THE GEOMETRY OF EMBEDDED CONSTANT MEAN CURVATURE TORI IN THE 3-SPHERE VIA INTEGRABLE SYSTEMS

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Abstract. We introduce the moduli space of spectral curves of constant mean curvature cylinders of finite type in the 3-sphere. Moduli space parameters are a hyperelliptic Riemann surface and a meromorphic one form. The subset of spectral curves of mean convex Alexandrov embedded cylinders is explicitly determined. We prove that all embedded \( \text{cmc} \) tori in the 3-sphere are surfaces of revolution using a combination of integrable systems methods and geometric analysis techniques.

Introduction

Imposing a combination of topological and geometric properties on a surface can severely restrict the number of examples. A particularly intriguing problem was posed by Lawson, who conjectured that the Clifford torus is the only minimal embedded torus in the 3-sphere. Recently Brendle [6] provided an ingenious analytical proof of this conjecture. Andrews and Li [3] have remarkably succeeded in extending these arguments and confirmed the conjecture by Pinkall and Sterling [32], which asserts that all embedded constant mean curvature (CMC) tori in \( \mathbb{S}^3 \) are surfaces of revolution. Subsequently Brendle [7] extended his result to Alexandrov embedded minimal tori.

The purpose of this paper is to provide conceptually very different proofs of the results by Brendle, Andrews and Li. We use integrable systems methods to reformulate the classification of constant mean curvature Alexandrov embedded tori into a problem of space moduli of hyperelliptic Riemann surfaces, and obtain independent proofs of the following two theorems:

Theorem 1. All mean convex Alexandrov embedded CMC tori in the 3-sphere are tori of revolution. In particular, all embedded CMC tori in the 3-sphere are tori of revolution.

Theorem 2. The Clifford torus is the only embedded minimal torus in the 3-sphere.

To prove these results, we develop moduli space techniques, which we hope can be useful in other situations to resolve uniqueness problems in integrable surface geometry (see e.g [15]). To explain our approach recall that Pinkall and Sterling [32] constructed CMC tori in \( \mathbb{R}^3 \) and independently Hitchin [17] harmonic maps from tori into \( \mathbb{S}^3 \) in terms of a real algebraic curve \( \Sigma \) and a holomorphic line bundle \( L \) on \( \Sigma \) via integrable systems theory. This algebro-geometric correspondence between pairs \((\Sigma, L)\) and geometric objects turned out to be a powerful tool for the construction of new examples. We want to enhance these methods in order to classify the corresponding geometric objects, focusing on embedded CMC tori in \( \mathbb{S}^3 \).

Consider a doubly periodic conformal immersion \( f : \mathbb{C} \to \mathbb{S}^3 \) with constant mean curvature \( H \). The Hopf differential \( Q = \langle f_z, f_{\bar{z}} \rangle dz d\bar{z} \) is holomorphic, and hence without zeroes on a torus. The exponent \( 2\omega \) in the induced metric \( ds^2 = \frac{1}{4(H^2 + 1)} e^{2\omega} dz \otimes d\bar{z} \) satisfies the sinh-Gordon equation

\[
\Delta \omega + \sinh(\omega) \cosh(\omega) = 0.
\]

Mathematics Subject Classification. 53A10, 37K10. September 17, 2013.
Infinitesimal variations that preserve the CMC condition require that in the normal direction the variation is prescribed by a function $u$ in the kernel of the linearized sinh-Gordon equation

$$\mathcal{L}u = \Delta u + u \cosh(2\omega) = 0.$$  

Pinkall-Sterling provide an iteration to obtain an infinite hierarchy of solutions $u_1, u_2, ...$ of the linearized sinh-Gordon equation $\mathcal{L}u_n = 0$. Applying this iteration to the trivial solution $u_0 \equiv 0$ yields the sequence of Jacobi fields

$$u_0 = 0, \quad u_1 = \omega_z, \quad u_2 = \omega_{zzz} - 2\omega_z^3, \quad u_3 = \omega_{zzzzz} - 10\omega_{zzz}\omega_z^3 - 10\omega_z^2\omega_z + 6\omega_z^5 \ldots .$$

The remarkable fact proven by Pinkall-Sterling is that for a doubly-periodic solution $\omega$ of the sinh-Gordon equation this iteration becomes stationary, so there exists an integer $g \geq 0$ and real constants $a_1, \ldots, a_g$ such that

$$u_g = a_1 u_1 + \ldots + a_{g-1} u_{g-1}.$$ 

Equivalently this means that the kernel of $\mathcal{L}$ is finite dimensional. This leads us to the definition of finite type: A solution of the sinh-Gordon equation is of finite type if the kernel of the linearized sinh-Gordon equation is finite dimensional. Thus the result by Pinkall-Sterling can be rephrased into the statement that all doubly-periodic real solutions of the sinh-Gordon equation are of finite type. In particular all CMC tori are thus said to be of finite type.

The image of a doubly periodic immersion restricted to a period parallelogram is a torus. The image of a strip obtained by extending one of the generators is then an infinite covering of the torus by a cylinder. Such CMC cylinders are also of finite type, but the class of CMC cylinders of finite type is much larger than such covers of CMC tori. A conformally immersed CMC cylinder is of finite type if there exists a conformal parametrization with constant Hopf-differential, and its conformal factor is a singly-periodic solution of the sinh-Gordon equation of finite type. When the curvature is uniformly bounded then the function $\omega$ is uniformly bounded. A theorem of Mazzeo and Pacard [27] states that an elliptic operator $\mathcal{L}u = \Delta u + gu$ on a cylinder $S^1 \times \mathbb{R}$ has for bounded continuous function $g$ a finite dimensional kernel in the space of uniformly bounded $C^2$ functions on $S^1 \times \mathbb{R}$. Hence for CMC cylinders of bounded curvature with constant Hopf differential the metric is of finite type in the meaning of Pinkall-Sterling. Equivalently we thus have that a conformally parameterized CMC cylinder with constant Hopf-differential and bounded curvature is of finite type.

Let $\mathcal{A}$ denote the set of such CMC-cylinders in $\mathbb{R}^3$ of finite type up to ambient isometries. Our main objective is the investigation of this space $\mathcal{A}$. To obtain nice parameters for $\mathcal{A}$ we invoke the algebro-geometric correspondence, which identifies a geometric object with an algebraic one. If the geometric object is a finite type CMC cylinder $f \in \mathcal{A}$, then the finite sequence $u_1, \ldots, u_d$ and the constants $a_1, \ldots, a_d$ above are encoded into a $2 \times 2$ matrix of polynomials, called a polynomial Killing field. A polynomial Killing field $\zeta$ is real analytic in $z$ and polynomial in $\lambda$. It solves a Lax equation, and thus its determinant $\lambda \mapsto a(\lambda)$ is a polynomial independent of $z$. Thus we associate with a cylinder $f \in \mathcal{A}$ a complex polynomial $a \in \mathbb{C}[\lambda]$ of degree $2g$, which in turn defines a hyperelliptic Riemann surface $\Sigma$ of genus $g$, and is called the spectral curve of $f$. The genus $g$ of $\Sigma$ is called the spectral genus. For $g = 0$, the corresponding cylinders are homogeneous, while for $g = 1$, the corresponding cylinders are in the family of Delaunay surfaces. The curve $\Sigma$ does not uniquely determine the cylinder, but there exists a finite dimensional compact subset of $\mathcal{A}$, all whose elements $f$ have the same spectral curve. We call this set the isospectral set.

Arguments of Hitchin [17] encode the non-trivial topology of $f \in \mathcal{A}$ in a meromorphic 1-form $dh$ on $\Sigma$. For given $\Sigma$ with polynomial $a$ the differential $dh$ is described by another polynomial $b$. Furthermore two unimodular complex numbers $\lambda_1$ and $\lambda_2$ parameterise the mean curvature and
the Hopf-differential. We call quadruples \((a, b, \lambda_1, \lambda_2)\) spectral data. For given spectral data \((a, b, \lambda_1, \lambda_2)\) the corresponding set of cylinders \(f \in \mathcal{A}\) build the isospectral set \(\mathcal{A}(a, b, \lambda_1, \lambda_2)\).

The Whitham equations define vector fields on the space of quadruples \((a, b, \lambda_1, \lambda_2)\) whose integral curves define deformation families in \(\mathcal{A}\). The corresponding flows simultaneously deform \(\Sigma, \text{dh}\) and the points \(\lambda_1\) and \(\lambda_2\). Along this flow the curvature can blow up. It turns out that the curvature stays uniformly bounded on the cylinder as long as the roots of \(a\) stay away from the poles of \(\text{dh}\). A typical limit of a curvature blow up is a chain of spheres touching each other at the limiting points of shrinking necks. Another accident of the flow are coalescing roots of \(a\). The limits are higher order roots of \(a\) and called singularities of \(\Sigma\), since the corresponding algebraic curve is not any more a complex manifold. The dimension of the isospectral set of \(\Sigma\) is the spectral genus, so equal to half the number of roots of \(a\). It turns out that in case of roots of \(a\) coalescing on \(S^1\) one dimension of the isospectral set shrinks to a point. In this case the limit of the isospectral sets coincides with the isospectral set of the desingularised curve \(\Sigma\) and there is no geometric accident. In case of roots of \(a\) coalescing at points away from \(S^1\) the limit of the isospectral sets is a union of the compact isospectral set of the desingularised curve and an extra higher-dimensional non-compact part. The lower-dimensional compact part is the closure of the higher-dimensional non-compact part and the union is still compact. The movement from the isospectral set of the desingularised curve to the extra non-compact part drastically changes the corresponding geometric cylinder. In this case there is a geometric accident.

As an application of the deformation of spectral data we establish global properties of the moduli space. We construct for all spectral data of \(f \in \mathcal{A}\) a path in the moduli space, which starts at the given spectral data and ends at spectral data of spectral genus zero. Along this path the curvature stays bounded and no geometric accidents happen (Theorem 3).

Since coverings of CMC tori are not embedded cylinders in \(S^3\), we relax the notion of embeddedness to the weaker notion of mean convex Alexandrov embeddedness, which turns out to be stable under continuous deformations in the space \(\mathcal{A}\). The spectral curves of homogeneous cylinders can effectively be classified by explicit calculation. In fact we determine all spectral data of spectral genus zero corresponding to mean convex Alexandrov embedded cylinders. Moreover, we extend the corresponding family to a two-dimensional family of spectral data of spectral genus at most one. The corresponding CMC cylinders are rotational mean convex Alexandrov embedded cylinders. We call this family of spectral data the rotational family. In Theorem 4 we show that the rotational family is connected with spectral curves outside of this family only by a movement from the lower-dimensional compact part of a non-singular spectral curve into the higher-dimensional extra non-compact part of a singular spectral curve as explained above. Here the desingularised curve of the latter spectral curve is the former spectral curve. As a consequence of the two theorems 3 and 4 we characterise in Theorem 5 the rotational family.

In the second part of the paper we apply our results on the moduli space and classify mean convex Alexandrov embedded cylinders of finite type in \(S^3\) - they are exactly the rotational cylinders. In section 8 and 9 we show that mean convex Alexandrov embeddedness is preserved by continuous deformations of the spectral data of mean convex Alexandrov embedded cylinders, as long as the degree of \(a\) is preserved. Using the non-compactness of cylinders instead of compact tori gives more space for deformations and allows to reduce the spectral genus to zero. But the control of the Alexandrov embeddedness during the deformation becomes more difficult. In general we have continuity only with respect to the topology defined by uniform distances on compact subsets. In order to preserve the Alexandrov embeddedness along such deformations, we need to localize the concept of Alexandrov embeddings and divide the cylinder into compact Alexandrov embedded pieces. Then we show first that along continuous deformations the compact pieces stay Alexandrov embedded. For this purpose we need a collar of the 3-manifold with boundary, whose depth is uniformly bounded from below. This follows from a maximum
principle at infinity. Secondly, we have to prove that these local Alexandrov embedded pieces can be glued to a global Alexandrov embedding. Here we show with the help of a chord arc bound, that inside large enough overlapping boundaries the chord distances of two such pieces are the same. This allows us to glue two overlapping Alexandrov embeddings from the boundary to the interior.

Let us briefly describe the organization of the paper. In the first section we outline all statements of the paper and present the architecture of the proof of all the main results of the paper. The two subsequent sections then recall the integrable systems theory of the construction of \( \text{cmc} \) cylinders immersed in \( \mathbb{S}^3 \). In particular in section 2 we encounter the sinh-Gordon equation, the notion of extended frame and the Sym-Bobenko formula. In section 3, we describe isospectral sets and the algebro-geometric correspondence for finite type cylinders. We then turn to an example in section 4 and describe the spectral data of rotational cylinders of Delaunay type. The purpose of section 5 is to adapt the isoperiodic deformation to our setting, and using the deformation of spectral data to flow through \( \text{cmc} \) cylinders. We develop general tools to deform spectral data. In section 6 we construct continuous paths from spectral data of arbitrary \( \text{cmc} \) cylinders of finite type to spectral data of genus zero. In section 7 we prove an isolated property in of the rotational family. In section 8 we provide a detailed analysis of mean convex Alexandrov embedded surfaces in \( \mathbb{S}^3 \). We prove with a maximum principle at infinity that for \( \text{cmc} \) mean convex Alexandrov embedded surfaces with bounded curvature the cut locus function is bounded from below (Proposition 8.1). We show that mean convex Alexandrov embedded surfaces obey a chord-arc bound, if the cut locus function has a positive lower bound, and the covariant derivative of the second fundamental form has a uniform upper bound (Proposition 8.2). Finally we explain how to proceed from local mean convex Alexandrov embeddings to global mean convex Alexandrov embeddings via local collar perturbations (Proposition 8.5). In the final section 9 we apply the results of section 8 to the cylinders \( f \in \mathcal{A} \). We show that mean convex Alexandrov embeddedness is preserved firstly under isospectral deformations (Proposition 9.1) and secondly under continuous deformations of spectral data (Proposition 9.2). This finishes the proof of theorems 1 and 2. In the appendix we prove Proposition 8.1 and Proposition 8.2 used in section 8.

1. Statement of the main Results with Applications

In this section we present our main results. We focus on the use of spectral data in our approach. Therefore we first give the definitions of surfaces and their spectral data. We postpone most technical details to the subsequent sections. In particular the integrable system machinery is used as a black box and will be explained in the two subsequent sections 2 and 3.

1.1. \textbf{cmc cylinders of finite type in} \( \mathbb{S}^3 \). We restrict our attention to conformal \( \text{cmc} \) immersions \( f : \mathbb{C}^* \to \mathbb{S}^3 \) from the parabolic cylinder to the round unit 3-sphere with some additional properties inherited from properties of \( \text{cmc} \) tori in \( \mathbb{S}^3 \).

\textbf{Definition 1.1.} A \textit{conformal} \( \text{cmc} \) immersion \( f : \mathbb{C}^* \to \mathbb{S}^3 \) from the parabolic cylinder to the three-dimensional round sphere is of finite type, if it has bounded curvature and if there exists a conformal parameter \( z \in \mathbb{C}/\tau \mathbb{Z} \simeq \mathbb{C}^* \) with \( \tau \in \mathbb{C}^* \) such that the Hopf differential is constant in this parametrization. Dividing out the ambient isometries \( \text{iso}(\mathbb{S}^3) \) we define

\[ \mathcal{A} = \{ \text{cmc-cylinders in } \mathbb{S}^3 \text{ of finite type} \} / \text{iso}(\mathbb{S}^3). \]

\textbf{Definition 1.2.} A \textit{mean convex Alexandrov embedding} in \( \mathbb{S}^3 \) is a smooth immersion \( f : M \to \mathbb{S}^3 \) from a connected surface \( M \) which extends as an immersion to a connected 3-manifold \( N \) with boundary \( M = \partial N \) with the following properties:
(i) The mean curvature of \( M \) in \( S^3 \) with respect to the inward normal is non-negative everywhere.

(ii) The manifold \( N \) is complete with respect to the metric induced by \( f \).

An immersion \( f : M \to S^3 \) just obeying condition (ii) is called an **Alexandrov embedding**. Let

\[
\mathcal{A}_{\text{Ae}} = \{ f \in \mathcal{A} \mid f \text{ is mean convex Alexandrov embedded} \}.
\]

**1.2. Spectral data.** For each \( f \in \mathcal{A} \) we construct a polynomial

\[
a \in \mathbb{C}^{2g}[\lambda]
\]

of degree \( 2g \), which obeys the reality conditions

\[
\lambda^{2g} \bar{a}(\bar{\lambda}^{-1}) = a(\lambda) \quad \text{and} \quad \lambda^{-g} a(\lambda) \leq 0 \text{ for all } \lambda \in S^1.
\]

This polynomial defines a hyperelliptic curve with three involutions:

\[
\begin{align*}
(1.2) \quad & \Sigma^* = \{ (\lambda, \nu) \in \mathbb{C}^* \times \mathbb{C} \mid \nu^2 = \lambda^{-1} a(\lambda) \}, \\
(1.3) \quad & \sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \quad \rho : (\lambda, \nu) \mapsto (\bar{\lambda}, \bar{\nu}), \quad \eta : (\lambda, \nu) \mapsto (\bar{\bar{\lambda}}, -\nu).
\end{align*}
\]

By construction we have a map \( \lambda : \Sigma^* \to \mathbb{C}^* \) of degree 2, which is branched at the \( 2g \)-pairwise distinct roots \( \{ \alpha_1, \ldots, \alpha_g, \bar{\alpha}_1, \ldots, \bar{\alpha}_g \} \) of the polynomial \( a \). By declaring the points over \( \lambda = 0, \infty \) to be two further branch points, we then have \( 2g + 2 \) branch points. This 2-point compactification \( \Sigma \) is the called the **spectral curve**. The genus \( g \) is called the **spectral genus**.

