Multivariate Multifractal analysis

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Based on joint works with

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Jubilee of Fourier Analysis and Applications:
A Conference Celebrating

John Benedetto’s 80th Birthday

Norbert Wiener Center  University of Maryland  September 19-21  2019
Don't judge each day by the harvest you reap but by the seeds that you plant.

Mark Twain
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Mark Twain
What is fractal geometry?

Geometry dealing with irregular sets

How can one quantify this irregularity?
Classification of fractals sets

Box dimension

Let $N(\varepsilon)$ be the minimal number of balls of radius $\varepsilon$ needed to cover the set $A$

$$N(\varepsilon) \sim \varepsilon^{-\text{dim}_B(A)}$$

Advantage:
Numerically computable through log-log plot regressions: $\log(N(\varepsilon))$ is plotted as a function of $\log(\varepsilon)$
The slope yields the dimension

$$\text{dim}_B = \frac{\log 2}{\log 3}$$

Triadic Cantor set

$$\text{dim}_B = \frac{\log 4}{\log 3}$$

Van Koch curve
Everywhere irregular functions

The existence of everywhere irregular functions was doubted by mathematicians until Weierstrass proposed his example in 1872.

\[ W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x) \]

\( 0 < H < 1 \)

C. Hermite: I turn my back with fright and horror to this appalling wound: Functions that have no derivative

H. Poincaré called such functions “monsters”
Everywhere irregular functions

Jean Perrin, in his book, “Les atomes” (1913), stated that irregular (nowhere differentiable) functions, far from being exceptional, are common in natural phenomena.

Fully developed turbulence

Internet traffic

Euro vs Dollar (2001-2009)

Density of population
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Jet turbulence Eulerian velocity signal (ChavarriaBaudetCiliberto95)

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Density of population

Multifractal analysis studies classification parameters for data (functions, measures, signals, images) based on regularity.
An orthonormal wavelet basis on $\mathbb{R}^d$ is generated by $2^d - 1$ smooth, well localized, oscillating functions $\psi^i$ such that the

$$2^{dj/2}\psi^i(2^j x - k),$$

$i = 1, \ldots 2^d - 1, j, k \in \mathbb{Z}^d$

form an orthonormal basis of $L^2(\mathbb{R}^d)$
Why use wavelet bases?

- Fast algorithms
- Sparse representations
- Characterization of regularity (global and pointwise)

\[
\int \psi(x) dx = \int x \psi(x) dx = \cdots = \int x^N \psi(x) dx = 0
\]

\[\implies \text{Wavelet analysis is blind to superimposed polynomial and (more generally) smooth trends}\]
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Wavelets translate hard problems on functions on (more) simple problems on sequences.
Wavelet structure functions

Notations:

Dyadic cubes: \( \lambda = \left[ \frac{k_1}{2^j}, \frac{k_1 + 1}{2^j} \right) \times \cdots \times \left[ \frac{k_d}{2^j}, \frac{k_d + 1}{2^j} \right) \)

Wavelet coefficients: \( c_\lambda = 2^{dj} \int_{\mathbb{R}^d} \int f(x, y) \psi^i \left( 2^j x - k \right) \, dx \)

Dyadic cubes at scale \( j \): \( \Lambda_j = \{ \lambda : |\lambda| = 2^{-j} \} \)
Wavelet structure functions

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\[ \forall p > 0, \quad S_f(p, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \]
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Wavelet structure functions:

\( \forall p > 0, \quad S_f(p, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \)

Wavelet scaling function:

\( \forall p > 0, \quad S_f(p, j) \sim 2^{-\zeta_f(p)j} \quad \text{when} \quad j \to +\infty \)
Wavelet scaling function

$\zeta_f(p) = p \cdot \sup\{s : f \in L^{p,s}\} = p \cdot \sup\{s : f \in B^{s,\infty}_p\}$
Wavelet scaling function

\[ \zeta_f(p) = p \cdot \sup \{ s : f \in L^{p,s} \} = p \cdot \sup \{ s : f \in B^{s,\infty}_p \} \]

Advantages of using the wavelet scaling function for classification:

- Effectively computable on experimental data through log-log plot regressions with respect to the scale parameter
- Independent of the (smooth enough) wavelet basis
- Invariant under the addition of polynomials or (smooth enough) trends
- “deformation invariant” (i.e. under a smooth change of coordinates)
- Deterministic for large classes of stochastic processes
Limitations of wavelet structure functions

Classification only based on structure functions proved insufficient in several occurrences (turbulence, ...)

