rfloor} (\tilde{D}_k)^2\right) \\
&\leq Ct_0\Delta t^2 \ .
\end{aligned}$$

In the same way:

$$\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\sum_{0\le k\le\lfloor t/\Delta t\rfloor} A_k\Delta\bar{B}_k^3\right)^2\right) \quad\le\quad t_0\Delta t^2 \ .$$

We also have:

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\sum_{0\le k\le\lfloor t/\Delta t\rfloor} R_k\right)^2\right) \quad&\le\quad \mathbb{E}\left(\sum_{0\le k,q\le\lfloor t_0/\Delta t\rfloor}|R_kR_q|\right)\\ &\le\quad Ct_0^2\Delta t^2 \ .\end{aligned}$$

Using the Lipschitz properties of v, w, w' and the Burkholder-Davis-Gundy inequality, we have:

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\int_0^t((ww')(\bar{\bar{F}}_{\eta(s)})-(ww')(\bar{\bar{F}}'_{\eta(s)}))\sigma^2(\bar{B}_s-\bar{B}_{\eta(s)})d\bar{B}_s\right)^2\right)\\ &\le C\int_0^{t_0}\mathbb{E}\left(\left(((ww')(\bar{\bar{F}}_{\eta(s)})-(ww')(\bar{\bar{F}}'_{\eta(s)}))\sigma^2(\bar{B}_s-\bar{B}_{\eta(s)})\right)^2\right)ds\\ &\le C\Delta t\int_0^{t_0}\mathbb{E}(\sup_{0\le u\le s}(\bar{\bar{F}}_{\eta(u)}-\bar{\bar{F}}'_{\eta(u)})^2)ds.\end{aligned}$$

Using again the Lipschitz properties of v, w, w' and the fact that $(u(F_t))_{t\ge0}$ is bounded in (4.15), we have:

$$\mathbb{E}(\sup_{0\le t\le t_0}(\bar{\bar{F}}_t-\bar{\bar{F}}'_t)^2) \quad\le\quad C(t_0+1)\int_0^{t_0}\mathbb{E}(\sup_{0\le v\le s}(\bar{\bar{F}}_v-\bar{\bar{F}}'_v)^2)ds + C(t_0\Delta t^2+t_0^2\Delta t^2).$$

We are now in position to apply Gronwall's Lemma and we get that:

$$\mathbb{E}(\sup_{0\le t\le t_0}(\bar{\bar{F}}_t-\bar{\bar{F}}'_t)^2)\le C(t_0+t_0^2)\Delta t^2\exp(C(t_0+1)t_0).$$

The process $(\bar{\bar{F}}'_t)_{t\ge0}$ is the Euler-Milstein scheme associated to $(F_t)_{t\ge0}$ so we know by [10] that:

$$\mathbb{E}(\sup_{0\le t\le t_0}(\bar{\bar{F}}_t-F_t)^2)\le C_{t_0}\Delta t^2$$

where C_{t_0} is a constant depending only on t_0 and the parameters of the problem. This implies that for all N:

$$\mathbb{E}(\sup_{0\le k\le N}(F_{k\Delta t}-\bar{F}_k)^2)\le C_{N\Delta t}\Delta t^2$$

where $C_{N\Delta t}$ is a constant depending only on $N\Delta t$ and the parameters of the problem. This finishes the proof.

□

4.3 The misspecified continuous filter : definition and control of the error

4.3.1 Definition

Suppose now that we take wrong coefficients $\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2$ instead of $\mu_1, \mu_2, \lambda_1, \lambda_2$.

As in (4.1), we may define the *misspecified continuous filter*, solution of:

$$\begin{aligned}\hat{F}_t \quad=\quad& F_0+\int_0^t\frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma^2}\hat{F}_s(1-\hat{F}_s)dX_s+\int_0^t\left(-\bar{\lambda}_1\hat{F}_s+\bar{\lambda}_2\left(1-\hat{F}_s\right)\right)ds\\ &-\int_0^t\frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma^2}\left(\bar{\mu}_1\hat{F}_s+\bar{\mu}_2\left(1-\hat{F}_s\right)\right)\hat{F}_s(1-\hat{F}_s)ds+\int_0^t\frac{\bar{\mu}_1-\bar{\mu}_2}{2}\hat{F}_s(1-\hat{F}_s)ds \ .\end{aligned}\tag{4.16}$$

We call $(\hat{F}_t)_{t\geq 0}$ the misspecified continuous filter. It is the filter one can compute with the available observations $(X_t)_{t\geq 0}$ and with the wrong coefficients.

Using (2.2), we can rewrite (4.16) in another form:

$$\begin{aligned}\hat{F}_t &= F_0 + \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma}\hat{F}_s(1-\hat{F}_s)d\bar{B}_s + \int_0^t \left(-\bar{\lambda}_1\hat{F}_s + \bar{\lambda}_2\left(1-\hat{F}_s\right)\right)ds \\ &\quad + \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma^2}\left\{(\mu_1 F_s + \mu_2(1-F_s)) - \left(\bar{\mu}_1\hat{F}_s + \bar{\mu}_2\left(1-\hat{F}_s\right)\right)\right\}\hat{F}_s(1-\hat{F}_s)ds.\end{aligned}$$

As in Section 2.2, we prove that the previous equation has a unique strong solution. Furthermore, this solution takes value in $(0,1)$.

Remark 4.8 (Parameter σ). In the above computations we assumed that we knew the exact value of σ. In fact σ can be well estimated in short time (provided we possess enough data). This is not the case for the other parameters.

Nevertheless, we could write an erroneous $\bar{\sigma}$ in the definition of the misspecified prediction filter and produce estimations in the same way as above. Here we made the choice not to do so in order to keep the computations more readable.

4.3.2 Control of the error

In the following, we will write C_{t_0} for a constant depending continuously on t_0 and the parameters λ_1, λ_2, μ_1, μ_2, σ, $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\mu}_1$, $\bar{\mu}_2$.

Lemma 4.9. *We have for all X_0, $F_0 \in (0,1)$ and $t_0 \geq 0$:*

$$\mathbb{E}\left[\sup_{0\leq t\leq t_0}|\hat{F}_t - F_t|^2\right] \leq C(t_0+1)\exp(C(t_0+1)t_0)\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|).$$

Proof. From the previous equations we deduce that

$$\begin{aligned}F_t - \hat{F}_t &= \underbrace{\int_0^t \left\{\frac{\mu_1-\mu_2}{\sigma}F_s(1-F_s) - \frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma}\hat{F}_s(1-\hat{F}_s)\right\}d\bar{B}_s}_{:=M_t} \\ &\quad + \int_0^t \underbrace{\left\{(-\lambda_1 F_s + \lambda_2(1-F_s)) - \left(-\bar{\lambda}_1\hat{F}_s + \bar{\lambda}_2\left(1-\hat{F}_s\right)\right)\right\}}_{:=\mathrm{err}_1(s)} ds \\ &\quad - \int_0^t \underbrace{\frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma^2}\{(\mu_1F_s+\mu_2(1-F_s)) - (\bar{\mu}_1\hat{F}_s + \bar{\mu}_2(1-\hat{F}_s))\}\hat{F}_s(1-\hat{F}_s)}_{:=\mathrm{err}_2(s)} ds\end{aligned}$$

In the following, we write C for a constant depending continuously on the parameters λ_1, λ_2, μ_1, μ_2, σ, $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\mu}_1$, $\bar{\mu}_2$ and which may change from line to line.

Control of err_1:

$$\mathrm{err}_1(t) = -(\lambda_1+\lambda_2)(F_t - \hat{F}_t) + \hat{F}_t(\bar{\lambda}_1 - \lambda_1 + \bar{\lambda}_2 - \lambda_2)$$

from which we deduce that

$$|\mathrm{err}_1(t)| \leq C|F_t - \hat{F}_t| + |\lambda_2 - \bar{\lambda}_2| + |\lambda_1 - \bar{\lambda}_1|$$

because the filter F is bounded by one (since it is a conditional probability).

