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Statistical estimation for reflected skew processes

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Abstract In this note, we construct estimators for the parameters that rule the law of reflected skewed diffusion processes. The convergence properties of these estimators rely on the ergodic properties of these processes.

Keywords Parameter estimation · Skew Brownian motion · Divergence form operator · One-dimensional diffusion · Local time

Mathematics Subject Classification (2000) 62M05 · 60J55

1 Introduction

The skew Brownian motion and more generally what one might call 'skewed processes' are often used in the modeling of discontinuous physical systems. Geophysical phenomenons, astrophysics, composite structures are some examples of such systems. See Lejay (2006) and the references therein for a survey of the various applications of skewed processes in physical models.

However, if the estimation of standard diffusion processes has been extensively studied, as far as the authors know, no result is available for the statistics of skewed processes. From a theoretical point of view, estimating the parameters of skewed processes is of particular

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interest since this problem is highly connected to the estimation of possible discontinuities in the diffusion coefficient of standard diffusion processes.

This article is a first attempt to calibrate the coefficients of some skewed process in the particular case where the process is reflected over a bounded interval. In the case where the process is not reflected and lives on the entire real line, one may use the explicit transition function of the skew Brownian motion and apply maximum likelihood technics in order to get estimators. In this article, we treat specifically the case of the reflected skew Brownian motion, but our results may be extended to more general reflected skewed diffusion processes.

Let $\left(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and let B be a $\mathbb{F}$-Brownian motion. We consider a skew Brownian motion reflected over the interval $[0, 1]$ and we set $0 < \gamma < 1$. This process X is defined as the unique weak solution of the following stochastic differential equation :

$$X_t = x + B_t + (2\alpha - 1)L_t^{\gamma}(X) - K_t^{[0,1]}(X),\ x \in [0, 1],\ \alpha \in]0, 1[\tag{1}$$

where $K^{[0,1]}(X) := L^0(X) - L^1(X)$ and the process $L^{\gamma}(X)$ is the symmetric local time of the process X (that is the local time defined from the Tanaka formula with the convention $\text{sign}(0) = 0$; for references about local times, see Revuz and Yor 1991).

The coefficient α is called the *skew coefficient* and the coefficient γ the *locus*. Our main interest concerns the estimation of these parameters from a single observation of the process X in the long run. Thus, we shall consider the statistical model

$$\left(\Omega, \mathcal{F}, \mathbb{F}, (X_t)_{0\leq t\leq T}, C([0, 1]), \mathcal{B}(C([0, 1])), \mathbb{P}_{(\alpha,\gamma)}, \alpha \in]0, 1[, \gamma \in]0, 1[\right),$$

where $\mathbb{P}_{(\alpha,\gamma)}$ is the law of the process X given α and γ.

The reflection over the bounded interval $[0, 1]$ by means of the process $K^{[0,1]}(X)$ ensures the ergodicity of the process X. This will be the key fact to construct our estimators.

The paper is organized as follows. In Sect. 2, we shall prove some results concerning the ergodicity of the process X and construct an efficient estimator of the parameter α supposing that the locus γ is known. In Sect. 3, we built up an estimator of the couple (α, γ) and prove its consistency. The construction of the estimator is based on the classical comparison of the empirical measure of the process and its limiting invariant measure. Section 4 is devoted to concluding remarks, including a brief discussion upon our approach.

2 Estimation of α when γ is known

In this first part, we shall assume that the locus of the asymmetry γ is known.

2.1 Main assumptions and results

Set

$$M_T := \frac{1}{T}\int_0^T \mathbb{1}_{[\gamma,1]}(X_s)\,ds,\ T > 0, \tag{2}$$

and

$$\hat{\alpha}_T := \frac{\gamma M_T}{M_T(2\gamma - 1) + (1 - \gamma)},\ T > 0. \tag{3}$$

Our main result is the following theorem.

Theorem 2.1.1 *The random variable $\hat{\alpha}_T$ is a strongly consistent and asymptotically normal estimator of the coefficient α.*

This theorem will appear as a corollary of two propositions. The first one states an ergodic theorem for the process X.

Proposition 2.1.1 *Let ν be the measure defined by*

$$\nu(dx) := \frac{1}{\alpha(1-\gamma)+(1-\alpha)\gamma}\left(\alpha\,\mathbb{1}_{[\gamma,\,1]}(x)+(1-\alpha)\,\mathbb{1}_{[0,\,\gamma]}(x)\right)dx \tag{4}$$

For any measurable function f in $L^1(\nu)$,

$$\frac{1}{T}\int_0^T f(X_s)\,ds \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s. } and\ L^1} \int f\,d\nu. \tag{5}$$

The second proposition of interest provides a central limit theorem.

Proposition 2.1.2 *For any measurable function f in $L^1(\nu)$,*

$$\sqrt{T}\left(\frac{1}{T}\int_0^T f(X_s)\,ds - \int f d\nu\right) \xrightarrow[T\to+\infty]{Law} \mathcal{N}(0,\sigma_f^2), \tag{6}$$

where $\mathcal{N}(0,\sigma_f^2)$ stands for the zero mean gaussian law with variance

$$\begin{aligned}\sigma_f^2 &= 4\int_0^\gamma\left(\int_y^\gamma \tilde{f}(z)\,dz\right)^2 \nu(dy) + 4\left(\frac{\alpha}{1-\alpha}\right)^2\int_0^\gamma\left(\int_\gamma^1 \tilde{f}(z)\,dz\right)^2 \nu(dy)\\ &\quad +8\frac{\alpha}{1-\alpha}\int_0^\gamma\left(\int_y^\gamma \tilde{f}(z)\,dz.\int_\gamma^1 \tilde{f}(z)\,dz\right)\nu(dy)+4\int_\gamma^1\left(\int_y^1 \tilde{f}(z)\,dz\right)^2\nu(dy)\end{aligned}$$

where $\tilde{f} := f - \int f d\nu$.

The next part of this paper is devoted to the proofs of these two propositions and of Theorem 2.1.1.

2.2 Proofs

Lemma 2.2.1 *The reflected skew Brownian motion X has a unique invariant measure ν defined by* (4).

Proof The idea of the proof is to apply the Itô-Tanaka formula to the process X and a function $\varphi_{p,q}$. The resulting process $Y := \varphi_{p,q}(X)$ is associated to a self-adjoint divergence form operator with Neumann conditions on the boundaries (or its corresponding Dirichlet form) so that the process Y admits the uniform measure as its invariant measure.

Let $\varphi_{p,q}$ be the continuous function defined by

$$\varphi_{p,q}(y) := p(y-\gamma)\mathbb{1}_{(y<\gamma)} + q(y-\gamma)\mathbb{1}_{(y\geq\gamma)} + \gamma,\ y\in\mathbb{R};\ p,\ q\in\mathbb{R}^*.$$

This function has an inverse $\varphi_{p,q}^{-1}$:

$$\varphi_{p,q}^{-1}(y) = \frac{1}{p}(y-\gamma)\mathbb{1}_{(y<\gamma)} + \frac{1}{q}(y-\gamma)\mathbb{1}_{(y\geq\gamma)} + \gamma,\ y\in\mathbb{R}.$$

