Rational dynamical systems on finite extensions of the field of $p$-adic numbers

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Introduction
I. The $p$-adic numbers

- $p \geq 2$ a prime number:
  \[ \forall n \in \mathbb{N}, \ n = \sum_{i=0}^{N} a_i p^i \ (a_i = 0, 1, \cdots, p-1). \]

- Ring $\mathbb{Z}_p$ of $p$-adic integers:
  \[ \mathbb{Z}_p \ni x = \sum_{i=0}^{\infty} a_i p^i. \]

- Field $\mathbb{Q}_p$ of $p$-adic numbers: fraction field of $\mathbb{Z}_p$:
  \[ \mathbb{Q}_p \ni x = \sum_{i=v(x)}^{\infty} a_i p^i, \ (\exists v(x) \in \mathbb{Z}). \]

Absolute value: $|x|_p = p^{-v(x)}$, metric: $d(x, y) = |x - y|_p$.

(So $\mathbb{Z}_p$ is the unit ball in $\mathbb{Q}_p$.)

- **ultrametric inequality**: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.
- **a fact**: $\overline{\mathbb{N}} = \mathbb{Z}_p$. 

Rational maps on finite extensions of the field of $p$-adic numbers
II. Arithmetic in $\mathbb{Q}_p$

Addition and multiplication: similar to the decimal way. "Carrying" from left to right.

Example: $x = (p - 1) + (p - 1) \times p + (p - 1) \times p^2 + \cdots$, then

- $x + 1 = 0$. So,

$$-1 = (p - 1) + (p - 1) \times p + (p - 1) \times p^2 + \cdots.$$

- $2x = (p - 2) + (p - 1) \times p + (p - 1) \times p^2 + \cdots$.

We also have substraction and division.

Then we can define polynomials and rational maps.

Remark: Development of numbers:

- $\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R}([-1, 1]) \rightarrow \mathbb{C}$

- $\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}_p(\mathbb{Z}_p) \rightarrow \mathbb{Q}_p^{a.c.} \rightarrow \mathbb{C}_p$
III. Projective line over $\mathbb{Q}_p$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}_p^2 \setminus \{(0,0)\}$, we say that $(x_1, y_1) \sim (x_2, y_2)$ if $\exists \lambda \in \mathbb{Q}_p^* \text{ s.t. } x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2$.

Projective line over $\mathbb{Q}_p$:

$$\mathbb{P}^1(\mathbb{Q}_p) := (\mathbb{Q}_p^2 \setminus \{(0,0)\}) / \sim$$

Spherical metric: for $P = [x_1, y_1], Q = [x_2, y_2] \in \mathbb{P}^1(\mathbb{Q}_p)$, define

$$\rho(P, Q) = \frac{|x_1y_2 - x_2y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}.$$

Viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \quad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1; \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$
Geometric representations of $\mathbb{P}^1(\mathbb{Q}_2)$ and $\mathbb{P}^1(\mathbb{Q}_3)$
IV. $p$-adic dynamical systems

A $p$-adic dynamical system is a couple $(X, f)$ where $X$ is a $p$-adic space and $f : X \to X$ is a transformation on $X$.

The beginning:
Oselies-Zieschang 1975: automorphisms of $\mathbb{Z}_p$,
Herman-Yoccoz 1983: complex $p$-adic dynamical systems,
Volovich 1987: $p$-adic string theory.

We are interested in the polynomials and rational maps considered as dynamical systems on $\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{P}^1(\mathbb{Q}_p)$.

As first investigations, we consider two families of dynamical systems:

1-Lipschitz dynamical systems and expanding dynamical systems.

Typical examples:
- $f(x) = x + 1$ on $\mathbb{Z}_p$ is minimal (every orbit is dense).
- $f(x) = \frac{x^p - x}{p}$ on $\mathbb{Z}_p$ is conjugate to the shift on $\{0, 1, \ldots, p - 1\}^\mathbb{N}$. 
V. 1-Lipschitz $p$-adic dynamical systems

Let $f \in \mathbb{Z}_p[x]$ be a polynomial of coefficients in $\mathbb{Z}_p$. Then it defines a dynamical system on $\mathbb{Z}_p$, denoted by $(\mathbb{Z}_p, f)$.

It is 1-Lipschitz and then equicontinuous.