Control of err_2:

The same type of calculations yields

$$|\mathrm{err}_2(t)| \le C(|F_t - \hat{F}_t| + |\mu_2 - \bar{\mu}_2| + |\mu_1 - \bar{\mu}_1|)$$

Control of the martingale term M:

Since F and $\hat{F}$ are almost surely bounded processes, we find that

$$\begin{aligned}
\mathbb{E}\left[|M_t|^2\right] &= \mathbb{E}\left[\int_0^t \left\{\frac{\mu_1-\mu_2}{\sigma}F_s(1-F_s) - \frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma}\hat{F}_s(1-\hat{F}_s)\right\}^2 ds\right] \\
&\le C\mathbb{E}\left[\int_0^t \left\{\frac{\mu_1-\mu_2}{\sigma} - \frac{\bar{\mu}_1-\bar{\mu}_2}{\sigma}\right\}^2 ds\right] + C\mathbb{E}\left[\int_0^t \left\{\frac{\mu_1-\mu_2}{\sigma}\left(F_s - \hat{F}_s\right)\right\}^2 ds\right] \\
&\le C((\mu_1-\bar{\mu}_1)^2 + (\mu_2-\bar{\mu}_2)^2) + C\int_0^t \mathbb{E}(F_s-\hat{F}_s)^2 ds
\end{aligned}$$

Conclusion:

From all the previous and making use of the trivial inequality ($i = 1, 2$)

$$\left(\int_0^t \mathrm{err_i}(s)ds\right)^2 \le t\int_0^t \mathrm{err_i^2}(s)ds,$$

we find that for all $t \le t_0$:

$$\mathbb{E}\left[\sup_{0\le s\le t}(F_s-\hat{F}_s)^2\right] \le C(t_0+1)\left(\int_0^t \mathbb{E}(\sup_{0\le u\le s}(F_u-\hat{F}_u)^2)ds + \sup_{i=1,2}(|\lambda_i-\bar{\lambda}_i| + |\mu_i-\bar{\mu}_i|)\right)$$

We are now in position to apply Gronwall's lemma, for all t in $[0, t_0]$,

$$\mathbb{E}\left[\sup_{0\le t\le t_0}(F_t-\hat{F}_t)^2\right] \le C(t_0+1)\exp(C(t_0+1)t_0)\sup_{i=1,2}(|\lambda_i-\bar{\lambda}_i| + |\mu_i-\bar{\mu}_i|).$$

□

4.4 The misspecified prediction filter

We define the misspecified prediction filter $(\hat{\bar{F}}_k)_{k\ge0}$ by recurrence, taking $\hat{\bar{F}}_0 = F_0$ and $\forall k$, $\hat{\bar{G}}_k = 1 - \hat{\bar{G}}_k$:

First step: selection

$$\begin{pmatrix}\hat{\bar{F}}_k \\ \hat{\bar{G}}_k\end{pmatrix} \longrightarrow \begin{pmatrix}\hat{\bar{F}}'_k = \hat{\bar{F}}_k \dfrac{1}{\sigma\sqrt{2\pi\Delta t}}\exp -\dfrac{\left(\Delta X_k - (\bar{\mu}_1 - \frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t} \\ \hat{\bar{G}}'_k = \hat{\bar{G}}_k \dfrac{1}{\sigma\sqrt{2\pi\Delta t}}\exp -\dfrac{\left(\Delta X_k - (\bar{\mu}_2 - \frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2\Delta t}\end{pmatrix}$$

Second step: normalization

$$\begin{pmatrix}\hat{\bar{F}}'_k \\ \hat{\bar{G}}'_k\end{pmatrix} \longrightarrow \begin{pmatrix}\hat{\bar{F}}''_k = \dfrac{\hat{\bar{F}}'_k}{\hat{\bar{F}}'_k + \hat{\bar{G}}'_k} \\ \hat{\bar{G}}''_k = \dfrac{\hat{\bar{G}}'_k}{\hat{\bar{F}}'_k + \hat{\bar{G}}'_k}\end{pmatrix}$$

Third step: evolution

$$\begin{pmatrix} \hat{\hat{F}}''_k \\ \hat{\hat{G}}''_k = 1 - \hat{\hat{F}}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{\hat{F}}_{k+1} = \hat{\hat{F}}''_k e^{-\bar{\lambda}_1 \Delta t} + \hat{\hat{G}}''_k (1 - e^{-\bar{\lambda}_2 \Delta t}) \\ \hat{\hat{G}}_{k+1} = \hat{\hat{G}}''_k e^{-\bar{\lambda}_2 \Delta t} + \hat{\hat{F}}''_k (1 - e^{-\bar{\lambda}_1 \Delta t}) \end{pmatrix}$$

Notice that $\forall k,\ \hat{\hat{F}}_k \in [0,1]$.

The following Lemma can be proved exactly like the Proposition 4.7 and we do not write its proof.

Lemma 4.10. *For all $N \in \mathbb{N}$, there exists a constant $C_{N\Delta t}$ (depending continuously on $N\Delta t$ and $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}_1, \bar{\mu}_2$) such that for any X_0, F_0:*

$$\boxed{\sup_{0 \le k \le N} \mathbb{E}((\hat{F}_{k\Delta t} - \hat{\hat{F}}_k)^2) \le C_{N\Delta t} \Delta t^2.}$$

5 Uniform convergence of the filters

5.1 A general result of approximation

Notation 5.1. *For all $0 \le s \le t$ and $y \in (0,1)$, we denote by $P_{s,t}y$ the value at time t of the solution of (2.3) whose value in s is equal to y. The operator P is a stochastic flow. For all t, we have $P_{0,t}F_0 = F_t$.*

For any stochastic flow $\tilde{P}$, we will use the following conventions:

- $\forall t,\ \tilde{P}_{t,t} = Id$
- *if* $s > t,\ \tilde{P}_{s,t} = Id$

Proposition 5.2. *Suppose we have a stochastic flow $(\tilde{P}_{s,t})_{0 \le s \le t}$ and $\epsilon > 0$ such that $\forall k \in \mathbb{N}$, $\forall y \in [0,1]$*

- $\forall s, (\tilde{P}_{s,t}y)_{t \ge s}$ *is a Markov process*
- $(\tilde{P}_{s,t}y)_{t \ge s}$ *is* $\left(\mathcal{F}^X_t\right)_{t \ge s}$ *adapted*
- $\forall t, \omega,\ \mathbb{E}(\sup_{s \in [0,1]} |P_{t,t+s}y - \tilde{P}_{t,t+s}y| | \mathbb{F}^X_t) \le \epsilon$

then

$$\sup_{t \ge 0} \mathbb{E}(|F_t - \tilde{P}_{0,t}F_0|) \le \frac{2\epsilon}{1 - e^{-\lambda_1 - \lambda_2}} \ .$$

Proof. We begin by showing that P is a contracting flow in some sense. Let us show that, $\forall x, x' \in [0,1]$ and $0 \le s \le t$:

$$\mathbb{E}(|P_{s,t}x - P_{s,t}x'|) \le e^{-(\lambda_1+\lambda_2)(t-s)}|x - x'| \ . \tag{5.1}$$

Suppose that $x \le x'$. Thanks to the almost surely comparison theorem, we have $\forall \omega$ and $\forall s \le t$, $P_{s,t}x \le P_{s,t}x'$. And so, thanks to the SDE (2.3), we have $\forall u \ge s$:

$$\begin{aligned} \mathbb{E}(|P_{s,u}x' - P_{s,u}x|) &= \mathbb{E}(P_{s,u}x' - P_{s,u}x) \\ &= -(\lambda_1 + \lambda_2) \int_s^u \mathbb{E}(P_{s,v}x' - P_{s,v}x) dv \ . \end{aligned}$$

So $\mathbb{E}(|P_{s,t}x' - P_{s,t}x|) = (x' - x)e^{-(\lambda_1+\lambda_2)(t-s)}$.