Following Hitchin [17] we encode the non-trivial topology of \( f \in \mathcal{A} \) in a meromorphic differential 1-form \( dh \) on \( \Sigma \). We write \( dh \) in terms of a polynomial

\[
b \in \mathbb{C}^{g+1}[\lambda]
\]

of degree \( g + 1 \) such that

\[
\lambda^{g+1} b(\bar{\lambda}^{-1}) = -b(\lambda) \quad \text{and} \quad dh = \frac{b(\lambda)d\lambda}{\nu \lambda^2}.
\]

The meromorphic differential \( dh \) has only second order poles without residues at \( \lambda = 0 \) and \( \lambda = \infty \). Moreover the exponent of the integral \( e^h \) has to be single-valued holomorphic function on \( \Sigma^* \). Finally two values \( \lambda_1 \neq \lambda_2 \in S^1 \) parameterise the mean curvature and the Hopf differential

\[
(1.5) \quad H = \frac{1}{16} \frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1} \quad \text{and} \quad Qdz^2 = \frac{i}{16} (\lambda_1^{-1} - \lambda_2^{-1}) dz^2.
\]

We call \( \lambda_1 \) and \( \lambda_2 \) **Sym points**. Putting this all together we obtain the spectral data of \( f \in \mathcal{A} \):

**Definition 1.3.** For all spectral genera \( g \in \mathbb{N} \cup \{0\} \) the spectral data of \( f \in \mathcal{A} \) consists of complex polynomials \( a \) and \( b \) of degree \( 2g \) and \( g + 1 \), and Sym points \( \lambda_1 \neq \lambda_2 \in S^1 \) such that:

(i) Reality conditions (1.1) and (1.4) hold.

(ii) \( \text{Re} \left( \int_{\alpha_i}^{1/\alpha_i} \frac{b(\lambda)}{\nu \lambda^2} d\lambda \right) = 0 \) for all roots \( \alpha_i \) of \( a \) where the integral is computed on the straight segment \( [\alpha_i, 1/\alpha_i]^\parallel \).

(iii) Let \( \gamma_i \subset \Sigma \) be the closed cycles over the straight segments \( [\alpha_i, 1/\alpha_i] \) and \( \Sigma = \Sigma \setminus \cup \gamma_i \). There exists a unique meromorphic function \( h \) on \( \Sigma \) such that

\[
dh = \frac{b(\lambda)}{\nu \lambda^2} \quad \text{and} \quad \sigma^* h(\lambda) = -h(\lambda).
\]

This function continuously extends to boundary cycles over the segments \( [\alpha_i, \bar{\alpha}_i^{-1}] \) and is assumed to take at all roots of \( a \) values in \( \pi i \mathbb{Z} \).
We shall use two properties of rotational family of finite type cylinders. They are surfaces of revolution. We call this family $A$ path-connected sets launay surfaces. The homogeneous have spectral genus $\text{cmc} = 1.3$.

Moreover at all roots $\alpha_i$ of $a(\lambda)$ we have $g(\alpha_i) = \pm 1$.

(v) $|a(0)| = \frac{1}{16}$ and $\lambda_2 = \lambda_1^{-1}$ with $\text{Im}(\lambda_1) < 0$.

Remark 1.6. Multiplying $a$ with positive constants and rotation $\lambda \mapsto e^{i\varphi} \lambda$ only changes the parametrisation of the cylinder. Condition (v) fixes these two trivial transformations.

Definition 1.3 is analogous to [14, Definition 5.10]. Conditions (i)-(ii) determine for all polynomials $a$ of degree $2g$ obeying (1.1) a real two-dimensional space of polynomials $b$ of degree $g + 1$. Condition (iii) is implicitly a condition on the polynomial $a$ and characterises spectral curves of periodic solutions of the corresponding integrable equation. Condition (iv) is an extra condition ensuring the periodicity of the immersed cylinder. Proposition 3.10 states the algebro-geometric correspondence for elements $f \in \mathcal{A}$. As explained in section 3 for spectral data $(a, b, \lambda_1, \lambda_2)$ with spectral genus $g \in \mathbb{N} \cup \{0\}$ there exists a finite-dimensional compact set $\mathcal{A}(a, b, \lambda_1, \lambda_2) \subset \mathcal{A}$. For each $f \in \mathcal{A}$ there exist spectral data $(a, b, \lambda_1, \lambda_2)$ with spectral genus $g \in \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}(a, b, \lambda_1, \lambda_2)$.

In Proposition 3.9 we show that the curvature is uniformly bounded as long as all roots of the polynomial $a$ (normalized by $|a(0)| = \frac{1}{16}$) are bounded away from $\lambda = 0$ and $\lambda = \infty$.

Definition 1.5. For $g \in \mathbb{N} \cup \{0\}$ denote

$$\mathcal{M}^g = \{ (a, b, \lambda_1, \lambda_2) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \times S^1 \times S^1 \mid (i)-(v) \text{ of definition 1.3 hold} \},$$

$$\mathcal{M} = \bigcup_{g \in \mathbb{N} \cup \{0\}} \mathcal{M}^g.$$ 

The subsets of spectral data of cmc cylinders with non-negative mean curvature are denoted by $\mathcal{M}^g_+ \subset \mathcal{M}^g$ and $\mathcal{M}_+ \subset \mathcal{M}$. Due to (1.5) the mean curvature is non-negative, if the arc of $S^1$ starting at $\lambda_1$ in the anti-clockwise direction and ending at $\lambda_2$ has length not larger than $\pi$. We call this arc the short arc. The arc starting at $\lambda_2$ in the anti-clockwise direction and ending at $\lambda_1$ is called the long arc.

Remark 1.6. The complex structure determines together with an orientation of $S^3$ a unique normal on the cylinder and the sign of $H$. The antipodal map is an isometry of $S^3$ which reverses the orientation. Due to (2.2) it corresponds to an interchange of the Sym points. Therefore spectral data with interchanged Sym points have in $\mathcal{A}$ the same isospectral sets. In particular, each $f \in \mathcal{A}$ is contained in $\mathcal{A}(a, b, \lambda_1, \lambda_2)$ for some $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_+$.

1.3. The rotational family. Rotational cmc cylinders in $\mathbb{S}^3$ are analogous to rotational cmc cylinders in $\mathbb{R}^3$: they are either homogeneous, or lie in the 1-parameter family of Delaunay surfaces. The homogeneous have spectral genus $g = 0$, while the Delaunay surfaces have spectral genus $g = 1$. In Section 4 we construct in analogy to the Delaunay surfaces two path-connected sets $\mathcal{M}^0_{\text{rot}} \subset \mathcal{M}_+^0$ and $\mathcal{M}^1_{\text{rot}} \subset \mathcal{M}_+^1$ of spectral data. The corresponding surfaces $\mathcal{A}_{\text{rot}} = \bigcup_{(a,b,\lambda_1,\lambda_2)\in \mathcal{M}_{\text{rot}}} \mathcal{A}(a,b,\lambda_1,\lambda_2)$ are mean convex Alexandrov embeddings $f : \overline{D} \times \mathbb{R} \to \mathbb{S}^3$ of finite type cylinders. They are surfaces of revolution. We call this family $\mathcal{M}_{\text{rot}} = \mathcal{M}^0_{\text{rot}} \cup \mathcal{M}^1_{\text{rot}}$ rotational family.

We shall use two properties of $\mathcal{M}_{\text{rot}}$ proven in Proposition 4.1. Firstly there exists an embedding

$$\mathcal{M}^0_{\text{rot}} \hookrightarrow \mathcal{M}^1_{\text{rot}}, \quad (a, b, \lambda_1, \lambda_2) \mapsto ((\lambda + 1)^2a, (\lambda + 1)b, \lambda_1, \lambda_2).$$
Secondly all mean convex Alexandrov embedded annuli of spectral genus 0 belong to $A_{\text{rot}}$:

\[(1.7) \quad \mathcal{M}^0_{\text{rot}} = \{(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^0_+ \mid A(a, b, \lambda_1, \lambda_2) \cap A_{\text{AE}} \neq \emptyset\} \]

\[= \{(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^0_+ \mid A(a, b, \lambda_1, \lambda_2) \subset A_{\text{AE}}\} \] .

Since $A(a, b, \lambda_1, \lambda_2) \subset A_{\text{AE}}$ for all $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^0_{\text{rot}}$ the first equality implies the second.

### 1.4. Higher order roots of $a$

For some $f \in A$ the polynomial $a$ of the corresponding spectral data $(a, b, \lambda_1, \lambda_2)$ has higher order roots at some $\lambda = \alpha \in \mathbb{C}^*$. In this case $\Sigma^*$ (defined in (1.2)) is not a submanifold of $\mathbb{C}^* \times \mathbb{C}$ at $(\lambda, \nu) \in (\alpha, 0)$, and $(\alpha, 0)$ is called a singularity of $\Sigma$. Hyperelliptic curves $\Sigma$ and $\bar{\Sigma}$, whose polynomials $\tilde{a} = p^2 a$ differ by the square of a polynomial $p$ have the same meromorphic functions. For all $\tilde{a}$ with higher order roots, there exists a polynomial $p$ and a polynomial $a$ with $\tilde{a} = p^2 a$ such that $a$ has only simple roots. In this way $\Sigma$ is considered as the desingularised $\bar{\Sigma}$ with the same meromorphic functions.

We next characterise pairs $(a, b, \lambda_1, \lambda_2)$ of spectral data in $\mathcal{M}$ for which $\tilde{a} = p^2 a$ and $\tilde{b} = pb$. We decorate the objects corresponding to $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2)$ with a tilde.

Suppose first $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}^g$. Choose any polynomial $p$ such that $p^2$ divides $\tilde{a}$, and

\[(1.8) \quad \chi_{\text{deg}, p} \bar{p}(\lambda^{-1}) = p(\lambda) \quad |p(0)| = 1.\]

Condition (iv) in Definition 1.3 implies that $\tilde{h}$ is holomorphic on $\bar{\Sigma}^*$ and $p$ divides $\tilde{b}$. For $a = \tilde{a}/p^2$ and $b = \tilde{b}/p$ we have $dh = \tilde{dh}$ with $\lambda = \tilde{\lambda}$. Then $(a, b, \lambda_1, \lambda_2)$ obeys conditions (i)-(v) with $f = p \tilde{f}$ and $g = \tilde{g}$.

Conversely, for $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g$ we set $\tilde{a} = p^2 a$ and $\tilde{b} = pb$. This implies $\tilde{h} = h$ with $\tilde{\lambda} = \lambda$. The relations $\sigma^* h = -h$ and $\sigma^* \nu = -\nu$ imply

\[g = \cosh(h) = \cosh(\tilde{h}) = \tilde{g}, \quad \frac{f}{p} = \frac{\sinh(h)}{\nu p} = \frac{\sinh(\tilde{h})}{\nu} = \tilde{f}.\]

For $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2)$ to satisfy condition (iv), $p$ must divide $\frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \frac{\sinh(h)}{\nu(\lambda - \lambda_1)(\lambda - \lambda_2)}$. Thus $\sinh(h)$ vanishes at the roots of $p$. Differentiation gives that the following meromorphic function is either holomorphic or has first order poles at the roots of $p$:

\[(1.9) \quad \frac{dh}{(\lambda - \lambda_1)(\lambda - \lambda_2)\nu p(\lambda)} = \frac{b(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)\lambda^2 a(\lambda)p(\lambda)}.\]

For such $p$ we have indeed $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}^g$. We summarise the discussion in the following

**Lemma 1.7.** For $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}^g$ choose any polynomial $p$ obeying (1.8) such that $p^2$ divides $\tilde{a}$. Then $p$ divides $\tilde{b}$ and $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^{g-\text{deg}, p}$ with $\tilde{a} = p^2 a$ and $\tilde{b} = pb$.

Conversely, suppose $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g$ and $p$ obeys (1.8). If in addition $\sinh(h)$ vanishes at the roots of $p$, and the function (1.9) has at the roots of $p$ at worst simple poles, then $(p^2 a, pb, \lambda_1, \lambda_2) \in \mathcal{M}^{g+\text{deg}, p}$.

In [14, Sections 4 and 6] the corresponding sets $A(p^2 a, pb, \lambda_1, \lambda_2)$ and $A(a, b, \lambda_1, \lambda_2)$ are investigated. For pairs $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g$ and $(p^2 a, pb, \lambda_1, \lambda_2) \in \mathcal{M}^{g+\text{deg}, p}$ as in Lemma 1.7 we have $A(a, b, \lambda_1, \lambda_2) \subset A(p^2 a, pb, \lambda_1, \lambda_2)$ with equality if all roots of $p$ belong to $S^1$. This implies $A(a, b, \lambda_1, \lambda_2) = A((\lambda + 1)^2 a, (\lambda + 1)b, \lambda_1, \lambda_2)$ for $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^0_{\text{rot}}$ in (1.6).

We interpret the supplementation of non-real singularities (away from $S^1$) as an enrichment of the complexity and the removal as a reduction of the complexity. Geometrically this corresponds to adding or removing bubbletons by a suitable Bianchi-Bäcklund transform. Adding or removing a unimodular singularity does not change the complexity. It will turn out, that the
enrichment of complexity destroys Alexandrov embeddedness, while the reduction of complexity preserves Alexandrov embeddedness.

**Definition 1.8.** A path in $\mathcal{M}$ is called piecewise continuous, if the path is continuous within one $\mathcal{M}^g$ besides finitely many jumps from spectral data $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g$ to $(p^2a, pb, \lambda_1, \lambda_2) \in \mathcal{M}^{g+\deg p}$ or vice versa from $(p^2a, pb, \lambda_1, \lambda_2) \in \mathcal{M}^{g+\deg p}$ to $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g$ as in Lemma 1.7.

1.5. **Paths to spectral genus zero.** We can now state the first main Theorem.

**Theorem 3.** At any $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_+^g$ there starts a compact piecewise continuous path $\gamma$ in $\mathcal{M}_+\to\mathcal{M}_0^g$. Along $\gamma$ the spectral genus increases at most by one at a multiple root of $a$ in $\mathbb{S}^1$.

The proof is given in Section 6. It uses deformations of spectral data introduced in Section 5. We integrate vector fields on the space of spectral data $(a, b, \lambda_1, \lambda_2)$, whose flows preserve the subset $\mathcal{M}_+^g$. Along this finite piecewise continuous path the roots of the polynomial $a$ will stay away from $\lambda = 0$ and $\lambda = \infty$ and the Sym points stay away from each other. Proposition 3.9 implies that the curvature remains uniformly bounded during the deformation.

The spectral data of spectral genus zero can be calculated explicitly. Therefore this Theorem allows to determine the connected components of the moduli space.

1.6. **Isolated property of $\mathcal{M}_{\text{rot}}$ in $\mathcal{M}_{\text{Ae}}$.** Now we state the second main Theorem.

**Theorem 4.** (i) For $g = 0, 1$: $\mathcal{M}_{\text{rot}}^g$ is open and closed in $\mathcal{M}_+^g$.

(ii) Let $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^1$ and $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^{1+\deg p}$ with $\deg p \geq 1$ be as in Lemma 1.7. At $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2)$ starts a piecewise continuous path in $\mathcal{M}_+$ to $\mathcal{M}_0^g \setminus \mathcal{M}_{\text{rot}}^g$ with decreasing $g$.

The first statement shows that a piecewise continuous path can only enter or leave $\mathcal{M}_{\text{rot}}$ at jumps described in Lemma 1.7 from $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^g$ to $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \not\in \mathcal{M}_{\text{rot}}^{g+\deg p}$. The second statement will imply in Theorem 5 that $\tilde{(a, \tilde{b}, \lambda_1, \lambda_2)}$ does not belong to $\mathcal{M}_{\text{Ae}}$ and that $\mathcal{M}_{\text{rot}}$ is isolated in $\mathcal{M}_{\text{Ae}}$. The proof of Theorem 4 is given in Section 7.

1.7. **Characterisation of $\mathcal{M}_{\text{rot}}$.** Theorems 3 and 4 yield a characterisation of $\mathcal{M}_{\text{rot}}$:

**Theorem 5.** Suppose the subsets $\mathcal{M}_{\text{Ae}}^g \subset \mathcal{M}_+^g$ have the following properties:

(i) $\mathcal{M}_{\text{Ae}}^0 = \mathcal{M}_{\text{rot}}^0$.

(ii) Let $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) = (p^2a, pb, \lambda_1, \lambda_2)$, $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_+$ and $p$ be as in Lemma 1.7.

- If $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{Ae}}^{g+\deg p}$, then $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{Ae}}^g$.

- If $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{Ae}}^g$ and all roots of $p$ belong to $\mathbb{S}^1$, then $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{Ae}}^{g+\deg p}$.

(iii) For all $g \in \mathbb{N} \cup \{0\}$ the subset $\mathcal{M}_{\text{Ae}}^g$ is closed and open in $\mathcal{M}_+^g$.

Then $\mathcal{M}_{\text{Ae}} = \bigcup_{g \in \mathbb{N} \cup \{0\}} \mathcal{M}_{\text{Ae}}^g = \mathcal{M}_{\text{rot}}$, i.e. $\mathcal{M}_{\text{Ae}}^0 = \mathcal{M}_{\text{rot}}^0$, $\mathcal{M}_{\text{Ae}}^1 = \mathcal{M}_{\text{rot}}^1$ and $\mathcal{M}_{\text{Ae}}^g = \emptyset$ for $g > 1$.

**Proof.** First we show for pairs $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}$ and $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \not\in \mathcal{M}_{\text{rot}}$ as in Lemma 1.7 that $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2)$ does not belong to $\mathcal{M}_{\text{Ae}}$. The map (1.6) guarantees either $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^1$ or $((\lambda + 1)^2 \tilde{a}, (\lambda + 1)\tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^1$. If in the second case $(\lambda + 1)^2$ does not divide $\tilde{a}$, then due to Lemma 1.7 $((\lambda + 1)^2 \tilde{a}, (\lambda + 1)\tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^{1+\deg p}$. Theorem 4 (ii) yields a piecewise continuous path either from $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \not\in \mathcal{M}_{\text{rot}}$ or from $((\lambda + 1)^2 \tilde{a}, (\lambda + 1)\tilde{b}, \lambda_1, \lambda_2) \not\in \mathcal{M}_{\text{rot}}$ to $\mathcal{M}_{\text{rot}}^g \setminus \mathcal{M}_{\text{rot}}^g$ with decreasing $g$. For $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{Ae}}$ this path stays due to properties (ii)-(iii) in $\mathcal{M}_{\text{Ae}}$ in contradiction to (i). This implies $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \not\in \mathcal{M}_{\text{Ae}}$.

Now we show $\mathcal{M}_{\text{Ae}} \subset \mathcal{M}_{\text{rot}}$. Due to Theorem 3 there starts at any element of $\mathcal{M}_{\text{Ae}}$ a piecewise continuous path to $\mathcal{M}_{\text{rot}}^0$. Due to properties (ii)-(iii) this path stays in $\mathcal{M}_{\text{Ae}}$. Due to property (i)
it ends in $M^0_{\text{rot}}$. Since $M^2_{\text{rot}}$ is closed in $M^2_+$ (Theorem 4) at some point the path enters the set $M^2_{\text{rot}}$ and stays in $M^2_{\text{rot}}$ until it reaches $M^0_{\text{rot}}$. Due to Theorem 4 (i) and the first argument this point has to be the initial element of $M_{\text{rot}}^0$ and all elements of $M_{Ae}$ belong to $M^2_{\text{rot}}$.

