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0.1 Multifractal analysis of (multiple) ergodic averages

For a dynamical system, we are interested in the multifractal analysis of the Birkhoff ergodic averages. Let \((X, \mathcal{B}, \mu, T)\) be an ergodic dynamical system, where \(\mathcal{B}\) is a sigma algebra on the space \(X\), \(T\) is a transformation measurable and invariant ergodic with respect to \(\mu\). In 1931, Birkhoff [Bir31] showed that for such a dynamical system, for any integrable function \(\varphi : X \to \mathbb{R}\), the ergodic averages

\[
A_n(\varphi, x) = \frac{\varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x)}{n},
\]

converge to \(\int_X \varphi(x) d\mu(x)\) for \(\mu\)-almost all \(x \in X\). The non-\(\mu\)-typical points having different limit properties for their ergodic averages, which are not considered in Birkhoff’s theorem, are interesting objects for multifractal analysis. Suppose that \(X\) is a metric space, the aim of multifractal analysis of Birkhoff’s ergodic averages is to calculate the Hausdorff dimensions of the level sets:

\[
\left\{ x \in X : \lim_{n \to \infty} A_n(\varphi, x) = \alpha \right\}, \quad \alpha \in \mathbb{R}.
\]

Such multifractal analysis dates back to a work of Besicovitch [B35] on the Hausdorff dimension of the sets of numbers whose frequency of digits in their dyadic expansions are prescribed. In fact, the result of [B35] can be viewed as a multifractal analysis of the Birkhoff averages of the indicator functions for the doubling map. Besicovitch’s work was significantly generalized by Eggleston [E49] and many others [Caj81, Dur97, Oli98, Oli00, Ols02, Ols03b, OW03, PS07, Vol58], for finite symbolic dynamical systems and interval maps with finitely many branches. However, the corresponding results for infinite symbolic dynamical systems and interval maps with infinitely many branches, like Gauss dynamical system associated to continued fractions, are quite few. In continued fractions, the multifractal analysis of the potential \(\log |T'|\) (with \(T\) being the Gauss dynamical system), whose Birkhoff ergodic averages are Lyapunov exponents, had been partially done by Pollicott and Weiss [PW99]. This analysis was later completed by our work


and independently by Kesseböhmer and Stratmann [KS07]. In [FLWW09], we have also done the multifractal analysis for the potential \(\log a_1\) (\(a_1\) being the function giving the first partial quotient of the continued fraction) whose Birkhoff ergodic averages are Khintchine exponents. The method of [FLWW09], which uses the Ruelle-Perron-Frobenius transfer operator theory to construct measures supporting on the level sets, can be applied to other piecewise continuous potentials.

On the other hand, in Gauss dynamical system, the multifractal analysis of a \(\mathbb{R}^N\)-valued potential : \(\varphi = (\mathbf{1}_{[1]}, \mathbf{1}_{[2]}, \ldots)\), where \(\mathbf{1}_{[j]}\) stands for the indicator function of the cylinder \([j]\) (i.e., the fundamental interval \([1/(j + 1), 1/j]\)) was done in


The corresponding level set is the set of numbers for which the frequency of the digits is prescribed in its continued fraction expansion. In fact, let \((p_k)_{k \geq 1}\) be a given probability vector,
i.e., \( p_k > 0 \) and \( \sum_{k \geq 1} p_k = 1 \), in [FLM10], we proved that the set of numbers whose frequency of the digit \( k \geq 1 \) in its continued fraction expansion is \( p_k \) is of Hausdorff dimension

\[
\max \left\{ \frac{1}{2} \sup \left\{ \frac{h_\mu}{2 \int \log x \, d\mu} : \mu \text{ is ergodic}, \mu\left(\left(\frac{1}{k+1}, \frac{1}{k}\right)\right) = p_k \right\} \right\},
\]

where \( h_\mu \) is the entropy of \( \mu \). Our work completes the work of Billingsley and Henningsen in 1975 where an upper bound of the Hausdorff dimension was obtained.


to \( \mathbb{R}^N \)-valued piecewise continuous potentials \( \varphi = (\varphi_1, \varphi_2, \cdots) \) in the interval map \( T \) of infinitely many branches which is defined on a collection of basic intervals \( \{I_i\}_{i \in \mathbb{N}} \), with \( \Lambda \) being its repeller. Among others, we show that if \( \varphi_i : X \to \mathbb{R} \) are all bounded then for all

\[
\gamma \in Z_0 = \left\{ \gamma \in \mathbb{R}^N : \exists \mu \text{ invariant}, \forall i \in \mathbb{N}, \int \varphi_i d\mu = \gamma_i \right\},
\]

the Hausdorff dimension of the level set

\[
\Lambda_\gamma = \left\{ x \in \Lambda : \forall i \in \mathbb{N}, \lim_{n \to \infty} A_n(\varphi_i, x) = \gamma_i \right\}, \quad \gamma = (\gamma_i)_{i \geq 1} \in \mathbb{R}^N,
\]
is given by