We now use some computation to get the result.

$$\begin{aligned}
F_t - \tilde{P}_{0,t}F_0 &= P_{0,t}F_0 - \tilde{P}_{0,t}F_0 \\
&= \sum_{k=0}^{\lfloor t \rfloor} P_{k,t}\tilde{P}_{0,k}F_0 - P_{k+1\wedge t,t}\tilde{P}_{0,k+1\wedge t}F_0 \\
&= \sum_{k=0}^{\lfloor t \rfloor} P_{k+1\wedge t,t}P_{k,k+1\wedge t}\tilde{P}_{0,k}F_0 - P_{k+1\wedge t,t}\tilde{P}_{k,k+1\wedge t}\tilde{P}_{0,k}F_0
\end{aligned}$$

So, by (5.1):

$$\begin{aligned}
\mathbb{E}(|F_t - \tilde{P}_{0,t}F_0|) &\leq \sum_{k=0}^{\lfloor t \rfloor} e^{-(\lambda_1+\lambda_2)(t-k+1\wedge t)}\mathbb{E}(|P_{k,k+1\wedge t}\tilde{P}_{0,k}F_0 - \tilde{P}_{k,k+1\wedge t}\tilde{P}_{0,k}F_0| \mid \mathbb{F}_k^X) \\
&\leq \sum_{k=0}^{\lfloor t \rfloor} e^{-(\lambda_1+\lambda_2)(t-k+1\wedge t)}\epsilon \\
&\leq \left(1 + \frac{1}{1-e^{-\lambda_1-\lambda_2}}\right)\epsilon \\
&\leq \frac{2\epsilon}{1-e^{-\lambda_1-\lambda_2}}.
\end{aligned}$$

□

5.2 Convergence of our filters

As a Corollary of Proposition 4.7, Lemmas 4.10, 4.9 and Proposition 5.2, we can state the following Theorem:

Theorem 5.3. *For all $t \geq 0$,*

$$\boxed{\mathbb{E}(|F_t - \bar{F}_{\lfloor t/\Delta t \rfloor}|) \leq C\sqrt{\Delta t}.} \tag{5.2}$$

$$\boxed{\mathbb{E}(|F_t - \hat{\bar{F}}_{\lfloor t/\Delta t \rfloor}|) \leq C(\sqrt{\Delta t} + \sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)).} \tag{5.3}$$

$$\boxed{\mathbb{E}(|F_t - \hat{F}_t|) \leq C(\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)).} \tag{5.4}$$

Proof. Equation (5.4) is a direct corollary of Lemma 4.9 and Proposition 5.2.

By Propositions 4.7 and 5.2: $\forall t$, $\mathbb{E}(|F_t - \bar{\bar{F}}_t|) \leq C\sqrt{\Delta t}$. For all k, $\bar{\bar{F}}_{k\Delta t} = \bar{F}_k$ and so, by (4.14), we have (5.2).

By Lemma 4.10, we have for all $k \leq 2/\Delta t$,

$$\mathbb{E}(|\hat{F}_{k\Delta t} - \hat{\bar{F}}_k|) \leq C\sqrt{\Delta t}.$$

So, by Lemma 4.9, we have for all $k \leq 2/\Delta t$,

$$\mathbb{E}(|F_{k\Delta t} - \hat{\bar{F}}_k|) \leq C(\sqrt{\Delta t} + \sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)).$$

Then, thanks to the SDE (2.3), for all $t \leq 1$,

$$\mathbb{E}(|F_t - \hat{F}_t|) \leq C(\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)).$$

And so (5.3) comes from Proposition 5.2.

□

5.3 Implementable strategy in the case of a Logarithmic Utility Function

We have seen in subsection 7.2 that, in the case of a Logarithmic Utility Function, the optimal allocation strategy is of the form $\pi_t^* = q^*(F_t)$. In practice, we are only able to act on the allocation of our wealth at discrete times and we may not know exactly $\lambda_1, \lambda_2, \mu_1, \mu_2$.

Suppose we approximate $(F_t)_{t\geq 0}$ by a misspecified prediction filter $(\hat{\bar{F}}_k)_{k\geq 0}$ based on a time discretization interval $\Delta t_1 > 0$ and "wrong" coefficients $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}_1, \bar{\mu}_2$ (see Remark 4.8 concerning the reasons that allow us to assume that we perfectly know the value of the parameter σ).

Suppose also that we change the allocation of our wealth at instants $0, \Delta t_2, 2\Delta t_2, \dots$ with Δt_2 a m-multiple of Δt_1. We denote by $\hat{\bar{W}}_t^x$ the wealth at time t (starting from x) obtained when we replace the optimal strategy $\pi_t^* = q^*(F_t)$ by $\hat{\pi}_{m\,k}^* = q^*(\hat{\bar{F}}_{m\,k})$ in the dynamics.

We have a formula similar to (3.2) for the growth rate of $\hat{\bar{W}}^x$:

$$\begin{aligned}\frac{1}{T}\log(\hat{\bar{W}}_t^x) &= \frac{1}{T}\log(x) + \frac{1}{T}\int_0^T \left(\left(1 - q^*(\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor})\right) r + q^*(\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor})\mu^{\text{opt}}(t) - \frac{\sigma^2}{2}\right) dt \\ &\quad - \frac{\sigma^2}{2T}\int_0^T q^*(\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor})^2 dt + \frac{\sigma}{T}\int_0^T q^*(\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor}) d\bar{B}_t.\end{aligned}$$

Equation (3.3) told us that the growth rate of the optimal wealth $W^{*,x}$ has an *a.s.* limit, the following theorem shows that the growth rate of $\hat{\bar{W}}^x$ is near that of $W^{*,x}$.

In the following, we write C for some constant depending continuously on the parameters of the problem. This constant may change from line to line.

Theorem 5.4. *For all $T \geq 1$*

$$\boxed{\mathbb{E}\left(\left|\frac{1}{T}\log(W_T^{*,x}) - \frac{1}{T}\log(\hat{\bar{W}}_t^x)\right|\right) \leq C\left(\sqrt{\Delta t_1} + \sqrt{\Delta t_2} + \sup_{i=1,2}(|\mu_i - \bar{\mu}_i| + |\lambda_i - \bar{\lambda}_i|)\right).}$$

Proof. The function q^* is Lipschitz and the process μ^{opt} is bounded. By Theorem 5.3, we get:

$$\begin{aligned}\mathbb{E}\left(\left|\frac{1}{T}\log(W_T^{*,x}) - \frac{1}{T}\log(\hat{\bar{W}}_t^x)\right|\right) &\leq \frac{C}{T}\int_0^T \mathbb{E}(|\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor} - F_t|)dt \\ &\leq \frac{C}{T}\int_0^T \mathbb{E}(|\hat{\bar{F}}_{\lfloor t/\Delta t_2\rfloor} - F_{\lfloor t/\Delta t_2\rfloor}|) + \mathbb{E}(|F_{\lfloor t/\Delta t_2} - F_t|)dt \\ &\leq C\left(\sqrt{\Delta t_1} + \sqrt{\Delta t_2} + \sup_{i=1,2}(|\mu_i - \bar{\mu}_i| + |\lambda_i - \bar{\lambda}_i|)\right).\end{aligned}$$

□

6 Numerical Experiments

6.1 Introduction

In this numerical section, we give the performances of the previous strategies. We also compare them to a strategy which does not need any mathematical model: a technical analysis technique based on the moving average indicator (see [3] for more details).

6.2 The Moving Average

At each discrete time, the trader computes the moving average of the prices:

$$M_t^{(\delta)} = \frac{1}{\delta} \int_{t-\delta}^{t} S_u du. \tag{6.1}$$

- If the prices are larger than the moving average, the trader estimates that the prices are in an increasing period: he/she buys the risky asset.
- If the prices are smaller than the moving average, the trader estimates that the prices are in a decreasing period: he/she sells the risky asset.

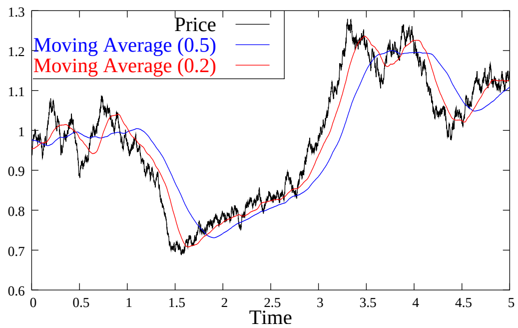
His/Her strategy can be sum up as:

$$\pi_t^{\mathrm{MA}} = \mathbb{1}_{\left\{ S_t > M_t^{(\delta)} \right\}}.$$

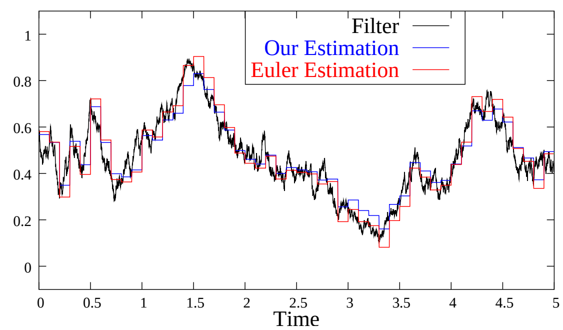
In this section, the values of the parameters are given in Table 1.