We consider the process $Y_t := \varphi_{p,q}(X_t)$. Since $L^{\gamma}_{\cdot}(X)$ denotes the symmetric local time of the process X at γ, the Itô-Tanaka formula leads to

$$\begin{aligned}
\varphi_{p,q}(X_t) &= \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(X_s)\, dX_s + \frac{q-p}{2} L^{\gamma}_t(X) \\
&\quad \text{with } \varphi'_{p,q}(y) := p\,\mathbb{1}_{(y<\gamma)} + q\,\mathbb{1}_{(y>\gamma)} + \frac{p+q}{2}\mathbb{1}_{(y=\gamma)}, \\
&= \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(X_s)\, dB_s + (2\alpha-1)\int_0^t \varphi'_{p,q}(X_s)\, dL^{\gamma}_s(X) \\
&\quad - \int_0^t \varphi'_{p,q}(X_s)\, dK^{[0,1]}_s(X) + \frac{q-p}{2} L^{\gamma}_t(X) \\
&= \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(X_s)\, dB_s + \left[(2\alpha-1)\frac{p+q}{2} + \frac{q-p}{2}\right] L^{\gamma}_t(X) \\
&\quad - \int_0^t \varphi'_{p,q}(X_s)\, dK^{[0,1]}_s(X).
\end{aligned}$$

We consider independently each integral in the former equation. First, note that $\varphi'_{p,q}\circ\varphi^{-1}_{p,q} \equiv \varphi'_{p,q}$ so that

$$\int_0^t \varphi'_{p,q}(X_s)\, dB_s = \int_0^t \varphi'_{p,q}(Y_s)\, dB_s.$$

So far, we have shown that

$$\begin{aligned}
Y_t = \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(Y_s)\, dB_s + \left[(2\alpha-1)\frac{p+q}{2} + \frac{q-p}{2}\right] L^{\gamma}_t(X), \\
- K^{[\varphi_{p,q}(0),\, \varphi_{p,q}(1)]}_t(Y).
\end{aligned} \tag{7}$$

with $K^{[\varphi_{p,q}(0),\, \varphi_{p,q}(1)]}_t(Y) := \int_0^t \varphi'_{p,q}(X_s)\, dK^{[0,1]}_s(X)$. Indeed, it is easy to check that this process is nothing else than

$$K^{[\varphi_{p,q}(0),\, \varphi_{p,q}(1)]}_t(Y) = L^{\varphi_{p,q}(0)}_t(Y) - L^{\varphi_{p,q}(1)}_t(Y).$$

We may define $\beta \in]0,1[$ such that Eq. 7 can be written as

$$Y_t = \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(Y_s)\, dB_s + \beta L^{\gamma}_t(Y) - K^{[\varphi_{p,q}(0),\, \varphi_{p,q}(1)]}_t(Y). \tag{8}$$

Applying the Itô-Tanaka formula to $\varphi^{-1}_{p,q}(Y_t) = X_t$ gives

$$X_t = \varphi^{-1}_{p,q}(Y_t) = x + B_t + \left[\beta\left(\frac{1}{2p} + \frac{1}{2q}\right) + \frac{q-p}{2} + \frac{p-q}{2pq}\right] L^{\gamma}_t(Y) - K^{[0,1]}_t(X). \tag{9}$$

Using (7), (8) and (9), (1) and identifying the bounded variation parts leads to

$$\begin{cases} \left((2\alpha-1)\frac{p+q}{2} + \frac{q-p}{2}\right) L^{\gamma}_t(X) = \beta L^{\gamma}_t(Y), \\ \left(\beta\left(\frac{1}{2p} + \frac{1}{2q}\right) + \frac{p-q}{2pq}\right) L^{\gamma}_t(Y) = (2\alpha-1)\, L^{\gamma}_t(X). \end{cases} \tag{10}$$

From (10), we get

$$((2\alpha-1)(p+q) + q - p)\,(\beta(q+p) + p - q) = 4\beta pq(2\alpha-1).$$

In particular, we may choose the constants p and q such that

$$\alpha = \frac{q}{p+q},$$

and with this choice, we see that the process Y satisfies the following SDE in the weak sense :

$$Y_t = \varphi_{p,q}(x) + \int_0^t \varphi'_{p,q}(Y_s)\, dB_s + \frac{q^2 - p^2}{q^2 + p^2} L_t^\gamma(Y) - K_t^{[\varphi_{p,q}(0),\, \varphi_{p,q}(1)]}(Y).$$

It is known that the process $(Y_t)_{t\geq 0}$ is associated (in the sense of Fukushima et al. 1994) to the symmetric Dirichlet form $\left(\mathcal{E}, H^1\left(]\varphi_{p,q}(0),\ \varphi_{p,q}(1)[\right)\right)$ defined by

$$\mathcal{E}(f, f) := \frac{1}{2}\int_{]\varphi_{p,q}(0),\, \varphi_{p,q}(1)[} \varphi'_{p,q}(x)^2 \left|\frac{df}{dx}(x)\right|^2 dx, \quad \forall f \in H^1\left(]\varphi_{p,q}(0),\ \varphi_{p,q}(1)[\right).$$

Therefore, it has a unique invariant measure: namely the uniform measure over the interval $[\varphi_{p,q}(0),\ \varphi_{p,q}(1)]$ (see Martinez 2004, Chap.4 for a proof). Consequently, the process $X_t = \varphi_{p,q}^{-1}(Y_t)$ has also a unique invariant measure $\nu(dx)$ and for any continuous and bounded function f, we have that

$$\begin{aligned}\int f\, d\nu &= \frac{1}{\varphi_{p,q}(1) - \varphi_{p,q}(0)} \int_{\varphi_{p,q}(0)}^{\varphi_{p,q}(1)} f \circ \varphi_{p,q}^{-1}(y)\, dy \\ &= \frac{1}{\varphi_{p,q}(1) - \varphi_{p,q}(0)} \int_0^1 f(y)\varphi'_{p,q}(y)\, dy.\end{aligned}$$

Therefore, as $\varphi_{p,q}(1) = q(1-\gamma) + \gamma, \varphi_{p,q}(0) = \gamma(1-p)$ and $\varphi'_{p,q}(y) = p\mathbb{1}_{(y\leq\gamma)} + q\mathbb{1}_{(y\geq\gamma)}$ a.e., we have

$$\begin{aligned}\nu(dx) &= \frac{1}{q - \gamma(p-q)}\left(p\mathbb{1}_{[0,\,\gamma]}(x) + q\mathbb{1}_{[\gamma,\,1]}(x)\right) dx \\ &= \frac{1}{\alpha(1-\gamma) + (1-\alpha)\gamma}\left((1-\alpha)\mathbb{1}_{[0,\,\gamma]}(x) + \alpha\mathbb{1}_{[\gamma,\,1]}(x)\right) dx\end{aligned}$$

and the lemma is demonstrated. □

The following corollary is a classical consequence of the former lemma.

Corollary 2.2.1 (Proposition 2.1.1) *For any measurable function f in $L^1(\nu)$,*

$$\frac{1}{T}\int_0^T f(X_s)\, ds \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s. and } L^1} \int f\, d\nu. \tag{11}$$

Proof The proof of this result is classical (see, e.g., Gihman and Skorohod 1972). The main assumption needed in the proof, the strong Markov property, holds for the process X (see Martinez 2004). □

This result also establishes the ergodicity of the process X. So far, we have then proved the strong law of large numbers. We now turn to the proof of the central limit theorem.

Proof (of Proposition 2.1.2) Once more, consider the function

$$\varphi_{p,q}(y) = p(y-\gamma)\mathbb{1}_{(y<\gamma)} + q(y-\gamma)\mathbb{1}_{(y\geq\gamma)} + \gamma,\ \ y \in \mathbb{R}.$$

We now choose the parameters p and q such that

$$\alpha = \frac{p}{p+q}.$$

Then, the process $\hat{Y}_t := \varphi_{p,q}(X_t)$ defined in (7) satisfies

$$d\hat{Y}_t = \varphi'_{p,q}(\hat{Y}_t)dB_t - dK_t^{[\varphi_{p,q}(0),\,\varphi_{p,q}(1)]}(\hat{Y}),\ t \in [0,T],\ \hat{Y}_0 = \varphi_{p,q}(x). \tag{12}$$

Note that applying the function $\varphi_{p,q}$ to $(X_t)_{t\geq 0}$ removes the generalized local time drift of X. As far as we know, this type of Zvonkin's transform was first used by Ouknine (1990) in the context of skew diffusions.