\[
\max \left\{ \inf \left\{ s \geq 0 : \sum_{i \in \mathbb{N}} \text{diam}(I_i)^s < \infty \right\}, \sup_{\mu \in \mathcal{M}(T)} \left\{ \frac{h_\mu}{\lambda_\mu} : \int \varphi d\mu = \gamma_i \forall i \in \mathbb{N}, h(\mu) < \infty \right\} \right\},
\]

where \( h_\mu \) and \( \lambda_\mu \) are the entropy and the Lyapunov exponent of the measure \( \mu \).

In continued fractions, the Birkhoff sum \( S_n(\varphi, x) := \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x) \) may go to infinity very fast. We are thus also interested in the points whose increasing rate of their Birkhoff sums is prescribed. We consider in [LR16a] L.M. Liao and M. Rams, Subexponentially increasing sum of partial quotients in continued fraction expansions, Math. Proc. Camb. Phil. Soc., 160 (2016), 401–412.

the set of such points for the potential \( \varphi = a_1 \). We show that

\[
\forall \gamma \in [1/2, 1), \quad \dim_H \left\{ x \in (0, 1) : \lim_{n \to \infty} \frac{a_1(x) + \cdots + a_n(x)}{\exp(n\gamma)} = 1 \right\} = \frac{1}{2}.
\]

Our work completes the works of Wu–Xu [WX11], and Xu [Xu08] who have studied the case where the increasing rate \( \exp\{n\gamma\} \) in the above formula is replaced by \( n^\gamma \) (\( \gamma > 1 \)), \( \exp\{n\gamma\} \) (\( \gamma \in (0, 1/2) \)) and \( \exp\{n\gamma\} \) (\( \gamma \geq 1 \)).

Consider a metric space \( X \) and a transformation \( T \) on it. Let \( f_1, \cdots, f_\ell \) (\( \ell \geq 2 \)) be \( \ell \) bounded functions from \( X \) to \( \mathbb{R} \). The multiple ergodic averages

\[
M_n(f_1, f_2, \cdots, f_\ell, x) := \frac{1}{n} \sum_{k=1}^n f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x)
\]

were well studied by Furstenberg [F77], Conze–Lesigne [CL84], Bourgain [Bou90], Host–Kra [HK05], Bergelson–Host–Kra [BHK05], et al. We study these multiple ergodic averages from a
multifractal analysis point of view. We are interested in the Hausdorff dimension of the level sets:

\[ \left\{ x \in X : \lim_{n \to \infty} M_n(f_1, f_2, \ldots, f_\ell, x) = \alpha \right\}, \quad \alpha \in \mathbb{R}. \]

As a first example, the full shift on \( \mathbb{D} = \{+1, -1\}^\mathbb{N} \) was studied in


We show

\[ \dim_H \left\{ x \in \mathbb{D} : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\} = 1 - \frac{1}{\ell} + \frac{1}{\ell} H \left( \frac{1 + \theta}{2} \right), \quad \theta \in [-1, 1], \]

where \( H(t) = -t \log_2 t - (1 - t) \log_2 (1 - t) \) is the entropy function. Our result was later developed to the full shift on \( \{0, 1\}^\mathbb{N} \) by Kenyon–Peres–Solomyak [KPS11, KPS12], Peres–Solomyak [PS11], Fan–Schmeling–Wu [FSW11, FSW16], Peres–Schmeling–Seuret–Solomyak [PSSS14] et al.

We further study in


some special multiple ergodic averages of a piecewise linear map \( T \) defined from two disjoint subintervals \( I_0, I_1 \subset [0, 1] \) onto \([0, 1] \), with slopes \( e^{\lambda_0} \) and \( e^{\lambda_1} \) \((\lambda_i > 0)\). We show that for all \( \theta \in [0, 1] \)

\[ \dim_H \left\{ x \in [0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_{I_{i_1}}(T^k x) 1_{I_{i_2}}(T^{2k} x) = \theta \right\} = \frac{\theta \log \frac{p(1-q)}{(1-p)q} - 2 \log(1-p)}{2\lambda_0}, \]

where \((p, q) \in [0, 1]^2\) is the unique solution of the system

\[ \begin{align*}
\theta(\lambda_1 - \lambda_0) \log \frac{p(1-q)}{(1-p)q} + \lambda_0 \log \frac{p^2(1-q)}{1-p} - 2\lambda_1 \log(1-p) &= 0, \\
2pq &= \theta(2 + p - q).
\end{align*} \]