6.3 A Nominal Trajectory

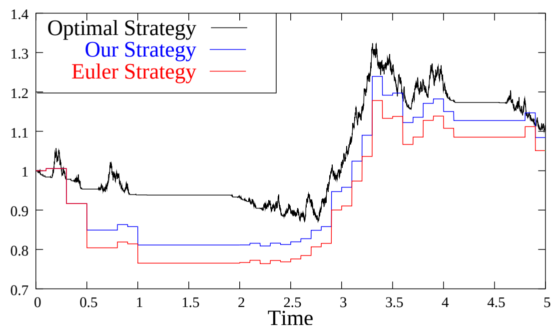
In Fig. 1, we present a typical trajectory with parameters given in Table 1.



(a) Prices and Moving Averages



(b) Exact filter, our Estimation, Estimation with Euler scheme



(c) Wealths with optimal strategy, with our estimation, with Euler scheme

Figure 1: A Nominal Trajectory

6.4 Comparison of Performances

In Fig. 2, we present the performances of traders using

1) The optimal allocation strategy

2) The allocation strategy using our estimation of the filter

3) The allocation strategy using the Euler's approximation of the filter

4) The moving average indicator with a window of 0.5 year.

We can remark that it is difficult to differentiate the performances of the second and the third traders.

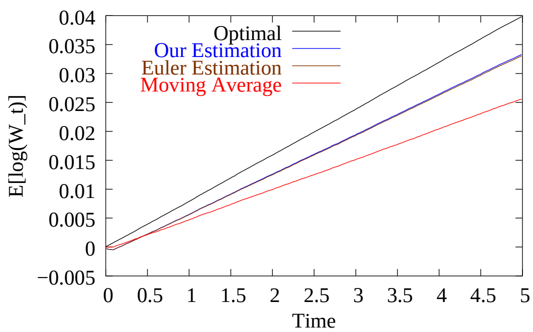


Figure 2: Comparison of Performances

6.5 Comparison of Performances with Errors on the Parameters

In Fig. 3, we present the performances of traders using

1) The optimal allocation strategy

2) The allocation strategy using our estimation of the filter with errors on the parameters

3) The allocation strategy using the Euler's approximation of the filter with errors on the parameters

4) The moving average indicator with a window of 0.5 year.

The misspecified parameters are given in Table 2. In this study, we do not use any estimation procedure. We suppose that the trader has his/her own estimation procedure that we do not describe here.

We can observe that, for this particular choice of parameters, the performances of the trader using the moving average indicator is between the performances of the trader using our estimation of the filter (with the calibration errors) and those using an Euler scheme.

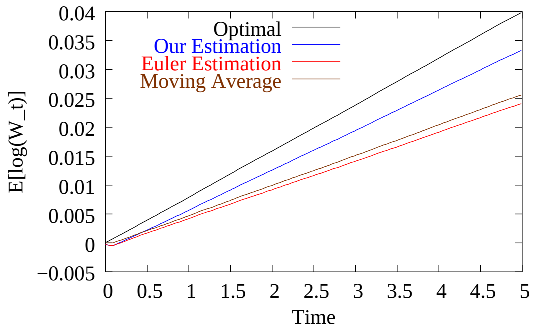


Figure 3: Comparison of Performances with Errors on the Parameters

Table 1: Values of the Parameters

μ_1	μ_2	λ_1	λ_2	σ	r
-0.1	0.1	1.0	1.0	0.15	0.0

Table 2: Estimated Values of the Parameters

$\bar{\mu}_1$	$\bar{\mu}_2$	λ_1	λ_2
-0.2	0.2	2.0	2.0

7 Appendix: Proof of Section 3

7.1 The case of General Utility Functions

7.1.1 An auxiliary unconstrained optimization problem

The proof is the same as in [3]. As in Karatzas-Shreve [6], we introduce an auxiliary unconstrained market $\mathcal{M}_\nu$ defined as follows:

we set $\mathcal{D}$ to be the subset of $\mathbb{F}^X$-progressively measurable processes $\nu : [0,T]\times\Omega \to \mathbb{R}$ such that

$$\mathbb{E}\left[\int_0^T \nu^-(t)dt\right] < \infty, \text{ where } \nu^-(t) := -\inf(0,\nu(t)).$$

The bond price process $S^{0,\nu}$ and the stock price S^ν satisfy (S^0 and S correspond to $\nu \equiv 0$)

$$\begin{cases} \dfrac{dS_t^{0,\nu}}{S_t^{0,\nu}} &= \dfrac{dS_t^0}{S_t^0} + \nu^-(t)dt, \\ S_0^{0,\nu} &= S_0^0 = 1. \end{cases}$$

$$\begin{cases} \dfrac{dS_t^{\nu}}{S_t^{\nu}} &= \dfrac{dS_t}{S_t} + (\nu^-(t)+\nu(t))dt, \\ S_0^{\nu} &= S_0. \end{cases}$$

For each auxiliary unconstrained market, let $\mathcal{A}(x,\nu)$ denote the set of admissible strategies, that is,

$$\begin{aligned} \mathcal{A}(x,\nu) \quad := \quad & \{\pi_. \ \mathbb{F}^X - \text{progressively measurable process s.t.} \\ & W_0^{\nu,\pi} = x, \ \ W_t^{\nu,\pi} > 0 \text{ for all } t > 0\}. \end{aligned}$$

where

$$\begin{cases} \dfrac{dW_t^{\nu,\pi}}{W_t^{\nu,\pi}} = \pi_t \dfrac{dS_t^\nu}{S_t^\nu} + (1-\pi_t)\dfrac{dS_t^{0,\nu}}{S_t^{0,\nu}}, \\ W_0^{\nu,\pi} = x. \end{cases}$$

We may write also this equation in the following manner

$$\begin{cases} \dfrac{dW_t^{\nu,\pi}}{W_t^{\nu,\pi}} = \dfrac{dW_t^{x,\pi}}{W_t^{x,\pi}} + (\nu^-(t) + \pi_t\nu(t))\,dt, \\ W_0^{\nu,\pi} = x. \end{cases} \tag{7.2}$$

We introduce the following optimization problem:

$$\mathcal{P}_\nu : \quad V(\nu,x) := \sup_{(\pi_t)_{0\le t\le T}\in\mathcal{A}(x,\nu)} \mathbb{E}\left[U\left(W_T^{\nu,\pi}\right)\right].$$

Notation: we note $\pi^{*,\nu}$ an optimal portfolio process for problem $\mathcal{P}_\nu$.

7.1.2 Link between the auxiliary unconstrained optimization problem $\mathcal{P}_\nu$ and the investor's constrained optimization problem $\mathcal{P}$

Proposition 7.1. *Suppose that for some $\nu_0 \in \mathcal{D}$ it is possible to find a portfolio-proportion process π^{*,ν_0} such that*

- *π^{*,ν_0} solves the auxiliary unconstrained problem $\mathcal{P}_{\nu_0}$:*

$$V(\nu_0,x) = \mathbb{E}\left[U\left(W_T^{\nu_0,\pi^{*,\nu_0}}\right)\right] = \sup_{(\pi_t)_{0\le t\le T}\in\mathcal{A}(x,\nu_0)} \mathbb{E}\left[U\left(W_T^{\nu_0,\pi}\right)\right],$$

- π^{*,ν_0} *is in the class* $\mathcal{A}(x)$ *and satisfies the constrained condition:*

$$\nu_0^-(t) + \pi_t^{*,\nu_0}\nu_0(t) = 0 \ \textit{ for Lebesgue-a.e. } t \in [0,T]. \tag{7.3}$$

Then, the portfolio-proportion process π^{*,ν_0} *is optimal for the investor's constrained optimization problem* $\mathcal{P}$ *and*

$$V^*(x) = V\left(\nu_0, x\right).$$

Furthermore,

$$V\left(\nu_0, x\right) = \inf_{\nu\in\mathcal{D}} V\left(\nu, x\right). \tag{7.4}$$