Consider a measurable function f in $L^1(\nu)$ and the following Poisson problem

$$\begin{cases} \frac{1}{2}\varphi'_{p,q}(y)^2 \frac{d^2u}{dy^2}(y) = -f \circ \varphi_{p,q}^{-1}(y) + \int f d\nu, \quad y \in (\varphi_{p,q}(0), \varphi_{p,q}(1)), \\ \frac{du}{dy}(\varphi_{p,q}(0)) = \frac{du}{dy}(\varphi_{p,q}(1)) = 0. \end{cases} \tag{13}$$

A solution u of this Poisson problem[1] is given by

$$\forall x \in [\varphi_{p,q}(0), \varphi_{p,q}(1)], \quad u(x) := 2\int_{\varphi_{p,q}(0)}^{x} \left(\int_z^{\varphi_{p,q}(1)} \frac{f \circ \varphi_{p,q}^{-1}(y) - \int f d\nu}{\varphi'_{p,q}(y)^2} dy \right) dz.$$

Indeed,

$$\begin{aligned} \frac{pq}{p+q}\frac{1}{\varphi'_{p,q}(x)} &= \frac{pq}{p+q}\left(\frac{1}{p}\mathbb{1}_{[0,\gamma]}(x) + \frac{1}{q}\mathbb{1}_{[\gamma,1]}(x)\right) \quad \text{a.e.} \\ &= \left(\frac{q}{p+q}\mathbb{1}_{[0,\gamma]}(x) + \frac{p}{p+q}\mathbb{1}_{[\gamma,1]}(x)\right) \quad \text{a.e.} \\ &= \left((1-\alpha)\mathbb{1}_{[0,\gamma]}(x) + \alpha\mathbb{1}_{[\gamma,1]}(x)\right) \quad \text{a.e.} \end{aligned}$$

and there exists a constant C (depending on p, q and γ) such that $\frac{C}{\varphi'_{p,q}(x)}dx = \nu(dx)$ a.e..

Consequently,

$$\begin{aligned} \frac{du}{dy}\left(\varphi_{p,q}(0)\right) &= 2\int_{\varphi_{p,q}(0)}^{\varphi_{p,q}(1)} \frac{f \circ \varphi_{p,q}^{-1}(y) - \int f d\nu}{\varphi'_{p,q}(y)^2} dy = 2\int_0^1 \frac{f(y) - \int f d\nu}{\left(\varphi'_{p,q} \circ \varphi_{p,q}(y)\right)^2}\varphi'_{p,q}(y)dy \\ &= 2\int_0^1 \frac{f(y) - \int f d\nu}{\varphi'_{p,q}(y)}dy \\ &= 2C\int_0^1 \left(f(y) - \int f d\nu\right)\nu(dy) = 0, \end{aligned}$$

and it is clear from the definition of u that it is a solution of the Poisson problem (13).

[1] The solution u is unique up to an additive constant.

Since u is twice differentiable, we can apply Itô's formula and we get

$$\begin{aligned}
u(\hat{Y}_T) &= u(\hat{Y}_0) + \frac{1}{2}\int_0^T \varphi'_{p,q}(\hat{Y}_s)^2 \frac{d^2u}{dy^2}(\hat{Y}_s)\,ds + \int_0^T \varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\,dB_s \\
&\quad - \int_0^T \frac{du}{dy}(\hat{Y}_s)\,dK_s^{[\varphi_{p,q}(0),\,\varphi_{p,q}(1)]}(\hat{Y}) \\
&= u(\hat{Y}_0) - \int_0^T f\circ\varphi_{p,q}^{-1}(\hat{Y}_s)\,ds + T\int f\,d\nu + \int_0^T \varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\,dB_s \\
&= u(\hat{Y}_0) - T\left(\frac{1}{T}\int_0^T f(X_s)\,ds - \int f\,d\nu\right) + \int_0^T \varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\,dB_s.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sqrt{T}\left(\frac{1}{T}\int_0^T f(X_s)\,ds - \int f\,d\nu\right) &= \frac{1}{\sqrt{T}}\int_0^T \varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\,dB_s \\
&\quad - \frac{1}{\sqrt{T}}\left(u(\hat{Y}_T) - u(\hat{Y}_0)\right).
\end{aligned}$$

Our aim is to apply the Central Limit Theorem for stochastic integrals in order to prove a convergence in law. The bracket of the stochastic integral appearing in the expression is

$$\frac{1}{T}\int_0^T \left(\varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\right)^2 ds.$$

We are now going to express this integral in terms of the process X and to apply the result of Proposition 2.1.1 :

$$\begin{aligned}
\sigma_f^2 &:= \text{a.s-}\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\right)^2 ds \\
&= \text{a.s-}\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi'_{p,q}(X_s)\left(\frac{du}{dy}\circ\varphi_{p,q}(X_s)\right)^2 ds \text{ as } \varphi'_{p,q}\circ\varphi_{p,q} = \varphi'_{p,q} \\
&= \int_{\mathbb{R}} \left(\varphi'_{p,q}(y)\frac{du}{dy}\circ\varphi_{p,q}(y)\right)^2 \nu(dy) \\
\sigma_f^2 &= 4\int_0^1 \left(\varphi'_{p,q}(y)\int_y^1 \frac{f(z)-\int f\,d\nu}{\varphi'_{p,q}(z)}\,dz\right)^2 \nu(dy) \;< +\infty.
\end{aligned}$$

Setting $\tilde{f} := f - \int f\,d\nu$, this last expression may also be written as

$$\begin{aligned}
\sigma_f^2 &= 4\int_0^\gamma \left(\int_y^\gamma \tilde{f}(z)\,dz\right)^2 \nu(dy) + 4\left(\frac{\alpha}{1-\alpha}\right)^2\int_0^\gamma \left(\int_\gamma^1 \tilde{f}(z)\,dz\right)^2 \nu(dy) \\
&\quad + 8\frac{\alpha}{1-\alpha}\int_0^\gamma \left(\int_y^\gamma \tilde{f}(z)\,dz.\int_\gamma^1 \tilde{f}(z)\,dz\right)\nu(dy) + 4\int_\gamma^1 \left(\int_y^1 \tilde{f}(z)\,dz\right)^2 \nu(dy)
\end{aligned}$$

so that, naturally, σ_f^2 does not depend on the choice of the constants p and q.