This motivated new developments based on seminal ideas introduced by Uriel Frisch and Giorgio Parisi, and led to the construction of new structure functions
Pointwise regularity

A function \( f \) is **continuous** at \( x_0 \) if, in a neighborhood of \( x_0 \),

\[
f(x) = f(x_0) + o(1)
\]
Pointwise regularity

A function $f$ is **continuous** at $x_0$ if, in a neighborhood of $x_0$,

$$f(x) = f(x_0) + o(1)$$

$f$ is **differentiable** at $x_0$ if there exists an affine function, i.e. a polynomial $P(x - x_0)$ of degree at most 1, such that in a neighborhood of $x_0$

$$f(x) = P(x - x_0) + o(|x - x_0|)$$
Pointwise regularity

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**Extension to non-integer orders of derivation :**

Let $f$ be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$; $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$f(x) = P(x - x_0) + O(|x - x_0|^\alpha)$$
**Pointwise regularity**

A function $f$ is **continuous** at $x_0$ if, in a neighborhood of $x_0$,

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The **Hölder exponent** of $f$ at $x_0$ is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$$
Functions with constant Hölder exponents

Fractional Brownian motions with Hölder exponents 0.3, 0.4, 0.5, 0.6 and 0.7
Functions with varying Hölder exponent

\[ h_f(x) = x \]

Constructions obtained by K. Daoudy, J. Lévy-Véhel and Y. Meyer
Wavelet leaders

Idea: In the scaling function, replace increments by quantities which encapsulate information on pointwise regularity
Wavelet leaders

Idea: In the scaling function, replace increments by quantities which encapsulate information on pointwise regularity.

Let $\lambda$ be a dyadic cube; $3\lambda$ is the cube of same center and three times wider.

Let $f$ be a locally bounded function; the wavelet leaders of $f$ are the quantities

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$
Computation of 2D wavelet leaders

Wavelet leaders allow to estimate pointwise H"older exponents. Let $\lambda_j(x_0)$ denote the dyadic cube of width $2^{-j}$ which contains $x_0$. $d_{\lambda_j}(x_0) = \sup_{\lambda_j' \subset 3 \lambda_j(x_0)} |c_{\lambda_j'}|$

Theorem: If $H_{\text{min}} > 0$, then $\forall x_0 \in \mathbb{R}^d$: $h_f(x_0) = \lim \inf_{j \to +\infty} \frac{\log(d_{\lambda_j}(x_0))}{\log(2^{-j})}$
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Difficulty to use directly the pointwise regularity exponent for classification

For classical models, such exponents are extremely erratic

Lévy processes

The function $h$ is random and everywhere discontinuous

$\implies$ Impossible to estimate numerically
Difficulty to use directly the pointwise regularity exponent for classification

For classical models, such exponents are extremely erratic.

Lévy processes

The function $h$ is random and everywhere discontinuous.

$\implies$ Impossible to estimate numerically

Goal: Recover some information on $h(x)$ from averaged quantities which would be:

- numerically computable by log-log plot regressions
- deterministic (independent of the sample path)
Back to scaling functions

“Improve” the scaling function by using quantities that incorporate pointwise regularity information.

\[ \sum_{\lambda \in \Lambda_j} |c_{\lambda}| \sim 2^{-\zeta_j} f(p) \]

\[ \sum_{\lambda \in \Lambda_j} |d_{\lambda}| \sim 2^{-\eta_j} f(p) \]

Advantages:

▶ Same as the wavelet scaling function

▶ \( \eta_j f(p) \) is also defined for \( p < 0 \)

▶ \( \eta_j f(p) \) encapsulates information on the inter-scales correlations of wavelet coefficients.
Back to scaling functions

“Improve” the scaling function by using quantities that incorporate pointwise regularity information

\( \Lambda_j \) denotes the set of dyadic cubes of width \( 2^{-j} \)

Wavelet scaling function

\[
2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)j}
\]

Leader scaling function

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2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j}
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Back to scaling functions

“Improve” the scaling function by using quantities that incorporate pointwise regularity information

$\Lambda_j$ denotes the set of dyadic cubes of width $2^{-j}$

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Advantages:

- Same as the wavelet scaling function +
- $\eta_f(p)$ is also defined for $p < 0$
- $\eta_f(p)$ encapsulates information on the inter-scales correlations of wavelet coefficients
Heuristic derivation of the multifractal formalism