Proof. On the one hand, for any $\nu \in \mathcal{D}$ and $\pi \in \mathcal{A}\left(x,\nu\right)$ such that $\nu^-(t) + \pi_t\nu(t) \ \geq \ 0$ a.e., we have that

$$W_T^{\nu,\pi} \geq W_T^{x,\pi},$$

and since U is an increasing function and $\mathcal{A}(x) \subset \mathcal{A}(x,\nu)$, we see that

$$V^*(x) \leq V\left(\nu, x\right) \quad \forall \nu \in \mathcal{D}. \tag{7.5}$$

On the other hand, since π^{*,ν_0} belongs to $\mathcal{A}(x)$ and because of the constrained condition (7.3), we have that

$$\begin{aligned} V^*(x) &\geq \mathbb{E}\left[U\left(W_T^{x,\pi^{*,\nu_0}}\right)\right] \\ &= \mathbb{E}\left[U\left(W_T^{\nu_0,\pi^{*,\nu_0}}\right)\right] \text{ because of (7.2) combined with (7.3),} \\ &= V\left(\nu_0, x\right) \text{ because } \pi^{*,\nu_0} \text{ is optimal for problem } \mathcal{P}. \end{aligned} \tag{7.6}$$

Combining this inequality with inequality (7.5), we finally deduce that

$$V^*(x) = V\left(\nu_0, x\right) = \inf_{\nu\in\mathcal{D}} V\left(\nu, x\right). \tag{7.7}$$

□

Remark 7.2. As we will see in the sequel, we can prove a converse of this result (see Proposition 7.8 for details): we will show in subsection 7.1.4 that if there exists a process $\hat{\nu} \in \mathcal{D}$ satisfying (7.4), then

$$V^*(x) = V\left(\hat{\nu}, x\right).$$

In other words, there is no *duality gap.*

7.1.3 Solution of the auxiliary unconstrained optimization problem $\mathcal{P}_\nu$

Let us introduce $(H_t^\nu)_{t\geq 0}$ the exponential martingale defined by

$$H_t^\nu := \exp\left(-\int_0^t \left(\frac{\mu^{\text{opt}}(s) + \nu(s) - r}{\sigma}\right) d\bar{B}_s - \frac{1}{2}\int_0^t \left(\frac{\mu^{\text{opt}}(s) + \nu(s) - r}{\sigma}\right)^2 ds\right).$$

We also need the following notations:

$$\mathcal{X}^\nu(y) = \mathbb{E}\left[H_T^\nu \mathrm{e}^{-rT-\int_0^T \nu^-(s)ds} I\left(y H_T^\nu \mathrm{e}^{-rT-\int_0^T \nu^-(s)ds}\right)\right] \tag{7.8}$$

The inverse of $\mathcal{X}^\nu$ will be denoted by $\mathcal{Y}^\nu$.

Proposition 7.3. *For each $\nu \in \mathcal{D}$, the optimal wealth is*

$$W_T^{\nu,*} = I\left(yH_T^\nu e^{-rT-\int_0^T \nu^-(s)ds}\right) \tag{7.9}$$

and

$$W_t^{\nu,*} = \frac{e^{rt+\int_0^t \nu^-(s)ds}}{H_t^\nu}\mathbb{E}\left[H_T^\nu e^{-rT-\int_0^T \nu^-(s)ds} I\left(yH_T^\nu e^{-rT-\int_0^T \nu^-(s)ds}\right)|\mathcal{F}_t^X\right],$$

where y stands for the Lagrange multiplier, that is, y $(y = \mathcal{Y}(x))$ is such that

$$\mathbb{E}\left[H_T^\nu e^{-rT-\int_0^T \nu^-(s)ds} I\left(yH_T^\nu e^{-rT-\int_0^T \nu^-(s)ds}\right)\right] = x. \tag{7.10}$$

Moreover, the optimal portfolio satisfies

$$\pi_t^{\nu,*} = \sigma^{-1}\left(\frac{\mu^{\text{opt}}(t) - r + \nu(t)}{\sigma} + \frac{\phi_t}{H_t^\nu W_t^{\nu,*} e^{-rt-\int_0^t \nu^-(s)ds}}\right).$$

where ϕ is a $\mathbb{F}^X$ adapted process which satisfies

$$\mathbb{E}\left[H_T^\nu e^{-rT-\int_0^T \nu^-(s)ds} I\left(yH_T^\nu e^{-rT-\int_0^T \nu^-(s)ds}\right)|\mathcal{F}_t^X\right] = x + \int_0^t \phi_s d\bar{B}_s.$$

Proof. We follow Karatzas's method (see for example Karatzas [6]).

For $\pi \in \mathcal{A}(x,\nu)$, we define $W^{\nu,\pi}$ by

$$\begin{aligned} \frac{dW_t^{\nu,\pi}}{W_t^{\nu,\pi}} &= (1-\pi_t)\frac{dS_t^{0,\nu}}{S_t^{0,\nu}} + \pi_t \frac{dS_t^\nu}{S_t^\nu} \\ &= \pi_t\left\{(\mu^{\text{opt}}(t) + \nu(t) - r)\,dt + \sigma d\bar{B}_t\right\} + (\nu^-(t) + r)\,dt \end{aligned}$$

Define

$$\bar{W}_t^{\nu,\pi} := W_t^{\nu,\pi} \exp\left(-rt - \int_0^t \nu^-(s)ds\right).$$

It follows that

$$\frac{d\bar{W}_t^{\nu,\pi}}{\bar{W}_t^{\nu,\pi}} = \pi_t\left(\mu^{\text{opt}}(t) + \nu(t) - r\right)dt + \pi_t\sigma d\bar{B}_t. \tag{7.11}$$

We note

$$\gamma_t := \mu^{\text{opt}}(t) + \nu(t) - r.$$

and we obtain

$$\bar{W}_t^{\nu,\pi} = x\exp\left(\int_0^t\left(\pi_s\gamma_s - \frac{1}{2}\sigma^2\pi_s^2\right)ds + \int_0^t \sigma\pi_s d\bar{B}_s\right).$$

We now search an exponential local martingale M_t (independent of π) such that $\tilde{W}_t^{\nu,\pi} = \bar{W}_t^{\nu,\pi} M_t$ is an exponential local martingale. Set

$$M_t = \exp\left(\int_0^t \varphi_s d\bar{B}_s - \frac{1}{2}\int_0^t \varphi_s^2 ds\right).$$

Then φ needs to satisfy

$$-\frac{1}{2}\left(\varphi_s + \sigma\pi_s\right)^2 = \pi_s\gamma_s - \frac{1}{2}\sigma^2\pi_s^2 - \frac{1}{2}\varphi_s^2,$$

from which $\varphi_s = -\frac{\gamma_s}{\sigma}$ and $M_t = H_t^\nu$. Thus, for all π,

$$d\tilde{W}_t^{\nu,\pi} = \tilde{W}_t^{\nu,\pi}\left(\sigma\pi_t - \frac{\gamma_t}{\sigma}\right)d\bar{B}_t. \tag{7.12}$$

Therefore, the process $\left\{\tilde{W}_t^{\nu,\pi}, 0 \le t \le T\right\}$ is non-negative $\left(\mathbb{F}^X, \mathbb{P}\right)$- local martingale and so a super martingale.

Notation 7.4.

$$\Gamma_t^\nu := H_t^\nu \exp\left(-rt - \int_0^t \nu^-(s)ds\right) \quad \textit{for } \nu \in \mathcal{D}.$$

Consequently,

$$\mathbb{E}\left[\Gamma_T^\nu W_T^{\nu,\pi}\right] \leq x. \tag{7.13}$$

We use a duality method: so, we introduce the *convex dual* of $U(.)$:

$$\tilde{U}(z) := \max_{0<u<\infty} \left[U(u) - uz\right], \quad z > 0.$$

In particular,

$$\forall z > 0, \forall x > 0, \quad U(x) \leq \tilde{U}(z) + xz. \tag{7.14}$$

Thanks to (7.13) and (7.14), we obtain: for all ν, $\pi \in \mathcal{A}(x,\nu)$, $y \geq 0$,

$$\begin{aligned} \mathbb{E}\left[U\left(W_T^{\nu,\pi}\right)\right] &\leq \mathbb{E}\left[\tilde{U}\left(y\Gamma_T^\nu\right)\right] + y\mathbb{E}\left[\Gamma_T^\nu W_T^{\nu,\pi}\right] \\ &\leq \mathbb{E}\left[\tilde{U}\left(y\Gamma_T^\nu\right)\right] + yx. \end{aligned}$$

In particular,

$$V(\nu,x) \leq \mathbb{E}\left[\tilde{U}\left(y\Gamma_T^\nu\right)\right] + yx \tag{7.15}$$

This inequality is an equality if only and if

$$\begin{cases} U'(W_T^{\nu,\pi}) = y\Gamma_T^\nu, \\ \mathbb{E}\left[\Gamma_T^\nu W_T^{\nu,\pi}\right] = x. \end{cases}$$

In order to get an equality, we choose y such that (7.10) is satisfied (y is the Lagrange Multiplier).