The results of [LR13a] were generalized in


to linear Cookie-Cutter dynamical systems, i.e., piecewise linear map defined from \( m \) disjoint subintervals \( I_0, \ldots, I_{m-1} \subset [0, 1] \) onto \([0, 1] \), with slopes \( e^{\lambda_0}, \ldots, e^{\lambda_{m-1}} \) \((\lambda_i > 0)\). Let \( \phi \) be a function defined on \([0, 1]^2\) taking real values. Assume that \( \phi \) is locally constant in the sense that \( \phi \) is constant on each hyper-rectangle \( I_{i_1} \times I_{i_2} \) \((0 \leq i_1, i_2 \leq m - 1)\). We show that the Hausdorff dimensions of the level set \((q \geq 2 \text{ is an integer})\)

\[ \left\{ x \in [0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi(T^k x, T^{kq} x) = \alpha \right\}, \]
is the second coordinate \( r \) of the unique solution to the system
\[
\begin{aligned}
P(s, r) &= \alpha s \\
\partial_s P(s, r) &= \alpha,
\end{aligned}
\]
where \( P(s, r) \) is the pressure function associated to a non-linear Ruelle operator.

We can also study simple and multiple ergodic averages at the same time. This involves calculating the Hausdorff dimension of an intersection of a level set of simple ergodic averages and a level set of multiple ergodic averages. As a first attempt in this direction, in [LR21] L.M. Liao and M. Rams, *Normal sequences with given limits of multiple ergodic averages*, Publicacions Matemàtiques, 65(1), (2021), 271 - 290.

we obtain the Hausdorff dimension of the set of normal sequences having a given limit of multiple ergodic means. Precisely, let \( \Sigma = \{0, 1\}^N \). A sequence \( \omega \in \{0, 1\}^N \) is called normal if for any \( n \in \mathbb{N} \), each word in \( \{0, 1\}^n \) has frequency \( 1/2^n \). Denote by \( \mathcal{N} \) the set of all normal sequences. Let
\[
A_\alpha := \{ (\omega_k)_{k=1}^\infty \in \Sigma : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \omega_k \omega_{2k} = \alpha \} \quad (\alpha \in [0, 1]).
\]
We prove that if \( \alpha \leq 1/2 \), then
\[
\dim_H(\mathcal{N} \cap A_\alpha) = \frac{1}{2} + \frac{1}{2} H(2\alpha),
\]
where \( H(t) = -t \log_2 t - (1-t) \log_2 (1-t) \); and if \( \alpha > 1/2 \), the set \( \mathcal{N} \cap A_\alpha \) is empty.

0.2 Diophantine approximation

The classical Diophantine approximation is the study of the approximation to a real number by rational numbers. In 1842, Dirichlet [D1842] demonstrated his fundamental theorem in Diophantine approximation: for every real number \( \theta \), for all real number \( Q \geq 1 \), there exists a rational \( p/q \) such that \( 1 \leq q \leq Q \) and \( |\theta - p/q| < 1/qQ \). As a corollary, for every real number \( \theta \), there exist infinitely many rational numbers \( p/q \) such that \( |\theta - p/q| < 1/q^2 \). Following Waldschmidt [Wal12], we call the approximation property in the Dirichlet theorem uniform approximation and that in its corollary asymptotic approximation.

We are interested in the sets of numbers which are approached in uniform or asymptotic way by rational numbers or by an orbit of a dynamical system with a given speed. In the literature, much more attention are paid in the asymptotic approximation. Denote by \( \|x\| \) the distance of \( x \) to the nearest integer. In 1924, Khinchine [K24] proved that for a continuous function \( \Phi : \mathbb{N} \to \mathbb{R}^+ \), if \( x \mapsto x \Phi(x) \) is non-increasing, then the set
\[
\{ \theta \in \mathbb{R} : \|n\theta\| < \Phi(n) \text{ for infinitely many } n \}
\]
has Lebesgue measure zero if the series \( \sum \Phi(n) \) converges and has full Lebesgue measure otherwise. The expected similar result by deleting the non-increasing condition on \( \Phi \) is the famous Duffin–Schaeffer conjecture [DS41]. One could find the recent progresses towards this conjecture in Haynes–Pollington–Velani [HPV12] and Beresnevich–Harman–Haynes–Velani [BHHV13]. The first result concerning Hausdorff dimension on the asymptotic approximation is due to Jarník [J29] who proved in 1929 that the set
\[
\{ \theta \in \mathbb{R} : \|n\theta\| < n^{-\tau} \text{ for infinitely many } n \}, \quad (\tau > 1),
\]
has Hausdorff dimension $2/(\tau + 1)$.