Conversely, due to properties (i)-(iii) and (1.6) $M_{Ae}$ contains $M^2_{\text{rot}}$, since $M^1_{\text{rot}}$ is connected. □

1.8. Mean convex Alexandrov embedded CMC cylinders of finite type in $S^3$. We use the characterisation of $M_{\text{rot}}$ to present a new proof of the Lawson conjecture and the Pinkall-Sterling conjecture. First we prove the following:

**Theorem 6.** Every mean convex Alexandrov embedded finite type CMC cylinder in $S^3$ is rotational.

**Proof.** We will prove that the spectral data of all $f \in A_{Ae}$ have the properties (i)-(iii) in Theorem 5. First we should define these spectral data. Any $f \in A$ belongs to $A(a, b, \lambda_1, \lambda_2)$ of many $(a, b, \lambda_1, \lambda_2) \in M_+$. If we replace $(a, b, \lambda_1, \lambda_2) \in M_+$ by $(p^2 a, pb, \lambda_1, \lambda_2) \in M^2_+$ as in Lemma 1.7, then $A(a, b, \lambda_1, \lambda_2) \subset A(p^2 a, pb, \lambda_1, \lambda_2)$. Due to Proposition 3.4 this is the only ambiguity of $(a, b, \lambda_1, \lambda_2)$. Indeed for all $f \in A$ there exists a unique $(a, b, \lambda_1, \lambda_2) \in M_+$, whose $A(a, b, \lambda_1, \lambda_2)$ is the smallest set containing $f$. More precisely, for all $f \in A$ there exists $(a, b, \lambda_1, \lambda_2) \in M_+$ such that all $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in M_+$ with $f \in A(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2)$ are of the form $(\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) = (p^2 a, pb, \lambda_1, \lambda_2)$ in Lemma 1.7. Proposition 9.1 shows that for all $f \in A_{Ae}$, these minimal sets $A(a, b, \lambda_1, \lambda_2)$ are completely contained in $A_{Ae}$. Therefore any $f \in A_{Ae}$ is contained in the set $A(a, b, \lambda_1, \lambda_2)$ of an element $(a, b, \lambda_1, \lambda_2) \in A_{Ae} \subset M_+$ as defined as follows:

**Definition 1.9.** Let $M^{0}_{Ae}$ denote the space of $(a, b, \lambda_1, \lambda_2) \in M^0_+$ with $A(a, b, \lambda_1, \lambda_2) \subset A_{Ae}$.

Proposition 4.1 shows (1.7) and property (i) in Theorem 5. Property (ii) follows from the properties of $A(a, b, \lambda_1, \lambda_2)$ and $A(p^2 a, pb, \lambda_2, \lambda_2)$ in the situation of Lemma 1.7. Due to [14, Lemma 4.7] we have in all cases $A(a, b, \lambda_1, \lambda_2) \subset A(p^2 a, pb, \lambda_1, \lambda_2)$ with equality, if all roots of $p$ lie in $S^1$. Finally property (iii) is proven in Proposition 9.2. This proves $M_{Ae} = M_{\text{rot}}$ and

$$A_{Ae} = \bigcup_{(a, b, \lambda_1, \lambda_2) \in M_{Ae}} A(a, b, \lambda_1, \lambda_2) = \bigcup_{(a, b, \lambda_1, \lambda_2) \in M_{\text{rot}}} A(a, b, \lambda_1, \lambda_2) = A_{\text{rot}}.$$ □

As a corollary we confirm a conjecture by Pinkall and Sterling [32] recently proven in [3].

**Theorem 1.** All mean convex Alexandrov embedded CMC tori in the 3-sphere are tori of revolution. In particular, all embedded CMC tori in the 3-sphere are tori of revolution.

**Proof.** Let $f : N \to S^3$ be a mean convex Alexandrov embedded CMC torus $M = \partial \tilde{N} \approx T^2$. Due to [24, Theorem 1] the homomorphism $\pi_1(M) \to \pi_1(N)$ is surjective and due to [31, Theorem 18.1] the boundary $\partial \tilde{N}$ of the universal covering $\tilde{N}$ of $N$ is a cylinder. Hence there exists a mean convex Alexandrov embedded CMC cylinder $\tilde{f} : \tilde{N} \to S^3$, which is the composition of $f$ with a covering map. The spectral data of $f$ and $\tilde{f}$ coincide. Due to Pinkall-Sterling [32] this cylinder is of finite type, and by Theorem 6 a surface of revolution. □

Hsiang-Lawson [19] prove that there are no embedded minimal tori of cohomogeneity one. Hence the Clifford torus is the only embedded minimal torus of revolution, see also [21]. By Theorem 1 the only embedded CMC tori are tori of revolution, and we thus obtain an independent proof of the Lawson conjecture, recently also proven by Brendle [6].

**Theorem 2.** The Clifford torus is the only embedded minimal torus in the 3-sphere.
Since all mean convex Alexandrov embedded CMC cylinders in \(S^3\) are surfaces of revolution around a closed geodesic, the ambient 3-manifold is diffeomorphic to \(\mathbb{D} \times \mathbb{R}\), where \(\mathbb{D}\) denotes the closed unit disk. We thus have the following generalisation of Lawson’s ‘unknottedness’ result [24].

**Theorem 7.** For all mean convex Alexandrov embedded CMC cylinders of finite type in the 3-sphere, the 3-manifold is diffeomorphic to the Cartesian product \(\mathbb{D} \times \mathbb{R}\).

## 2. Conformal cmc immersions into \(S^3\)

This section recalls the relationship between CMC immersed surfaces in \(S^3\) and solutions of the sinh-Gordon equation, before considering the special case of CMC cylinders in \(S^3\), the notion of monodromy and the period problem.

### 2.1. The sinh-Gordon equation

We identify the 3-sphere \(S^3 \subset \mathbb{R}^4\) with \(S^3 \cong SU_2\). The Lie algebra of the matrix Lie group \(SU_2\) is \(su_2\), equipped with the commutator \([\cdot, \cdot]\). We denote by \(\langle \cdot, \cdot \rangle\) the bilinear extension of the Ad-invariant inner product \((X, Y) \mapsto -\frac{1}{4}\text{tr}(XY)\) of \(su_2\) to \(su^2_c = su_2(\mathbb{C})\). For a conformal map \(f : \mathbb{C} \to S^3\) let \(v\) denote the conformal factor, \(Qdz^2\) the Hopf differential and \(H\) the mean curvature. The double cover of the isometry group \(SO(4)\) is \(SU_2 \times SU_2\) via the action \((F, G) \mapsto FXG^{-1}\). We define maps \(F, G : \mathbb{C} \to SU_2\) such that

\[
F = FG^{-1}, \quad F_z = F(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})G^{-1}, \quad f = F(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})G^{-1}, \quad N = F(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})G^{-1}.
\]

Here \(N : \mathbb{C} \to su_2 \subset \mathbb{R}^4\) denotes the normal of \(f\). The maps \(F, G\) are unique up to the transformation \((F, G) \mapsto (-F, -G)\). A direct calculation shows that the corresponding \(su_2\)-valued 1-forms \(F^{-1}dF\) and \(G^{-1}dG\) can be calculated in terms of \(v, Q, H\) and their derivatives. The Gauß-Codazzi equations are the integrability conditions of these 1-forms.

If \(f\) has constant mean curvature, then we can define a \(su_2\)-valued 1-form \(\alpha_0\) and two \(sl_2\)-valued 1-forms \(\alpha_{\pm}\) on \(\mathbb{C}\) and two **Sym points** \(\lambda_1 \neq \lambda_2 \in S^1\) with

\[
F^{-1}dF = \alpha_{\lambda_1}, \quad G^{-1}dG = \alpha_{\lambda_2}, \quad \alpha_\lambda = \lambda^{-1}\alpha_- + \alpha_0 + \lambda \alpha_+.
\]

Moreover, the 1-form \(\alpha_\lambda\) solves the Maurer-Cartan equation \(2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0\) for all \(\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}\) and takes for \(\lambda \in S^1\) values in \(su_2\). We can integrate \(\alpha_\lambda\) and obtain

\[
dF_\lambda = F_\lambda \alpha_\lambda, \quad F_\lambda(0) = 1
\]

the corresponding **extended frame** \(F_\lambda\) parameterised by \(\lambda \in \mathbb{C}^*\). For given \(\alpha_\lambda\) the map \(\lambda \mapsto F_\lambda\) is holomorphic on \(\mathbb{C}^*\) and has essential singularities at \(\lambda = 0, \infty\). The corresponding immersion \(f\) is by definition of \(F_\lambda\) up to an isometry of \(S^3\) given by the **Sym-Bobenko-formula**

\[
f = F_{\lambda_1}F_{\lambda_2}^{-1}.
\]

The CMC condition implies that the Hopf differential is a holomorphic quadratic differential [18]. On the cylinder \(\mathbb{C}^*\) there is an infinite dimensional space of holomorphic quadratic differentials, large classes of which can be realized as Hopf differentials of CMC cylinders [20]. On a CMC torus the Hopf differential is constant (and non-zero). Since we are ultimately interested in tori, we restrict our attention to CMC cylinders considered via Proposition 2.1 which have constant non-zero Hopf differentials on the universal covering \(\mathbb{C}\) of \(\mathbb{C}^*\).

If Hopf differential and mean curvature are constant, then the 1-form \(\alpha_\lambda\) may be written as

\[
\alpha_\lambda = \frac{1}{4} \begin{pmatrix}
2\omega z & 2\omega z & i\lambda^{-1}\omega d\bar{z} + i\omega d\bar{z} \\
i\omega^{-1} d\bar{z} + i\lambda\omega d\bar{z} & -2\omega z & 2\omega z \end{pmatrix}.
\]

Here \(\omega : \mathbb{C} \to \mathbb{R}\) is a solution of the sinh-Gordon equation

\[
8\omega_{zz} + \sinh(2\omega) = 0.
\]
The corresponding mean curvature \( H \), Hopf differential \( Qdz^2 \) are as in (1.5), and the conformal factor \( v \) is
\[
(2.5) \quad v^2 = 2\langle f^{-1}f_z, f^{-1}f_{\bar{z}} \rangle = \frac{e^{2\omega}}{8} (\lambda_1^{-1} - \lambda_2^{-1})(\lambda_1 - \lambda_2) = \frac{e^{2\omega}}{4(H^2 + 1)}.
\]

Putting all this together we have the following version of [5, Theorem 14.1]

**Proposition 2.1.** [5] Conformal cmc immersions \( f : \mathbb{C} \to S^3 \) with constant Hopf differentials are in one-to-one correspondence to solutions \( \omega : \mathbb{C} \to \mathbb{R} \) of (2.4) together with \( \lambda_1 \neq \lambda_2 \in S^1 \).

More precisely, given such \( f : \mathbb{C} \to S^3 \) there exist unique \( c > 0, \lambda_1 \neq \lambda_2 \in S^1 \), such that the mean curvature and the Hopf differential of the immersion \( z \mapsto f(cz) \) obey (1.5). Equation (2.5) defines due to the Gauß-Codazzi equations a solution \( \omega \) of (2.4). The corresponding \( \alpha \) solves the Maurer-Cartan equation and defines an extended frame \( F_\alpha \) (2.1). Finally, (2.2) gives \( z \mapsto f(cz) \) up to an isometry of \( S^3 \).

Conversely, for a solution \( \omega : \mathbb{C} \to \mathbb{R} \) of (2.4) and \( \lambda_1 \neq \lambda_2 \in S^1 \) the corresponding extended frame \( F_\lambda \) (2.1) with \( \alpha \) defines by (2.2) a conformal cmc immersion with (1.5) and (2.5).

### 2.2. Monodromy and periodicity condition

Let \( F_\lambda \) be an extended frame for a cmc immersion \( f : \mathbb{C} \to S^3 \) such that (2.2) holds for two distinct unimodular numbers \( \lambda_1, \lambda_2 \). Let \( \tau : \mathbb{C} \to \mathbb{C}, z \mapsto z + \tau \) be a translation, and assume that \( \alpha = F_\lambda^{-1}dF_\lambda \) has period \( \tau \), so that \( \tau^*\alpha = \alpha \circ \tau = \alpha \). Then we define the **monodromy** of \( F_\lambda \) with respect to \( \tau \) as
\[
(2.6) \quad M_\lambda(\tau) = \tau^*(F_\lambda) F_\lambda^{-1}.
\]
Periodicity \( \tau^*f = f \) in terms of the monodromy is then \( \tau^*f = M_{\lambda_1}(\tau) f M_{\lambda_2}^{-1}(\tau) \). Hence \( \tau^*f = f \) holds for surfaces not contained in the fixed point set of an isometry if and only if
\[
(2.7) \quad M_{\lambda_1}(\tau) = M_{\lambda_2}(\tau) = \pm 1.
\]

### 3. Finite type solutions of the sinh-Gordon equation

In this section we discuss the solutions of the sinh-Gordon equation which are called finite type solutions. They are parameterised by \( 2 \times 2 \)-matrix polynomials with respect to the spectral parameter \( \lambda \), which parameterises families of solutions of the Maurer-Cartan-equations. More precisely, the finite type solutions of the sinh-Gordon equation are in one-to-one correspondence with maps called polynomial Killing fields from \( \mathbb{C} \) to these matrix polynomials. These polynomial Killing fields themselves solve a non-linear partial differential equation, but they are uniquely determined by one of their values. We shall call these values potentials. The Symes method calculates the solutions in terms of the potentials with the help of a loop group splitting. The eigenvalues of these matrix polynomials define a real hyperelliptic curve (1.2).

One spectral curve corresponds to a whole family of finite type solutions of the sinh-Gordon equation. We call the sets of finite type solutions (or their potentials), which belong to the same spectral curve, isospectral sets.

#### 3.1. Polynomial Killing fields

For some aspects of the theory untwisted loops are advantageous, and avoiding the additional covering map \( \lambda \mapsto \sqrt{\lambda} \) simplifies for example the description of Bianchi-Bäcklund transformations by the simple factors [37, 22]. For the description of polynomial Killing fields on the other hand, the twisted loop algebras as in [8, 9, 10, 11, 26] are better suited, but we remain consistent and continue working in our ‘untwisted’ setting.

**Definition 3.1.** A solution \( \omega \) of the sinh-Gordon equation (2.4) is called a **finite type solution** if and only if for the corresponding \( \alpha \) in (2.3), the Lax equation
\[
d\Phi = [\Phi, \alpha]
\]
has a polynomial solution \( \Phi = \sum_d \lambda^d \Phi_d \) with smooth \( \Phi_d : \mathbb{C} \to \mathfrak{sl}_2(\mathbb{C}) \).

The finite type solutions can be constructed by elements of finite dimensional vector spaces. Finite type solutions of the sinh-Gordon equation give rise to algebraic objects which we call potentials. They are defined as follows.

**Definition 3.2.** For \( g \in \mathbb{N} \cup \{0\} \) define

\[
\mathcal{P}_g = \left\{ \xi = \sum_{d=-1}^g \xi_d \lambda^d \mid \xi_{-1} \in \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \xi_d = -\xi_{g-1-d}^T \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } d = -1, \ldots, g \right\}.
\]

Clearly \( \mathcal{P}_g \) is a real \( 3g + 2 \)-dimensional vector space and has up to isomorphism a unique norm \( \| \cdot \| \). These Laurent polynomials define smooth mappings from \( \lambda \in \mathbb{S}^1 \) into \( \mathfrak{sl}_2(\mathbb{C}) \). Note that \( \sqrt{\lambda} \mapsto \lambda^{1/2} \xi \) belongs to the loop Lie algebra \( \mathfrak{Asu}_2 \) of the loop group \( \text{Asu}_2 \). In order to construct solutions of the sinh-Gordon equation we need in addition the conditions \( \xi_{-1} \neq 0 \) and \( \text{tr}(\xi_{-1} \xi_0) \neq 0 \). Conjugation by diagonal matrices in \( \text{SU}_2 \) act on \( \mathcal{P}_g \). We remove this trivial degree of freedom (see also [14, Remarks Isometric normalisation 1-4]) and define

\[
\mathcal{P}_g = \left\{ \xi \in \mathcal{P}_g \mid \xi_{-1} \in \mathbb{R}^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \text{tr}(\xi_{-1} \xi_0) \neq 0 \right\}.
\]

An element \( \xi \in \mathcal{P}_g \) is called a potential. Expanding a map \( \zeta : \mathbb{C} \to \mathcal{P}_g \) as

\[
\zeta(z) = \begin{pmatrix} 0 & \beta_{-1}(z) \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0(z) & \beta_0(z) \\ \gamma_0(z) & -\alpha_0(z) \end{pmatrix} \lambda^0 + \ldots + \begin{pmatrix} \alpha_g(z) & \beta_g(z) \\ \gamma_g(z) & -\alpha_g(z) \end{pmatrix} \lambda^g
\]

we associate a matrix 1-form defined by

\[
\alpha(\zeta) = \begin{pmatrix} \alpha_0(z) & \beta_{-1}(z) \lambda^{-1} \\ \gamma_0(z) & -\alpha_0(z) \end{pmatrix} dz - \begin{pmatrix} \beta_0(z) & \gamma_0(z) \\ \beta_{-1}(z) \lambda & -\alpha_0(z) \end{pmatrix} dz.
\]

There is a correspondence between finite type solutions of the sinh-Gordon equation and solutions \( \zeta : \mathbb{C} \to \mathcal{P}_g \) of a Lax equation. A polynomial Killing field is a map \( \zeta : \mathbb{C} \to \mathcal{P}_g \) which solves

\[
d\zeta = [\zeta, \alpha(\zeta)] \quad \text{with} \quad \zeta(0) = \xi.
\]

This equation gives a solution of the sinh-Gordon equation by setting \( \alpha(\zeta) = \alpha_\lambda \) as in (2.3), see e.g. [14] Proposition 3.2.