The system $(X, T)$ is equicontinuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s. t. } d(T^n x, T^n y) < \epsilon \ (\forall n \geq 1, \forall d(x, y) < \delta).$$

**Theorem**

Let $X$ be a compact metric space and $T : X \to X$ be an equicontinuous transformation. Then the following statements are equivalent:

1. $T$ is minimal (every orbit is dense).
2. $T$ is ergodic with respect to the Haar measure.
VI. Expanding dynamical systems on $\mathbb{Q}_p$

- $f : X (\subset \mathbb{Q}_p) \to \mathbb{Q}_p$ is continuously differentiable at $a \in X$ if

$$\lim_{(x,y) \to (a,a), x \neq y} \frac{f(x) - f(y)}{x - y} =: f'(a) \text{ exists.}$$

- is analytic if $f$ can be written as a power series

$$f(x) = \sum_{n \geq 0} a_n (x - x_0)^n, \text{ with } x_0 \in X, \quad \left( f'(x) = \sum_{n \geq 1} na_n (x - x_0)^{n-1} \right).$$

**Lemma (Local rigidity lemma)**

Let $U$ be a clopen (close and open) set and $a \in U$. Suppose $f : U \to \mathbb{Q}_p$ is continuously differentiable, and $f'(a) \neq 0$. Then there exists $r > 0$ such that $B_r(a) \subset U$ and

$$\forall x, y \in B_r(a), \quad |f(x) - f(y)|_p = |f'(a)|_p |x - y|_p.$$

**Remark**: If further $f$ is analytic, the above $r > 0$ can be estimated by

$$\left| \frac{f^{(k)}(x_0)}{k!} \right|_p r^{k-1} < |f'(x_0)|_p, \quad \forall k \geq 2, \forall x, y \in D(x_0, r).$$
1-Lipschitz dynamical systems in $\mathbb{Q}_p$
I. Polynomial dynamical systems on $\mathbb{Z}_p$

Theorem (Ai-Hua Fan, L.; 2011) minimal decomposition

Let $f \in \mathbb{Z}_p[x]$ with $\deg f \geq 2$. The space $\mathbb{Z}_p$ can be decomposed into three parts:

$$\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where

- $\mathcal{P}$ is the finite set consisting of all periodic orbits;
- $\mathcal{M} := \bigsqcup_{i \in I} \mathcal{M}_i$ ($I$ finite or countable)
  - $\mathcal{M}_i$ : finite union of balls,
  - $f : \mathcal{M}_i \to \mathcal{M}_i$ is minimal;
- $\mathcal{B}$ is attracted into $\mathcal{P} \sqcup \mathcal{M}$.

**Remark**: Similar decomposition theorem holds for

- power series in a finite extension of $\mathbb{Q}_p$ (S.Fan–L 2015)
Two typical decompositions of $\mathbb{Z}_p$
II. One application in Number Theory

Proposition (Fan-Li-Yao-Zhou 2007)

Let \( k \geq 1 \) be an integer, and let \( a, b, c \) be three integers in \( \mathbb{Z} \) coprime with \( p \geq 2 \). Let \( s_k \) be the least integer \( \geq 1 \) such that \( a^{s_k} \equiv 1 \pmod{p^k} \).

(a) If \( b \not\equiv a^j c \pmod{p^k} \) for all integers \( j \) \( (0 \leq j < s_k) \), then \( p^k \nmid (a^n c - b) \), for any integer \( n \geq 0 \).

(b) If \( b \equiv a^j c \pmod{p^k} \) for some integer \( j \) \( (0 \leq j < s_k) \), then we have

\[
\lim_{N \to +\infty} \frac{1}{N} \text{Card}\{1 \leq n < N : p^k \mid (a^n c - b)\} = \frac{1}{s_k}.
\]

Remark: Consider \( T : x \mapsto ax \). Then

\[
p^k \mid (a^n c - b) \iff |T^n(c) - b|_p \leq p^{-k} \iff T^n(c) \in \overline{B}(b, p^{-k}).
\]

Coelho and Parry 2001: Ergodicity of \( p \)-adic multiplications and the distribution of Fibonacci numbers.
Expanding dynamical systems in $\mathbb{Q}_p$
I. \( p \)-adic repeller