Now, for this choice of y, we will construct a portfolio such that the process

$$Z_t := \frac{1}{\Gamma_t^\nu}\mathbb{E}\left[\Gamma_T^\nu I\left(y\Gamma_T^\nu\right)|\mathcal{F}_t^X\right]$$

is its wealth process. We use a martingale representation property of the Brownian filtration in order to find the optimal strategy π^*. Indeed, there exists a predictable process ϕ such that

$$\mathbb{E}\left[\Gamma_T^\nu I\left(y\Gamma_T^\nu\right)|\mathcal{F}_t^X\right] = x + \int_0^t \phi_s d\bar{B}_s.$$

In particular, with the notation $\tilde{Z}_t = Z_t\Gamma_t^\nu$, we obtain

$$d\tilde{Z}_t = \phi_t d\bar{B}_t.$$

Consider the strategy

$$\pi_t^* = \sigma^{-1}\left(\frac{\mu^{\text{opt}}(t) - r + \nu(t)}{\sigma} + \frac{\phi_t}{Z_t\Gamma_t^\nu}\right).$$

In view of (7.12), we have

$$\frac{d\tilde{W}_t^{\nu,\pi^*}}{\tilde{W}_t^{\nu,\pi^*}} = \frac{\phi_t}{\tilde{Z}_t}d\bar{B}_t = \frac{d\tilde{Z}_t}{\tilde{Z}_t}.$$

Using uniqueness argument, we obtain $\tilde{Z}_t = \tilde{W}_t^{\nu,\pi^*}$ and so $Z_t = W_t^{\nu,\pi^*}$. □

Corollary 7.5. *For each $\nu \in \mathcal{D}$ and each $x > 0$, we have*

$$V(\nu,x) = \mathbb{E}\left[U \circ I\left(\mathcal{Y}^\nu(x)\Gamma_T^\nu\right)\right] \tag{7.16}$$

For technical reasons, we will use the *convex dual* of the value function $V(\nu, x)$:

$$\tilde{V}(\nu, y) := \max_{0<x<\infty} \left[V(\nu, x) - xy\right].$$

Corollary 7.6. *For each $\nu \in \mathcal{D}$ and each $y \in \mathbb{R}_+$, we have the two identities:*

$$\tilde{V}(\nu, y) = \mathbb{E}\left[\tilde{U}\left(y\Gamma_T^\nu\right)\right], \tag{7.17}$$

$$\tilde{V}(\nu, y) = V(\nu, \mathcal{X}(y)) - \mathcal{X}(y)y. \tag{7.18}$$

Proof. On the one hand, we remark that

$$\forall z > 0, \quad U(I(z)) = \tilde{U}(z) + zI(z).U(I(z)) = \tilde{U}(z) + zI(z). \tag{7.19}$$

For all y,

$$U(I(y\Gamma_T^\nu)) = \tilde{U}(y\Gamma_T^\nu) + y\Gamma_T^\nu I(y\Gamma_T^\nu).$$

Taking the expectation and thanks to (7.16) and (7.8),

$$V(\nu, \mathcal{X}(y)) = \mathbb{E}\left[\tilde{U}(y\Gamma_T^\nu)\right] + y\mathcal{X}(y).$$

And so

$$\mathbb{E}\left[\tilde{U}(y\Gamma_T^\nu)\right] \leq \tilde{V}(\nu, u).$$

On the other hand, thanks to (7.15), we have

$$\forall z > 0, \quad V(\nu, z) \leq \mathbb{E}\left[\tilde{U}(y\Gamma_T^\nu)\right] + yz.$$

It follows immediately that

$$\tilde{V}(\nu, y) \leq \mathbb{E}\left[\tilde{U}(y\Gamma_T^\nu)\right]$$

and (7.17) is proved.

Thanks to (7.19),

$$\tilde{V}(\nu, y) = \mathbb{E}\left[U \circ I\left(y\Gamma_T^\nu\right)\right] - \mathbb{E}\left[y\Gamma_T^\nu I(y\Gamma_T^\nu)\right].$$

We now use (7.8) and (7.16) and we obtain (7.18). □

7.1.4 Solution of the investor's constrained optimization problem $\mathcal{P}$

We will need the following lemma:

Lemma 7.7. *For any given $\mathbb{F}^X$-progressively measurable process $\pi : [0,T] \times \Omega \to \mathbb{R}$, there exists a $\mathbb{R}$ valued $\mathbb{F}^X$-progressively measurable process ℓ such that*

$$|\ell(t)| \leq 1, \quad 0 \leq t \leq T, \quad a.s.$$

and for all $t \in [0,T]$, we have that

$$\begin{array}{l} \pi_t \in [0,1] \Leftrightarrow \ell(t) = 0 \;\; a.s., \\ \pi_t \notin [0,1] \Leftrightarrow \ell^-(t) + \pi_t \ell(t) < 0 \;\; a.s.. \end{array} \tag{7.20}$$

Proof. For every $t \in [0,T]$, we may choose

$$\ell(t) := \begin{cases} -1 & \text{if } \pi_t > 1, \\ 0 & \text{if } \pi_t \in [0,1], \\ 1 & \text{if } \pi_t < 0, \end{cases}$$

and (7.20) is clearly satisfied. □

Proposition 7.8. *If there exists $\hat{\nu}$ such that*

$$V(\hat{\nu}, x) = \inf_{\nu \in \mathcal{D}} V(\nu, x), \tag{7.21}$$

then the optimal portfolio $\pi^{*,\hat{\nu}}$ for the unconstrained problem $\mathcal{P}_{\hat{\nu}}$ is also an optimal portfolio for the constrained original problem $\mathcal{P}$ and

$$W_t^* = W_t^{\hat{\nu},\pi^{*,\hat{\nu}}}. \tag{7.22}$$

Proof. We follow Karatzas-Shreve p. 276.

1. The dual problem

If $y = \mathcal{Y}^{\hat{\nu}}(x)$, thanks to (7.18),

$$\begin{aligned}\tilde{V}(\hat{\nu}, y) &= V(\hat{\nu}, \mathcal{X}^{\hat{\nu}}(y)) - y\mathcal{X}^{\hat{\nu}}(y)\\ &= V(\hat{\nu}, x) - yx\\ &\leq V(\nu, x) - yx\\ &\leq \tilde{V}(\nu, y).\end{aligned}$$

So, we have proved:

$$\tilde{V}(\hat{\nu}, y) = \inf_{\nu \in \mathcal{D}} \tilde{V}(\nu, y). \tag{7.23}$$

2. The perturbation method: outline of the proof

Since the utility function U may tend to minus infinity at 0, we introduce $\{\tau_n\}_{n\in\mathbb{N}}$, a nondecreasing sequence of localization stopping times converging up to T.

The idea is to perform a kind of derivation of $\tilde{V}(\nu, x)$ with respect to ν, using the condition (7.21).

More precisely, for any $\beta \in \mathcal{D}$ and any $\varepsilon \in]0,1[$, it is easily seen that

$$\nu_{\varepsilon,n}(t) := \begin{cases} (1-\varepsilon)\hat{\nu}(t) + \varepsilon\beta(t), & 0 \leq t \leq \tau_n \\ \hat{\nu}(t) \quad \tau_n < t \leq T, \end{cases}$$

belongs to $\mathcal{D}$. We study the limiting behavior of

$$\frac{\tilde{V}(\nu_{\varepsilon,n}, y) - \tilde{V}(\hat{\nu}, y)}{\varepsilon y} \geq 0$$

as ε tends to 0.

