Some interplays between multifractal analysis and Diophantine approximation

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based on joint works with

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Diophantine approximation, fractal geometry and related topics

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\[
\sum_{n=1}^{\infty} (nx)^2
\]
where \((x)\) is the one periodic function satisfying
\[
\forall x \in \left[-\frac{1}{2}, \frac{1}{2}\right), (x) = x
\]
Bernhard Riemann's habilitation

Jump of amplitude \(\pi^2/8\) at rationals \(p/q\) where \(p \wedge q = 1\)

Continuous at other rationals and at irrationals
$1854$
\[ \mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \]

where \((x)\) is the one periodic function satisfying

\[ \forall x \in [-1/2, 1/2), \quad (x) = x \]

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1854

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R(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}
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where \((x)\) is the one periodic function satisfying

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Bernhard Riemann’s habilitation

- Jump of amplitude \(\frac{\pi^2}{8q^2}\) at rationals \(\frac{p}{2q}\) where \(p \land 2q = 1\)
- Continuous at other rationals and at irrationals
Pointwise Hölder regularity

Let $f$ be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$

$f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of $f$ at $x_0$ is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$$
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Cusp: \(C_H(x) = |x - x_0|^H\)

Chirp: \(C_{H,\beta} = |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)\)
Jump lemma: Let $f$ be a function discontinuous on a dense set of points $J$ where it has a right and a left limit; $s(r)$ denotes the jump of $f$ at $r \in J$. 

Let $x_0 \in \mathbb{R}$; then 

$$ h_f(x_0) \leq \liminf_{r \to x_0, r \in J} \log|s(r)| \log|r - x_0| := j_s(x_0) \text{ jump exponent of } s \text{ at } x_0. $$

$$ R(x) = \infty \sum_{n=1}^{\infty} \frac{n^2}{x_0 n^2} $$

Let $x_0/\in \mathbb{Q}$, 

$$ \tau_n(x_0) = \left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n} \tau_n(x_0) $$

Let $\tau(x_0) = \limsup_{n \to \infty} \tau_n(x_0)$ (irrationality exponent).

$R$ has a jump of amplitude $\pi/8 q_n^2$ at $p_n^2 q_n = \Rightarrow h_R(x_0) \leq 1/\tau(x_0)$. 
**Jump lemma**: Let $f$ be a function discontinuous on a dense set of points $J$ where it has a right and a left limit; $s(r)$ denotes the jump of $f$ at $r \in J$. Let $x_0 \in \mathbb{R}$; then

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$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

Let $x_0 \notin \mathbb{Q}$, $r_n = \frac{p_n}{q_n}$ be the convergents of $x_0$,

$$\tau_n(x_0) : \left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n(x_0)}} \quad \text{Let } \tau(x_0) = \limsup_{n \to \infty} \tau_n(x_0) \quad \text{(irrationality exponent)}$$
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$\mathcal{R}$ has a jump of amplitude $\frac{\pi^2}{8q_n^2}$ at $\frac{p_n}{2q_n} \implies h_{\mathcal{R}}(x_0) \leq 1/\tau(x_0)$
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\( R \) has a jump of amplitude \( \frac{\pi^2}{8q_n^2} \) at \( \frac{p_n}{2q_n} \). Actually:

\[
h_R(x_0) = \frac{2}{\tau(x_0)}
\]
Hecke’s functions

The sawtooth function is \( \{ x \} = x - [x] - 1/2 \)

The function \( \{ x \} \) is odd, one-periodic, has zero average, and has a jump of magnitude one at integers.
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E. Hecke (1921) :

\[ \mathcal{H}_a(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^a} \]

It is a Dirichlet series in \( a \), and its analytic continuation depends on the irrationality exponent of the parameter \( x \)
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If \( a \in \mathbb{R} \) and \( a > 1 \), then the jumps at \( p/q \) is \( s(p/q) = \zeta(a)/q^s \)

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\forall x_0 \in \mathbb{R}, \quad h_R(x_0) = s/\tau(x_0)
\]