$E_f(H)$ is the set of points where $h_f(x) = H$ 
$D_f(H)$ denotes its Hausdorff dimension (i.e. the multifractal spectrum) 
$\Lambda_j$ denotes the set of dyadic cubes of width $2^{-j}$

Leader structure function: $T_f(p, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j}$
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Leader structure function: \( T_f(p, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_{\lambda}|^p \sim 2^{-\eta_f(p)j} \)

Estimation of the contribution to \( T_f(p, j) \) of the cubes of length \( 2^{-j} \) containing a point where the Hölder exponent takes the value \( H \):

On such a cube \( |d_{\lambda}| \sim 2^{-Hj} \) and there are \( \sim 2^{\mathcal{D}_f(H)j} \) such cubes

Thus the contribution is

\[ \sim 2^{-dj} \cdot (2^{-Hj})^p \cdot (2^{-j})^{\mathcal{D}_f(H)} = (2^{-j})^{d+Hp-\mathcal{D}_f(H)} \]
Heuristic derivation of the multifractal formalism

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Thus the contribution is

\[\sim 2^{-dj} \cdot (2^{-Hj})^p \cdot \left(2^{-j}\right)^{\mathcal{D}_f(H)} = (2^{-j})^{d + Hp - \mathcal{D}_f(H)}\]

In the limit \(j \to +\infty\), the main contribution comes from the smallest exponent, so that : \(\eta_f(p) = \inf_H (d + Hp - \mathcal{D}_f(H))\)

Thus \(\eta_f\) is expected to be the Legendre transform of \(\mathcal{D}_f\)
The Leader Legendre Spectrum

If $D_f$ is concave, it should be recovered from $\eta_f$ through an inverse Legendre transform

$$D_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

The Leader Legendre Spectrum is

$$L_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$
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$$L_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

Theorem: If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then

$$\forall H \in \mathbb{R}, \; D_f(H) \leq L_f(H)$$

When $D_f(H) = L_f(H)$ the multifractal formalism is satisfied
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When $D_f(H) = L_f(H)$ the multifractal formalism is satisfied

There are two kinds of validity theorems:

- Generic results (Baire and prevalence)
- Particular models
Model refutation: Fully developed turbulence
(joint work with Bruno Lashermes)

Jet turbulence Eulerian velocity signal (ChavarriaBaudetCiliberto95)

Log-normal vs. Log-Poisson model
Construction of a measure $\mu$ on the interval $[0, 1]$ : A quantity of total mass 1 of sand is poured at the top.
Construction of a measure $\mu$ on the interval $[0, 1]$ : A quantity of total mass 1 of sand is poured at the top

At each node,

1/4 of the remaining sand falls on the left and 3/4 on the right

$\mu(I) =$ quantity of sand falling inside $I$
Quantity of sand falling inside intervals of length $2^{-n}$:
Quantity of sand falling inside intervals of length $2^{-n}$:

\[
\begin{align*}
\text{far left} & : \mu(l) = \left(\frac{1}{4}\right)^n = |l|^2 \\
\text{far right} & : \mu(l) = \left(\frac{3}{4}\right)^n = |l|^{\log(4/3)/\log 2} \\
\text{average} & : \mu(l) = \left(\frac{1}{4}\right)^{n/2} \left(\frac{3}{4}\right)^{n/2} = |l|^{\log(4/\sqrt{3})/\log 2}
\end{align*}
\]

Exponents fluctuate from point to point.
Cascade models : Binomial cascade

Repartition function of the measure $\mu$:

$$f(x) := \mu([0, x])$$

= amount of sand falling in $[0, x]$
Cascade models: Binomial cascade

Repartition function of the measure $\mu$:

$$f(x) := \mu([0, x])$$

= amount of sand falling in $[0, x]$

$$f(x + \delta) - f(x) = \mu([x, x + \delta]) \sim \delta^{h_f(x)}$$

$$h_f(x) \in \left[ \frac{\log(4/3)}{\log 2}, 2 \right]$$
Cascade models: Binomial cascade

Repartition function of the measure $\mu$:

$$f(x) := \mu([0, x]) = \text{amount of sand falling in } [0, x]$$

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$f$ is a multifractal function
Cascade models: Binomial cascade

Repartition function of the measure \( \mu \):

\[
f(x) := \mu([0, x]) = \text{amount of sand falling in } [0, x]
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\[
f(x + \delta) - f(x) = \mu([x, x + \delta]) \sim \delta^{h_f(x)}
\]

\[
h_f(x) \in \left[\frac{\log(4/3)}{\log 2}, 2\right]
\]

\( f \) is a multifractal function

Rule of thumb:
The multifractal formalism holds for “homogeneous data”
Why multivariate analysis?