In 1907 Minkowski [M1907] proved that for any irrational $\theta$, for any real $y$ which is not equal to any $m\theta + \ell$ with $m, \ell \in \mathbb{N}$, there exist infinitely many integers $n$ such that $\|n\theta - y\| < \frac{1}{n^\tau}$. Such asymptotic approximation concerning an inhomogeneous term $y$ is called inhomogeneous Diophantine approximation. The inhomogeneous analogous of Khintchine’s theorem on the Lebesgue measure of the set

$$L_\star(y) := \{ \theta \in \mathbb{R} : \|n\theta - y\| < n^{-\tau} \text{ for infinitely many } n \}, \quad (\tau > 1),$$

has Hausdorff dimension $2/(\tau + 1)$ which is the same as that of $L_\star(0)$ obtained by Jarník [J29].

In inhomogeneous approximation, for a fixed irrational number $\theta$, we are also interested in the set

$$L_\Phi(\theta) := \{ y \in \mathbb{R} : \|n\theta - y\| < \Phi(n) \text{ for infinitely many } n \},$$

where $\Phi : \mathbb{N} \to \mathbb{R}^+$ is a function decreasing to zero. The fullness of the Lebesgue measure of $L_\Phi[\theta]$ was widely studied. In 1955, Kurzweil [Kur55] showed that, if the irrational $\theta$ is of bounded type (the partial quotients of the continued fraction of $\theta$ is bounded), then for a monotone decreasing function $\Phi : \mathbb{N} \to \mathbb{R}^+$, with $\sum \Phi(n) = \infty$, the set $L_\Phi[\theta]$ has full Lebesgue measure. In 1957, Cassels [Cas57] proved that for almost all $\theta$, the set $L_\Phi[\theta]$ has full Lebesgue measure if $\sum \Psi(n) = \infty$. More new results in this direction can be found in the recent works Laurent–Nogueira [LN12], Kim [Kim14], and Fuchs–Kim [FK16].

We estimate the Hausdorff dimension for the set $L_\Phi[\theta]$ in


We prove that for any $\theta$ with irrationality exponent $w(\theta) = \sup\{ \beta \geq 1 : \liminf_{n \to \infty} n^\beta \|n\theta\| = 0 \} \in [1, \infty]$, 

$$\min\left\{ u_\Phi, \max\left\{ l_\Phi, \frac{1 + u_\Phi}{1 + w(\theta)} \right\} \right\} \leq \dim_H(L_\Phi[\theta]) \leq u_\Phi,$$

where

$$u_\Phi := \limsup_{n \to \infty} \frac{\log n}{-\log \Phi(n)} \quad l_\Phi := \liminf_{n \to \infty} \frac{\log n}{-\log \Phi(n)}.$$

Our result is a generalization of that of Bugeaud [Bug03], and Schmeling–Troubetzkoy [TS03] who independently obtained the Hausdorff dimension of $L_\Phi[\theta]$ for $\Phi(n) = 1/n^\tau$ ($\tau > 1$).

By replacing “$n\theta$” in the definition of $L_\Phi[\theta]$ by the orbit $T^nx$ of a point $x$ under an expanding finite Markov interval map $T$ on $[0, 1]$, in


we calculate the Hausdorff dimension of

$$L^\delta(x) := \{ y \in [0, 1] : |T^nx - y| < n^{-\delta} \text{ for infinitely many } n \in \mathbb{N} \}.$$
Let $\mu_\phi$ be the Gibbs measure associated to a Hölder continuous potential $\phi$. Denote $\alpha_{\text{max}} := \int_{[0,1]} (\phi) d\mu_{\text{max}} / \int_{[0,1]} \log |T'| d\mu_{\text{max}}$, where $\mu_{\text{max}}$ is the Gibbs measure associated with the potential $-\log |T'|$. It turns out that the Hausdorff dimension of is strongly related to the multifractal spectrum of $\mu_\phi$:

$$D_{\mu_\phi} : \alpha \geq 0 \mapsto \dim_H \left\{ y \in [0,1] : \lim_{r \to 0} \frac{\log \mu_\phi(B(y,r))}{\log r} = \alpha \right\}. $$

In fact, among others, we prove that for $\mu_\phi$-almost every $x \in [0,1]$, the Hausdorff dimension of $L^\delta(x)$ is

$$\dim_H L^\delta(x) = \begin{cases} 
1/\delta & \text{if } 0 < 1/\delta \leq \dim_H \mu_\phi, \\
D_{\mu_\phi}(1/\delta) & \text{if } \dim_H \mu_\phi < 1/\delta \leq \alpha_{\text{max}}, \\
1 & \text{if } 1/\delta > \alpha_{\text{max}}.
\end{cases} $$

Our result generalizes that of Fan–Schmeling–Troubetzkoy [FST13] for the doubling map. It is recently generalized to piecewise expanding interval maps by Persson–Rams [PR17].