- \( f : X \to \mathbb{Q}_p, \ X \subset \mathbb{Q}_p \) compact open.
- Assume that
  1. \( f^{-1}(X) \subset X \);
  2. \( X = \bigsqcup_{i \in I} B_{p^{-\tau}}(c_i) \) (with some \( \tau \in \mathbb{Z} \)), \( \forall i \in I, \ \exists \tau_i \in \mathbb{Z} \) s.t.

\[
|f(x) - f(y)|_p = p^{\tau_i} |x - y|_p \quad (\forall x, y \in B_{p^{-\tau}}(c_i)). \tag{1}
\]

- Define **Filled Julia set** :

\[
J_f = \bigcap_{n=0}^{\infty} f^{-n}(X).
\]

We have \( f(J_f) \subset J_f \). \((X, J_f, f)\) is called

\( \rightarrow \) a **\( p \)-adic weak repeller** if all \( \tau_i \geq 0 \) in (1), but at least one \( \tau_i > 0 \).

\( \rightarrow \) a **\( p \)-adic repeller** if all \( \tau_i > 0 \) in (1).
II. Description by subshift of finite type

- For any $i \in I$, let
  
  \[ I_i := \{ j \in I : B_j \cap f(B_i) \neq \emptyset \} = \{ j \in I : B_j \subset f(B_i) \}. \]

- Define $A = (A_{i,j})_{I \times I}$:
  
  \[ A_{ij} = 1 \text{ if } j \in I_i; \quad A_{ij} = 0 \text{ otherwise}. \]

- If $A$ is irreducible, we say that $(X, J_f, f)$ is **transitive**.

- Let $(\Sigma_A, \sigma)$ be the corresponding subshift.


Let $(X, J_f, f)$ be a transitive $p$-adic weak repeller with matrix $A$. Then the dynamics $(J_f, f)$ is topologically conjugate to the shift dynamics $(\Sigma_A, \sigma)$. 
III. Examples and applications

Let $a \in \mathbb{Z}_p$, $a \equiv 1 \pmod{p}$ and $m \geq 1$ be an integer. Consider $f_{m,a} : \mathbb{Q}_p \to \mathbb{Q}_p$:

$$f_{m,a}(x) = x^p - ax \frac{p^m}{p^m}.$$

- $I_{m,a} = \{0 \leq k < p^m : k^p - ak \equiv 0 \pmod{p^m}\}$
- $X_{m,a} = \bigcup_{k \in I_{m,a}} (k + p^m \mathbb{Z}_p)$, $J = \bigcap_{n=0}^{\infty} f_{m,n}^{-n}(X)$.


$(J, f_{m,a})$ is conjugate to $(\{0, \ldots, p - 1\}^\mathbb{N}, \sigma)$.

**Theorem (Woodcock and Smart 1998)**

$(J, f_{1,1})$ is conjugate to $(\{0, \ldots, p - 1\}^\mathbb{N}, \sigma)$.

Rational maps without wildly recurrent Julia critical point
I. Julia sets from the big space to the small space

For a dynamical system \((X, f)\), the **Fatou set** is the set of points in \(X\) having a neighborhood on which \(\{f^n\}_{n=1}^{\infty}\) is equicontinuous. and the **Julia set** \(J_X(f)\) is the complement of of \(F_X(f)\).

- \(K\) : a finite extension of the field \(\mathbb{Q}_p\)
- \(\mathbb{C}_p\) : the field of \(p\)-adic complex numbers.
- \(f \in K(z)\) : a rational map (which induces dynamical systems on the projective space \(\mathbb{P}^{1}_{\mathbb{C}_p}\) and \(\mathbb{P}^{1}_K\)).

Let \(z_0\) be a critical point of \(f\) \((f'(z_0) = 0)\). It is called
  - **wild** if \(p\) divides its local degree \(d\) (locally \(f(z) = (z - z_0)^d g(z)\) with \(g(z_0) \neq 0\)),
  - **recurrent** if it is contained in its \(\omega\)-limit set.