By the Symes method [9], see also [14, Proposition 3.2], the extended framing \( F_\lambda : \mathbb{C} \to \text{Asu}_2 \) of a CMC immersion of finite type is given by the unitary factor of the Iwasawa decomposition

\[
\exp(z \xi) = F_\lambda B
\]

for some \( \xi \in \mathcal{P}_g \) with \( g \in \mathbb{N} \cup \{0\} \). Due to Pressley-Segal [33], the Iwasawa decomposition is a diffeomorphism between the loop group \( \text{Asl}_2(\mathbb{C}) \) of \( \text{SL}_2(\mathbb{C}) \) into point-wise products of elements of \( \text{Asu}_2 \) with elements of the loop group \( \Lambda^+ \text{SL}_2(\mathbb{C}) \) of holomorphic maps from \( \lambda \in \mathbb{D} \) to \( \text{SL}_2(\mathbb{C}) \), which take at \( \lambda = 0 \) values in the subgroup of \( \text{SL}_2(\mathbb{C}) \) of upper-triangular matrices with positive real diagonal entries. For every \( \xi \in \mathcal{P}_g \) there exists a unique \( \mathfrak{su}_2 \)-valued 1-form \( \alpha(\xi) \) on \( \mathbb{C} \), such that \( \xi dz - \alpha(\xi) \) takes values in the Lie algebra of \( \Lambda^+ \text{SL}_2(\mathbb{C}) \) of the right hand factor in the Iwasawa decomposition (3.5).

For each potential \( \xi \in \mathcal{P}_g \), there exists a unique polynomial Killing field given by

\[
\zeta = B \xi B^{-1} = F_\lambda^{-1} \xi F_\lambda \quad \text{with} \quad F_\lambda \text{ and } B \text{ as in (3.5)}.
\]

Thus \( \det(\zeta(z)) = \det(\xi) \) does not depend on the variable \( z \). For a potential \( \xi = \xi(0) \) the polynomial \( \alpha(\lambda) = -\lambda \det(\xi) \) defines the spectral curve. For \( \xi \in \mathcal{P}_g \) with \( \text{tr}(\xi_{-1} \xi_0) = -\frac{1}{16} \) the corresponding 1-form \( \alpha(\xi) \) is the \( \alpha_\lambda \) in (2.3) for that particular solution \( \omega \) of the sinh-Gordon equation corresponding to the extended frame \( F_\lambda \) of (3.5). For general \( \xi \in \mathcal{P}_g \) with \( \xi_{-1} \neq 0 \) and \( \text{tr}(\xi_1 \xi_0) \neq 0 \) the corresponding \( \alpha(\xi) \) differs from (2.3) by multiplication of \( \lambda \) and \( dz \) with
constant non-zero complex numbers. Given a polynomial Killing field \( \zeta \), we set the potential \( \xi = [\zeta]_0 \) in (3.5). Thus \( \zeta \), or the potential \( \xi \), gives rise to an extended frame, and thus to an isometric family of finite type CMC surfaces in \( S^3 \).

### 3.2. Roots of polynomial Killing fields.

If a potential \( \xi \) has a root at some \( \lambda = \alpha \in \mathbb{C}^* \), then the corresponding polynomial Killing field has a root at the same \( \lambda \) for all \( z \in \mathbb{C} \). In this case we may reduce the order of \( \xi \) and \( \zeta \) without changing the corresponding \( F_\lambda \) (3.5). Let

\[
(3.7) \quad p(\lambda) = \begin{cases} 
\sqrt{-\alpha \lambda} + \sqrt{-1} & \text{for } \alpha \bar{\alpha} = 1 \\
(\bar{\alpha} \lambda - 1)(\lambda - \alpha) & \text{for all } \alpha \in \mathbb{C} 
\end{cases} \quad \lambda^{\deg(p)} \bar{p}(\lambda^{-1}) = p(\lambda).
\]

If the polynomial Killing field \( \zeta \) with potential \( \xi \in \mathcal{P}_g \) has a simple root at \( \lambda = \alpha \in \mathbb{C}^* \), then \( \zeta/p \) does not vanish at \( \alpha \) and is the polynomial Killing field with potential \( \xi/p \in \mathcal{P}_{g-\deg(p)} \). Furthermore, obviously \( \zeta \) and \( \zeta/p \) commute, and we next show that both polynomial Killing fields \( \zeta \) and \( \zeta/p \) give rise to the same extended frame \( F_\lambda \) (3.5) (compare [14, Proposition 4.4]).

**Proposition 3.3.** If a polynomial Killing field \( \zeta \) with potential \( \xi \in \mathcal{P}_g \) has zeroes in \( \lambda = \alpha \in \mathbb{C}^* \), then there is a polynomial \( p(\lambda) \), such that the following two conditions hold:

(i) \( \zeta/p \) is the polynomial Killing field with potential \( \xi/p \in \mathcal{P}_{g-\deg(p)} \), which gives rise to the same associated family as \( \zeta \).

(ii) \( \zeta/p \) has no zeroes in \( \lambda = \alpha \in \mathbb{C}^* \).

**Proof.** An appropriate rotation of \( \lambda \) transforms any root \( \alpha \in \mathbb{C}^* \) into a negative root. For such negative roots the corresponding potentials \( \xi \) and \( \xi/p \) are related by multiplication with a polynomial of respect to \( \lambda \) with positive coefficients. In the Iwasawa decomposition (3.5) this factor is absorbed in \( B \). Hence the corresponding extended frames coincide. \( \square \)

Hence amongst all polynomial Killing fields that give rise to a particular CMC surface of finite type there is one of smallest possible degree (without adding further poles), and we say that such a polynomial Killing field has minimal degree. A polynomial Killing field has minimal degree if and only if it has neither roots nor poles in \( \lambda \in \mathbb{C}^* \). We recall three statements of [14]. The first part restates [14, Theorem 2.4], the second part follows immediately from Section 2 in [14], and the third part is a variant of [14, Proposition 4.5].

**Proposition 3.4.** (i) A periodic bounded solution \( \omega \) of (2.4) is of finite type.

(ii) A solution \( \omega \) of (2.4) is of finite type, if and only if there exists \( \xi \in \mathcal{P}_g \) with \( g \in \mathbb{N} \cup \{0\} \), whose polynomial Killing field has the same \( \alpha \lambda \) (2.3).

(iii) There exists a unique polynomial Killing field of minimal degree that gives rise to \( \omega \). Thus every CMC immersion \( f : \mathbb{C} \to S^3 \) of finite type is up to isometry determined by a unique triple \( (\xi, \lambda_1, \lambda_2) \) with \( \xi \in \mathcal{P}_g \) without roots and \( g \in \mathbb{N} \cup \{0\} \).

### 3.3. Spectral curves I.

Due to (3.6) the zero set of the characteristic polynomial \( \det(\nu I - \zeta) \) of a polynomial Killing field \( \zeta \) with potential \( \xi \in \mathcal{P}_g \) does not depend on \( z \in \mathbb{C} \). This zero set agrees with (1.2) for \( a = -\lambda \det \xi \). For \( \xi \in \mathcal{P}_g \) the polynomial \( a = -\lambda \det \xi \) of degree \( 2g \) satisfies the reality conditions (1.1)

### 3.4. Isospectral sets.

We recall the isospectral action and investigate the isospectral sets.

**Definition 3.5.** (Isospectral action). Let \( \xi \in \mathcal{P}_g \) and \( t = (t_0, \ldots, t_{g-1}) \in \mathbb{C}^g \), and

\[
\exp \left( \xi \sum_{i=0}^{g-1} \lambda^{-i} t_i \right) = F_\lambda(t)B(t)
\]
Define the map \(\pi(t) : \mathcal{P}_g \to \mathcal{P}_g\) by
\[
\pi(t) \xi = B(t) \xi B^{-1}(t) = F^{-1}_\lambda(t) \xi F_\lambda(t)
\]
In [14, Section 4] this group action is investigated. This action is commutative. It preserves the following sets:

**Definition 3.6.** For a polynomial \(a\) of degree \(2g\) obeying (1.1) the isospectral set is defined as
\[
\mathcal{I}(a) = \{\xi \in \mathcal{P}_g \mid \det \xi = -\lambda^{-1} a(\lambda)\}.
\]
Due to Proposition 3.4 (iii) cmc immersions of finite type \(f : \mathbb{C} \to S^3\) are determined by triples \((\xi, \lambda_1, \lambda_2)\). In section 1 we denote for given spectral data \((a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g\) by \(A(a, b, \lambda_1, \lambda_2)\) the set of all cylinders \(f \in \mathcal{A}\) corresponding to the triples \(\{(\xi, \lambda_1, \lambda_2) \mid \xi \in \mathcal{I}(a)\}\).

The sets \(\mathcal{I}(a)\) build the fibres of the following map:

**Definition 3.7.** Let \(\mathcal{M}^g\) denote the space of triples \((a, \lambda_1, \lambda_2)\) of \(a \in \mathbb{C}^{2g}\) obeying (1.1) together with two Sym points \(\lambda_1 \neq \lambda_2 \in S^1\). Then we consider the following map:
\[
A : \{ (\xi, \lambda_1, \lambda_2) \in \mathcal{P}_g \times S^1 \times S^1 \mid \lambda_1 \neq \lambda_2 \} \to \mathcal{M}^g\quad (\xi, \lambda_1, \lambda_2) \mapsto (-\lambda \det \xi, \lambda_1, \lambda_2).
\]

**Lemma 3.8.** The map \(A(3.9)\) is open and proper.

**Proof.** The Laurent coefficients of \(\xi = \sum_{d=-1}^g \lambda^d \xi_d\) are
\[
\xi_d = \frac{1}{2\pi i} \int_{S^1} \lambda^{-d} \xi \frac{d\lambda}{\lambda}.
\]
Using a norm yields
\[
\|\xi_d\| \leq \frac{1}{2\pi i} \int_{S^1} \|\lambda^{-d} \xi\| \frac{d\lambda}{\lambda} \leq \sup_{\lambda \in S^1} \sqrt{-\lambda^{-g} a(\lambda)}.
\]
Thus each entry \(\xi_d\) of \(\xi\) is bounded if \(\sqrt{-\lambda^{-g} a(\lambda)}\) is bounded on \(S^1\). For polynomials \(a\) obeying the reality conditions (1.1) this follows from the roots of \(a\) and \(|a(0)|\) being bounded. Hence \(A^{-1}[K]\) is a bounded subset of the real finite-dimensional vector space \(\mathcal{P}_g \times \mathbb{C} \times \mathbb{C} \supset \mathcal{P}_g \times S^1 \times S^1\), if \(K\) is a compact subset of \(\mathcal{M}^g\). By continuity of \(A\) it is also closed, and therefore compact.

Due to [14, Proposition 4.12 and Theorem 6.8] the orbits of the group action (3.8) are the subsets of \(\mathcal{I}(a)\) of all elements \(\xi\) with the same roots on \(\mathbb{C}^*\) counted with multiplicities. For any polynomial \(a\) of degree \(2g\) obeying (1.1), an off-diagonal potential
\[
\xi = \begin{pmatrix}
0 \\
\gamma(\lambda) \\
\lambda^{-1} \beta(\lambda) \\
0 
\end{pmatrix}
\]
belongs to \(\mathcal{I}(a)\), if and only if the polynomials \(\beta\) and \(\gamma\) of degree \(g\) obey \(\beta(\lambda)\gamma(\lambda) = a(\lambda)\) and \(\gamma(\lambda) = -\lambda^g \tilde{\beta}(\lambda^{-1})\). The roots of \(\beta\) are \(g\) roots of \(a\), which are mapped by \(\lambda \mapsto \lambda^{-1}\) onto the remaining \(g\) roots of \(a\). For any choice of such roots, \(\beta\) and \(\gamma\) are determined up to multiplication by inverse unimodular numbers. At higher order roots \(\alpha \in \mathbb{C}^* \setminus S^1\) of \(a\) we can choose the multiplicity of the root of \(\beta\) at \(\alpha\) between zero and the multiplicity of the root \(\alpha\) of \(a\). The sum of the multiplicities of the roots of \(\beta\) at \(\alpha\) and at \(\alpha^{-1}\) has to be equal to the multiplicity of the root of \(a\) at \(\alpha\). Therefore there exists in every orbit of the isospectral group action (3.8) at least one off-diagonal \(\xi\). Due to the relation between the polynomials \(a\) and \(\beta\) and \(\gamma\), the map \(A(3.9)\) is open at off-diagonal \(\xi\). Since the isospectral action (3.8) acts by diffeomorphisms on \(\mathcal{P}_g\) and preserves the fibres of the map \(A\), this map is (globally) open.

The map which associates to the data \((\xi, \lambda_1, \lambda_2)\) the corresponding cmc immersion is smooth. As in [15, Proposition 4.4] any bound on the coefficients of \(a\) implies a uniform curvature bound and a bound on the covariant derivative of the second fundamental from:
Proposition 3.9. Let \( \mathcal{K} \subset \mathbb{C}^g[\lambda] \times \mathbb{S}^1 \times \mathbb{S}^1 \) be a set of polynomials \( a \) obeying the reality condition (1.1) together with two Sym points \( \lambda_1 \neq \lambda_2 \). If in \( \mathcal{K} \) all roots of \( a \) and \( |a(0)| \) are uniformly bounded and \( \lambda_1 \) and \( \lambda_2 \) are uniformly bounded away from each other, then all CMC immersions \( f: \mathbb{C} \to \mathbb{S}^3 \) corresponding to \((\xi, \lambda_1, \lambda_2) \in A^{-1}[\mathcal{K}] \) have uniformly bounded \( C^{k,\alpha} - \text{norm} \) for fixed \( k, \alpha \). In particular all these immersions have uniformly bounded curvature and second fundamental forms with uniformly bounded covariant derivatives.

Proof. To prove this proposition it suffices to show a uniform \( C^{k,\alpha} \) bound on the corresponding solutions \( \omega: \mathbb{C} \to \mathbb{R} \) of (2.4). The closure of \( \mathcal{K} \) in \( \mathbb{C}^g[\lambda] \times \mathbb{S}^1 \times \mathbb{S}^1 \) is a compact subset of \( \mathcal{M}^0 \). Lemma 3.8 gives a uniform bound on \( \omega \). Schauder estimates bound the \( C^{k,\alpha} \) norms on \( \mathbb{C} \).  

3.5. Spectral curves II. We utilise the description of finite type CMC surfaces in \( \mathbb{S}^3 \) via spectral curves due to Hitchin [17], and relate to our previous definition of spectral curves due to Bobenko [5]. While Hitchin defines the spectral curve as the characteristic equation for the holonomy of a loop of flat connections, Bobenko defines the spectral curve as the characteristic equation of a polynomial Killing field. We shall use both of these descriptions, and briefly recall their equivalence: Due to (2.6), the monodromy \( C^* \to \text{SL}_2(\mathbb{C}) \), \( \lambda \mapsto M_\lambda \) is a holomorphic map with essential singularities at \( \lambda = 0, \infty \). By construction the monodromy takes values in \( \text{SU}_2 \) for \( |\lambda| = 1 \). The monodromy depends on the choice of base point, but its conjugacy class and hence eigenvalues \( \mu, \mu^{-1} \) do not. With \( \Delta(\lambda) = \text{tr}(M_\lambda) \) the spectral variety is given by

\[
\{ (\lambda, \mu) \in C^* \times C^* | \mu^2 - \Delta(\lambda) \mu + 1 = 0 \}.
\]

The eigenspace of \( M_\lambda \) depends holomorphically on \((\lambda, \mu)\) and defines the eigenbundle on the spectral variety. Let us compare the previous definition of a spectral curve (1.2) of periodic finite type solutions of the sinh-Gordon equations. Let \( \zeta \) be a polynomial Killing field with potential \( \xi \in \mathcal{P}_g \), with period \( \tau \) so that \( \zeta(z + \tau) = \zeta(z) \) for all \( z \in \mathbb{C} \). Then also the corresponding \( \alpha(\zeta) \) is \( \tau \)-periodic. Let \( dF_\lambda = F_\lambda \alpha(\zeta), F_\lambda(0) = 1 \) and \( M_\lambda(\tau) = F_\lambda(\tau) \) be the monodromy with respect to \( \tau \). Then for \( z = 0 \) we have \( \xi = \zeta(0) = \zeta(\tau) = F_\lambda^{-1}(\tau) \xi F_\lambda(\tau) = M_\lambda^{-1}(\tau) \xi M_\lambda(\tau) \) and thus

\[
[ M_\lambda(\tau), \xi ] = 0.
\]

All eigenvalues of holomorphic \( 2 \times 2 \) matrix-valued functions depending on \( \lambda \in \mathbb{C}^g \) and commuting point-wise with \( M_\lambda(\tau) \) or \( \xi \) define the sheaf of holomorphic functions of the spectral curve. Hence the eigenvalues of \( \xi \) and \( M_\lambda(\tau) \) are different functions on the same Riemann surface. Furthermore, on this common spectral curve the eigenspaces of \( M_\lambda(\tau) \) and \( \xi \) coincide point-wise. Consequently the holomorphic eigenbundles of \( M_\lambda(\tau) \) and \( \xi \) coincide.

We obtain the following Proposition (compare with [4, 5, 14]).

Proposition 3.10. Let the spectral data \((a, b, \lambda_1, \lambda_2)\) obey (i)–(iv) in Definition 1.3. Then all \( \xi \in \mathcal{I}(a) \) together with \( \lambda_1, \lambda_2 \) correspond to CMC immersions \( f: \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^3 \) of finite type.

Conversely, if \( \xi \in \mathcal{P}_g \) without roots on \( \lambda \in \mathbb{C}^* \) together with \( \lambda_1 \neq \lambda_2 \in \mathbb{S}^1 \) corresponds to a CMC immersion \( f: \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^3 \) of finite type with period \( \tau \in \mathbb{C} \), then besides \( a = -\lambda \det \xi \) there exists a unique polynomial \( b \), such that \((a, b, \lambda_1, \lambda_2)\) obey (i)–(iv).

4. Example: Spectral data of rotational cylinders

We next construct a two-dimensional family of spectral data \((a, b, \lambda_1, \lambda_2)\), for which all triples \((\xi, \lambda_1, \lambda_2) \in A^{-1}[\{(a, \lambda_1, \lambda_2)\}] \) correspond to mean convex Alexandrov embedded cylinders. We first construct in Proposition 4.1 a one-parameter family of homogeneous mean convex Alexandrov embedded CMC cylinders, and a two-parameter family of mean convex Alexandrov embedded rotational CMC cylinders of spectral genus \( g = 1 \). We then prove that all mean
convex Alexandrov embedded cylinders of spectral genus zero belong to the one-dimensional family of homogeneous cylinders.