We shall perform the perturbation of $\hat{\nu}$ with two particular choices of β:

a) $\beta := \hat{\nu} + \ell$ for some $\ell \in \mathcal{D}$. When we derive in this direction (for a precise choice of ℓ), we finally get that $\pi^{*,\hat{\nu}}$ takes values in $[0,1]$ ($\pi^{*,\hat{\nu}}$ is in $\mathcal{A}(x)$) and so

$$\hat{\nu}^-(t) + \pi_t^{*,\hat{\nu}}\hat{\nu}(t) \geq 0 \qquad \forall t \in [0,T]. \tag{7.24}$$

b) $\beta \equiv 0$. As we will see in the sequel, when we derive in this direction, we finally get that

$$\hat{\nu}^-(t) + \pi_t^{*,\hat{\nu}}\hat{\nu}(t) = 0 \qquad \forall t \in [0,T]. \tag{7.25}$$

The assumptions of Proposition 7.1 are satisfied and the result follows immediately.

Hence, the whole proof of the proposition relies in deducing (7.24) and (7.25) from (7.21).

Let us define

$$\xi(s) := \begin{cases} -\hat{\nu}^-(s), \text{ if } \beta = 0, \\ \ell^-(s), \text{ if } \beta = \hat{\nu}(s) + \ell(s) \text{ for some } \ell \in \mathcal{D}. \end{cases}$$

We begin to show that (7.24) and (7.25) can be easily deduced from the following inequality:

$$\mathbb{E}\left[\int_0^{\tau_n} \Gamma_t^{\hat{\nu}} W_t^{\hat{\nu},\pi^{*,\hat{\nu}}} \left(\pi_t^{*,\hat{\nu}}(\beta(t) - \hat{\nu}(t)) + \xi(t)\right) dt\right] \geq 0, \quad n = 1, 2, \ldots. \tag{7.26}$$

- Proof of (7.24): we invoke Lemma 7.7 to obtain a process $\ell \in \mathcal{D}$ satisfying (7.20) with $\pi = \pi^{*,\hat{\nu}}$. We take $\beta := \hat{\nu} + \ell$ (so that $\xi = \ell^-$). In this case, inequality (7.26) becomes

$$\mathbb{E}\left[\int_0^{\tau_n} \Gamma_t^{\hat{\nu}} W_t^{\hat{\nu},\pi^{*,\hat{\nu}}} \left(\pi_t^{*,\hat{\nu}} \ell(t) + \ell^-(t)\right) dt\right] \geq 0, \quad n = 1, 2, \dots$$

and from (7.20), we see that $\pi^{*,\hat{\nu}}$ takes values in $[0,1]$. Thus, it follows also that (7.24) holds true.

- Proof of (7.25): we take $\beta = 0$. In this case, $\xi = -\hat{\nu}^-$ and (7.26) becomes

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\int_0^{\tau_n} \Gamma_t^{\hat{\nu}} W_t^{\hat{\nu},\pi^{*,\hat{\nu}}} \left(-\pi_t^{*,\hat{\nu}} \hat{\nu}(t) - \hat{\nu}^-(t)\right) dt\right] \\ &= \mathbb{E}\left[-\int_0^{\tau_n} \Gamma_t^{\hat{\nu}} W_t^{\hat{\nu},\pi^{*,\hat{\nu}}} \left(\pi_t^{*,\hat{\nu}} \hat{\nu}(t) + \hat{\nu}^-(t)\right) dt\right], \quad n = 1, 2, \dots \end{aligned}$$

and since $\Gamma_\cdot^{\hat{\nu}} W_\cdot^{\hat{\nu},\pi^{*,\hat{\nu}}} \geq 0$ and (7.24), we can deduce directly (7.25).

Hence, we see that the result of the proposition will be proved if we are able to prove (7.26). The rest of the proof is dedicated to the proof of (7.26).

3. First step for the proof of (7.26): derivation

We introduce the following notations:

$$N_t := \int_0^t \frac{\beta(s) - \hat{\nu}(s)}{\sigma} d\bar{B}_s + \int_0^t \frac{(\beta(s) - \hat{\nu}(s))\left(\mu^{\text{opt}}(s) + \hat{\nu}(s) - r\right)}{\sigma^2} ds,$$

$$L_t := \int_0^t \xi(s) ds,$$

$$\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}} := \exp\left(-rt - \int_0^t \hat{\nu}^-(s) ds\right) W_t^{\hat{\nu},\pi^{*,\hat{\nu}}}.$$

In order to ensure that the terms in the exponentials are well defined, we choose $\{\tau_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} \tau_n \;:=\; & \inf\Big\{t \in [0,T] \text{ s.t. } |N_t| + \langle N \rangle_t + |L_t| \geq n, \\ & \text{or } \int_0^t \frac{\left(\mu^{\text{opt}}(s) + \hat{\nu}(s)\right)^2}{\sigma^2} ds \geq n, \\ & \text{or } \int_0^t |\bar{W}_s^{\hat{\nu},\pi^{*,\hat{\nu}}}|^2 \left(\frac{\beta(s) - \hat{\nu}}{\sigma}\right)^2 ds \geq n, \\ & \text{or } \int_0^t (N_s + L_s)^s \, |\pi^{*,\hat{\nu}}|^2 ds \geq n \Big\} \wedge T. \end{aligned}$$

Clearly, τ_n converges to T almost surely as n tends to infinity.

Because $\hat{\nu}$ minimizes $\tilde{V}(\nu, x)$ over $\mathcal{D}$, we have that

$$0 \leq \frac{\tilde{V}(\nu_{\varepsilon,n}, y) - \tilde{V}(\hat{\nu}, y)}{\varepsilon y}. \tag{7.27}$$

Thanks to (7.17), we have

$$\frac{\tilde{V}(\nu_{\varepsilon,n}, y) - \tilde{V}(\hat{\nu}, y)}{\varepsilon y} = \mathbb{E}(Y_{\varepsilon,n})$$

where

$$Y_{\varepsilon,n} := \frac{\tilde{U}\left(y\Gamma_T^{\nu_{\varepsilon,n}}\right) - \tilde{U}\left(y\Gamma_T^{\hat{\nu}}\right)}{\varepsilon y}.$$

We first remark that $\tilde{U}'(y) = -I(y)$ (it can be easily deduced from (7.19)). Remember our choice of $\{\tau_n\}_{n\in\mathbb{N}^*}$ and that I is a decreasing function. Applying the finite increments inequality, we get the inequality

$$\tilde{U}\left(y\Gamma_T^{\nu_{\varepsilon,n}}\right) - \tilde{U}\left(y\Gamma_T^{\hat{\nu}}\right) \leq y\Gamma_T^{\hat{\nu}} I\left(y\mathrm{e}^{-\varepsilon n}\Gamma_T^{\hat{\nu}}\right)\left(1-\Lambda_T^{\varepsilon,n}\right)^+ , \tag{7.28}$$

where

$$\Lambda_t^{\varepsilon,n} := \frac{\Gamma_t^{\nu_{\varepsilon,n}}}{\Gamma_t^{\hat{\nu}}}.$$

Thanks to the definition of Γ and using the subadditivity of $\nu \to \nu^-$,

$$\Lambda_t^{\varepsilon,n} \geq \exp\left[-\varepsilon N_{t\wedge\tau_n} - \frac{\varepsilon^2}{2}\langle N\rangle_{t\wedge\tau_n} - \varepsilon\int_0^{t\wedge\tau_n}\xi(s)ds\right]. \tag{7.29}$$

From the definition of $\{\tau_n\}_{n\in\mathbb{N}}$, we can easily see that

$$\Lambda_t^{\varepsilon,n} \geq \mathrm{e}^{-\varepsilon n} \quad 0\leq t\leq T,$$

so that

$$\frac{1-\Lambda_t^{\varepsilon,n}}{\varepsilon} \leq K_n, \quad 0 \leq t\leq T, \tag{7.30}$$

with $K_n := \sup_{0<\varepsilon<1}\frac{1}{\varepsilon}\left(1-\mathrm{e}^{-\varepsilon n}\right)$, which is a finite number.