The assumptions of the Central Limit Theorem for Brownian martingales are satisfied (see Basawa and Rao 1980) and therefore

$$\frac{1}{\sqrt{T}}\int_0^T \varphi'_{p,q}(\hat{Y}_s)\frac{du}{dy}(\hat{Y}_s)\,dB_s \xrightarrow[T\to+\infty]{Law} \mathcal{N}\left(0,\sigma_f^2\right).$$

Moreover, as the function u is continuous over the compact set $[\varphi_{p,q}(0), \varphi_{p,q}(1)]$, u is bounded and so,

$$\frac{1}{\sqrt{T}}\left(u(\hat{Y}_T) - u(\hat{Y}_0)\right) \xrightarrow[T\to+\infty]{\mathbb{P}} 0.$$

Finally, we apply Slutsky's lemma to deduce the convergence in law

$$\sqrt{T}\left(\frac{1}{T}\int_0^T f(X_s)\,ds - \int f\,d\nu\right) \xrightarrow[T\to+\infty]{Law} \mathcal{N}(0, \sigma_f^2).$$

□

Remark Note that σ_f^2 can be written as $\sigma_f^2 = \int \left|\frac{d\tilde{u}}{dx}(x)\right|^2 \nu(dx)$ where

$$\tilde{u}(x) := 2\int_0^x \varphi'_{p,q}(y) \int_y^1 \frac{f(z) - \int f d\nu}{\varphi'_{p,q}(z)}\,dz\,dy$$

is a solution of the Poisson problem related to the infinitesimal generator of the reflected skew Brownian motion:

$$\begin{cases} \frac{1}{2}\frac{d^2\tilde{u}}{dy^2}(y) = -f + \int f d\nu, \quad y \in]0,\gamma[\cup]\gamma,1[\\ (1-\alpha)\frac{d\tilde{u}}{dy}(\gamma-) = \alpha\frac{d\tilde{u}}{dy}(\gamma+) = 0 \\ \frac{d\tilde{u}}{dy}(0) = \frac{d\tilde{u}}{dy}(1) = 0. \end{cases} \tag{14}$$

This result is consistent with the classical result of diffusion theory (see Bhattacharya 1982).

We have now all the elements to prove the main theorem of this section.

Proof (of Theorem 2.1.1) Remember the definition of the estimator $\hat{\alpha}_T$:

$$\hat{\alpha}_T := \frac{M_T\gamma}{M_T(2\gamma-1) + (1-\gamma)}$$

with

$$M_T := \frac{1}{T}\int_0^T \mathbb{1}_{(X_s\geq\gamma)}\,ds,\ T > 0.$$

From Proposition 2.1.1, we know that the process M converges almost surely,

$$M_T \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s.}} \int_\gamma^1 d\nu.$$

This limit can be restated as a function of α :

$$\int_\gamma^1 d\nu = \frac{\alpha(1-\gamma)}{\alpha(1-\gamma) + (1-\alpha)\gamma}.$$

Then, the strong consistency of the estimator : $\hat{\alpha}_T \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s.}} \alpha$ results from a straightforward computation.

In order to prove the asymptotic normality, we have to rewrite $\hat{\alpha}_T$ in a slightly different way :

$$\sqrt{T}\left(\hat{\alpha}_T - \alpha\right) = \sqrt{T}\frac{M_T\gamma - \alpha M_T\,(2\gamma-1) - \alpha(1-\gamma)}{M_T\,(2\gamma-1) + (1-\gamma)}$$
$$= \frac{(1-\alpha)\gamma + \alpha(1-\gamma)}{M_T\,(2\gamma-1) + (1-\gamma)} \times \sqrt{T}\left(M_T - \int_\gamma^1 d\nu\right).$$

Using the results of the Propositions 2.1.1 and 2.1.2, we know that, on the one hand,

$$M_T \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s.}} \int_\gamma^1 d\nu,$$

and, on the other hand,

$$\sqrt{T}\left(M_T - \int_\gamma^1 d\nu\right) \xrightarrow[T\to+\infty]{Law} \mathcal{N}\left(0, \sigma^2_{\mathbb{1}_{(.\geq\gamma)}}\right).$$

Therefore, applying Slutsky's lemma once again, we can conclude that our estimator is asymptotically normal :

$$\sqrt{T}\left(\hat{\alpha}_T - \alpha\right) \xrightarrow[T\to+\infty]{Law} \mathcal{N}\left(0, v^2\right)$$

with variance

$$v^2 := \left(\frac{(1-\alpha)\gamma + \alpha(1-\gamma)}{\int_\gamma^1 d\nu\,(2\gamma-1) + (1-\gamma)}\sigma_{\mathbb{1}_{(.\geq\gamma)}}\right)^2.$$

This ends the proof of our main theorem. □

2.3 Estimation of α for general skew processes

We now turn to the general case of skew diffusion processes.

Let Y be the unique strong solution to the stochastic differential equation

$$Y_t = y + \int_0^t d(Y_s)\,ds + \int_0^t \sigma(Y_s)\,dB_s + (2\alpha-1)L_t^{\tilde{\gamma}}(Y) - K_t^{[l,r]}(Y),$$
$$y \in [l, r],\ \alpha \in]0,\ 1[. \tag{15}$$

Once again our purpose is to estimate the skew coefficient α from a single observation of the process Y in the long run (when we suppose that the locus $\tilde{\gamma}$ is known).

Assumptions

- The functions d and σ are supposed to be Lipschitz.
- There exists a constant $\lambda > 0$ such that

$$\sigma \geq \lambda > 0.$$

- The functions d and σ are supposed to be known.

A scale function of the symmetric process ($\alpha = \frac{1}{2}$) is

$$s(y) := \int_l^y \exp\left(-2\int_l^z \frac{d(u)}{\sigma^2(u)}\,du\right) dy, \quad l \le y \le r. \tag{16}$$

As the function s is twice differentiable, the Itô-Tanaka formula applied to this function and the process Y leads to the following representation for the process $Y^s := s(Y)$.

$$Y_t^s = s(y) + \int_0^t (\sigma s') \circ s^{-1}(Y_u^s)\,dB_u + (2\alpha - 1)L_t^{s(\tilde{\gamma})}(Y^s) - \int_0^t s' \circ s^{-1}(Y_u^s) dK_u^{[l,r]}(Y).$$

Now, let us define the function $\sigma^s := (\sigma s') \circ s^{-1}$ and the process

$$K_t^{[s(l),\,s(r)]}(Y^s) := \int_0^t s' \circ s^{-1}(Y_u^s) dK_u^{[l,r]}(Y).$$

One can easily check that the process Y^s satisfies the following SDE

$$Y_t^s = s(y) + \int_0^t \sigma^s(Y_u^s)\,dB_u + (2\alpha - 1)L_t^{s(\tilde{\gamma})}(Y^s) - K_t^{[s(l),\,s(r)]}(Y^s),$$

where the process $K^{[s(l),\,s(r)]}(Y^s)$ induces a reflection over the interval $[s(l),\ s(r)]$.

Moreover, it is possible to perform an invertible change of time $(T_t)_{t\ge 0}$ and to normalize the process Y so that

$$\frac{Y_{T_t}^s - s(l)}{s(r) - s(l)} = X_t, \quad \mathbb{P} - \text{ps} \ \ \forall t \ge 0,$$

where X is a solution of (1) with $x := \frac{s(y)-s(l)}{s(r)-s(l)}$ and $\gamma := \frac{s(\tilde{\gamma})-s(l)}{s(r)-s(l)}$ (see Martinez 2004, Chap. 5 for details). X is precisely the SDE whose solution has been studied in the first part of the paper. Following exactly the demonstration of the previous section, it is then a simple matter to prove the following result :

Corollary 2.3.1 *The estimator of the parameter* α *defined by*

$$\hat{\alpha}_T := \frac{M_T \int_l^{\tilde{\gamma}} \frac{dz}{(\sigma^2 s')(z)}}{M_T\left(\int_l^{\tilde{\gamma}} \frac{dz}{(\sigma^2 s')(z)} - \int_{\tilde{\gamma}}^r \frac{dz}{(\sigma^2 s')(z)}\right) + s(r)}, \quad T > 0,$$

where

$$M_T := \frac{1}{T}\int_0^T (\sigma s')^2(Y_s)\,\mathbb{1}_{(Y_s \ge \tilde{\gamma})}\,ds, \quad T > 0,$$

is strongly consistent and asymptotically normal.