**Proposition 4.1.** There exists a one-dimensional family \( \mathcal{M}_\text{rot}^0 \subset \mathcal{M}_+^0 \) of spectral data of mean convex Alexandrov embedded CMC cylinders parameterised by the mean curvature \( H \in [0, \infty) \). For all elements of \( \mathcal{M}_\text{rot}^0 \) the function \( \mu^2 - 1 \) has only three roots on \( S^1 \) at the two Sym points \( \lambda_1 \) and \( \lambda_2 \) and at \( \lambda = -1 \), and no other root on \( S^1 \). This family contains all spectral data of mean convex Alexandrov embedded CMC cylinders of spectral genus zero.

The family \( \{(\lambda + 1)^2 a, (\lambda + 1), \lambda_1, \lambda_2 \} \mid (a, b, \lambda_1, \lambda_2) \in \mathcal{M}_\text{rot}^0 \) extends to a two-dimensional family \( \mathcal{M}_\text{rot}^0 \subset \mathcal{M}_+^0 \) parameterised by \( (H, \alpha) \in [0, \infty) \times [2, \infty) \). The former family corresponds to \( \alpha = 2 \). For all elements of \( \mathcal{M}_\text{rot}^0 \) the function \( \mu^2 - 1 \) vanishes only at \( \lambda_1 \) and \( \lambda_2 \).

**Proof.** For spectral genus zero, the polynomial \( a \) is equal to \( a(\lambda) = -\frac{1}{16} \). In this case \( h = \ln \mu \) in Definition 1.3 (iii) is a meromorphic function with simple poles at \( \lambda = 0, \lambda = \infty \):

\[
h = \frac{\beta_0 \lambda + \tilde{\beta}_0}{4 \lambda \nu}, \quad \nu^2 = -\frac{1}{16 \lambda}, \quad dh = \frac{\beta_0 \lambda - \tilde{\beta}_0}{8 \lambda^2 \nu} d\lambda, \quad b(\lambda) = \frac{\beta_0 \lambda - \tilde{\beta}_0}{8}.
\]

We chose the Sym points as complex conjugate \( \lambda_2 = \bar{\lambda}_1 \). They are determined by \( H \) (1.5):

\begin{align*}
\lambda_1 &= \frac{H - i}{\sqrt{H^2 + 1}}, \quad \lambda_2 = \frac{H + i}{\sqrt{H^2 + 1}} \quad \text{or} \quad \lambda_1 = -\frac{H - i}{\sqrt{H^2 + 1}}, \quad \lambda_2 = -\frac{H + i}{\sqrt{H^2 + 1}}.
\end{align*}

At \( \lambda = \lambda_1, \lambda_2 \) we require the closing conditions that \( h \in \pi i \mathbb{Z} \). Thus we have

\begin{align*}
h|_{\lambda_1} &\in \mathbb{Z} \iff \beta_0^2 \lambda_1 + \tilde{\beta}_0^2 \bar{\lambda}_1 = \pi^2 m^2 - 2 \beta_0 \tilde{\beta}_0, \\
h|_{\lambda_2} &\in \mathbb{Z} \iff \beta_0^2 \lambda_2 + \tilde{\beta}_0^2 \bar{\lambda}_2 = \pi^2 n^2 - 2 \beta_0 \tilde{\beta}_0,
\end{align*}

for some integers \( m, n \in \mathbb{Z} \) whose difference \( m - n \in 2 \mathbb{Z} \) is even. We make the following claim:

If a cylinder is mean convex Alexandrov embedded then \( m = n = \pm 1 \) (for this ensures that the surface is simply wrapped with respect to the rotational period). To prove this claim first note that any spectral genus zero cylinder is a covering of a homogeneous embedded torus [21]. The complement of this homogeneous embedded torus with respect to \( S^3 \) consists of two connected components \( \mathcal{D}_\pm \), both diffeomorphic to \( S^1 \times S^1 \). Assume the mean curvature vector points into \( \mathcal{D}_+ \). For a mean convex Alexandrov embedded CMC cylinder the extension \( f : N \rightarrow S^3 \) is a surjective immersion onto the closure \( \bar{\mathcal{D}}_+ \) of \( \mathcal{D}_+ \). Hence this map is a covering map. The fundamental group of \( \bar{\mathcal{D}} \times S^1 \) is isomorphic to \( \mathbb{Z} \). Now the covers of a topological space are in one-to-one correspondence with subgroups of the fundamental group [35, §14], and all proper subgroups of \( \pi_1(\bar{\mathcal{D}} \times S^1) \cong \mathbb{Z} \) correspond to compact covers. Hence the only non-compact cover of \( \bar{\mathcal{D}} \times S^1 \) is the universal cover \( \bar{\mathcal{D}} \times \mathbb{R} \). Therefore \( f \) is the universal covering map. In particular the period of the cylinder is a primitive period in the kernel of

\[
H_1(S^1 \times S^1, \mathbb{Z}) \rightarrow H_1(\bar{\mathcal{D}} \times S^1, \mathbb{Z}).
\]

These are primitive periods of closed one-dimensional subgroups of the isometry group of \( S^3 \), which fixes a great circle (rotation axis) in \( S^3 \). In the group \( SU_2 \times SU_2 \) such subgroups belong to the diagonal. This implies \( m = n = \pm 1 \) and proves the claim.

Returning to the proof of the theorem, then (4.2) with \( m^2 = n^2 = 1 \) and (4.1) simplifies to

\[
\beta_0^2 = \frac{\pi^2 \sqrt{H^2 + 1}}{2 \sqrt{H^2 + 1} + 2H} \quad \text{or} \quad \beta_0^2 = \frac{\pi^2 \sqrt{H^2 + 1}}{2 \sqrt{H^2 + 1} - 2H}.
\]

Now we claim that only the first solution corresponds to mean convex Alexandrov embedded cylinders. In fact, we have seen above that the period of the cylinder should correspond to an element in the kernel of \( H_1(\partial \mathcal{D}_+, \mathbb{Z}) \rightarrow H_1(\mathcal{D}_+, \mathbb{Z}) \). The Möbius transformation \( \lambda \mapsto -\lambda \) interchanges the two solutions. Hence we may consider both families of homogeneous cylinders as two copies of a family of tori considered as cylinders with respect to different periods. In the
limit $H \to \infty$ the length of the period of the first solution stays bounded, and the length of the period of the second solution tends to infinity. Hence the period of the second solution are the rotation period of $D_-$, i.e. a primitive element of the kernel of $H_1(\partial D_-, \mathbb{Z}) \to H_1(D_-, \mathbb{Z})$ and therefore the translation period of $D_+$. The period of the first solution is the rotation period of $D_+$. Therefore only the first solution corresponds to mean convex Alexandrov embedded cylinders. The corresponding family of spectral data $(a, b, \lambda_1, \lambda_2)$ parameterised by $H \in [0, \infty)$ is denoted by $\mathcal{M}_\text{rot}^1$.

A root of $\mu^2 - 1$ is determined by the equation

$$\beta^2_0 \lambda + \beta^2_0 \lambda^{-1} = \pi^2 l^2 - 2 \beta_0 \bar{\beta}_0$$

with $l \in \mathbb{Z}$.

The first solution obeys $\beta^2_0 \leq \frac{\pi^2}{4}$. Hence the first solution has on $S^1$ only the solutions $\lambda = \lambda_1, \lambda_2$ with $l = \pm 1$ and $\lambda = -1$ with $l = 0$. Due to condition (iii) in Definition 1.3 the former has to be preserved. If we deform the latter into two different roots of $a$, then $h$ remains a meromorphic function on the genus one spectral curve. We thus obtain another family $\mathcal{M}_\text{rot}^1$ parameterised by $H \in [0, \infty)$ and $\alpha \in [2, \infty)$:

\[
\begin{align*}
\alpha(\lambda) &= \frac{-\lambda^2 + \alpha \lambda + 1}{16}, \\
\nu^2 &= \frac{-\lambda + \alpha + \lambda^{-1}}{16}, \\
h &= \frac{\beta_1 \nu}{4}, \\
dh &= \frac{\beta_1(1 - \lambda^2)}{8 \nu \lambda^2} \, d\lambda, \\
b(\lambda) &= \frac{\beta_1(1 - \lambda^2)}{8}.
\end{align*}
\]

At $\lambda_1, \lambda_2$ the closing conditions $h = \pm \pi i$ must hold and thus

$$\beta^2_1 = \frac{\pi^2 H^2 + 1}{2H + \alpha \sqrt{H^2 + 1}} \leq \frac{\pi^2}{\alpha}$$

for $H \geq 0$.

Therefore roots of $\mu^2 - 1$ at $\lambda$ have to satisfy

$$\beta^2_1 (\lambda + \alpha + \lambda^{-1}) = \pi^2 l^2$$

with $l \in \mathbb{Z}$.

For $\alpha \in (2, \infty)$ and $\lambda \in S^1$ we have $0 < \beta^2_1 (\lambda + \alpha + \lambda^{-1}) \leq 2\pi^2$. Thus there are no solutions on $S^1$ besides $\lambda = \lambda_1, \lambda_2$ with $l = \pm 1$.

The corresponding frames can be easily calculated, and the corresponding surfaces are surfaces of revolution around a closed geodesic [21]. In the proof of Theorem 6 we mainly use properties (1.6) and (1.7). We do not make use of the fact that all cylinders $f \in \bigcup_{(a,b,\lambda_1,\lambda_2) \in \mathcal{M}_\text{rot}^1} A(a, b, \lambda_1, \lambda_2)$ are mean convex Alexandrov embedded. In fact, this follows from Theorem 6.

The boundary of $\mathcal{M}_\text{rot}^1$ consists of

- **homogeneous cylinders in $S^3$:** $H \in [0, \infty)$, $\alpha = 2$,
- **minimal cylinders in $S^3$:** $H = 0$, $\alpha \in [2, \infty)$,

In addition to these boundary components there exists two limiting cases: When $H = \infty$, $\alpha \in [2, \infty)$ we obtain unduloidal cmc cylinders in $\mathbb{R}^3$; When $H \in [0, \infty)$, $\alpha = \infty$, the resulting surfaces are **chain of spheres**. As a consequence of Proposition 4.1 for $2 < \alpha$ the only way to increase the genus is to open two conjugate pairs of double roots of $a$ in $\mathbb{C}^+ \setminus S^1$.

5. **Deformation of spectral data**

We now derive vector fields on open sets of spectral data $(a, b, \lambda_1, \lambda_2)$ and show that their integral curves are differentiable families of spectral data of periodic finite type solutions of the sinh-Gordon equation. We parameterise such families by one or more real parameters, which we will denote by $t$. We view the functions on the corresponding families of spectral curves $\Sigma_t$ locally as two-valued functions depending on $\lambda$ and $t$. From conditions (ii)-(iii) in Definition 1.3
we conclude that \( \partial_t h \) is meromorphic on the corresponding family of spectral curves, which is anti-symmetric with respect to the hyperelliptic involution \( \sigma \) (1.3). Furthermore, this function \( \partial_t h \) can only have poles at the branch points, or equivalently at the zeroes of \( a \), and at \( \lambda = 0 \) and \( \lambda = \infty \). If we assume that for such a family of spectral curves the genus \( g \) is constant, then \( \partial_t h \) can have at most first order have poles at simple roots of \( a \). In general we have

\[
\partial_t h = \frac{c(\lambda)}{\nu \lambda}
\]

with a polynomial \( c \) of degree \( g + 1 \) obeying the reality condition

\[
\lambda^{g+1} c(\lambda^{-1}) = c(\lambda).
\]

The corresponding vector field on the space of spectral data is derived from the equality of both mixed second derivatives with respect to \( \lambda \) and \( t \). For this purpose we write the derivative of \( h \) with respect to \( \lambda \) as (compare Definition 1.3 (iii))

\[
\partial_{\lambda}^2 h = \partial_{\lambda} \frac{c'}{\nu \lambda} - \frac{c}{\nu \lambda^2} = \frac{2 \lambda a(\lambda)c'(\lambda) - a(\lambda)c(\lambda) - \lambda a'(\lambda)c(\lambda)}{2 \nu \lambda^2 a(\lambda)},
\]

\[
\partial_{\lambda}^2 h = \partial_t \frac{b}{\nu \lambda^2} = \frac{b}{\nu \lambda^2} - \frac{b \nu}{\nu^2 \lambda^2} = \frac{2ab - b\nu}{2\nu \lambda^2 a}.
\]

Second partial derivatives commute if and only if

\[
2ab - b\nu = 2\lambda ac' - ac - \lambda a'c.
\]

Both sides in the last formula are polynomials of at most degree \( 3g + 1 \) which satisfies a reality condition. This corresponds to \( 3g + 2 \) real equations. Choosing a polynomial \( c \) which satisfies the reality condition (5.2) we thus obtain a vector field on polynomials \( a \) and \( b \).

**Remark 5.1.** If \( a \) and \( b \) have common roots, the vector field defined in (5.5) by a polynomial \( c \) has a singularity. In [15, Proposition 9.5] we construct integral curves passing through these singularities for polynomials \( c \), which does not vanish at the common roots of \( a \) and \( b \).

For spectral data of CMC cylinders in \( S^3 \) we have to deform in addition to the polynomials \( a \) and \( b \) the two Sym points, such that the closing condition (iii) of Definition 1.3 is preserved. As long as \( \lambda_1 \neq \lambda_2 \), and thus \( |H| < \infty \), we preserve the closing condition if \( \frac{d}{dt} h(\lambda_j(t), t) = 0 \), which holds precisely when \( \partial\lambda h(\lambda_j(t), t) \dot{\lambda}_j + \partial_t h(\lambda_j(t), t) = 0 \). Using equations (5.3) and (5.1), the closing conditions are therefore preserved if and only if

\[
\dot{\lambda}_1 = \frac{c(\lambda_1)}{b(\lambda_1)} \quad \text{and} \quad \dot{\lambda}_2 = -\frac{c(\lambda_2)}{b(\lambda_2)}.
\]

The equations (5.5)-(5.6) define rational vector fields on the space of spectral data \( (a, b, \lambda_1, \lambda_2) \) obeying conditions (i)-(iv) in Definition 1.3. These vector fields are smooth as long as \( a \) and \( b \) do not have common roots, and \( b \) does not vanish at \( \lambda_1 \) and \( \lambda_2 \). The normalisation \( \lambda_2 = \lambda_1^{-1} \) in Definition 1.3 (v) is preserved if

\[
\frac{c(\lambda_1)}{b(\lambda_1)} + \frac{c(\lambda_2)}{b(\lambda_2)} = 0.
\]

The function \( \mu = e^h \) is defined on the spectral curve, which is a two-sheeted covering over \( \lambda \in \mathbb{C} \). Hence \( \mu \) is a two-valued function depending on \( \lambda \in \mathbb{C}^* \). Instead of \( \mu \) the function

\[
\Delta : \mathbb{C}^* \to \mathbb{C}, \quad \lambda \mapsto \Delta = \mu + \mu^{-1}
\]
is single-valued and determines the corresponding values $\mu_{1,2} = \frac{1}{2}(\Delta \pm \sqrt{\Delta^2 - 4})$ of the function $\mu$. The range of (5.8) is called $\Delta$-plane and the domain is called $\lambda$-plane. Then $\mu \in S^1$ is equivalent to $\Delta \in [-2, 2]$. We shall construct in Proposition 5.2 a one-dimensional family of spectral data, such that the values of $\Delta$ at the $g + 1$ simple roots of $b$ are prescribed. These values lie on given curves in the $\Delta$-plane. If these curves do not intersect $\Delta = \pm 2$, then the roots of $b$ stay away from the roots of $a$.

Let $\Delta_0$ be the function (5.8) corresponding to the initial spectral data $(a_0, b_0, \lambda_{1,0}, \lambda_{2,0}) \in M^g$. We choose curves $\beta_1, \ldots, \beta_{g+1}$ in the $\lambda$-plane. They define the prescribed curves $t \mapsto \Delta_0(\beta_1(t)), \ldots, t \mapsto \Delta_0(\beta_{g+1}(t))$ in the $\Delta$-plane. We impose the following conditions:

(i) $\beta_1, \ldots, \beta_{g+1}:[-1, 1] \to \{\lambda \in \mathbb{C}^* \mid a_0(\lambda) \neq 0, \lambda \neq \lambda_{1,2}\}$ are either constant (see Figure 1) or smooth embedded curves (see Figures 2-3).

(ii) The initial values $\beta_1(0), \ldots, \beta_{g+1}(0)$ are the roots of the initial polynomial $b_0$.

(iii) Each curve $\beta_i([-1, 1])$ is either disjoint from the other (see Figure 1-2) or intersects exactly one other curve $\beta_j([-1, 1])$ in one point $\beta_i(1) = \beta_j(1)$ with $\beta_i(1) \neq \pm \beta_j(1)$ (see Figure 3). We assume that there exists a single-valued injective branch of $h$ on $\beta_i[0, 1]$ and $\beta_j[0, 1]$ respectively, which obeys (5.9) along $\beta_i$ and $\beta_j$:

$$h(\beta_i(-t)) = h(\beta_i(t)) \quad \text{for all } t \in [0, 1].$$

Such curves $\beta_i$ exist since the simple root $\beta_i(0)$ of $b$ is a first order branch point of $h$.

(iv) The curves respect the reality condition, so that two not necessarily different curves $\beta_i$ and $\beta_j$ with conjugated initial roots $\beta_j(0) = \overline{\beta_i(0)}$ obey $\beta_j(t) = -\overline{\beta_i(t)}$ for all $t \in [0, 1]$.

**Proposition 5.2.** Let $(a_0, b_0, \lambda_{1,0}, \lambda_{2,0}) \in M^g$ and let $\Delta_0$ (5.8) take $g + 1$ pairwise different values in $\mathbb{C} \setminus \{-2, 2\}$ at the roots of $b_0$. For given smooth curves $\beta_1, \ldots, \beta_{g+1}$ obeying (i)-(iv) there exists a continuous family $(a_t, b_t, \lambda_{1,t}, \lambda_{2,t})_{t \in [0, 1]}$ in $M^g$ such that $\Delta_t$ (5.8) takes at the roots of the polynomial $b_t$ the values $\Delta_0(\beta_1(t)), \ldots, \Delta_0(\beta_{g+1}(t))$.