Except for the study of the approximation of a single orbit, we also investigate how close two different orbits of a dynamical system will be. Let $(X, T, \mu)$ be a measure-preserving dynamical system where the space $X$ is equipped with a distance $d$. For two different points $x, y \in X$, and a positive integer $n$, we define the shortest distance $m_n(x, y)$ between the orbits $T^i x$ and $T^j y$ with $0 \leq i, j \leq n$ as:

$$m_n(x, y) = \min_{i,j=0,\ldots,n-1} (d(T^i x, T^j y)).$$

We study the asymptotic behavior of $m_n(x, y)$. The correlation dimension $C_\mu$ of the measure $\mu$ is defined by

$$C_\mu = \lim_{r \to 0} \frac{\log \int_X \mu(B(\mu(x), r))} {\log r} \text{ if the limit exists.}$$


we prove that if the dynamical system $(X, T, \mu)$ has exponential decay of correlation, then $\mu \times \mu$-almost surely

$$\lim_{n \to +\infty} \frac{\log m_n(x, y)} {n} = \frac{2}{C_\mu}. $$

For the irrational rotations, the behavior is different. The irrationality exponent of the rotation angle plays an important role.

In the direction of uniform approximation, the known results are in higher dimensional spaces on the singular vectors. A vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is called singular if for every $\epsilon > 0$, there exists $T_0$ such that for all $T > T_0$ the system of inequalities

$$\max_{1 \leq i \leq d} |qx_i - p_i| < \epsilon \frac{1}{T^{1/d}} \text{ and } 0 < q < T$$

admits an integer solution $(p, q) \in \mathbb{Z}^d \times \mathbb{Z}$. When $d = 1$, only the rational numbers are singular. The existence of singular vectors that do not lie in a rational subspace was proved by Khintchine [K26] for $d \geq 2$. Davenport and Schmidt [DS70] showed that the set of singular vectors is of Lebesgue measure zero. However, the calculation of the Hausdorff dimension of singular vectors is far from easy. The set of singular vectors in dimension two was proved to have Hausdorff
dimension $4/3$ by Cheung [Ch11]. In general dimension $d \geq 3$, Cheung and Chevallier [CC16] proved that the Hausdorff dimension of the singular vectors is $d^2/(d + 1)$.


We compute the Hausdorff dimension of the set $(\theta \in \mathbb{R} \setminus \mathbb{Q}, \tau \geq 1$ are fixed)

$$\mathcal{U}_\tau[\theta] := \{y \in \mathbb{R} : \text{ for all } Q, 1 \leq m \leq Q \text{ such that } ||n\theta - y|| < Q^{-\tau}\}.$$  

By examining the detailed Cantor structure of $\mathcal{U}_\tau[\theta]$, we show that if $\theta$ is an irrational with irrationality exponent $w(\theta) > 1$, then $\mathcal{U}_\tau[\theta] = \mathbb{T}$ if $\tau < 1/w(\theta)$; $\mathcal{U}_\tau[\theta] = \{i\theta \in \mathbb{T} : i \geq 1, i \in \mathbb{Z}\}$ if $\tau > w(\theta)$; and

$$\dim_H(\mathcal{U}_\tau[\theta]) = \begin{cases} 
\lim_{k \to \infty} \frac{\log \left(\frac{n_k^{1+1/\tau} \prod_{j=1}^{k-1} n_j^{1/\tau} ||n_j \theta||}{\log(n_k ||n_k \theta||^{-1})}\right)}{\log \left(\frac{\prod_{j=1}^{k-1} n_j ||n_j \theta||^{1/\tau}}{\log(n_k ||n_k \theta||^{-1})}\right)}, & 1/\tau < \tau < 1, \\
\lim_{k \to \infty} \frac{\log \left(\frac{\prod_{j=1}^{k-1} n_j ||n_j \theta||^{1/\tau}}{\log(n_k ||n_k \theta||^{-1})}\right)}{\log \left(\frac{n_k^{1+1/\tau} \prod_{j=1}^{k-1} n_j^{1/\tau} ||n_j \theta||}{\log(n_k ||n_k \theta||^{-1})}\right)}, & 1 < \tau < w(\theta),
\end{cases}$$

where $n_k$ is the (maximal) subsequence of the denominator $(q_k)$ of the convergent of continued fraction of $\theta$, such that $n_k \parallel n_k \theta \parallel^{1/\tau} < 1$ if $1/\tau < \tau < 1$ and $n_k \parallel n_k \theta \parallel < 2$, if $1 < \tau < w(\theta)$.