**Theorem 1 (S.Fan–L–Nie–Wang, in preparation)**

Let \(f \in K(z)\) be a rational map of degree at least 2. Suppose \(f\) has no wildly recurrent Julia critical points in \(\mathbb{P}^{1}_K\). Then

\[
J_K(f) = J_{\mathbb{C}_p}(f) \cap \mathbb{P}^{1}_K.
\]
II. Dynamics on the Julia set

Denote by $\text{Crit}_J(f)$ the set of the critical points in $J_K(f)$. A rational map $f \in K(z)$ is \textit{geometrically finite} if every critical point in $\text{Crit}_J(f)$ has finite forward orbit.

Denote by

$$\text{GO}(\text{Crit}_J(f)) := \{y \in K : \exists x \in \text{Crit}_J(f), \exists m, n \in \mathbb{N}, \text{s.t. } f^n(y) = f^m(x)\}$$

the \textit{grand orbit} of the critical points in $\text{Crit}_J(f)$ and write

$$I_K(f) := J_K(f) \setminus \text{GO}(\text{Crit}_J(f)).$$

\textbf{Theorem 2 (S. Fan–L–Nie–Wang, in preparation)}

Let $f \in K(z)$ be a geometrically finite rational map of degree at least two. Then there exist a \textbf{countable states Markov shift} $(\Sigma_A, \sigma)$ and a bijection $h : J_K(f) \rightarrow \Sigma_A$ such that $(I_K(f), f)$ is topologically conjugate to $(h(I_K(f)), \sigma)$ via $h$. 
III. Tools for the proof of Theorem 1

Rivera-Letelier 2003 (Classification theorem)

Let $U$ be a periodic Fatou component of a rational map of degree at least 2. Then $U$ is
- either attracting,
- or indifferent.

Benedetto 2000 (No wandering domain)

Let $\phi \in \overline{\mathbb{Q}}_p(z)$ be a rational map with no wildly critical recurrent Julia points. Then $\phi$ has no wandering domains in $\mathbb{P}^1_{\mathbb{C}_p}$.

Benedetto 2000 (Finite period theorem)

Let $\phi \in \overline{\mathbb{Q}}_p(z)$ be a rational map with no wildly critical recurrent Julia points. Then for any finite extension $K$ of $\mathbb{Q}_p$ which contains the coefficients of $\phi$, the number of periodic Fatou components in $\mathbb{P}^1_{\mathbb{C}_p}$ which contains points in $K$ is finite.
IV. Ideas of the proof for Theorem 1

By definition, \( J_K(\phi) \subset J_{\mathbb{C}p}(\phi) \). We need only to prove \( J_{\mathbb{C}p}(\phi) \subset J_K(\phi) \).

We distinguish the following two cases:

1. \((\mathcal{O}_\phi(x))' \cap \text{Crit}_{\mathbb{C}p}(\phi) = \emptyset\) : we apply the following lemma

**Lemma 1**

Let \( \phi \in K(z) \) be a rational map of degree \( \deg \phi \geq 2 \). For a point \( x \in \mathbb{P}^1_K \), assume that \( \mathcal{O}_\phi(x) \cap \text{Crit}_{\mathbb{C}p}(\phi) = \emptyset \). Then \( x \in J_{\mathbb{C}p}(\phi) \) if and only if there exists \( \epsilon_0 > 0 \) such that for any \( y \in \mathbb{P}^1_{\mathbb{C}p} \) such that \( x \neq y \),

\[
\sup_{n \geq 0} \rho(\phi^n(x), \phi^n(y)) > \epsilon_0.
\]

2. \((\mathcal{O}_\phi(x))' \cap \text{Crit}_{\mathbb{C}p}(\phi) \neq \emptyset\) : we need the following lemma whose proofs are based on the tools in the precedent slide.

**Lemma 2**

Let \( \phi \in K(z) \) be a rational map of degree at least 2 with no recurrent critical Julia point. If \( x \in \partial J_{\mathbb{C}p}^K(\phi) \), then \( x \in J_K(\phi) \).
An example:

\[ x \mapsto \frac{9}{4}x(x - 1)^2 \text{ on } \mathbb{P}^1(\mathbb{Q}_2) \]
1. Fixed points and critical orbits

Consider $f : \mathbb{P}^1_{\mathbb{Q}_2} \to \mathbb{P}^1_{\mathbb{Q}_2}$ given by

$$f(x) = \frac{9}{4}x(x - 1)^2.$$ 

- Fixed points: 0, 1/3 and $\infty$.
- Critical points: 1, 1/3 and $\infty$.
- 1/3 and $\infty$ are the super-attracting.
- 0 is the repelling fixed point with multiplier 9/4.