The proof is similar to the case $\sigma \equiv 1$ and $d \equiv 0$. It relies on change of time techniques and the use of the scale function s [see for example Rogers et al. (2000) for elements of the proof in the case $\alpha = \frac{1}{2}$].

3 Estimation of parameters α and γ when they are both unknown

Let $(X_t)_{t\ge 0}$ be the strong solution of

$$X_t = x_0 + B_t + (2\alpha - 1)L_t^{\gamma}(X) - K_t^{[0,1]}. \tag{17}$$

We denote by $\phi_{\alpha,\gamma}$ the function $y \mapsto (y-\gamma)1\!\!1_{(y<\gamma)} + q(y-\gamma)1\!\!1_{(\gamma\le y)} + \gamma$ with $q \in \mathbb{R}^{*,+}$ satisfying $\alpha + (\alpha-1)q = 0$ (i.e. $\alpha = \frac{q}{1+q}$). (Note that $\phi_{\alpha,\gamma}$ is equal to the function $\varphi_{1,q}$ in our previous section). Then $\{\phi_{\alpha,\gamma}(X_t); t \ge 0\}$ is a process whose invariant measure $U_{\alpha,\gamma}$ is simply the uniform probability measure over $[\phi_{\alpha,\gamma}(0), \phi_{\alpha,\gamma}(1)]$ (See proof of Lemma 2.2.1 in the previous section for details).

Our aim is to give consistant estimators of the parameters. Though we do not work in a semiparametric context, we follow the ideas of Bordes, Delmas, and Vandekerkhove developed in Bordes et al. (2006).

Let $(\hat\alpha, \hat\gamma) \in]0,1[\times]0,1[$ be an estimator of (α,γ). We introduce $\phi_{\hat\alpha,\hat\gamma}$ the function $y \mapsto (y-\hat\gamma)1\!\!1_{(y<\hat\gamma)} + \hat q(y-\hat\gamma)1\!\!1_{(\hat\gamma\le y)} + \hat\gamma$ with $\hat q \in \mathbb{R}^{*,+}$ subject to the condition $\hat\alpha + (\hat\alpha-1)\hat q = 0$. Using the identity $\phi_{\hat\alpha,\hat\gamma} = \phi_{\hat\alpha,\hat\gamma} \circ \phi^{-1}_{\alpha,\gamma} \circ \phi_{\alpha,\gamma}$ it is easy to find the invariant measure $\mu_{\hat\alpha,\hat\gamma}$ of the transformed process $\{\phi_{\hat\alpha,\hat\gamma}(X_t); t \ge 0\}$:

$$\mu_{\hat\alpha,\hat\gamma}(dy) = \frac{1}{\phi_{\alpha,\gamma}(1) - \phi_{\alpha,\gamma}(0)} \times \frac{\phi'_{\hat\alpha,\hat\gamma}}{\phi'_{\alpha,\gamma}} \circ \phi^{-1}_{\hat\alpha,\hat\gamma}(y)1\!\!1_{[\phi_{\hat\alpha,\hat\gamma}(0),\phi_{\hat\alpha,\hat\gamma}(1)]}(y)dy. \tag{18}$$

(With this notation $\mu_{\alpha',\gamma'}$ coincides with the uniform measure $U_{\alpha',\gamma'}$ only when (α',γ') is equal to (α,γ)).

3.1 Construction and convergence of a right-hand sided estimator

3.1.1 Boundary problems: construction of the estimation set $\mathcal{K}_\eta$

Note that $\phi_{\hat\alpha,\hat\gamma}$ is not defined on the entire compact set $[0,1]\times[0,1]$. Let us explain why.

1. $\hat\gamma = 0$ or $\hat\gamma = 1$: even though $\phi_{\hat\alpha,\hat\gamma}$ may be extended to this case (when $\hat\alpha \ne 1$ and $\hat\alpha \ne 0$) we do not include it in our estimation set for technical reasons. Indeed, if we estimate γ with $\hat\gamma = 1$ for example, this means that we think that the diffusion process solution of (17) is led by a parameter γ equal to 1. In particular, the terms $K^{[0,1]}_{\cdot}$ and $(2\alpha-1)L^1_{\cdot}(X)$ would interfere with one another and we cannot guarantee that there is a strong solution to (17) in this case. In fact, if $\alpha > \frac{1}{2}$ the process may pass through the barrier 1 because the local time $K^{[0,1]}$ ceases to be strong enough to 'push' the process towards the interior of [0, 1].

2. $\hat\alpha = 1$ or $\hat\alpha = 0$: these cases would lead to a choice of $\hat q = +\infty$ or $\hat q = 0$ meaning that we think that the process is not reflected on [0, 1] but rather on $[\gamma, 1]$ or $[0, \gamma]$. This is not reasonable, since the model ensures that we 'know' somehow that the process is reflected over the entire interval [0, 1].

In order to avoid these problems we introduce the estimation set $\mathcal{K}_\eta$. Fix $\eta = (\eta_1,\eta_2) \in]0,\frac{1}{2}[^2$: our estimation set $\mathcal{K}_\eta$ is given by

$$\mathcal{K}_\eta = [\eta_1, 1-\eta_1] \times [\eta_2, 1-\eta_2]. \tag{19}$$

In particular, $\mathcal{K}_\eta$ is a compact subset of $]0,1[\times]0,1[$.

3.1.2 Estimation

The Kolmogorov distance d between two measures μ and ν in $\mathcal{M}_1(\mathbb{R})$ is defined as

$$d(\mu, \nu) = \sup_{x \in \mathbb{R}} |F_\mu(x) - F_\nu(x)|, \tag{20}$$

where F_μ (resp. F_ν) is the continuous c.d.f corresponding to μ (resp. ν).

Let us introduce two new measures. First, we introduce $U_{\hat{\alpha},\hat{\gamma}}$, the uniform measure over $[\phi_{\hat{\alpha},\hat{\gamma}}(0), \phi_{\hat{\alpha},\hat{\gamma}}(1)]$:

$$U_{\hat{\alpha},\hat{\gamma}}(dy) := \frac{1}{\phi_{\hat{\alpha},\hat{\gamma}}(1) - \phi_{\hat{\alpha},\hat{\gamma}}(0)} 1\!\!1_{[\phi_{\hat{\alpha},\hat{\gamma}}(0), \phi_{\hat{\alpha},\hat{\gamma}}(1)]}(y)(dy).$$

Second, we introduce the empirical measure $\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}$ of the transformed process $\{\phi_{\hat{\alpha},\hat{\gamma}}(X_t); t \geq 0\}$: for any $A \in \mathcal{B}(\mathbb{R})$,

$$\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}(A) = \frac{1}{T} \int_0^T 1\!\!1_A(\phi_{\hat{\alpha},\hat{\gamma}}(X_s)) ds.$$

Our estimator is given by

$$(\bar{\alpha}^T, \bar{\gamma}^T) = \arg \min_{(\hat{\alpha},\hat{\gamma}) \in \mathcal{K}_\eta} d(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}). \tag{21}$$

(Note that, since $\{s \in [0, T] : \phi_{\hat{\alpha},\hat{\gamma}}(X_s) = x\}$ is a negligible set w.r.t Lebesgue measure, $F_{\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}}(.)$ is continuous and (21) makes sense).

Remark We call $(\bar{\alpha}^T, \bar{\gamma}^T)$ a right-hand sided estimator of the skew coefficient α and the locus γ because it is based on the use of $\phi_{\alpha,\gamma}(y) = (y - \gamma) 1\!\!1_{y<\gamma} + q(y - \gamma) 1\!\!1_{\gamma \leq y} + \gamma$; this means that our contrast function estimates asymmetry considering rather what happens over γ than what happens below. Of course, it is possible to construct other estimators of (α, γ) based on $\varphi_{p,q} : y \mapsto p(y - \gamma) 1\!\!1_{y<\gamma} + q(y - \gamma) 1\!\!1_{\gamma \leq y} + \gamma$ (with p and q such that $\alpha = \frac{q}{p+q}$), but these estimators would lead basically to the same results as those exposed here.