The Hölder exponent coincides with the jump exponent
Davenport series

H. Davenport (1936) considered series of the form

\[ \sum_{n=1}^{\infty} a_n \{nx\} \quad a_n \in \mathbb{R} \]

where \( \{x\} = x - \lfloor x \rfloor \). The M"obius inversion formula implies that

\[ a_n = -\pi \sum_{d|n} \mu(d) c_{d/n} \]

where \( \mu(n) = 0 \) if \( n \) is not squarefree; otherwise, \( \mu(n) = (-1)^{\omega(n)} \), with \( \omega(n) \) being the number of prime factors of \( n \). Properties of Davenport series have been studied by P. Hartman, R. de la Bretèche, G. Tenenbaum, S. Nicolay, J. Bremont, ...
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\[ \sum_{n=1}^{\infty} a_n \{nx\} \quad a_n \in \mathbb{R} \]

\[ \sum_{n=1}^{\infty} a_n \{nx\} = \sum_{m=1}^{\infty} c_m \sin(2\pi mx) \quad \text{where} \quad c_m = -\frac{1}{\pi m} \sum_{n|m} na_n \]

which follows from

\[ \{x\} = -\sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{\pi m}. \]
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Möbius inversion formula implies that

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Jumps of Davenport series and multifractal analysis

We now assume that \((a_n) \in l^1\) so that the series is normally convergent.

If \(p \wedge q = 1\), then the jump of \(\sum_{n=1}^{\infty} a_n \{nx\}\) at \(r = \frac{p}{q}\) is
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s(r) = \sum_{l=1}^{\infty} a_{lq}
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Consider the same questions in the more general setting of pure jumps functions.
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For Hecke’s functions, \(\forall x, h_f(x) = j_s(x)\) and the jump function
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Find a condition under which \( \forall x, h_f(x) = j_s(x) \)
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Consider the same questions in the more general setting of pure jumps functions.
Multifractal analysis

Let $f$ be a locally bounded function. The **isoregularity sets** of $f$ are the sets

$$E_H = \{ x_0 : h_f(x_0) = H \}$$

The **multifractal spectrum** of $f$ is

$$D_f(H) = \dim (E_H)$$

where $\dim$ denotes the Hausdorff dimension.
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▶ Find Davenport series with different spectra
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- Find Davenport series with different spectra
- Find a simple condition under which spectra are linear except perhaps at one point
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Extensions of Davenport series

Davenport series in several variables

\[ n = (n_1, \ldots, n_d) \sum_{n \neq 0}^{\infty} a_n \{ n \cdot x \} \quad a_n \in \mathbb{R} \]

where \( n \cdot x \) is the usual scalar product in \( \mathbb{R}^d \)
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Multivariate Davenport series

We consider two functions \( f_1(t) \) and \( f_2(t) \)

\[ E(H_1, H_2) = \{ x : h_{f_1}(x) = H_1 \text{ and } h_{f_2}(x) = H_2 \} = E_1(H_1) \cap E_2(H_2) \]

The bivariate multifractal spectrum is

\[ D_{f_1, f_2}(H_1, H_2) = \text{dim}(E(H_1, H_2)) = \text{dim}(E_1(H_1) \cap E_2(H_2)) \]

where \( \text{dim} \) denotes the Hausdorff dimension
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The bivariate multifractal spectrum is

\[ \mathcal{D}_{f_1, f_2}(H_1, H_2) = \dim(E(H_1, H_2)) = \dim(E_1(H_1) \cap E_2(H_2)) \]

where \( \dim \) denotes the Hausdorff dimension

Determine the bivariate spectrum of \( \mathcal{H}_a(\cdot) \) and \( \mathcal{H}_a(\cdot + y) \)
Brjuno’s function

Let \( x \notin \mathbb{Q} \quad x = [a_0; a_1, \cdots a_n, \cdots] \)

The convergents \( p_n/q_n \) of \( x \) are

\[
\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}}
\]

Brjuno’s function is

\[
B(x) = \sum_{n=0}^{\infty} |p_{n-1} - q_{n-1}x| \log \left( \frac{p_{n-1} - x q_{n-1}}{q_n x - p_n} \right)
\]

where \((p_{-1}, q_{-1}) = (1, 0), \quad (p_0, q_0) = (0, 1), \text{ and } (p_1, q_1) = (1, a_1)\)