We have the following critical portrait of $f$:

\[
\begin{array}{c}
0 \quad \xrightarrow{2:1} \quad \frac{1}{3} \quad \xleftarrow{1:1} \quad 1:1 \quad \xrightarrow{2:1} \quad \infty \quad \xleftarrow{3:1}
\end{array}
\]

**Remark**: the map $f$ is the unique cubic postcritically finite polynomial with rational coefficients such that the corresponding $p$-adic Julia set contains a critical point.
II. Basic facts of the map $x \mapsto \frac{9}{4} x(x - 1)^2$

For a point $a \in F(f)$, denote $\Omega_a$ the Fatou component of $f$ containing $a$. For $a = \infty$, we have

- $\mathbb{P}^1(\mathbb{Q}_2) \setminus \mathbb{Z}_2 \subset \Omega_\infty$.
- $f(2 + 4\mathbb{Z}_2) \subset \Omega_\infty$.
- $1 + 2 + 4\mathbb{Z}_2 \subset \Omega_{1/3}$.
- $2 + 4\mathbb{Z}_2 \cup 1 + 2 + 4\mathbb{Z}_2 \subset F(f)$.

The map $f$ is a scaling on $4\mathbb{Z}_2$ with scaling ratio 4. Moreover, for $n \geq 1$ and $a, b \in \{0, 1\}$,

- $f(2^{n+1}\mathbb{Z}_2) = 2^{n-1}\mathbb{Z}_2$.
- $f(2^{n+1} + 2^{n+2}\mathbb{Z}_2) = 2^{n-1} + 2^n\mathbb{Z}_2$.
- $f(2^{n+1} + a2^{n+2} + 2^{n+3}\mathbb{Z}_2) = 2^{n-1} + a2^n + 2^{n+1}\mathbb{Z}_2$.
- $f(2^{n+1} + a2^{n+2} + b2^{n+3} + 2^{n+4}\mathbb{Z}_2) = 2^{n-1} + a2^n + b2^{n+1} + 2^{n+2}\mathbb{Z}_2$. 
III. Basic facts of the map $x \mapsto \frac{9}{4}x(x - 1)^2$, continued

For $n \geq 1$,

- $f(2^{2n} + 2^{2n+3}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n+1}\mathbb{Z}_2$. In particular,
  
  $f(2^2 + 2^5\mathbb{Z}_2) = \{1\} \cup \bigcup_{m \geq 2} (1 + 2^{m+1} + 2^{m+3}\mathbb{Z}_2 \cup 1 + 2^{m+1} + 2^{m+2} + 2^{m+3}\mathbb{Z}_2)$.

- $f(2^{2n} + 2^{2n+2} + 2^{2n+3}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n} + 2^{2n+1}\mathbb{Z}_2$.

- $f(2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = 2^{2n-1} + 2^{2n}\mathbb{Z}_2$. In particular,
  
  $f^{n+1}(2^{2n-1} + 2^{2n}\mathbb{Z}_2) \subset \Omega_{\infty}$.

- $f(2^{2n} + 2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n-1} + 2^{2n}\mathbb{Z}_2$. In particular,
  
  $f^n(2^{2n} + 2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = \Omega_{1/3}$.

- $f(1 + 2^n\mathbb{Z}_2) \subset 2^{2n-2}\mathbb{Z}_2$.

- $f(1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2) = 2^{2n} + 2^{2n+3}\mathbb{Z}_2$. In particular,
  
  $f$ is a scaling on $1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2$ with scaling ratio $2^{-n}$.

- $f(1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2) = 2^{2n} + 2^{2n+3}\mathbb{Z}_2$. In particular,
  
  $f$ is a scaling on $1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2$ with scaling ratio $2^{-n}$. 