**Proof.** We use the methods of [15, Section 9.2]. We compactify the $\lambda$-planes of the functions $\Delta_t$ (5.8) corresponding to the spectral data $(a_t, b_t, \lambda_{1,t}, \lambda_{2,t})$ to $\mathbb{C}P^1$. For any $t \in (0, 1]$ we remove from the compactified $\lambda$-plane of $\Delta_t$ the curves $\beta_1, \ldots, \beta_{g+1}$. Along this open subset of $\mathbb{C}P^1$ we glue deformed small tubular neighbourhoods $W_{1,t}, \ldots, W_{M,t}$ of the curves $\beta_1, \ldots, \beta_{g+1}$ as explained below. The result is a compact simply connected Riemann surface. By the uniformisation theorem there exists a global parameter $\lambda_t$ on this compact Riemann surface, which takes the value $\lambda_t = 0$ and $\lambda_t = \infty$ at the two points $\lambda = 0$ and $\lambda = \infty$ in the $\lambda$-plane of $\Delta_0$. The appropriately normalised parameter $\lambda_t$ identifies the compact Riemann surface with the compactified $\lambda$-plane of $\Delta_t$. Due to the gluing rules $\Delta_0$ extends to a holomorphic function $\Delta_t$ with respect to $\lambda_t \in \mathbb{C}$ and defines the spectral data $(a_t, b_t, \lambda_{1,t}, \lambda_{2,t})$.

![Figure 1. Constant curve](image1.png)

![Figure 2. Single curve](image2.png)

![Figure 3. Two curves](image3.png)

Due to conditions (i) and (iii), $\beta_1([-1, 1]) \cup \ldots \cup \beta_{g+1}([-1, 1])$ is the disjoint union $S_1 \cup \ldots \cup S_M$ of connected curves in the $\lambda$-plane of $\Delta_0$. Each $S_m$ contains either one curve like in Figure 1...
and 2, or contains two connected curves like in Figure 3. For each \( m = 1, \ldots, M \) the branch of \( h \) specified in condition (iii) has on \( S_m \) vanishing derivative only at the roots of \( b_0 \). This branch takes due to (5.9) on all curves \( \beta_i \) in \( S_m \) the same values at \( \beta_i(t) \) and \( \beta_i(-t) \).

We choose simply connected tubular neighbourhoods \( V_1, \ldots, V_M \) of \( S_1, \ldots, S_M \). Each \( V_m \) is sufficiently small not to contain a root of \( a_0 \). Therefore each \( U_m = \{(\lambda, \nu) \in \Sigma^* \mid \lambda \in V_m\} \) has two connected components. The branch of \( h \) in condition (iii) extends to one of these components. We choose integers \( n_1, \ldots, n_M \) such that \( h - 2n_m\pi i \) does not vanish on this component of \( U_m \). We extend the branch of \( h \) to the other component of \( U_m \) by setting \( \sigma^*(h - 2n_m\pi i) = -h + 2n_m\pi i \). With this choice \( (h - 2n_m\pi i)^2 \) is invariant with respect to the hyperelliptic involution and therefore is well defined on \( V_m \). The only critical points of \( (h - 2n_m\pi i)^2 \) on \( V_m \) are the roots of \( b_0 \) in \( S_m \).

We define polynomials \( A_m \) with the same critical values as \( (h - 2n_m\pi i)^2 \) on \( S_m \). After a change of coordinate \( z \mapsto cz + x_0 \) the highest coefficient is one and the second highest coefficient vanishes. So \( A_m \) takes one of the following forms:

\[
\begin{align*}
A_m(z_m) &= z_m^2 + a_m & \text{crit.pt. } z_m = 0 & \text{crit.value } a_m \\
A_m(z_m) &= z_m^3 + b_m z_m + a_m & \text{crit.pts. } z_m = \pm(\frac{-b_m}{3})^{1/2} & \text{crit.values } a_m \pm 2(\frac{-b_m}{3})^{3/2}.
\end{align*}
\]

On each simply connected \( V_m \) there exists a holomorphic function \( z_m \) such that \( (h - 2n_m\pi i)^2 \) is equal to a \( A_m(z_m) \). The roots of \( b_0 \) are the critical points of \( (h - 2n_m\pi i)^2 \) and correspond to the critical points of \( A_m \). Due to condition (iii) on sufficiently small tubular neighbourhoods \( V_1, \ldots, V_M \) the functions \( z_1, \ldots, z_M \) are biholomorphic maps onto simply connected open subsets \( W_1, \ldots, W_M \subset \mathbb{C} \). The images of \( S_1, \ldots, S_M \) in these subsets are denoted by \( T_1, \ldots, T_M \). For \( m = 1, \ldots, M \) we obtain a biholomorphic map \( S_m \subset V_m \simeq T_m \subset W_m \) (left hand side of (5.12)).

Now we deform these discs \( W_1, \ldots, W_M \). For this purpose we deform the polynomials \( A_m \) into a continuous family of polynomials \( \{A_{m,t}\}_{t \in [0,1]} \) of the form (5.10) with coefficients \( a_{m,t} \) and \( b_{m,t} \), respectively. Their critical values are the values of \( (h - 2n_m\pi i)^2 \) at \( \beta_j(t) \) and \( \beta_j(-t) \) and eventually at \( \beta_j(t) \) and \( \beta_j(-t) \). Due to (5.10) the coefficient \( a_{m,t} \) depend smoothly on \( t \in [0,1] \).

In case of two intersecting curves like in Figure 3 the difference of the critical values has at \( t = 1 \) a first order root with respect to \( t \). In this case \( b_{m,t} \) depends smoothly on \( t = 1 - (1-t)^2 \). For any \( z_m \in W_m \setminus T_m \) we consider a curve

\[
[0,1] \to \mathbb{C} \quad t \mapsto z_{m,t} \quad \text{with} \quad A_{m,t}(z_{m,t}) = A_m(z_m).
\]

Since \( A_{m,t}(z_{m,t}) \) is constant along the curve, it solves the differential equation

\[
\dot{z}_{m,t} = -\frac{A_{m,t}(z_{m,t})}{A'_{m,t}(z_{m,t})},
\]

with

\[
z_{m,0} = z_m.
\]

Figure 4. Deformation of a single curve

As long as \( z_{m,t} \) does not meet a critical point of \( A_{m,t} \) it is smooth. Along the interval \( t \in [0,1] \) the critical values of \( A_{m,t} \) move along the values of \( A_m \) on \( T_m \). This shows that for \( z_m \in W_m \setminus T_m \) these curves are smooth. Furthermore, for any \( t \in [0,1] \) the maps \( z_m \mapsto z_{m,t} \) are biholomorphic maps from \( W_m \setminus T_m \) onto the complement \( W_m,t \setminus T_m,t \) of the union of finitely many compact curves \( T_{m,t} \) in an open subset \( W_{m,t} \). On the deformations \( T_{m,t} \) of \( T_m \) the deformed polynomial
\( A_{m,t} \) takes the same values as the undeformed polynomial \( A_m \) on \( T_m \). For \( t \neq 0,1 \), two curves of \( T_{m,t} \) intersect each other at the critical points of \( A_{m,t}(z_m,t) \) as in Figure 4.

We glue the deformed subsets \( W_{1,t}, \ldots, W_{M,t} \) of \( \mathbb{C} \) parameterised by \( z_{m,t} \) in such a way with the undeformed \( \mathbb{CP}^1 \setminus (S_1 \cup \ldots \cup S_M) \), that the values of the polynomials \( A_{m,t}(z_{m,t}) \) coincide on \( W_{m,t} \setminus T_m \) with the values of the initial polynomials \( A_m(z_m) \) on \( W_m \setminus T_m \simeq V_m \setminus S_m \):

\[
\begin{align*}
S_m \subset V_m \ni \lambda &\simeq z_m \in T_m \subset W_m \\
(\Lambda + 2n_m \pi i)^2 \sqrt{A_m} &\subset C \\
W_m \setminus T_m \ni z_m &\simeq z_{m,t} \in W_{m,t} \setminus T_{m,t} \\
A_m \setminus \sqrt{A_{m,t}} &\subset \mathbb{C}
\end{align*}
\]

(5.12)

The initial function \( \Delta_0 \) on \( \mathbb{CP}^1 \setminus (T_1 \cup \ldots \cup T_M) \) is on \( W_m \setminus T_m \) equal to \( 2 \cosh(\sqrt{A_m(z_m)}) \). It extends as \( 2 \cosh(\sqrt{A_{m,t}(z_{m,t}))} \) holomorphically to \( W_{1,t}, \ldots, W_{M,t} \). We obtain a new copy of \( \mathbb{CP}^1 \) with two marked points \( \lambda = 0 \) and \( \lambda = \infty \) and a holomorphic map \( \Delta_t \) from the complement of these marked points to \( \mathbb{C} \). Due to condition (iv) the anti-holomorphic involution \( \lambda \mapsto \bar{\lambda}^{-1} \) of \( \mathbb{CP}^1 \setminus (T_1 \cup \ldots \cup T_M) \) extends to a holomorphic function \( \Delta_t \) depending on the global parameter \( \lambda_t \in \mathbb{C}^* \), which is on \( W_{m,t} \setminus T_{m,t} \) equal to \( 2 \cosh(\sqrt{A_{m,t}(z_{m,t}))} \).

We may fix the degree of freedom of rotations, by assuming that the values \( \lambda_{1,t} \) and \( \lambda_{2,t} \) at the Sym points \( \lambda_1 \) and \( \lambda_2 \) in the \( \lambda \)-plane of \( \Delta_0 \) obey \( \lambda_{1,t} \lambda_{2,t}^{-1} = \lambda_1^{-1} \lambda_2 \). This normalisation preserves condition (v) in Definition 1.3. Let \( a_t \) be the unique polynomial obeying (1.1), whose roots are the values of \( \lambda_t \) at the roots of \( a_0 \) in the \( \lambda \)-plane \( \mathbb{CP}^1 \setminus (T_1 \cup \ldots \cup T_M) \) of \( \Delta_0 \). Since \( a \) has \( 2g \) roots in \( \mathbb{C}^* \setminus (T_1 \cup \ldots \cup T_M) \), \( a_t \) has \( 2g \) roots in \( \mathbb{C}^* \). By construction, the \( 2 \)-valued function \( \mu \) with \( \Delta_0(\lambda) = \mu + \mu^{-1} \) on \( \mathbb{CP}^1 \setminus (T_1 \cup \ldots \cup T_M) \) extends to a unique \( 2 \)-valued function \( \mu_t \) depending on \( \lambda_t \in \mathbb{CP}^1 \) with \( \mu_t + \mu_t^{-1} = \Delta_t(\lambda_t) \). The corresponding 1-form \( \partial dh \) is meromorphic on the spectral curve induced by \( a_t \) of the form (5.3) with a unique polynomial \( b_t \). Together with \( \lambda_{1,t} \) and \( \lambda_{2,t} \) we obtain a family of spectral data \( (a_t,b_t,\lambda_{1,t},\lambda_{2,t}) \in [0,1] \in M^g \). By construction \( \lambda_t \mapsto a_t(\lambda_t)b_t(\lambda_t)(\lambda_t - \lambda_{1,t})(\lambda_t - \lambda_{2,t}) \) has pairwise different roots for \( t \neq 1 \).

Finally we show that this family of spectral data is smooth, since it solves an ordinary differential equation with smooth coefficients. The \( t \)-derivative of \( \mu(\beta(t),t) \) at a simple root \( \beta(t) \) of \( b \) is equal to \( \frac{\partial}{\partial t} \partial \beta(t) \partial \mu(\beta(t),t) = \partial \mu(\beta(t),t) \partial t + \mu(t,\beta(t),t) = \frac{c(\beta(t))}{\partial \beta(t)} d \beta(t) \). It does not depend on \( \beta(t) \) and is determined by \( c(\beta(t)) \). For roots \( \beta \) of \( b \) on \( \mathbb{S}^3 \) the polynomial \( c(\lambda) = \frac{\lambda^{\alpha + \beta}}{\lambda^{1-\beta} - \lambda^{1-\beta}} b(\lambda) \) vanishes at all other roots of \( b \). For roots \( \beta \in \mathbb{C} \setminus \mathbb{S}^1 \) the polynomial \( c(\lambda) = \frac{\lambda^{\alpha + \beta}}{\lambda^{1-\beta} - \lambda^{1-\beta}} b(\lambda) \) vanishes at all roots of \( b \) besides \( \beta \) and \( \beta^{-1} \). Therefore we may change the values of \( \mu \) at the simple roots of \( b \) independently. Furthermore, the values of \( c \) at the roots of \( b \) determine \( c \) up to adding to \( c \) a real multiple of \( ib \). The vector field corresponding to \( c = ib \) describes the rotations \( \lambda \mapsto e^{it} \lambda \). After adding to \( c \) a real multiple of \( ib \) the sum of the values of \( \xi \) at \( \lambda_1 \) and \( \lambda_2 \) vanishes. Due to (5.6) this normalisation preserves \( \lambda_1 \lambda_2^{-1} \). In particular, the smooth curves \( t \mapsto \Delta_0(\beta_{1}(t)), \ldots, t \mapsto \Delta_0(\beta_{g+1}(t)) \) determine together with the normalisation \( \frac{\partial}{\partial t} \lambda_1 \lambda_2^{-1} = 0 \) smooth \( t \)-dependent \( e \) on the space of spectral data \( (a,b,\lambda_1,\lambda_2) \) with pairwise different roots of \( \lambda \mapsto a(\lambda)b(\lambda)(\lambda - \lambda_1)(\lambda - \lambda_2) \) such that the values of \( \Delta(5.8) \) at the roots of \( b \) of the corresponding solution of (5.5)-(5.6) follow the given curves \( t \mapsto \Delta_0(\beta_{1}(t)), \ldots, t \mapsto \Delta_0(\beta_{g+1}(t)) \). This implies that this family of spectral data is for \( t \neq 1 \) indeed smooth. In case of two intersecting curves \( \beta_0 \) and \( \beta_j \) like in Figure 3 at \( t = 1 \) the critical values of \( A_{m,t} \) at the two coalescing roots of \( b_t \) does not depend smoothly on \( t \). But this family depends together with the coefficients of \( A_{1,t}, \ldots, A_{M,t} \) for \( t = 1 - (1-t)^2 \) smoothly on \( t \in [0,1] \).
The proof even shows that there exists a family of spectral data \((a_t, b_t, \lambda_{1,t}, \lambda_{2,t})_{t \in T} \in \mathcal{M}^g\) parameterised by tuples \(t = (t_1, \ldots, t_{g+1}) \in T \subset [0, 1]^{g+1}\) with the values \(h(\beta_i(t_j)) = h(\beta_i(-t_j))\) at the roots of \(b_t\). Here \(T\) is due to condition (iv) characterised by the equations \(t_i = t_j\) on the parameters of conjugated initial roots \(\beta_j(0) = \beta_i^{-1}(0)\).

The following Lemma is used in Section 7 and carries over [15, Section 9.2] to the present situation.

**Lemma 5.3.** Let \(\tilde{\mathcal{M}}^g\) be the space of \((a, b, \lambda_1, \lambda_2) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \times \mathbb{S}^1 \times \mathbb{S}^1\) obeying (i)-(iv) in Definition 1.3 without the inequality in (1.1), such that \(\lambda_1 \neq \lambda_2, b(\lambda_1) \neq 0 \neq b(\lambda_2)\) and \(\frac{b}{\alpha}\) has first order poles at all common roots of \(a\) and \(b\). It is a real \(g + 1\)-dimensional submanifold. The equations (5.5) and (5.6) identify \(T\mathcal{M}^g\) with \(c \in \mathbb{C}^{g+1}[\lambda]\) obeying (5.2) and (5.7).

**Proof.** The space of \((a, b, \lambda_1, \lambda_2)\) obeying condition (i) in Definition 1.3 without the inequality in (1.1) is a real \((2g + 1) + (g + 2) + 2 = 3g + 5\) dimensional space. Condition (ii) states that the integral of \(dh\) along \(g\) independent cycles of \(\Sigma\) vanishes. This contains \(g\) linear independent conditions. The normalization (vi) contains two linear independent conditions. Hence conditions (i)-(ii) and (vi) describe a \(2g + 3\)-dimensional subspace. Condition (iii) states that the integrals of \(dh\) along \(g\) further independent cycles of \(\Sigma\) takes values in \(2\pi i\mathbb{Z}\). Condition (iv) assumes that the integrals of \(dh\) along two paths connecting both points over \(\lambda_1\) and \(\lambda_2\) takes values in \(2\pi i\mathbb{Z}\). Hence \(\tilde{\mathcal{M}}^g\) is the subset of a \(2g + 3\)-dimensional space defined by \(g + 2\) real functions taking values in \(2\pi i\mathbb{Z}\). We show that all tangent vectors in the kernel of the derivatives of these functions form a \(g + 1\)-dimensional space and apply the implicit function theorem.

For a tangent vector \((\hat{a}, \hat{b}, \hat{\lambda}_1, \hat{\lambda}_2)\) at \((a, b, \lambda_1, \lambda_2) \in \tilde{\mathcal{M}}^g\) in the kernel of the derivatives of these functions the integral of the 1-form \(\hat{h} \, dh\) in Definition 1.3 (iii) along any closed cycle of \(\Sigma\) vanishes. The derivative \(\hat{h}\) defines a single-valued meromorphic function on the corresponding spectral curve \(\Sigma\). Due to the properties of \(h\) it is of the form (5.1) with a polynomial \(c\) of degree \(g + 1\) obeying (5.2) and (5.5). Equation (5.6) determines \(\hat{\lambda}_1\) and \(\hat{\lambda}_2\) in terms of \(c\). If \(a\) and \(b\) do not have a common root, then (5.5) uniquely determines \(\hat{a}\) and \(\hat{b}\) in terms of \(c\). Due to the implicit function theorem \(\tilde{\mathcal{M}}^g\) is at \((a, b, \lambda_1, \lambda_2)\) a submanifold with \(T\tilde{\mathcal{M}}^g\) isomorphic to the \(g + 1\) dimensional space of polynomials \(c\) obeying (5.2) and (5.7).

**Remark 5.4.** The relation (5.5) between \(c\) and \((\hat{a}, \hat{b})\) does not encode Definition 1.3 (iv). Without this condition we can add the polynomial \(a\) without changing \(h\) a double root \(\alpha \in \mathbb{S}^1\) or a pair of double roots \(\alpha, \alpha^{-1} \in \mathbb{C}^* \setminus \mathbb{S}\) together with the same simple roots of \(b\). The movement of such double roots is not controlled by \(c\). Due to condition (iv) we can add such roots of \(a\) and \(b\) only at points with \(\mu = \pm 1\). If \(\frac{b}{\alpha}\) has simple poles at common roots, then \(c\) determines \((\hat{a}, \hat{b})\).