The whole proof of the consistency of the estimator defined in (21) is based on the following Glivenko-Cantelli type result :

Lemma 3.1 *For any* $(\hat{\alpha}, \hat{\gamma}) \in \mathcal{K}_\eta$,

$$d\left(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, \mu_{\hat{\alpha},\hat{\gamma}}\right) \xrightarrow[T \to +\infty]{\mathbb{P}-\text{a.s.} \textit{and t.v.}} 0$$

where t.v. stands for the convergence in total variation.

Proof As the limiting measure $F_{\mu_{\hat{\alpha},\hat{\gamma}}}$ is absolutely continuous (see Lemma 2.2.1 of the previous section), this result follows from a result of Bosq and Davydov (1999). □

We now study the consistency of the right-hand sided estimator defined in (21).

3.1.3 Identifiability

Proposition 3.1 *Suppose* $\alpha \neq \frac{1}{2}$ *(i.e.* $q \neq 1$*). Then for any* $(\hat{\alpha}, \hat{\gamma}) \in \mathcal{K}_\eta$,

$$|\hat{\alpha} - \alpha| + |\hat{\gamma} - \gamma| > 0 \Longrightarrow d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) > 0.$$

Proof We have that

$$d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) = c_{\alpha,\gamma} \sup_{x\in\mathbb{R}} \left| \left\{ \int_{\phi_{\hat{\alpha},\hat{\gamma}}(0)}^{x} \frac{\phi'_{\hat{\alpha},\hat{\gamma}}}{\phi'_{\alpha,\gamma}} \circ \phi^{-1}_{\hat{\alpha},\hat{\gamma}}(y) - \frac{\phi_{\alpha,\gamma}(1) - \phi_{\alpha,\gamma}(0)}{\phi_{\hat{\alpha},\hat{\gamma}}(1) - \phi_{\hat{\alpha},\hat{\gamma}}(0)} \right\} \mathbb{1}_{[\phi_{\hat{\alpha},\hat{\gamma}}(0),\phi_{\hat{\alpha},\hat{\gamma}}(1)]}(y)dy \right|$$

where $c_{\alpha,\gamma} = \frac{1}{\phi_{\alpha,\gamma}(1)-\phi_{\alpha,\gamma}(0)} > 0$. So that,

$$\begin{aligned} d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) &\geq c_{\alpha,\gamma} \left| \int_{\phi_{\hat{\alpha},\hat{\gamma}}(0)}^{x} \left\{ \left(\left((\hat{q}-1)\mathbb{I}_{\hat{\gamma}<\gamma} + \frac{1-q}{q}\mathbb{I}_{\gamma<\hat{\gamma}} \right) \mathbb{1}_{\gamma\wedge\hat{\gamma}\leq y<\gamma\vee\hat{\gamma}} + \frac{\hat{q}-q}{q}\mathbb{1}_{\gamma\vee\hat{\gamma}\leq y} \right) \circ \phi^{-1}_{\hat{\alpha},\hat{\gamma}}(y) \right. \right. \\ &\quad \left. \left. + \frac{(\hat{q}-q) + \hat{\gamma}(1-\hat{q}) - \gamma(1-q)}{\phi_{\hat{\alpha},\hat{\gamma}}(1) - \phi_{\hat{\alpha},\hat{\gamma}}(0)} \right\} \mathbb{1}_{[\phi_{\hat{\alpha},\hat{\gamma}}(0),\phi_{\hat{\alpha},\hat{\gamma}}(1)]}(y)dy \right|. \end{aligned} \tag{22}$$

We distinguish three cases ; since the arguments are basically the same for the three cases, we only treat the first case in detail.

1. $\hat{\gamma} < \gamma$; We integrate first for $x = \phi_{\hat{\alpha},\hat{\gamma}}(\gamma)$ and then for $x \geq \phi_{\hat{\alpha},\hat{\gamma}}(1)$ in (22). We obtain two linear equations w.r.t. both differences $\Delta\gamma := \gamma - \hat{\gamma}$ and $\Delta q := q - \hat{q}$. Namely, define β as

$$\beta := \frac{\phi_{\hat{\alpha},\hat{\gamma}}(\gamma) - \phi_{\hat{\alpha},\hat{\gamma}}(0)}{\phi_{\hat{\alpha},\hat{\gamma}}(1) - \phi_{\hat{\alpha},\hat{\gamma}}(0)} = \frac{\hat{q}\Delta\gamma + \hat{\gamma}}{\hat{q}(1-\hat{\gamma}) + \hat{\gamma}}.$$

   We find a system $\mathcal{S}$ of linear equations

$$\mathcal{S} : \begin{cases} l_1\Delta q + l_2\Delta\gamma = 0 \\ l_3\Delta q + l_4\Delta\gamma = 0 \end{cases}$$

   where

$$\begin{cases} l_1 := -\beta(1-\gamma) \neq 0 \\ l_2 := (\hat{q}+\beta)(\hat{q}-1) \\ l_3 := -(1-\gamma)(1+\frac{\hat{q}}{q}) \neq 0 \\ l_4 := (\hat{q}^2 - 1). \end{cases}$$

   Studying the linear system $\mathcal{S}$ yields

$$d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) = 0 \Leftarrow: \mathcal{S} \quad \text{is satisfied} \quad \Longleftarrow: \begin{cases} \Delta\gamma = 0, \quad \Delta q = 0 \\ \text{(excluded because} \quad \hat{\gamma} < \gamma) \\ \text{or} \\ \hat{q} = q = 1 \end{cases}$$

   and we retrieve the symmetric case $\alpha = \frac{1}{2}$ excluded from our hypothesis.
2. $\hat{\gamma} > \gamma$;
   Once again integrating first for $x = \phi_{\hat{\alpha},\hat{\gamma}}(\hat{\gamma})$ and then for $x \geq \phi_{\hat{\alpha},\hat{\gamma}}(b)$ in (22). Both equations yield in this case

$$d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) = 0 \Leftarrow: \begin{cases} \Delta\gamma = 0, \quad \Delta q = 0 \quad \text{(excluded)} \\ \text{or} \\ \hat{q} = q = 1 \end{cases}$$

   and we retrieve the symmetric case excluded from our hypothesis.