For the case $w(\theta) = 1, \infty$, the estimations of $\dim_H \mathcal{U}_\tau[\theta]$ are also obtained. We remark that our results give an answer for the case of dimension one of Problem 3 in Bugeaud and Laurent [BL05].


which is a generalization of the result of Cheung [Ch11] on the weighted singular vectors. Let $w = (w_1, w_2)$ be a pair of positive real numbers such that $w_1 + w_2 = 1$. We show that the set of $w$-singular vector, i.e., the vectors $x = (x_1, x_2) \in \mathbb{R}^2$ satisfying for every $\varepsilon > 0$ there exists $T_0 > 1$ such that for all $T > T_0$ the system of inequalities

$$|qx_1 - p_1| < \varepsilon^{w_1} T^{-w_1}, |qx_2 - p_2| < \varepsilon^{w_2} T^{-w_2},$$

admits an integer solution $(p, q) \in \mathbb{Z}^2 \times \mathbb{Z}$, is of Hausdorff dimension

$$1 - \frac{1}{1 + \max(w_1, w_2)}.$$ 

We remark that $w$-singular vectors have some geometric meaning. It was observed by Dani [Dan85] that $w$-singular vectors correspond to certain divergent trajectories in the space $\mathcal{L}_3$ of unimodular lattices in $\mathbb{R}^3$ with respect to the one-parameter semi-group

$$A^+ = \{a_t = \text{diag}(e^{w_1 t}, e^{w_2 t}, e^{-t}) : t \geq 0\}.$$ 

In the above study on uniform approximation, we may also replace “$n \theta$” by an orbit of a dynamical system, for example, $T^m_\beta x$, with $T_\beta$ being the $\beta$-transformation on $[0, 1]$. This leads to the results in [BL16] Y. Bugeaud, L. Liao, Uniform Diophantine approximation related to $b$-ary and $\beta$-expansions, Ergodic Theory and Dynamical Systems, 36, no. 1, (2016), 1-22.
We show that for \( v \in [0, 1] \),
\[
\dim_H \{ x \in [0, 1] : \forall N \gg 1, \exists 1 \leq n \leq N, T^n_{\beta}(x) < (\beta^N)^{-v} \} = \left( \frac{1 - v}{1 + v} \right)^2,
\]
and
\[
\dim_H \{ \beta > 1 : \forall N \gg 1, \exists 1 \leq n \leq N, T^n_{\beta}(1) < (\beta^N)^{-v} \} = \left( \frac{1 - v}{1 + v} \right)^2.
\]
To compare with the asymptotic approximation, we remark that Shen–Wang [SW13] showed
\[
\dim_H \{ x \in [0, 1] : T^n_{\beta}(x) < (\beta^n)^{-v}, \text{ for infinitely many } n \} = \frac{1}{1 + v},
\]
and Persson–Schmeling [PS08] proved
\[
\dim_H \{ \beta > 1 : T^n_{\beta}(1) < (\beta^n)^{-v}, \text{ for infinitely many } n \} = \frac{1}{1 + v}.
\]
The square in the above dimensional formula in uniform approximation comes from a maximizing process of the lower bounds of the Hausdorff dimension.

We can further replace \( n \theta \) in the uniform approximation by the sequence \( \beta^n \) with \( \beta > 1 \) in the unit circle \( \mathbb{R}/\mathbb{Z} \). We remind that if \( \beta > 1 \) is not an integer, then \( \beta^n \) is different to \( T^n_{\beta}(1) \).

For a real number \( B > 1 \) and a sequence of real numbers \( y = (y_n)_{n \geq 1} \) in \( [0, 1] \), set
\[
F(B, y) := \{ \beta > 1 : \text{ for all large integer } N, \| \beta^n - y_n \| < B^{-N} \text{ has a solution } 1 \leq n \leq N \}.
\]
Among other things, we prove in


that for any \( v > 1 \), we have
\[
\lim_{\epsilon \to 0} \dim_H([v - \epsilon, v + \epsilon] \cap F(B, y)) \geq \left( \frac{\log v - \log B}{\log v + \log B} \right)^2.
\]

0.3 Dynamical systems on the field of \( p \)-adic numbers and negative beta transformation

The field \( \mathbb{Q}_p \) of \( p \)-adic numbers was introduced by Hensel in 1897. It has been studied in more detail in the field of Number Theory. Volovich ([V87]) applied the \( p \)-adic numbers to establish his \( p \)-adic string theory. This work ([V87]) aroused interest in the \( p \)-adic models in Physics and stimulated the study of \( p \)-adic dynamical systems. The first work of dynamical systems on \( \mathbb{Q}_p \) is probably due to Oselies and Zieschang [OZ75]. Later, we can find the work of Anashin [A94], Coelho and Parry [CP01], Gundlach, Khrennikov and Lindahl [GKL01], Larin [La02], et al. In another direction, the dynamics on the field \( \mathbb{C}_p \) of \( p \)-adic complex numbers, and on the Berkovich space associated with the projective line of \( \mathbb{C}_p \), have been extensively studied by Herman–Yoccoz [HY83], Hsia [Hsi00], Benedetto [Ben01], Rivera-Letelier [RL03], Favre–Rivera-Letelier [FRL06] et al.