It remains to include the case of common roots of \(a\) and \(b\) with simple poles of \(\frac{a}{b}\). We describe \(\Sigma\) by the function \(\Delta\) (5.8) and use the methods of Proposition 5.2 (see [15, Section 9.2]). We remove small disjoint discs \(V_m \subset \mathbb{C}^*\) around the roots of \(b\) from the compactified \(\lambda\)-plane \(\mathbb{C}P^1\) of \(\Delta\) (5.8). Each \(V_m\) contains only one root of \(\lambda = a(\lambda)b(\lambda)(\lambda - \lambda_1)(\lambda - \lambda_2)\). On each \(V_m\) we choose a single-valued branch of \(h\) and an integer \(n_m\) such that \(h - n_m\pi i\) vanishes only at the common root of \(a\) and \(b\) in \(V_m\) (if there is one). Due to (1.9) \(\frac{h - n_m\pi i}{\mu^m}\) is holomorphic on \(V_m\) without roots. Therefore the roots of \(h - n_m\pi i\) coincide with the roots of \(a\) in \(V_m\). The discs \(V_m\) have unique coordinates \(z_m\) which vanish at the root of \(b\) in \(V_m\) with

\[
A_m(z_m) = z_m^{d_m} + a_{m,d_m} = (h - n_m\pi i)^2.
\]

Here \(d_m - 1\) is the order of the root of \(b\) in \(V_m\). For common roots, \(d_m\) is the order of the root of \(a\). \(V_m\) is mapped by \(z_m\) biholomorphically onto small discs \(W_m \subset \mathbb{C}\).

For all \((\hat{a}, \hat{b}, \hat{\lambda}_1, \hat{\lambda}_2)\) in a sufficiently small neighbourhood \(O \subset \tilde{\mathcal{M}}^g\) of \((a, b, \lambda_1, \lambda_2)\) the corresponding branches \(\hat{h} - n_m\pi i\) are well defined on the deformed subsets \(\hat{V}_1, \ldots, \hat{V}_M\) of the
compactified \( \lambda \)-plane of the deformed \( \tilde{\Delta} \) \((5.8)\). Furthermore the function \((\tilde{h} - n_m \pi i)^2\) takes for all \( m = 1, \ldots, M \) on the boundary \( \partial \tilde{V}_m \) the same values as \( A_m \) on the boundary of \( W_m \). We glue the complement \( \mathbb{CP}^1 \setminus W_m \) of the compactified \( z_k \in \mathbb{CP}^1 \) plane in such a way along the boundary of \( \tilde{V}_m \) that \( A_m(z_m) \) coincides on \( \partial W_m \) with \((\tilde{h} - n_m \pi i)^2\) on \( \partial \tilde{V}_m \). This yields a new copy of \( \mathbb{CP}^1 \). By uniformisation there exists a global coordinate \( \tilde{z}_m \) on the new copy of \( \mathbb{CP}^1 \) with a pole at the pole of \( z_m \) in \( \mathbb{CP}^1 \setminus W_m \). It is unique up to Möbius transformations fixing \( \tilde{z}_m = \infty \). The function \( A_m(z_m) = (h - n_m \pi i)^2 \) \((5.13)\) on \( \mathbb{CP}^1 \setminus W_m \) extends as \((\tilde{h} - n_m \pi i)^2\) to \( W_m \). This function is meromorphic on the new copy of \( \mathbb{CP}^1 \) with a single pole at the pole of \( \tilde{z}_m \) of degree \( d_m \). Therefore this function is a polynomial \( A_m \) with respect to \( \tilde{z}_m \) of degree \( d_m \).

After a unique Möbius transformation of the coordinate \( \tilde{z}_m \) fixing \( \tilde{z}_k = \infty \) we have
\[
(5.14) \\
\tilde{A}_m(\tilde{z}_m) = \tilde{z}_m^{d_m} + \tilde{a}_{m,2} \tilde{z}_m^{d_m-2} + \ldots + \tilde{a}_{m,d_m} = (\tilde{h} - n_m \pi i)^2.
\]

The coefficients \( \tilde{a}_{m,2}, \ldots, \tilde{a}_{m,d_m} \) belong to a small neighbourhood of the corresponding coefficients of \( A_m \). The map \( \lambda \mapsto \lambda^{-1} \) interchanges the roots of \( b \) and therefore also the discs \( V_1, \ldots, V_M \) and the polynomials \( A_1, \ldots, A_M \). Since this symmetry is preserved, the coefficients of those not necessarily different polynomials have complex conjugate coefficients, which are interchanged by this symmetry. The sum of the real degrees of freedom of the coefficients of \( \tilde{A}_1, \ldots, \tilde{A}_M \) is \( \deg b = g + 1 \). We obtain a space of polynomials \((\tilde{A}_1, \ldots, \tilde{A}_M) \in B\) of small perturbations of \((A_1, \ldots, A_M)\) parameterised by a small ball in \( \mathbb{R}^{g+1} \). This yields a map
\[
\Phi : O \rightarrow B \subset \mathbb{R}^{g+1} \quad (\tilde{a}, \tilde{b}, \tilde{\lambda}_1, \tilde{\lambda}_2) \mapsto (\tilde{A}_1, \ldots, \tilde{A}_M).
\]

In order to apply the implicit function theorem it remains to show that \( \Phi' \) maps those tangent vectors \((\tilde{a}, \tilde{b}, \tilde{\lambda}_1, \tilde{\lambda}_2)\) at \((a, b, \lambda_1, \lambda_2)\) injectively to \( T_{(\tilde{A}_1, \ldots, \tilde{A}_M)} B = \mathbb{R}^{g+1} \) which correspond to polynomials \( c \) of degree \( g + 1 \) obeying \((5.2)\) and \((5.7)\). Due to \((5.1), (5.3)\) and \((5.14)\) we have
\[
\frac{\partial \tilde{h}}{\partial \lambda} = \frac{\lambda c}{b} = \frac{\tilde{A}_m}{A'_m} \frac{d\lambda}{dz_m}
\]
on \( V_m \). Since \( \lambda \mapsto z_m \) is biholomorphic from \( V_m \) onto \( W_m \), \( \frac{d\lambda}{dz_m} \) does not vanish on \( V_n \). Hence the singular parts of \( \frac{\lambda c}{b} \) at the roots of \( b \) are in one-to-one correspondence to \( \tilde{A}_1, \ldots, \tilde{A}_M \). \( \square \)

6. Proof of Theorem 3

In this section we construct a piecewise continuous path (see Definition 1.8) in \( \mathcal{M}_\pm \) connecting an arbitrary starting point with an endpoint in \( \mathcal{M}_\pm^0 \). In the proof we control the deformation of the spectral data by choosing piecewise appropriate polynomials \( c \).

We prove this theorem in seven steps. In each step we choose an appropriate polynomial \( c \), such that the corresponding solution of the ordinary differential equations \((5.5)-(5.6)\) describes a path with specified properties. For this purpose we exhibit how the choice of \( c \) controls the movement of the roots of \( a \), the roots of \( b \) and the Sym points \( \lambda_1 \) and \( \lambda_2 \). Let us assume that there is given a path of spectral data parameterised by \( t \), which solves \((5.5)\) with some smooth family of \( c \)'s. The corresponding equations \((1.2)\) define a family of hyperelliptic algebraic curves. These curves are two-sheeted coverings over \( \lambda \in \mathbb{CP}^1 \). We consider the functions \( \mu \) on these two sheeted coverings as two-valued functions depending on \( \lambda \) and the deformation parameter \( t \). The hyperelliptic involution \( \sigma \) \((1.3)\) interchanges the two branches of \( \mu(\lambda(t), t) \) and acts by \( \mu \mapsto \mu^{-1} \). Now let \( t \mapsto \lambda(t) \) be a path in \( \mathbb{C}^* \), such that the function \( t \mapsto \mu(\lambda(t), t) \) is constant along the former path of spectral data. Due to \((5.1)\) and \((5.3)\) the latter path \( t \mapsto \lambda(t) \) obeys
\[
(6.1) \quad \frac{d\mu(\lambda(t), t)}{dt} = \frac{\partial \mu(\lambda(t), t)}{\partial \lambda} \dot{\lambda}(t) + \frac{\partial \mu(\lambda(t), t)}{\partial \lambda} = 0, \quad \frac{\dot{\lambda}(t)}{\lambda(t)} = -\frac{c(\lambda(t))}{b(\lambda(t))}.
\]
The roots of $a$ and the Sym points (5.6) move along paths on which $\mu$ is constant. Finally, simple roots of $b$, which are also roots of $c$, also move along paths on which $\mu$ is constant. In this case the right hand side in the differential equation should be replaced by the unique holomorphic extension of the quotient $-\xi$ to the common roots of $b$ and $c$. Thus (6.1) describes the movement of several significant points. The $t$-derivative of $\mu(\beta(t))$ at a simple root $\beta(t)$ of $b$ is equal to $\frac{d}{dt}\mu(\beta(t), t) = \partial_\lambda \mu(\beta(t), t) \dot{\beta}(t) + \partial_t \mu(\beta(t), t) = \frac{c(\beta(t))}{c(\beta(t))}$. It does not depend on $\dot{\beta}(t)$ and is determined by $c(\beta(t))$. Changing the values of $\mu$ at the roots of $b$ implicitly changes the relative positions of the roots of $a$, the roots of $b$ and the Sym points. At the roots of $a$ and the Sym points we have $\mu = \pm 1$. We can avoid that the roots of $b$ meet the roots of $a$ and the Sym points by ensuring that the values of $\mu$ (or $\Delta$) at the roots of $b$ stay away from $\mu = \pm 1$ (or $\Delta = \pm 2$).

The function $\mu$ transforms as $\sigma^* \mu = \mu^{-1}$ and $\rho^* \mu = \bar{\mu}^{-1}$. It takes on the fixed point set $\lambda \in \mathbb{S}^1$ of the involution $\rho$ (1.3) unimodular values. Therefore the argument of $\mu$ is on $\mathbb{S}^1$ a function with values in $\mathbb{R}/2\pi\mathbb{Z}$. The two branches of $\arg \mu$ are the negative of each other. The critical points of $\arg \mu$ on $\mathbb{S}^1$ are the roots of $b$ on $\mathbb{S}^1$. In particular, at simple roots of $b$ on $\mathbb{S}^1$, one branch of this function has a local maximum and the other branch a local minimum.

We first apply in steps 1-3 small deformations to achieve that $a$ and $b$ have pairwise different simple roots in $\mathbb{C}^* \setminus \{\lambda_1, \lambda_2\}$. In step 4 we achieve in addition that $\mu$ takes at the roots of $b$ in $\mathbb{S}^1$ pairwise different values on $\mathbb{S}^1 \setminus \{-1, 1\}$ and at the other roots pairwise different values on $\mathbb{C}^* \setminus \mathbb{S}^1$.

1. In step 1 we divide the polynomials $a$ and $b$ by polynomials $p^2$ and $p$, if $a$ has higher order roots, as described in Lemma 1.7. We will meet such $a$ again only in the last step.

2. In step 2 we choose a small deformation, which decreases the length of the short arc and eventually separates the roots of $b$ from the Sym points $\lambda_1$ and $\lambda_2$. This step is only required if initially $H = 0$ or if $b$ vanishes at one Sym point. For polynomials $c$ obeying (5.2) the meromorphic function $-\xi$ takes imaginary values on $\mathbb{S}^1$. We choose $c$, which does not vanish at the Sym points $\lambda_1$ and $\lambda_2$, such that $-\xi$ has positive imaginary part at the end of the short arc nearby $\lambda_1$ and negative imaginary part at the end of the short arc nearby $\lambda_2$. Since $c$ has degree $g + 1 \geq 2$ such polynomials exist. If $b$ does not vanish at the Sym points, then due to (5.6) with this choice of $c$ the Sym points are moved towards the interior of the short arc.

Since $\arg \mu$ has only isolated critical points on $\mathbb{S}^1$, the restriction of one branch to the short arc has at the boundary of the short arc a local maximum, and the restriction of the other branch has a local minimum. Due to (5.1) the branch of $\arg \mu$ with a local maximum at the end of the short arc strictly increases (for increasing $t$) nearby the Sym point, and the other branch strictly decreases.

In Figure 6 a family of graphs of the function $\arg \mu$ on $\mathbb{S}^1$ with a local maximum at $\arg \lambda_0$ for $t = t_0$ is shown. The corresponding pre-image $\{\arg(\lambda, t) \mid \mu(\lambda, t) = \mu_0\}$ is shown in Figure 7. At $(\lambda_0, t_0)$ we see a bifurcation into two different paths $t \mapsto \lambda(t)$ in the pre-image $\{(\lambda, t) \mid \mu(\lambda, t) = \mu_0\}$. If the Sym point is a root of $b$, then the pre-image $\{(\lambda, t) \mid \mu(\lambda, t) = \mu_0\}$ might contain several paths ending and starting at the Sym point. Any such path in $\mathbb{S}^1$ (if they exist) can be chosen as the corresponding Sym point along the path $t \mapsto (a_t, b_t)$. With our choice of $c$ there always exist two continuous paths $t \mapsto \lambda_1(t)$ and $t \mapsto \lambda_2(t)$ for $t \geq 0$ moving the two Sym points toward the interior of the short arc. By choosing these paths as the Sym points along the path $t \mapsto (a_t, b_t)$, we obtain a path of spectral data with strictly decreasing length of the short arc. This shows that with our choice of $c$ an integral curve of (5.5) yields
a path in $\mathcal{M}_d^g$ with strictly increasing mean curvature. If $a$ and $b$ do not have common roots, then the smooth vector field (5.5) always has a local solution. If $a$ and $b$ have common roots, then we may assume that $c$ does not vanish at the common roots of $a$ and $b$. We apply [15, Proposition 9.5] as described in Remark 5.1.

Figure 6. Family of graphs

Figure 7. Pre-image of $\mu = \mu_0$

Figure 8. Coalescing roots of $a$

3. In step 3 we choose a small deformation which separates the roots of $b$ from each other. In step 2 we achieved that the short arc has length smaller then $\pi$. Hence we can follow any deformation for a short time. The values of $\partial_t \partial_\lambda h$ (5.1) at some higher order root $\beta \in \mathbb{C}^*$ of $b$ depends linearly on $c(\beta)$ and $c'(\beta)$ (5.4). Since $c$ has degree $g + 1 \geq 1$ there exist $c$ such that $\partial_t \partial_\lambda h$ does not vanish at $\beta$. With the following lemma we may deform successively all higher order roots of $b$ into simple roots.

**Lemma 6.1.** Let $c \in \mathbb{C}^{g+1}[\lambda]$ obey (5.2) such that the 1-form $d\frac{\mu}{\mu\lambda}$ (5.1) does not vanish at a higher order root of $b$ which is not a root of $a$. Then the flow of the vector field (5.5) separates the higher order root of $b$ after arbitrarily short time into several simple roots. On $S^1$ the 1-form $d\frac{\mu}{\mu\lambda}$ is purely imaginary. One choice of sign of $d\frac{\mu}{\mu\lambda}$ at a double root of $b$ on $S^1$ separates the double root of $b$ along $S^1$. The other choice of sign separates the double root off $S^1$.

**Proof.** At higher order roots of $b$ there exists a local coordinate $z$ such that $\partial_\lambda h = z^n$ with $n \geq 2$. If $\partial_t \partial_\lambda h$ takes the value $C \in \mathbb{C}^*$ there, then $\partial_\lambda h$ is for small $t$ nearby $z = 0$ of the form

$$\partial_\lambda h = z^n + C t + O(t^2) + O(z).$$

For small $t$ any such function has $n$ distinct simple roots near $z = 0$.

The function $h$ and the 1-form $dh$ are purely imaginary on $S^1$. Therefore $\partial_t dh$ is also purely imaginary on $S^1$. Due to (5.1) the 1-form $d\frac{\mu}{\mu\lambda}$ is equal to $\partial_t dh$ and purely imaginary on $S^1$. Nearby a double root of $dh$ on $S^1$ there exists a local coordinate $z$ which is real on $S^1$ such that

$$dh|_{t=0} + t \partial_t dh + O(t^2) = -iz^2dz + t (icdz + O(z)dz) + O(t^2) \quad \text{with} \quad C \in \mathbb{R}.$$  

For small $t > 0$ the 1-form $dh$ has for $C > 0$ no root and for $C < 0$ two roots in $z \in \mathbb{R}$ near $z = 0$. \hfill \Box

4. After steps 1-3 we can assume that the short arc has length smaller than $\pi$, and all roots of $b$ and $a$ are simple and pairwise distinct, and lie away from the Sym points. In step 4 we achieve in addition that $\mu$ takes at the roots of $b$ in $S^1$ pairwise different values on $S^1 \setminus \{0,1\}$ and at the other roots pairwise different values on $\mathbb{C}^* \setminus S^1$. The $t$-derivative of $h(\beta(t), t)$ at a simple root $\beta(t)$ of $b$ does not depend on $\dot{\beta}(t)$ and is determined by $c(\beta(t))$ (see (6.1)). For roots $\beta$ on $S^1$ the polynomial $c(\lambda) = \frac{1+\beta}{1-\beta} b(\lambda)$ vanishes at all other roots of $b$. For roots $\beta \in \mathbb{C}^* \setminus S^1$ the polynomials $c(\lambda) = (\frac{C}{1-\beta} - \frac{CA}{1-\beta\lambda}) b(\lambda)$ vanishes at all roots of $b$ besides $\beta$ and $\beta^{-1}$. Therefore we may change the values of $\mu$ at the simple roots of $b$ independently.

5. After the first four steps we have achieved arrive that $a$ and $b$ have pairwise different simple roots in $\mathbb{C}^* \setminus \{\lambda_1, \lambda_2\}$, and such that $\mu$ takes at the roots of $b$ pairwise different values in
In step 5 we remove all roots of $b$ from the short arc with possibly one exception. Here we use two Lemmata.

**Lemma 6.2.** Let $\beta \in S^1$ be a simple root of $b$ in the interior of the short arc. We choose the sign of the polynomial $c = \pm \frac{\lambda + i}{\lambda - \beta}$, whose vector field decreases $\arg \mu$ at the branch having on $S^1$ at $\beta$ a local maximum, and increases $\arg \mu$ at the branch having on $S^1$ at $\beta$ a local minimum, respectively. Then this vector field decreases the length of the short arc.

If $\beta$ belongs to the interior of the long arc, then the same conclusion holds for the sign, which increases $\arg \mu$ at the branch having on $S^1$ at $\beta$ a local maximum.