3. $\hat{\gamma} = \gamma$;
   Choosing $x \geq \phi_{\hat{\alpha},\hat{\gamma}}(1)$ we see directly in this case that $d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}})$ is equal to 0 if and only if $\hat{\alpha} = \alpha$. □

3.1.4 Consistency

Lemma 3.2 *Suppose that $(\alpha, \gamma) \in \mathcal{K}_\eta$. We have that*

$$d(\hat{\mu}^T_{\bar{\alpha}^T,\bar{\gamma}^T}, U_{\bar{\alpha}^T,\bar{\gamma}^T}) \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s.}} 0.$$

Proof Let $(\hat{\alpha}, \hat{\gamma}) \in \mathcal{K}_\eta$. The definition of $\left(\bar{\alpha}^T, \bar{\gamma}^T\right)$ and the triangular inequality imply that

$$d(\hat{\mu}^T_{\bar{\alpha}^T,\bar{\gamma}^T}, U_{\bar{\alpha}^T,\bar{\gamma}^T}) \leq d(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}) \leq d(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, \mu_{\hat{\alpha},\hat{\gamma}}) + d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\alpha,\gamma}) + d(U_{\alpha,\gamma}, U_{\hat{\alpha},\hat{\gamma}}).$$

From Lemma 3.1, we have that

$$d(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, \mu_{\hat{\alpha},\hat{\gamma}}) \xrightarrow[T\to+\infty]{\mathbb{P}-\text{a.s.}} 0.$$

From (18) and the fact that $\phi_{\alpha,\gamma}(0) = \phi_{\hat{\alpha},\hat{\gamma}}(0)$, we have that for any $x \in \mathbb{R}$,

$$\begin{aligned}
&\left|F_{\mu_{\hat{\alpha},\hat{\gamma}}}(x) - F_{U_{\alpha,\gamma}}(x)\right| \\
&\leq c_{\alpha,\gamma} \int_{\phi_{\alpha,\gamma}(0)}^{\phi_{\hat{\alpha},\hat{\gamma}}(1)\vee\phi_{\alpha,\gamma}(1)} \left| \frac{\phi'_{\hat{\alpha},\hat{\gamma}}}{\phi'_{\alpha,\gamma}} \circ \phi^{-1}_{\hat{\alpha},\hat{\gamma}}(y) 1\!\!1_{[\phi_{\alpha,\gamma}(0),\phi_{\hat{\alpha},\hat{\gamma}}(1)]}(y) - 1\!\!1_{[\phi_{\alpha,\gamma}(0),\phi_{\alpha,\gamma}(1)]}(y)\right| dy \\
&\leq c_{\alpha,\gamma} \left\{ |\hat{q}(\hat{q}-1)||\gamma - \hat{\gamma}|\mathbb{I}_{\hat{\gamma}<\gamma} + \left|\frac{q-1}{q}\right| |\gamma - \hat{\gamma}|\mathbb{I}_{\gamma<\hat{\gamma}} \right. \\
&\left. + \left|\frac{\hat{q}}{q}\right| |b - \gamma \vee \hat{\gamma}||\hat{q} - q| + |1 - \gamma \wedge \hat{\gamma}||q - \hat{q}| + |\gamma - \hat{\gamma}| \right\} \\
&\leq C(|\hat{\alpha} - \alpha| + |\gamma - \hat{\gamma}|)
\end{aligned}$$

(where $c_{\alpha,\gamma} = \frac{1}{\phi_{\alpha,\gamma}(1)-\phi_{\alpha,\gamma}(0)} > 0$ and C is a positive constant). In particular,

$$\min_{(\hat{\alpha},\hat{\gamma})\in\mathcal{K}_\eta} d(\mu_{\hat{\alpha},\hat{\gamma}}, U_{\alpha,\gamma}) = d(\mu_{\alpha,\gamma}, U_{\alpha,\gamma}) = 0.$$

It is clear that

$$\min_{(\hat{\alpha},\hat{\gamma})\in\mathcal{K}_\eta} d(U_{\hat{\alpha},\hat{\gamma}}, U_{\alpha,\gamma}) = d(U_{\alpha,\gamma}, U_{\alpha,\gamma}) = 0.$$

Gathering these results together proves the lemma. □

Theorem 3.1 *Suppose that $(\alpha, \gamma) \in \mathcal{K}_\eta$.*
$(\bar{\alpha}^T, \bar{\gamma}^T)$ converges almost surely to (α, γ) as T tends to $+\infty$.

We begin with a continuity lemma :

Lemma 3.3 *Let $(T_n)_{n\in\mathbb{N}^*}$ be a strictly increasing sequence of real numbers such that $T_n \xrightarrow[n\to+\infty]{} \infty$.*

Suppose that a random sequence $(\alpha^{T_n}, \gamma^{T_n}) \in \mathcal{K}_\eta$ converges almost surely to (α', γ') as n tends to $+\infty$. We have that

$$d(\hat{\mu}^{T_n}_{\alpha^{T_n},\gamma^{T_n}}, U_{\alpha^{T_n},\gamma^{T_n}}) \xrightarrow[n\to+\infty]{\mathbb{P}-\text{a.s.}} d(\mu_{\alpha',\gamma'}, U_{\alpha',\gamma'}). \tag{23}$$

Proof We start from the inequality

$$\begin{aligned}
&|d(\hat{\mu}^{T_n}_{\alpha^{T_n},\gamma^{T_n}}, U_{\alpha^{T_n},\gamma^{T_n}}) - d(\mu_{\alpha',\gamma'}, U_{\alpha',\gamma'})| \\
&\quad \leq (\hat{\mu}^{T_n}_{\alpha^{T_n},\gamma^{T_n}}, \mu^{T_n}_{\alpha',\gamma'}) + d(\mu^{T_n}_{\alpha',\gamma'}, \mu_{\alpha',\gamma'}) + d(U_{\alpha',\gamma'}, U_{\alpha^{T_n},\gamma^{T_n}}) \\
&\quad = I_1^n + I_2^n + I_3^n.
\end{aligned}$$

From the convergence of the empirical measure w.r.t the supremum norm (Lemma 3.1), it is quite easy to see that I_2^n converges almost surely to 0 as n goes to infinity. The same holds for I_3^n. We only concentrate the proof on the first term I_1^n. We have to deal with

$$I_1^n := d(\hat{\mu}^{T_n}_{\alpha^{T_n},\gamma^{T_n}}, \mu^{T_n}_{\alpha',\gamma'}).$$

Let us fix $\epsilon > 0$. We introduce a mollifying function ζ_ϵ for the regularization of the indicator function. We have that

$$\begin{aligned}
I_1^n &= \sup_{x\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \mathbb{1}_{]-\infty,x[}(\phi_{\alpha^{T_n},\gamma^{T_n}}(X_s)) - \mathbb{1}_{]-\infty,x[}(\phi_{\alpha',\gamma'}(X_s))ds \right| \\
&\leq \sup_{x\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \zeta_\epsilon * \mathbb{1}_{]-\infty,x[}(\phi_{\alpha^{T_n},\gamma^{T_n}}(X_s)) - \zeta_\epsilon * \mathbb{1}_{]-\infty,x[}(\phi_{\alpha',\gamma'}(X_s))ds \right| \\
&\quad + \sup_{x\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \mathbb{1}_{]-\infty,x[}(\phi_{\alpha^{T_n},\gamma^{T_n}}(X_s)) - \zeta_\epsilon * \mathbb{1}_{]-\infty,x[}(\phi_{\alpha^{T_n},\gamma^{T_n}}(X_s))ds \right| \\
&\quad + \sup_{x\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \zeta_\epsilon * \mathbb{1}_{]-\infty,x[}(\phi_{\alpha',\gamma'}(X_s)) - \mathbb{1}_{]-\infty,x[}(\phi_{\alpha',\gamma'}(X_s))ds \right| \\
&:= J_\epsilon^{1,n} + J_\epsilon^{2,n} + J_\epsilon^{3,n}.
\end{aligned}$$

Because we are working with the L^∞ norm, it is easy to see that for all $n \in \mathbb{N}^*$:

$$\begin{aligned}
J_\epsilon^{2,n} + J_\epsilon^{3,n} &\leq 2\sup_{x\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \zeta_\epsilon * \mathbb{1}_{]-\infty,x[}(X_s) - \mathbb{1}_{]-\infty,x[}(X_s)ds \right| \\
&\leq 2\sup_{x\in\mathbb{R}} \frac{1}{T_n}\int_0^{T_n} \mathbb{1}_{]x,x+\epsilon[}(X_s)ds.
\end{aligned}$$

Then, from Lemma 3.1,

$$\sup_{x,\epsilon\in\mathbb{R}} \left| \frac{1}{T_n}\int_0^{T_n} \mathbb{1}_{]x,x+\epsilon[}(X_s)ds - \int \mathbb{1}_{]x,x+\epsilon[}(y)\nu(dy) \right| \xrightarrow[n\to+\infty]{\mathbb{P}-\text{a.s.}} 0.$$

Thus, there exists a possibly random N such that for any $n \geq N$,

$$J_\epsilon^{2,n} + J_\epsilon^{3,n} \leq C_\eta^{(1)}\epsilon$$

where $C_\eta^{(1)}$ is a uniform constant w.r.t n and ϵ but that may depend on the choice of η (recall that η is the bi-dimensional parameter that rules the quality of our compact estimation set $\mathcal{K}_\eta$).