MEG recordings
Why multivariate analysis?

MEG recordings

A collection of signals are recorded simultaneously.
Multivariate analysis

Regularity exponents \( h_1(x), \ldots, h_m(x) \) are associated with each signal \( y_i(t) \)

Each exponent is associated with a *multiresolution quantity* \( d_{\lambda}^i \) through the formula

\[
\text{pour } i = 1, 2, \quad \forall x_0 \in \mathbb{R}^d \quad h_i(x_0) = \liminf_{j \to +\infty} \frac{\log \left( d_{\lambda}^i(x_0) \right)}{\log(2^{-j})}
\]

Let \( E(H_1, H_2) = E(H_1) \cap E(H_2) \)

\( E(H_1, H_2) \) is the set of points where \( h_1(x) = H_1 \) and \( h_2(x) = H_2 \)

The *joint multifractal spectrum* is (for \( m = 2 \))

\[
\mathcal{D}(H_1, H_2) = \dim( E(H_1, H_2) )
\]
The multivariate multifractal formalism

The multivariate structure function is

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_1^\lambda)^p (d_2^\lambda)^q \]

The multivariate scaling function is

\[ \forall p, q \in \mathbb{R}, \quad \eta(p, q) = \lim_{j \to +\infty} \inf \frac{\log(T_f(p, q, j))}{\log(2^{-j})} \]
The multivariate multifractal formalism

The multivariate structure function is

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$$\forall p, q \in \mathbb{R}, \, \eta(p, q) = \lim_{j \to +\infty} \inf \frac{\log(T_f(p, q, j))}{\log(2^{-j})}$$

The multivariate Legendre spectrum is

$$\mathcal{L}(H_1, H_2) = \inf_{(p, q) \in \mathbb{R}^2} (d + H_1 p + H_2 q - \eta(p, q))$$

The multivariate multifractal formalism holds if

$$\mathcal{D}(H_1, H_2) = \mathcal{L}(H_1, H_2)$$
Heuristic derivation of the multifractal formalism

\[ E(H_1, H_2) \] is the set of points where \( h_1(x) = H_1 \) and \( h_2(x) = H_2 \)

\[ D(H_1, H_2) \] is the Hausdorff dimension of \( E(H_1, H_2) \)

Structure functions:

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_1^\lambda)^p (d_2^\lambda)^q \]
Heuristic derivation of the multifractal formalism

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Structure functions :
\[
T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \mathcal{L}_j} (d^1_\lambda)^p (d^2_\lambda)^q
\]

We estimate the contribution to \( T_f(p, q, j) \) of dyadic cubes of width \( 2^{-j} \) which contain a point where \( h_1(x) = H_1 \) and \( h_2(x) = H_2 \) :
Heuristic derivation of the multifractal formalism

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For such a cube \( |d_{\lambda}^1| \sim 2^{-H_1j} \), and \( |d_{\lambda}^2| \sim 2^{-H_2j} \)
Heuristic derivation of the multifractal formalism

$E(H_1, H_2)$ is the set of points where $h_1(x) = H_1$ and $h_2(x) = H_2$

$\mathcal{D}(H_1, H_2)$ is the Hausdorff dimension of $E(H_1, H_2)$

Structure functions:

$$T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_\lambda^1)^p (d_\lambda^2)^q$$

We estimate the contribution to $T_f(p, q, j)$ of dyadic cubes of width $2^{-j}$ which contain a point where $h_1(x) = H_1$ and $h_2(x) = H_2$:

For such a cube $|d_\lambda^1| \sim 2^{-H_1 j}$, and $|d_\lambda^2| \sim 2^{-H_2 j}$

There are $\sim 2^{\mathcal{D}(H_1, H_2) j}$ cubes of this type.
Heuristic derivation of the multifractal formalism

\( E(H_1, H_2) \) is the set of points where \( h_1(x) = H_1 \) and \( h_2(x) = H_2 \)