Moreover, from (7.29), we can see that

$$\limsup_{\varepsilon\to 0}\frac{1-\Lambda_t^{\varepsilon,n}}{\varepsilon} \leq N_{t\wedge\tau_n} + L_{t\wedge\tau_n}, \quad 0\leq t\leq T. \tag{7.31}$$

Applying Fatou's lemma in (7.28) (justified because of (7.30)) and using (7.31) and (7.27), we can conclude that

$$\begin{aligned}
0 &\leq \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon y}\mathbb{E}\left[\tilde{U}\left(y\Gamma_T^{\nu_{\varepsilon,n}}\right) - \tilde{U}\left(y\Gamma_T^{\hat{\nu}}\right)\right] \\
&\leq \mathbb{E}\left[\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon y}\left(\tilde{U}\left(y\Gamma_T^{\nu_{\varepsilon,n}}\right) - \tilde{U}\left(y\Gamma_T^{\hat{\nu}}\right)\right)\right] \\
&\leq \mathbb{E}\left[\Gamma_T^{\hat{\nu}} I\left(y\Gamma_T^{\hat{\nu}}\right)\left(N_{\tau_n}+L_{\tau_n}\right)\right] \\
&= \mathbb{E}\left[\Gamma_T^{\hat{\nu}} W_T^{\hat{\nu},\pi^{*,\hat{\nu}}}\left(N_{\tau_n}+L_{\tau_n}\right)\right],
\end{aligned} \tag{7.32}$$

where the last equality comes from (7.9) and the definition of function I.

Thus, we see that we directly retrieve (7.26) from (7.32) by showing that in fact, for any $n\in\mathbb{N}^*$:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_n}\Gamma_t^{\hat{\nu}} W_t^{\hat{\nu},\pi^{*,\hat{\nu}}}\left(\pi_t^{*,\hat{\nu}}\left(\beta(t)-\hat{\nu}(t)\right)+\xi(t)\right)dt\right] \\
&= \mathbb{E}\left[\Gamma_T^{\hat{\nu}} W_T^{\hat{\nu},\pi^{*,\hat{\nu}}}\left(N_{\tau_n}+L_{\tau_n}\right)\right], \quad n=1,2,\ldots.
\end{aligned} \tag{7.33}$$

We now turn to the proof of (7.33).

4. Second step for the proof of equality (7.26) : integration by parts and proof of (7.33)

Thanks to (7.11),

$$\frac{d\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}}{\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}} = \pi_t^{*,\hat{\nu}}\left[\sigma d\bar{B}_t + \left(\mu^{\mathrm{opt}}(t)+\hat{\nu}(t)-r\right)dt\right].$$

In particular, a differentiation by parts gives:

$$\begin{aligned} d\left(\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}(L_t+N_t)\right) &= \bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}(dL_t+dN_t)+(L_t+N_t)\,d\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}} \\ &\quad +\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}\pi_t^{*,\hat{\nu}}(\beta(t)-\hat{\nu}(t))\,dt. \end{aligned}$$

When we integrate this equality, we get that

$$\begin{aligned} \bar{W}_{\tau_n}^{\hat{\nu},\pi^{*,\hat{\nu}}}(L_{\tau_n}+N_{\tau_n}) &= \int_0^{\tau_n}\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}\left(\pi_t^{*,\hat{\nu}}(\beta(t)-\hat{\nu}(t))+\xi(t)\right)dt \\ &\quad +\int_0^{\tau_n}\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}\left[\frac{\beta(t)-\hat{\nu}(t)}{\sigma^2}+(L_t+N_t)\,\pi_t^{*,\hat{\nu}}\right] \\ &\quad \times\left(\sigma d\bar{B}_t+\left(\mu^{\text{opt}}(t)+\hat{\nu}(t)-r\right)dt\right). \end{aligned} \tag{7.34}$$

Now, according to the Girsanov and Novikov theorems, the process

$$\forall 0\le t\le T\quad \bar{B}_t+\int_0^t\frac{\mu^{\text{opt}}(s)+\hat{\nu}(s)-r}{\sigma}ds$$

is a Brownian motion under the probability measure

$$\mathbb{P}_{\hat{\nu},n}(A):=\mathbb{E}\left[H_{\tau_n}^{\hat{\nu}}\mathbb{1}_A\right],\qquad A\in\mathcal{F}_T.$$

Note that

$$\Gamma_t^{\hat{\nu}}W_t^{\hat{\nu},\pi^{*,\hat{\nu}}}=\bar{W}_t^{\hat{\nu},\pi^{*,\hat{\nu}}}H_t^{\hat{\nu}}$$

and from proposition 7.3 and the optional sampling theorem for martingales,

$$\Gamma_{\tau_n}^{\hat{\nu}}W_{\tau_n}^{\hat{\nu},\pi^{*,\hat{\nu}}}=\mathbb{E}\left[\Gamma_T^{\hat{\nu}}W_T^{\hat{\nu},\pi^{*,\hat{\nu}}}|\mathcal{F}_{\tau_n}^X\right].$$

Thus, taking expectations with respect to $\mathbb{P}_{\hat{\nu},n}$ in (7.34) finally yields (7.33), and the proposition is proved. □

7.2 The particular Case of the Logarithmic Utility Function

Proposition 7.9. *If $U(.)=\log(.)$ and the initial endowment is x, then the optimal wealth process and strategy are*

$$W_t^{*,x}=\frac{x\exp\left(rt+\int_0^t\hat{\nu}^-(s)ds\right)}{H_t^{\hat{\nu}}} \tag{7.35}$$

$$\pi_t^*=\frac{\mu^{\text{opt}}(t)-r+\hat{\nu}(t)}{\sigma^2}, \tag{7.36}$$

where

$$\hat{\nu}(t)=\begin{cases} -\left(\mu^{\text{opt}}(t)-r\right) & \text{if } \frac{\mu^{\text{opt}}(t)-r}{\sigma^2}<0, \\ 0 & \text{if } \frac{\mu^{\text{opt}}(t)-r}{\sigma^2}\in[0,1], \\ \sigma^2-\mu^{\text{opt}}(t)+r & \text{otherwise}, \end{cases} \tag{7.37}$$

and, as above,

$$\hat{\nu}^-(t):=-\inf(0,\hat{\nu}(t)).$$

Remark 7.10. Note that this proposition gives the result announced in the introduction: the best allocation strategy π^* is given in the particular case of a logarithmic Utility Function by

$$\pi_t^*=\text{proj}_{[0,1]}\left\{\frac{\mu^{\text{opt}}(t)-r}{\sigma^2}\right\}.$$

Proof. If $U(x) = \log(x)$, then for each $\nu \in \mathcal{D}$, we find that the solution of the unconstrained problem is

$$\pi_t^{\nu,*} := \left(\frac{\mu^{\text{opt}}(t) - r + \nu(t)}{\sigma^2}\right),$$

$$W_t^{\nu,*} = \frac{\exp^{rt+\int_0^t \nu^-(s)ds}}{H_t^\nu},$$

$$V(\nu, x) = \log(x) + rT + \mathbb{E}\left[\int_0^T \nu^-(t)dt\right] + \mathbb{E}\left[\int_0^T \frac{1}{2}\left(\frac{\mu^{\text{opt}}(t) - r + \nu(t)}{\sigma}\right)^2 dt\right].$$

Set $\theta(t) := \frac{\mu^{\text{opt}}(t)-r}{\sigma}$ and $u(\nu, \theta, \sigma) := \nu^- + \frac{1}{2}\left(\theta + \frac{\nu}{\sigma}\right)^2$. Then the process $\hat{\nu}$ defined by (7.37) satisfies

$$u(\hat{\nu}(t), \theta(t), \sigma) = \min_{\nu \in \mathcal{D}} \left\{\nu^-(t) + \frac{1}{2}\left(\frac{\mu^{\text{opt}}(t) - r + \nu(t)}{\sigma}\right)^2\right\}.$$

Moreover,

$$\mathbb{E}\left[\int_0^T \hat{\nu}^-(t)dt\right] + \mathbb{E}\left[\int_0^T \frac{1}{2}\left(\frac{\mu^{\text{opt}}(t) - r + \hat{\nu}(t)}{\sigma}\right)^2 dt\right] \leq \mathbb{E}\left[\int_0^T \frac{3}{2}\left(\frac{\mu^{\text{opt}}(t) - r + \hat{\nu}(t)}{\sigma}\right)^2 dt\right] < \infty,$$

and thus, $V(\hat{\nu}, x) < \infty$ for all x. □

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