Using the Lipschitz constant of the ϵ-regularization of the indicator function gives

$$\begin{aligned}
J_\epsilon^{1,n} &\leq \frac{1}{2\epsilon}\left|\phi_{\alpha^{T_n},\gamma^{T_n}}(X_s) - \phi_{\alpha',\gamma'}(X_s)\right| \\
&\leq \frac{C_\eta^{(2)}}{\epsilon}\left(\left|\alpha^{T_n} - \alpha'\right| + \left|\gamma^{T_n} - \gamma'\right|\right)
\end{aligned}$$

where $C_\eta^{(2)}$ is another constant that may depend on the choice of η.

In conclusion, we have for any $n \geq N$, for any $\epsilon > 0$

$$I_1^n \leq C_\eta^{(1)}\epsilon + \frac{C_\eta^{(2)}}{\epsilon}\left(\left|\alpha^{T_n} - \alpha'\right| + \left|\gamma^{T_n} - \gamma'\right|\right).$$

Thus, making ϵ depend on n so that $\epsilon_n := \sqrt{\left|\alpha^{T_n} - \alpha'\right| + \left|\gamma^{T_n} - \gamma'\right|}$, we clearly see that $I_1^n \xrightarrow[n\to+\infty]{\mathbb{P}-\text{a.s.}} 0$, and the lemma is proved. □

Proof (of Theorem 3.1)

Let $(T_n)_{n\in\mathbb{N}^*}$ be a strictly increasing sequence of real numbers such that $\lim_{n\to+\infty} T_n = +\infty$ and suppose that $(\bar{\alpha}^{T_n}, \bar{\gamma}^{T_n})$ does not converge almost surely to (α, γ) as n tends to $+\infty$. Since $\left\{(\bar{\alpha}^{T_n}, \bar{\gamma}^{T_n}) : n \in \mathbb{N}^*\right\}$ is a sequence in the compact set $\mathcal{K}_\eta$, this means that we may find a subsequence (still denoted abusively by $\left\{(\bar{\alpha}^{T_n}, \bar{\gamma}^{T_n}) : n \in \mathbb{N}^*\right\}$) converging towards some $(\alpha', \gamma') \neq (\alpha, \gamma)$. From Proposition 3.1, we have that $d(\mu_{\alpha',\gamma'}, U_{\alpha',\gamma'}) > 0$. From the result of Lemma 3.3,

$$d(\hat{\mu}^{T_n}_{\bar{\alpha}^{T_n},\bar{\gamma}^{T_n}}, U_{\bar{\alpha}^{T_n},\bar{\gamma}^{T_n}}) \xrightarrow[n\to+\infty]{\mathbb{P}-\text{a.s.}} d(\mu_{\alpha',\gamma'}, U_{\alpha',\gamma'}) > 0,$$

in contradiction with the result of Lemma 3.2. □

3.2 Estimation of asymmetry and locus parameters for general reflected skew processes

We now turn to the general case of skew diffusion processes. Let Y be the unique strong solution to the stochastic differential equation

$$Y_t = y + \int_0^t d(Y_s)\,ds + \int_0^t \sigma(Y_s)\,dB_s + (2\alpha - 1)L_t^{\tilde{\gamma}}(Y) - K_t^{[l,r]}(Y),$$
$$y \in [l, r], \alpha \in]0,\ 1[. \tag{24}$$

Our purpose is to estimate the skew coefficient α together with the locus $\tilde{\gamma}$ from a single observation of the process Y in the long run.

Let us assume that the assumptions of Sect. 2.3 are satisfied and remember the definition of the scale function s (for $\alpha = \frac{1}{2}$) given in (16).

Once again, it is possible to perform an invertible change of time $(T_t)_{t\geq 0}$ so that

$$s(Y_{T_t}) = X_t, \quad \mathbb{P} - \text{ps} \ \ \forall t \geq 0,$$

where X is a solution of (1) with $x := s(y), \gamma := s(\tilde{\gamma})$, and $[a, b] := [s(l), s(r)]$ (see Martinez 2004, Chap. 5 for details). Define

$$(\bar{\alpha}^T, \bar{\gamma}^T) := \arg\min_{(\hat{\alpha},\hat{\gamma})\in\mathcal{K}_\eta} d(\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}, U_{\hat{\alpha},\hat{\gamma}}). \tag{25}$$

where $\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}$ now denotes the empirical measure of the transformed process $\left\{\phi_{\hat{\alpha},\hat{\gamma}} \circ s(Y_t);\ \ t \geq 0\right\}$ s.t for any $A \in \mathcal{B}(\mathbb{R})$,

$$\hat{\mu}^T_{\hat{\alpha},\hat{\gamma}}(A) = \frac{1}{T}\int_0^T \mathbb{1}_A(\phi_{\hat{\alpha},\hat{\gamma}} \circ s(Y_s))ds.$$

Define $\tilde{\gamma}^T := s^{-1}(\bar{\gamma}^T)$. Our estimator of $(\alpha, \tilde{\gamma})$ is then given by $(\bar{\alpha}^T, \tilde{\gamma}^T)$.

We have the following Theorem :

Theorem 3.2 *Suppose that* $(\alpha, \tilde{\gamma}) \in \mathcal{K}_\eta$.
Then $(\bar{\alpha}^T, \tilde{\gamma}^T)$ *converges almost surely to* $(\alpha, \tilde{\gamma})$ *as* T *tends to* $+\infty$.

The proof follows from the case $\sigma \equiv 1$ and $d \equiv 0$. It relies on change of time techniques and the use of the scale function s.

4 Conclusion

4.1 Discussion upon our approach

It is worth pointing out that the measures of the processes $X(\alpha, \gamma)$ are singular for different values of α and γ. Thus, at least theoretically and in our continuous time setting, as soon as the trajectory of the process $X(\alpha, \gamma)$ hits the locus of asymmetry γ, estimators of the parameters with 'almost immediate' speed of convergence might be built. Nevertheless, in practice, one has to deal with discrete data. Thus, such estimators, which should be founded on the estimation of the local time at every point of the domain and the underlying excursion measure of the process, may behave very badly. Meanwhile, our approach is robust to discretization. Even if the long run is not theoretically necessary in this estimation problem, we think that such an asymptotic has to be considered for practical purposes.

4.2 Perspectives

This article is a first attempt to estimate the parameters of skewed processes. So far, we have provided a method to consistently estimate both the asymmetry coefficient and the locus of the asymmetry in a one dimensional setting. Much remains to do. Of course, the extension to the multidimensional setting is the great matter. But the lack of uniqueness of the notion of local time in higher dimension seems to forbid a general treatment of this question. Another path to explore is the study of kernel estimators to handle the functional case, where the asymmetry is defined by $\alpha(\gamma(t))L_t^{\gamma(t)}$ or $\alpha(L_t^\gamma)$ (see Barlow et al. 2000, for an introduction to such processes).

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