We are interested in the dynamical systems on \( \mathbb{Q}_p \). The first examples are polynomials and rational maps. Denote by \( \mathbb{Z}_p \) the ring of \( p \)-adic integers. Based on the ideas of Desjardins [DZ] and Zieve [Z-thesis], we prove in
that for a polynomial \( \phi \in \mathbb{Z}_p[x] \) of degree greater than 2, \( \mathbb{Z}_p \) is decomposed into three parts:

\[
\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},
\]

where \( \mathcal{P} \) is the finite set consisting of all periodic points of \( \phi \), \( \mathcal{M} = \sqcup_i \mathcal{M}_i \) is the union of all (at most countably many) clopen invariant sets such that each \( \mathcal{M}_i \) is a finite union of balls and each subsystem \( \phi : \mathcal{M}_i \to \mathcal{M}_i \) is minimal, and each point in \( \mathcal{B} \) lies in the attracting basin of a periodic orbit or of a minimal subsystem. This decomposition of space is called minimal decomposition.

Even though we have a general minimal decomposition theorem for the polynomials in \( \mathbb{Z}_p[x] \), it is not an easy thing to give the precise decomposition for a given polynomial. In


we obtain such minimal decomposition for the square mapping \( f : x \mapsto x^2 \) on \( \mathbb{Z}_p \), and in


we obtain the detailed minimal decomposition for Chebyshev polynomials on \( \mathbb{Z}_2 \).

A similar minimal decomposition is obtained in


for convergent series over a finite extension of \( \mathbb{Q}_p \).

For the rational maps, we first study in


the most simple case: rational maps of degree one, i.e., the homographic maps on \( \mathbb{Q}_p \):

\[
\phi(x) = \frac{ax + b}{cx + d} \quad (a, b, c, d \in \mathbb{Q}_p, \ ad – bc \neq 0).
\]

If \( \phi \) has one (resp. two) fixed points in \( \mathbb{Q}_p \), then \( \phi \) is conjugate to a translation (resp. multiplication) and the minimal decomposition has already been done in [FF11]. We show that if \( \phi \) has no fixed point in \( \mathbb{Q}_p \), and \( \phi^n \neq id \) for all integers \( n > 0 \), then \( \phi \) is consider as a dynamics on the projective line \( \mathbb{P}^1(\mathbb{Q}_p) \) of \( \mathbb{Q}_p \) and the system \( (\mathbb{P}^1(\mathbb{Q}_p), \phi) \) can be decomposed into a finite number of minimal subsystems. These minimal subsystems are topologically conjugate to each other. We also obtain the necessary and sufficient conditions under which the dynamics \( (\mathbb{P}^1(\mathbb{Q}_p), \phi) \) is minimal.

For rational maps of higher degree, in


we study the rational maps with good reduction. We show that if \( \phi \in \mathbb{Q}_p(z) \) is a rational map of degree \( \geq 2 \) with good reduction, then

\[
\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B}
\]

where \( \mathcal{P} \) is the finite set consisting of all periodic points of \( \phi \), \( \mathcal{M} = \sqcup_i \mathcal{M}_i \) is the union of all (at most countably many) clopen invariant sets such that each \( \mathcal{M}_i \) is a finite union of balls
and each subsystem $\phi : M_i \to M_i$ is minimal, and points in $B$ lie in the attracting basin of a periodic orbit or of a minimal subsystem. The minimal criterion for rational maps with good reduction is also given. In the case $p = 2$, we obtain some necessary/sufficient conditions for the minimality on $\mathbb{Z}_2$ in terms of the coefficients of $\phi$. In particular, if $\phi \in \mathbb{Q}_2(z)$ is a rational map of degree 2, 3 or 4 with good reduction, then the dynamical system $\langle P^1(\mathbb{Q}_2), \phi \rangle$ is not minimal.

All the above minimal decomposition results are for 1-Lipschitz dynamical systems on $\mathbb{Q}_p$ or $\mathbb{P}^1(\mathbb{Q}_p)$. However, there are many polynomials and rational maps with coefficients in $\mathbb{Q}_p$ which have expanding property. To describe these dynamics, we often use subshifts of finite type.


we prove that a transitive $p$-adic weak repeller in $\mathbb{Q}_p$ with incidence matrix $A$ is topologically conjugate to the subshift of finite type $(\Sigma_A, \sigma)$. This result holds also for the so-called generalized $p$-adic repeller. Many polynomials and rational maps have subsystems as (generalized) $p$-adic repellors. In


and in


we describe the dynamical structures of the rational map $x \mapsto ax + 1/x$, where $a \in \mathbb{Q}_p$ for $p \geq 3$ and $p = 2$ respectively. We give precise description for the Julia set and the Fatou set of the rational map $x \mapsto ax + 1/x$. The subshifts of finite type conjugate to the $p$-adic repellors contained in its Julia set are also given.