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For such a cube \(|d_1^\lambda| \sim 2^{-H_1j}|\), and \(|d_2^\lambda| \sim 2^{-H_2j}|\)

There are \( \sim 2^{\mathcal{D}(H_1,H_2)j} \) cubes of this type. So this contribution is

\[
\sim 2^{-dj} \cdot (2^{-H_1j})^p (2^{-H_2j})^q \cdot 2^{\mathcal{D}(H_1,H_2)j} = (2^{-j})^{d+H_1p+H_2q-\mathcal{D}(H_1,H_2)}
\]

When \( j \to +\infty \), the main contribution is given by the smallest exponent, so that

\[
\eta(p, q) = \inf_{H} (d + H_1p + H_2q - \mathcal{D}(H_1, H_2))
\]
Heuristic derivation of the multifractal formalism

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T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d^1_\lambda)^p (d^2_\lambda)^q
\]

We estimate the contribution to \( T_f(p, q, j) \) of dyadic cubes of width \( 2^{-j} \) which contain a point where \( h_1(x) = H_1 \) and \( h_2(x) = H_2 \):

For such a cube \( |d^1_\lambda| \sim 2^{-H_1 j} \), and \( |d^2_\lambda| \sim 2^{-H_2 j} \)

There are \( \sim 2^{\mathcal{D}(H_1,H_2)j} \) cubes of this type. So this contribution is

\[
\sim 2^{-dj} \cdot (2^{-H_1 j})^p (2^{-H_2 j})^q \cdot 2^{\mathcal{D}(H_1,H_2)j} = (2^{-j})^{d+H_1 p+H_2 q-\mathcal{D}(H_1,H_2)}
\]

When \( j \to +\infty \), the main contribution is given by the smallest exponent, so that

\[
\eta(p, q) = \inf_H (d + H_1 p + H_2 q - \mathcal{D}(H_1, H_2))
\]

The bivariate multifractal spectrum is recovered by an inverse Legendre transform

\[
\mathcal{D}(H_1, H_2) = \inf_{p,q} (d + H_1 p + H_2 q - \eta(p, q))
\]
Inspecting the formula

**An extreme case:** \( f_1(x) = f_2(x) \) and \( h_1(x) = h_2(x) \)

Both spectra are carried by the diagonal

\[
\mathcal{D}(H_1, H_2) = \mathcal{D}(H_1) \quad \text{if} \quad H_1 = H_2
\]
\[
= -\infty \quad \text{else}
\]

\[
\mathcal{L}(H_1, H_2) = \mathcal{L}(H_1) \quad \text{if} \quad H_1 = H_2
\]
\[
= -\infty \quad \text{else}
\]

Red : Theoretical multifractal spectrum

Blue : Estimated Legendre spectrum

Multivariate multifractal analysis of a binomial cascade with itself
Independent processes

Assumption: Wavelet leaders are stationary with short range correlations only

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_\lambda^1)^p (d_\lambda^2)^q \sim \mathbb{E} \left( (d_\lambda^1)^p (d_\lambda^2)^q \right) \]

If the signals are independent, then

\[ T_f(p, q, j) = \mathbb{E} \left( (d_\lambda^1)^p \right) \mathbb{E} \left( (d_\lambda^2)^q \right) \]
Independent processes

Assumption: Wavelet leaders are stationary with short range correlations only

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\[ T_f(p, q, j) = T_f(p, j) T_f(q, j) \quad \text{and} \quad \eta(p, q) = \eta(p) + \eta(q) \]

\[ \mathcal{L}(H_1, H_2) = \mathcal{L}(H_1) + \mathcal{L}(H_1) - d \]
Independent processes

Assumption: Wavelet leaders are stationary with short range correlations only

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_1^\lambda)^p (d_2^\lambda)^q \sim \mathbb{E} \left( (d_1^\lambda)^p (d_2^\lambda)^q \right) \]

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\[ \mathcal{L}(H_1, H_2) = \mathcal{L}(H_1) + \mathcal{L}(H_2) - d \]

This is similar to the codimension formula for intersections

\[ \dim_H(A \cap B) = \dim_H(A) + \dim_H(B) - d \]

codimensions add up
Independent processes

Assumption: Wavelet leaders are stationary with short range correlations only

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d_1^\lambda)^p (d_2^\lambda)^q \sim E \left( (d_1^\lambda)^p (d_2^\lambda)^q \right) \]