J.-C. Yoccoz : petits diviseurs en dimension 1, Astérisque 231 (1995)
An equivalent definition

Let

- \( \{x\} \) denote the fractional part of \( x \)
- Gauss map:
  \[ \alpha : ]0, 1[ \rightarrow ]0, 1[ \]
  \[ \alpha(x) = \left\{ \frac{1}{x} \right\} \]
- \( \alpha_n(x) \) the iterates of \( \alpha(x) \)
An equivalent definition

Let

- \( \{ x \} \) denote the fractional part of \( x \)
- Gauss map :
  \[
  \alpha : ]0, 1[ \rightarrow ]0, 1[ \setminus \mathbb{Q}
  \]
  \[
  \alpha(x) = \left\{ \frac{1}{x} \right\}
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- \( \alpha_n(x) \) the iterates of \( \alpha(x) \)

\[
B(x) = \sum_{n=0}^{\infty} \alpha_0(x) \alpha_1(x) \cdots \alpha_{n-1}(x) \log \left( \frac{1}{\alpha_n(x)} \right)
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B(x) = \sum_{n=0}^{\infty} \alpha_0(x) \alpha_1(x) \cdots \alpha_{n-1}(x) \log \left( \frac{1}{\alpha_n(x)} \right)
\]

An irrational number \( x \) is a \textit{Brjuno number} if the series which defines \( B(x) \) is convergent
Some history

Let $f$ be a holomorphic function which can be written near 0

$$f(z) = e^{2i\pi \alpha} z + O(z^2)$$

for example $P_\alpha(z) = e^{2i\pi \alpha} z (1 + z)$

$f$ is linearisable near 0 if it is conjugate with its linear part $R_\alpha(z) = e^{2i\pi \alpha} z$:

$$\exists \phi : \quad \phi \circ R_\alpha = f \circ \phi$$

If $\alpha \notin \mathbb{Q}$, then $f$ is “formally linearisable” : There exists a formal entire series $\phi(z) = z + \cdots$ such that:

$$\phi \circ R_\alpha = f \circ \phi$$

Let $r(f)$ be the radius of convergence of $\phi$

$f$ is linearisable iff $r(f) > 0$
Some history

- $P_\alpha$ is linearisable iff $B(\alpha) < \infty$ (Yoccoz 1995)

- Yoccoz Conjecture: $B(\alpha) + \log(r(P_\alpha))$ is bounded (and even continuous): 
  X. Buff and A. Ch´eritat: The Brjuno function continuously estimates the size of quadratic Siegel disk, Ann. of Math., 2006

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\[ B(\alpha) + \log(r(P_\alpha)) \]
Functional equation : $B(x) = -\log x + xB(\alpha(x))$
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\[
B(x) = \sum_{n=0}^{\infty} \alpha_0(x) \alpha_1(x) \cdots \alpha_{n-1}(x) \log \left( \frac{1}{\alpha_n(x)} \right)
\]

S. Marmi: From small divisors to Brjuno functions

\( B \) is not locally bounded but belongs to BMO and thus to \( L^p \) spaces

\[ \forall p < \infty \quad f \in BMO \text{ if } \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty \]

(\( f_I : \text{the average value of } f \text{ on } I \))

$T^{p}_\alpha(x_0)$ regularity

Pointwise Hölder regularity: $f \in C^\alpha(x_0)$ if

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

Drawback: This definition requires $f$ to be locally bounded
\( T^p_\alpha(x_0) \) regularity

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Clue: The definition of pointwise Hölder regularity can be rewritten

\[
f \in C^\alpha(x_0) \iff \sup_{B(x_0,r)} |f(x) - P(x - x_0)| \leq Cr^\alpha
\]
$T^p_\alpha(x_0)$ regularity

**Pointwise Hölder regularity**: $f \in C^\alpha(x_0)$ if

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**Clue**: The definition of pointwise Hölder regularity can be rewritten

$$f \in C^\alpha(x_0) \iff \sup_{B(x_0, r)} |f(x) - P(x - x_0)| \leq Cr^\alpha$$

**Definition (Calderón & Zygmund, 1961)**: Let $f \in L^p(\mathbb{R}^d)$; $f \in T^p_\alpha(x_0)$ if there exists a polynomial $P$ such that for $r$ small enough,

$$\left(\frac{1}{r^d} \int_{B(x_0, r)} |f(x) - P(x - x_0)|^p dx\right)^{1/p} \leq Cr^\alpha$$