As another important family of arithmetic dynamics, in


we examine the negative beta-transformation defined by

$$T_{-\beta} : (0, 1] \to (0, 1], \quad x \mapsto -\beta x + \lfloor \beta x \rfloor + 1.$$ 

This transformation is a natural modification of the beta-transformation on which many mathematicians have had contributions: Rényi [R57], Gelfond [Ge59], Parry [P60], Bertrand-Mattis [BM86], Blanchard [Bla89] et al. Using $T_{-\beta}$, we can write any number in $(0, 1]$ as an expansion in base $(-\beta)$, called $(-\beta)$-expansion.

Different to the $\beta$-transformation, the density of the absolutely continuous invariant measure of the $(-\beta)$-transformation may be zero on some subintervals of $[0, 1]$. We fully describe these gaps. For each $n \geq 1$, let $\gamma_n$ be the (unique) positive real number defined by $\gamma_n^{g_n+1} = \gamma_n + 1$, with $g_n = \lfloor 2^{n+1}/3 \rfloor$, and set $\gamma_0 = \infty$. We prove that for any $\gamma_n + \beta < \gamma_n$, $n \geq 0$, the set of gaps of the transformation $T_{-\beta}$ consists of $g_n = \lfloor 2^{n+1}/3 \rfloor$ disjoint non-empty intervals. The $(-\beta)$-expansion of 1 determines the dynamics of $T_{-\beta}$, thus is quite important. We show that the $(-\gamma_n)$-expansion of 1 is given by $\varphi^{n+1}(2\mathbb{T})$, where $\mathbb{T} = 111 \cdots$ and $\varphi$ is the substitution on $\{1, 2\}^\mathbb{N}$ defined by $\varphi : 1 \mapsto 2$, $2 \mapsto 211$. Further, we prove that for every $\beta > 1$, $T_{-\beta}$ is locally eventually onto, thus topologically mixing, on $(0, 1]$ deleting the gaps. As applications,
we can confirm a conjecture of Gora [Go07] claiming that all \((-\beta)\)-transformations are exact with respect to the unique absolutely continuous invariant measure. We can also deduce that any \((-\beta)\)-transformation has a unique maximal entropy measure, which completes a study of Faller [Fa08].

0.4 Fuglede conjecture in the field of \(p\)-adic numbers

Let \(G\) be a locally compact abelian group and \(\Omega \subset G\) be a Borel set of positive and finite Haar measure. We call \(\Omega\) a spectral set if there exists a set \(\Lambda\) of continuous characters of \(G\) which form a Hilbert basis of \(L^2(\Omega)\). We say that \(\Omega\) tiles \(G\) by translation if there exists a set of translates \(T \subset G\) such that \(\sum_{t \in T} 1_{\Omega}(x-t) = 1\) for almost all \(x \in G\). The Fuglede conjecture for the group \(G\) states that \(\Omega\) is a spectral set if and only if \(\Omega\) tiles \(G\).

This conjecture was first proposed by Fuglede [Fug74] in 1974 for the case \(G = \mathbb{R}^d\). The origin of the Fuglede conjecture is a Segal problem (1958) which aims at characterizing the domains \(\Omega \subset \mathbb{R}^d\) such that the differential operators \(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d}\) defined on their common definition domain \(C_c^\infty(\Omega)\) have commutative self-adjoint extensions in \(L^2(\Omega)\). Fuglede showed that this is the case if and only if \(\Omega\) is a spectral set. This thus leads to the problem of characterizing the spectral sets on \(\mathbb{R}^d\). Fuglede also showed that this is indeed the case if \(\Omega\) tiles \(\mathbb{R}^d\) by a lattice \(A\mathbb{Z}^d\) with \(A\) being a non-singular matrix.

The Fuglede conjecture has attracted considerable attention during the last decades. Many positive results are obtained in particular cases under different restrictions. However, in 2004, Tao [Tao04] proved that the direction “spectral \(\Rightarrow\) tiling” is false in case \(\mathbb{R}^d\) with \(d \geq 5\). Later, Matolcsi [Mat05], Matolcsi–Kolountzakis [KM06a, KM06b], Farkas–Gy [FG06], Farkas–Matolcsi–Móra [FMM06] gave a series of counterexamples showing that the conjecture is false for both directions for \(d \geq 3\). The validity of the conjecture for cases of \(d = 1, 2\) remains unknown. Iosevich, Katz, and Tao [IKT03] showed that the conjecture is true for the convex subset of \(\mathbb{R}^2\).


the Fuglede conjecture in the case \(G = \mathbb{Q}_p\), i.e., in a one-dimensional \(p\)-adic space. Note that the convexity on \(\mathbb{Q}_p\) is not properly defined. This difference gives rise to difficulties in the research for \(p\)-adic spaces. On the other hand, the non-Archimedean property of the \(p\)-adic spaces simplifies the tiling structure. In fact, the tilings are described by \(p\)-homogenous trees.
Bibliography