If the signals are independent, then

\[ T_f(p, q, j) = E \left( (d_1^\lambda)^p \right) E \left( (d_2^\lambda)^q \right) \]

\[ T_f(p, q, j) = T_f(p, j) T_f(q, j) \quad \text{and} \quad \eta(p, q) = \eta(p) + \eta(q) \]

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codimensions add up

Generically true for fractal sets (P. Mattila et al.) under “reasonable” assumptions
Independent processes

Assumption: Wavelet leaders are stationary with short range correlations only

\[ T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (d^1_{\lambda})^p (d^2_{\lambda})^q \sim \mathbb{E}\left((d^1_{\lambda})^p (d^2_{\lambda})^q\right) \]

If the signals are independent, then

\[ T_f(p, q, j) = \mathbb{E}\left((d^1_{\lambda})^p\right) \mathbb{E}\left((d^2_{\lambda})^q\right) \]

\[ T_f(p, q, j) = T_f(p, j) T_f(q, j) \quad \text{and} \quad \eta(p, q) = \eta(p) + \eta(q) \]

\[ \mathcal{L}(H_1, H_2) = \mathcal{L}(H_1) + \mathcal{L}(H_1) - d \]

This is similar to the codimension formula for intersections

\[ \dim_H(A \cap B) = \dim_H(A) + \dim_H(B) - d \]

codimensions add up

Generically true for fractal sets (P. Mattila et al.) under “reasonable” assumptions

This is usually not true for the sets \( E(H_1, H_2) = E(H_1) \cap E(H_2) \)
Binomial cascades of parameters $p$ and $q$

$p$ and $q$ on the same side of $1/2$

Theoretical Legendre spectrum vs. its estimation on a sample path
Binomial cascades of parameters $p$ and $q$

$p$ and $q$ are on opposite sides of $1/2$

Theoretical Legendre spectrum vs. its estimation on a sample path
Independent lacunary wavelet series

Sample paths of two independent lacunary wavelet series

Theoretical multifractal spectra computation on a sample path $D(H)$ in blue

Theoretical $D(H)$ recalled in blue

$\mathcal{L}(H)$ in red

Computation on a sample path theoretical $D(H)$ recalled in blue

$\mathcal{L}(H)$ in red
Why is the multivariate multifractal formalism so wrong?

Multifractal spectra of many models and processes do not follow the codimension formula

\[ D(H_1, H_2) = \begin{cases} 
D(H_1) + D(H_2) - d & \text{if } D(H_1) + D(H_2) - d \geq 0 \\
-\infty & \text{else}
\end{cases} \]

but instead the large intersection formula

\[ D(H_1, H_2) = \min(D(H_1), D(H_2)) \]

Whereas, the Legendre spectrum of independent processes follows the codimension formula
Why is the multivariate multifractal formalism so wrong?

Multifractal spectra of many models and processes do not follow the codimension formula

\[ D(H_1, H_2) = \begin{cases} D(H_1) + D(H_2) - d & \text{if } D(H_1) + D(H_2) - d \geq 0 \\ -\infty & \text{else} \end{cases} \]

but instead the large intersection formula

\[ D(H_1, H_2) = \min(D(H_1), D(H_2)) \]

Whereas, the Legendre spectrum of independent processes follows the codimension formula

Generic results of validity of the multivariate multifractal formalism have been proved in products of function spaces (Mourad Ben Slimane et al.)
Intuitions and questions

- **Univariate miracle**: The leader structure functions
  \[ T_{p,j} = 2^{-d_j} \sum_{\lambda \in \Lambda_j} |d_{\lambda}|^p \sim 2^{-\eta_j(p)j} \]
  simultaneously have a function space and a probabilistic interpretation, whereas the multivariate structure functions
  \[ S_{p,q,j} = 2^{-d_j} \sum_{\lambda \in \Lambda_j} (d_{\lambda}^1)^p (d_{\lambda}^2)^q \]
  have no function space interpretation and have a probabilistic interpretation only in the independent case.

- Are there “natural” function spaces which “encode” some correlation between wavelet leaders?

- The multivariate structure functions do not take into account cross-scale correlations between wavelet leaders.

- Which information does the multivariate structure functions yield?

- Work out more examples to get some intuition!

- What is John staring at?
A few references


- **Multifractal formalisms for multivariate analysis**, S. Jaffard, S. Seuret, H. Wendt, R. Leonarduzzi, P. Abry, To appear Proceedings Royal Society A

Thank you for your attention!