The $p$-exponent of $f$ at $x_0$ is $h_p(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$
The $p$-exponent of Brjuno’s function

**Theorem:** For any $p \in [1, +\infty]$, the $p$-exponent of Brjuno’s function is independent of $p$ and given by

$$h_B^p(x_0) = \begin{cases} 0 & \text{if } x_0 \in \mathbb{Q} \\ \frac{1}{\tau(x_0)} & \text{else} \end{cases}$$
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**Idea of the proof for ($p = 1$)**

Ingredients: sharp estimates on the mean value of $B$ on “cells” (adjacents points of the Farey tree)

*M. Balazard, B. Martin* : *Comportement local moyen de la fonction de Brjuno, Fund. Math.*(2012)

Allows to prove regularity hence to get a lower bound on the 1-exponent
The $p$-exponent of Brjuno’s function: Irregularity

**Definition:** Let $\psi$ be a bounded, compactly supported, function such that

$$\sup_{x \in \mathbb{R}} |\psi(x)| \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} \psi(x) \, dx = 0.$$ 

The continuous wavelet transform of $f$ is

$$\forall a > 0, \ b \in \mathbb{R}, \quad C_f(a, b) = \frac{1}{a} \int_{\mathbb{R}} f(x) \psi \left( \frac{x - b}{a} \right) \, dx$$

**Proposition:** Let $p \in [1, +\infty]$ and $f \in L^p_{\text{loc}}(\mathbb{R})$ If $f \in T^p_\alpha(x_0)$, then

$$\text{supp} \left( \psi_{a,b} \right) \subset [x_0 - \rho, x_0 + \rho] \implies |C_f(a, b)| \leq C \frac{\rho^{\alpha+1}}{a}.$$
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One uses Haar’s wavelet : $\psi = 1_{[0,1/2]} - 1_{[1/2,1]}$

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One uses Haar’s wavelet: $\psi = 1_{[0,1/2]} - 1_{[1/2,1]}$.

The continuous wavelet transform of $f$ is the difference of mean values of $f$. Mean value estimates on $B$ allow to get a lower bound on this transform.
Variants and extensions

Extensions of the Brjuno function have been proposed, based on variants of the Gauss map or of the continuous fraction algorithm.
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Extensions of the Brjuno function have been proposed, based on variants of the gauss map or of the continuous fraction algorithm.

Their pointwise regularity is a the subject of intensive investigations (M. Balazard, T. Lamby, S. Marmi, B. Martin, S. Nicolay, T. Schindler...).
Brjuno’s function and Davenport series

Let \( \{x\} = x - [x] - 1/2 \)

Davenport series are of the form \( \sum_{n=1}^{\infty} a_n \{nx\} \)
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The Hilbert transform: \((Hf)(x) = \int_{\mathbb{R}} \frac{1}{x-y} f(y) dy\)

The Hilbert transform preserves the \( p \)-exponent for \( p \in (1, +\infty) \)

Hilbert transform of Brjuno’s function
Riemann’s nondifferentiable function

\[ R_2(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \]
Riemann’s nondifferentiable function

\[ \mathcal{R}_2(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \]

Let \( x_0 \not\in \mathbb{Q} \), \( \frac{p_n}{q_n} \) be the convergents of \( x_0 \),

\[ \tau_n(x_0) : \quad \left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n(x_0)}} \]

Let \( \tau'(x_0) = \limsup_{n \to +\infty} \tau_n(x_0) \), where the sequence \( n \) is restricted to the values such that \( p_n \) and \( q_n \) are not both odd.