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IV. Fatou set and Julia set of the map $x \mapsto \frac{9}{4}x(x - 1)^2$

**Proposition**

The Julia set of $f$ is

$$J(f) = \bigcap_{n \geq 1} f^{-n}(4\mathbb{Z}_2 \cup 1 + 4\mathbb{Z}_2).$$

**Collorary**

The Fatou set of $f$ is

$$F(f) = \bigcup_{n \geq 0} f^{-n}(\Omega_\infty \cup \Omega_{1/3}).$$
IV. Fatou set and Julia set of the map \( x \mapsto \frac{9}{4}x(x - 1)^2 \)

For \( n \geq 1 \), define

\[
    A_n := 2^{2n} + 2^{2n+3}\mathbb{Z}_2, \\
    A'_n := 2^{2n} + 2^{2n+2} + 2^{2n+3}\mathbb{Z}_2, \\
    B_n := 1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2, \\
    B'_n := 1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2.
\]

Then under \( f \), we have the following portrait:
V. Countable states Markov shift

Set

\[ \alpha_n := A_n \cap J(f), \quad \alpha'_n := A'_n \cap J(f), \quad \beta_n := B_n \cap J(f), \quad \beta'_n := B'_n \cap J(f), \]

and define

\[ \alpha_{\infty} := \{0\} \quad \text{and} \quad \beta_{\infty} := \{1\}. \]

Denote

\[ \mathcal{A} := \{\alpha_{\infty}, \beta_{\infty}, \alpha_1, \alpha'_1, \beta_1, \beta'_1, \ldots, \alpha_n, \alpha'_n, \beta_n, \beta'_n, \ldots\}. \]

Let the matrix \( A = (A_{\gamma_i, \gamma_j})_{\gamma_i, \gamma_j \in \mathcal{A}} \) be given by \( A_{\gamma_i, \gamma_j} = 1 \) if \( \gamma_j \subset f(\gamma_i) \), and otherwise, \( A_{\gamma_i, \gamma_j} = 0 \).

Note that

\[ f(\alpha_1) = \beta_{\infty} \cup \bigcup_{n \geq 2} \{\beta_n, \beta'_n\}. \]

and

\[ f(\beta_{\infty}) = \alpha_{\infty}, \quad \text{and} \quad f(\alpha_{\infty}) = \alpha_{\infty}. \]
The matrix $A$ has a unique irreducible component $A'$. It corresponds to the symbol set $A' = \{\alpha_1, \alpha_2, \beta_2, \beta'_2, \ldots, \alpha_n, \beta_n, \beta'_n, \ldots\}$. The corresponding subsystem $(\Sigma'_{A'}, \sigma')$ has the following matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \beta_2 & \beta'_2 & \alpha_3 & \beta_3 & \beta'_3 & \alpha_4 & \beta_4 & \beta'_4 & \alpha_5 & \beta_5 & \beta'_5 & \ldots \\
\alpha_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta'_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\alpha_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta'_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\alpha_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta'_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\alpha_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\beta'_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]
VII. The Gurevich entropy

The Gurevich entropy $h_G(\Sigma_B, \sigma)$ is defined to be the $\sup h_{top}(\Sigma'_B, \sigma)$, where the supremum is over all subsystems $(\Sigma'_B, \sigma)$ formed by restricting to a finite subset of symbols.

Suppose $B$ is irreducible. For any $\gamma_{i_0} \in \Sigma$, a first-return loop of length $n \geq 1$ at $\gamma_{i_0}$ is a path $\{\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_n}\}$ such that $\gamma_{i_0} = \gamma_{i_n}$, $\gamma_{i_k} \neq \gamma_{i_0}$ for $1 \leq k \leq n - 1$ and $B_{\gamma_{i_k}, \gamma_{i_{k+1}}} = 1$ for $0 \leq k \leq n - 1$. Denote $\delta_{\gamma_{i_0}}(n)$ the number of first return loops at $\gamma_{i_0}$ of length $n$, and set

$$S_{\gamma_{i_0}}(z) = \sum_{n \geq 1} \delta_{\gamma_{i_0}}(n) z^n.$$

If $1 - S_{\gamma_{i_0}}(z)$ has a real root $R > 0$ such that $S_{\gamma_{i_0}}(z)$ converges and is not 1 on $|z| < R$, then $h_G(\Sigma_B, \sigma) = -\log R$. (If such $R$ exists, then it is independent of $\gamma$.)

In our case,

$$S_{\alpha_1}(z) = \sum_{n \geq 3} 2z^n = \frac{2z^3}{1 - z},$$

and

$$h_G(\Sigma'_A, \sigma') = -\log R \approx 0.22933.$$