If \( x_0 \not\in \mathbb{Q} \), then:

\[ h_{\mathcal{R}_2,x_0} = \frac{1}{2} + \frac{1}{2\tau'(x_0)} \]
Riemann’s nondifferentiable function and turbulence

A new and surprising connexion between Riemann’s function
\[ R(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2} \]
and turbulence was recently uncovered between Riemann’s function
and turbulence (V. Banica, A. Boritchev, D. Eceizabarrena, C. J.
Garcia-Cervera, F. de la Hoz, R. L. Jerrard, D. Smets, L. Vega, and V.
Vilaça Da Rocha):

The complex-valued version
\[ \phi(x) = i\pi x + \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 x}}{n^2} \]
appears as the trajectory
of the corners of polygonal
vortex filaments that follow
the binormal flow

Credit: A. Boritchev, D. Eceizabarrena, V. Vilaça Da Rocha
Open problems concerning the regularity of Riemann’s function

\[ R_s(t, x) = \sum_{n=1}^{\infty} \frac{e^{i\pi(nt+n^2x)}}{n^s} \]

determine the \( p \)-exponents of these functions (as a two-variable function, or for specific values of \( t \)). There are partial results on this problem:

- S. Seuret and A. Ubis for \( p = 2 \) and \( t = 0 \)
- V. Banica and L. Vega for \( s = 2 \) and \( p = \infty \)

determine their multifractal spectra.
Open problems concerning the regularity of Riemann’s function

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Determine their multifractal spectra
A problem posed by Hilbert

In 1873 Du Bois-Reymond proved that the Fourier series of a periodic continuous function may diverge at some points

Hilbert asked his PhD student Alfred Haar to determine if this drawback is inherent to any orthonormal basis, or if some bases can behave “better” than the trigonometric system

In 1909, Haar constructed the Haar basis to solve this problem
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In 1909, Haar constructed the Haar basis to solve this problem.

\[
\begin{align*}
\varphi(x) &= 1 \quad \text{if } x \in (0, 1), \\
\varphi(0) &= \varphi(1) = 1/2, \\
\varphi(x) &= 0 \quad \text{else}
\end{align*}
\]

\[
\psi(x) = \varphi(2x) - \varphi(2x - 1)
\]

The \[
\begin{align*}
\varphi(x) \\
2^{j/2}\psi(2^j x - k), \quad j \geq 0, \quad k = 0, \ldots, 2^j - 1
\end{align*}
\]
form an orthonormal basis of \(L^2([0, 1])\).
The Haar basis

Theorem (A. Haar, 1919) : If $f$ is continuous on $[0, 1]$. Let

$$C_0 = \int_0^1 f(t) dt \quad \text{and} \quad C_{j,k} = 2^j \int_0^1 f(t) \psi(2^j t - k) dt$$

then the partial sums

$$f_J(x) = C_0 \varphi(x) + \sum_{j \leq J} \sum_k C_{j,k} \psi(2^j x - k)$$

converge uniformly to $f$
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converge uniformly to \( f \)

Theorem: (G. Bourdaud, 1995) The Haar basis is an unconditional basis of the Besov spaces \( B^{s,p}_p \) if and only if

\[
\frac{1}{p} - 1 < s < \min \left( \frac{1}{p}, 1 \right)
\]

Haar-wavelet characterization of Besov spaces: Let \( p > 0 \)

\[
f \in B^{s,p}_p \quad \text{if} \quad \exists C, \forall j : \quad 2^{(sp-1)j} \sum_{j,k} |C_{j,k}|^p \leq C
\]

Limitation of the Haar basis: Lack of regularity
Let $H$ be a Hilbert space. A frame is a sequence of elements of $H$ $(e_n)_{n \in \mathbb{N}}$ such that

$$\exists C, C' > 0, \forall f \in H, \quad C \| f \|^2_H \leq \sum_{n} |\langle f | e_n \rangle|^2 \leq C' \| f \|^2_H$$

The notion of Frame was introduced by Duffin and Schaeffer in 1952 in the context of nonharmonic Fourier series.
Tight frames

Frame: \( \exists C, C', \forall f, \quad C \| f \|_H^2 \leq \sum_n |\langle f | e_n \rangle|^2 \leq C' \| f \|_H^2 \)

Tight frame: \( \exists C, \forall f, \quad \sum_n |\langle f | e_n \rangle|^2 = C \| f \|_H^2 \)

Consequence

\[
f = \frac{1}{C} \sum_n \langle f | e_n \rangle e_n
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Advantages and drawbacks:

- Little flexibility for the choice of the analyzing wavelet
- Wavelet type regularity characterizations are the same as for orthonormal wavelet bases
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Finite unions of orthonormal bases constitute tight frames
Uniform Hölder regularity

Let $\alpha \in ]0, 1[$; a function $f$ belongs to $C^\alpha(\mathbb{R})$ if $f$ is bounded and

$$\exists C, \ \forall x, y \in \mathbb{R}, \quad |f(x) - f(y)| \leq C|x - y|^\alpha$$
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$$\exists C, \forall x, y \in \mathbb{R}, \quad |f(x) - f(y)| \leq C|x - y|^\alpha$$

**Proposition** : Let $\alpha \in (0, 1)$. Let $C_{j,k}$ denote the wavelet coefficients of $f$ on a wavelet frame

If $f \in C^\alpha([0, 1])$ then $|C_{j,k}| \leq C2^{-\alpha j}$

This criterium supplies a characterization if the wavelet system used is an orthonormal basis of smooth wavelets
Haar-Weierstrass functions

Let $\beta > 0$; the Haar-Weierstrass functions are defined by

$$\forall x \in \mathbb{R}, \quad H_\beta(x) = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} 2^{-\beta j} \psi_{j,k}(x)$$

Credit: Guillaume Saes

Haar-Weierstrass function

$\psi$ is Haar’s wavelet

Meyer-Weierstrass function

$\psi$ is Meyer’s $C^\infty$ wavelet
The Krim-Pesquet tight frames

One picks the union of $N$ orthonormal wavelet bases, translated by $k/N$

$$\varphi_k(x) = \varphi \left( x - \frac{k}{N} \right), \text{ and } \psi_{j,k}(x) = \psi \left( 2^j x - \frac{k}{N} \right), \quad j, k \in \mathbb{Z}$$

The $\varphi_k$ and $2^{j/2}\psi_{j,k}$ constitute a tight frame

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|_{L^2}^2 = \frac{1}{N} \left( \sum_k |\langle f | \varphi_k \rangle|^2 + \sum_{j,k} |\langle f | 2^{j/2} \psi_{j,k} \rangle|^2 \right).$$
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**Advantage**: improves the lack of translation invariance of the dyadic grid used for orthonormal bases (initial motivation of Krim-Pesquet for radar detection) and Coifman-Donoho (denoising)
Haar Frames

The Haar basis is used in situations where other wavelets are not available. Its main limitation is its inability to measure regularity

\[ I_k(x) = \varphi \left( x - \frac{k}{3} \right), \text{ and } H_{j,k}(x) = \psi \left( 2^j x - \frac{k}{3} \right), \quad j, k \in \mathbb{Z} \]

The \( I_k \) and \( 2^{j/2} H_{j,k} \) are a union of three orthonormal bases and therefore constitute a tight frame.
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The \( I_k \) and \( 2^{j/2}H_{j,k} \) are a union of three orthonormal bases and therefore constitute a \textit{tight frame}

\textbf{Theorem} : Let \( \alpha \in (0, 1) \) and let \( f \) be a locally bounded function \( f \in C^\alpha(\mathbb{R}) \) if and only if its Haar frame coefficients

\[ C_k = \int f(x) I_k(x) \, dx \quad \text{and} \quad c_{j,k} = 2^j \int f(x) H_{j,k}(x) \, dx \]

satisfy

\[ \exists C, \quad \forall k, \quad |C_k| \leq C \quad \text{and} \quad \forall j, k, \quad |c_{j,k}| \leq C2^{-\alpha j} \]

An analogue of this theorem holds for pointwise regularity
The proof of this theorem lies on the fact that, at any scale, any point cannot be jointly well approximated by points of the dyadic and the 1/3 shifted dyadic grid.
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Open problems:

Consider a collection of functions $(a_n)^{1/2}H(a_n x + b_n)$ which constitutes an $L^2$ frame. Under which conditions on the sequence $(a_n, b_n)$ does the same theorem holds? or “almost holds”? Extend these result to Daubechies wavelets
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Extend these result to Daubechies wavelets.
Thank you for your attention!