**Proof.** Let $\beta \in S^1$ be a simple root of $b$ in the interior of the short arc. The vector field corresponding to $c = ib$ describes a rotation of $\lambda$ and does not change the length of the short and the long arc. Therefore we can add to $c$ a real multiple of $ib$ without changing the derivative of the length of the short arc. After adding to $c$ an appropriate multiple of $ib$ the quotient $\frac{1}{c}$ will have a unique root in the interior of the long arc. In this case, due to the formula (5.6), the sign of the imaginary parts of $\lambda_1/\lambda$ and $\lambda_2/\lambda$ are different. The branch of $\arg \mu$ having on $S^1$ at $\beta$ a local maximum is monotonically decreasing for $\arg \lambda > \arg \beta$. Consequently, $\arg \lambda_2$ changes in the same direction as $\arg \mu$ at the maximum over $\beta$. This implies the claim. \qed

**Lemma 6.3.** Let $\beta \in S^1$ be a simple root of $b$ along the integral curve of the vector field corresponding to $c = \pm \frac{\lambda + i}{\lambda - \beta}$. Two roots of $a$ can only coalesce at $\beta$ at some point of the integral curve, if the branch of $\arg \mu$ having on $S^1$ at $\beta$ a local maximum increases, and the branch of $\arg \mu$ having on $S^1$ at $\beta$ a local minimum decreases.

**Proof.** The number by which a prescribed value is attained by the function $\mu$ inside a given domain $\Omega \subset \mathbb{C}^*$ cannot change, as long as this values is not attained on the boundary $\partial \Omega$. The function $\mu$ takes at the roots of $a$ a value $\mu_0 \in \{-1, 1\}$ independent of $t$. If two roots coalesce at a simple root $\beta \in S^1$, then $\arg \mu$ takes the corresponding value $\arg \mu_0 \in 2\pi\mathbb{Z}$ twice on $S^1$, once with multiplicity one at both branches of $\arg \mu$. Therefore $\arg \mu$ takes on a neighbourhood of $\beta$ in $S^1$ values in the complement of $(\arg \mu_0 - \epsilon, \arg \mu_0 + \epsilon)$, as long as $\arg \mu_0$ is taken at two roots of $a$ in a neighbourhood of $\beta$ away from $S^1$. More precisely the branch of $\arg \mu$ having at $\beta$ a local maximum takes at the maximum a value smaller than $\arg \mu_0$. The other branch takes at the minimum a value larger then $\arg \mu_0$. Consequently the values of $\arg \mu$ increases on the branch having at $\beta$ a local maximum before the roots of $a$ coalesce at $\beta$, and the values of $\arg \mu$ decreases on the branch having at $\beta$ a local minimum. In Figure 8 the graphs of the two branches of $\arg \mu$ nearby a simple root of $b$ in $S^1$ for coalescing roots of $a$ are shown. \qed

**Continuation of the proof of the Theorem 3:** Every simple root $\beta_i$ of $b$ in the short arc is a local extremum of the restriction of $\arg \mu$ to $S^1$. First we assume that the short arc contains more than one root of $b$. Then two of the roots of $b(\lambda)(\lambda - \lambda_1)(\lambda - \lambda_2)$ on $S^1$ are adjacent neighbours of $\beta_i$. The value $\arg \mu_1$ of $\arg \mu$ at one of these two neighbours is closer to the value $\arg \mu_0$ of $\arg \mu$ at $\beta_i$ than the other value $\arg \mu_2$. Then there exists a smooth embedding $\beta_i : [-1, 1] \hookrightarrow S^1$ whose restriction to $[0, 1]$ moves the root $\beta_i$ to this neighbour and obeys (5.9). This path together with the constant paths of all other roots of $b$ obey conditions (ii)-(iv) of Proposition 5.2. If the neighbour is a Sym point, then condition (i) is violated for $t = 1$. The construction of Proposition 5.2 applies to this situation and yields a path of spectral data. At the end point of the path the equations (5.6) becomes singular. As we have seen in step 2 such singularities are bifurcation points of the movement of the Sym points. The corresponding path of spectral data is an integral curve of the vector field described in Lemma 6.2 and increases the mean curvature. This deformation changes the values of $\mu$ at the simple root at $\beta_i$ of $b$ and fixes the values of $\mu$ at all other roots of $b$, $a$ and at the Sym points. For $t = 1$ the root $\beta_i$ either
meets one of the Sym points $\lambda_1$ and $\lambda_2$ or another root of $b$. If $\beta_i$ meets only one Sym point, then the deformation described in step 2 moves $\beta_i$ into the long arc. If $\beta_i$ meets another root $\beta_j = \beta_i(1)$ of $b$, then we move with Lemma 6.1 $\beta_i$ and $\beta_j$ by a small deformation away from $S^1$. Afterwards we continue with the deformation of another root of $b$ in the short arc.

After finitely many such deformations we arrive at spectral data with at most one root of $b$ in the short arc. Moreover we can assume that along the short arc the integral of $dh$ vanishes (and a branch of $b$ takes at $\lambda_1$ and $\lambda_2$ the same values).

If the integral of $dh$ along the short arc does not vanish, we can move the last root of $b$ inside the short arc into the long arc without shrinking the short arc to zero. If finally the short arc does not contain any root of $b$, then the integral of $dh$ along the short arc does not vanish. Since the integral of $dh$ has to be a multiple of $2\pi i$, there exists one point in the short arc, such that the integral from both Sym points to this point of $dh$ are equal. At this point $\mu$ has to be equal to $\pm 1$. With Lemma 1.7 we add to $a$ a double root and to $b$ a simple root at this point in $S^1$. We obtain new spectral data in $M^+$. With Lemma 5.3 we deform the double root of $a$ on $S^1$ into two roots away from $S^1$ and the root of $b$ staying on $S^1$ (compare Lemma 6.3).

After this deformation the sign of $dh$ changes at the root of $b$. Since initially the integrals of $dh$ along both segments of the short arc are equal up to sign, the integral of $dh$ along the short arc will be zero afterwards. Therefore we end with exactly one root of $b$ in the short arc and the integral of $dh$ over the short arc vanishes as we assume in the preceding paragraph.

6. In step 6 we move all roots of $b$ with the exception of the single root in the short arc to the long arc. To all pairs of roots in $C^* \setminus S^1$ we apply successively the following lemma.

**Lemma 6.4.** Let the short arc contain one root and the integral of $dh$ along the short arc vanish. If there exists a pair of simple roots of $b$ interchanged by $\lambda \rightarrow \lambda^{-1}$ with values of $\mu$ not in $S^1$, then we can move one of these pairs of roots of $b$ to a point $\lambda_0$ in the long arc.

**Proof.** We apply Proposition 5.2. First we construct a path $\beta_i : [-1, 1] \rightarrow \overline{B(0, 1)}$, which together with the path $\beta_j(t) = \beta_i^{-1} t$ obeys conditions (i)-(iv) in Proposition 5.2. We consider for all $C > 1$ the sets

$$S_C = \{(\lambda, \nu) \in \Sigma^* \mid \lambda \in B(0, 1) \text{ and } |\mu(\lambda, \nu)| = C\}.$$ 

If $S_C$ does not contain roots of $b$, then it is a one-dimensional submanifold of $\Sigma^*$. On a punctured disc in $\Sigma$ around the point $(\nu, \lambda) = (\infty, 0)$ the function $h$ has a unique single-valued injective branch with $\sigma^* h = -h$. For large $C > 1$, $S_C$ is completely contained in this punctured disc and has only one connected component. If we decrease $C$, then $S_C$ remains a connected component as long as $S_C$ does not contain a root of $b$. By assumption $S_C$ contains roots of $b$ for some $C > 1$. There exists a largest $C_1 > 0$, such that $S_{C_1}$ contains roots of $b$. Now we choose a path $\gamma$ in $\overline{B(0, 1)}$ starting at some point $\lambda_0$ in the interior of the long arc, which intersects these submanifolds $(S_C)_{C > 1}$ transversally. For $C > C_1$ the sets $S_C$ are connected. Moreover the connected component of $S_{C_1}$ which intersects $\gamma$ contains a root of $b$. Therefore there exists a smallest $C_2 > 1$ such that the connected component of $S_{C_2}$ which intersects $\gamma$ contains a root of $b$. For $1 < C < C_2$ the connected components of $S_C$ intersecting $\gamma$ contain no root of $b$ and are either diffeomorphic to $\text{arg } \mu \in \mathbb{R}$ or to $\text{arg } \mu \in \mathbb{R}/2n\pi \mathbb{Z}$ for some $n \in \mathbb{N}$. Therefore there exists a smooth path intersecting these lines transversally from the root $\beta_1$ of $b$ in $S_{C_2}$ to $\lambda_0$. We may choose this path in such a way that any branch of $h$ maps this path to a straight line in the $h$-plane. In this way we get a smooth embedded path $\beta_i : [0, 1] \rightarrow \overline{B(0, 1)}$ from a root $\beta_i = \beta_i(0)$ of $b$, with values of $\mu$ not in $S^1$, to $\lambda_0 = \beta_i(1) \in S^1$.

We extend this path smoothly to $[-1, 0]$ in such a way that (5.9) is satisfied. If this extension meets another root of $b$, then we slightly change $\lambda_0$ on $S^1$ and the straight line $\beta_i([0, 1])$ in $h$-plane from $\lambda_0$ to $\beta_1$ such that the unique extension to $[-1, 0]$ obeying (5.9) does not meet another
root of \( b \). The map \( h \) map \( \beta_t([-1,0]) \) to the segment \( \beta_t([0,1]) \) \((h(\beta_t(-t)) = h(\beta_t(t))) \). Now we claim that \( \beta_t(-1) \) does not belong to \( S^1 \) but to an other component of \( \{\mu = 1\} \). Otherwise the path \( \beta_t([-1,1]) \) divides \( B(0,1) \) into two connected components. But in any neighbourhood of a point of \( S_C \) with \( 1 < C < C_2 \) there starts a straight half-line in the \( h \)-plane parallel to the segments \( \beta_t([0,1]) \) and \( \beta_t([-1,0]) \) which does not contain roots of \( b \) and ends at \( \lambda = 0 \). Such half-lines do not intersect \( \beta_t([-1,1]) \). Hence \( \beta_t([-1,1]) \) does not divide \( B(0,1) \) and \( \beta_t(-1) \notin S^1 \).

Now we apply Proposition 5.2 to prove the Lemma. For the conjugated root of \( b \) we choose the conjugated path \( \beta_t(t) = \bar{\beta}_t^{-1}(t) \). Along the deformation it could happen that the mean curvature goes to zero i.e. the length of the short arc comes to equal to \( \pi \). To avoid this case we move the simple root in the short arc. For the unique root of \( b \) in the short arc we choose a path \( \beta_k : [-1,1] \to S^1 \) along the short arc connecting both Sym points \( \beta_k(-1) = \lambda_1 \) and \( \beta_k(1) = \lambda_2 \) and obeying (5.9). Besides these three paths \( \beta_i \), \( \beta_j \) and \( \beta_k \) we choose constant paths for all other roots of \( b \). For any \( s \in [0,1) \) we evaluate \( \beta_k \) at \( st \) and restrict the path \( \beta_k \) to \([-s,s) \). These paths obey conditions (i)-(iv). The Proposition 5.2 yields a two-dimensional family of deformed spectral data \( (a_{s,t}, b_{s,t}, \lambda_1, \lambda_2, t) \mid (t,s) \in [0,1] \times [0,1] \in M^g \), such that \( \Delta (5.8) \) takes at the deformed roots \( \beta_i \), \( \beta_j \) and \( \beta_k \) of \( b_{s,t} \) the values \( \Delta_0(\beta_i(t)) \), \( \Delta_0(\beta_j(t)) \) and \( \Delta_0(\beta_k(s)) \), respectively. This family is again unique and for \( t \in [0,1) \) smooth, if we normalise \( \lambda_1 \lambda_2^{-1} \) independent of \( s \) and \( t \). Let \( L(s,t) \) denote the corresponding length of the short arc. Due to Lemma 6.2 \( s \mapsto L(s,t) \) is for all \( t \in [0,1] \) monotonically decreasing. With \( L_{\min} = \min \{L(0,t) \mid t \in [0,1]\} \) there exists for all \( t \in [0,1] \) a unique \( s(t) \in [0,1] \) such that \( L(s,t) = L_{\min} \). First we take the path of spectral data parameterised by \( (s,0) \) with \( s \in [0,s(0)] \) and then the path of spectral data parameterised by \( (t,s(t)) \) with \( t \in [0,1] \). We obtain a continuous path of spectral data in \( M^g_+ \), which deforms the conjugated pair \( (\beta_i, \beta_j) \) of roots of \( b \) into a pair of double roots on \( S^1 \).

7. In step 7 we finally decrease the genus. First we show

**Lemma 6.5.** Suppose \( a \in \mathbb{C}^{2g}[\lambda] \) has \( 2g \) simple roots, and \( b \in \mathbb{C}^{g+1}[\lambda] \) has \( g + 1 \) simple roots in \( S^1 \), and they obey conditions (i)-(iii) in Definition 1.3. Then the set of \( \lambda \in \mathbb{P}^1 \) with \( h \in i\mathbb{R} \) is the union of \( S^1 \) with finitely mutually disjoint smooth curves intersecting \( S^1 \) exactly once in a root of \( b \). Each curve connects a pair of roots of \( a \) which are exchanged by \( \lambda \mapsto \lambda^{-1} \) or \( (\lambda = 0, \lambda = \infty) \) with each other.

**Proof.** The set of \( \lambda \in \mathbb{C}^* \) where \( h \) takes values in \( i\mathbb{R} \) is away from the roots of \( a \) and \( b \) a submanifold of \( \mathbb{C}^* \). At each root of \( a \) the function \( (h - ni\pi)^2 \) vanishes for some \( n \in \mathbb{Z} \) and is a local coordinate. Hence all roots of \( a \) are endpoints of this submanifold. At the roots of \( b \) two such submanifolds intersect transversely. Furthermore \( h^{-2} \) is a local coordinate at the marked points \( \lambda = 0 \) and \( \lambda = \infty \). They are also endpoints of this submanifold in \( \mathbb{P}^1 \). Therefore the intersection with \( B(0,1) \subset \mathbb{C}^* \) is a disjoint union of smooth curves, which end either at \( \lambda = 0 \), or at the roots of \( a \) inside of \( B(0,1) \), or at the roots of \( b \) in \( \partial B(0,1) \). In every connected component of the complement of these paths in \( B(0,1) \) which does not contain \( \lambda = 0 \) the real part of \( h \) vanishes by the maximum principle.

Therefore the complement of these paths in \( B(0,1) \) is connected. Hence there is no path, which connects a root of \( b \) in \( \partial B(0,1) \) with another such root. Then all \( g + 1 \) roots of \( b \) in \( \partial B(0,1) \) are connected either with \( \lambda = 0 \) or with one of the \( g \) roots of \( a \) inside of \( B(0,1) \). The involution \( \lambda \mapsto \lambda^{-1} \) maps these paths in \( B(0,1) \) onto the corresponding paths in \( \mathbb{C}^* \setminus B(0,1) \). This proves the claim.

**Conclusion of the proof of the Theorem 3:** We thus arrive at spectral data \( (a,b,\lambda_1, \lambda_2) \) such that the short arc contains exactly one root of \( b \) and the integral of \( dh \) along the short arc
vanishes. With Lemma 6.4 we move successively the pairs of roots of \( b \) away from \( S^1 \) to double roots in the long arc and separate with Lemma 6.1 the double root into two different roots on \( S^1 \). Afterwards we continue with the deformation of another pair of roots of \( b \) in \( \mathbb{C}^* \setminus S^1 \).

We are now in the situation where all roots of \( b \) are simple roots inside the interior of the long arc, and one simple root lies in the interior of the short arc. Due to Lemma 6.5 all roots of \( b \) correspond to a pair of branch points. For any root \( \beta_i \) of \( b \) in the long arc, which does not correspond to the pair \((\lambda = 0, \lambda = \infty)\), there exists a smooth curve \( \beta_t : [-1, 1] \to \mathbb{C}^* \) from the root \( \beta_i(-1) \) in \( B(0, 1) \) of \( a \) to the root \( \beta_i(1) \) outside of \( B(0, 1) \) along the curve described in Lemma 6.5. Furthermore there exists a branch \( h \) along this path and a unique \( n_i \in \mathbb{Z} \) such that \( h - n_i\pi i \) vanishes at both end points by the reality condition, but not in between, since \( dh \) has only one simple root on this path at the zero of \( b \). Finally we may parameterise this path in such a way that (5.9) is fulfilled. Together with the constant paths for all other roots of \( b \) this path obeys conditions (ii)-(iv). The path meets two roots of \( a \) and does not obey (i). The function \((h - n_i\pi i)^2 \) depends on \( \lambda \) and does not have critical points at the roots of \( a \). The whole construction of the proof of Proposition 5.2 carries over. But in this case the tubular neighbourhood \( V_i \) of the path \( \beta_i([-1, 1]) \) contains two roots of \( a \) at the roots of \( A_i = (h - n_i\pi i)^2 \) like in Lemma 5.3. During the whole deformation these two roots of \( A_i,t \) are in \( W_i,t \). For \( t = 1 \) these two roots form a double root at \( \beta_i(1) \), which is a single root for \( t = 0 \). Due to Lemma 6.2 the mean curvature increases along this path, and this path stays in \( \mathcal{M}_{\mp}^0 \). At the end point we may reduce with Lemma 1.7 the genus by one. We apply this deformation to all roots of \( b \) in the long arc with the exception of possibly the root \( \beta_j \) which is connected by the curve of Lemma 6.5 with the pair \((\lambda = 0, \lambda = \infty)\). The spectral genus is successively reduced to at most one.

If finally the unique simple root \( \beta_k \) in the short arc is connected by the curve of Lemma 6.5 with a pair of finite branch points, we apply the analogous deformation to \( \beta_k \). Due to Lemma 6.2 along this deformation the length of the short arc increases, and the end point has spectral genus zero but might not belong to \( \mathcal{M}_{\mp}^0 \). In this case we increase afterwards \( \text{arg}(\mu) \) at the local maximum over the last root \( \beta_j \) of \( b \) in the long arc. Since the spectral genus is zero \( h \) is a meromorphic function and has two roots over \( -\beta_j \). If we increase the argument of \( \mu \) at the local maximum over \( \beta_j \) by a large real number \( s \) then \( b(0) \) becomes very large and both \( \text{Sym} \) points move near to the roots of \( h \) at \( -\beta_j \in S^1 \). Therefore the short arc becomes very small and the path enters again \( \mathcal{M}_{\mp}^0 \). Finally we interchange the order of these two deformations. We first increase the argument of \( \mu \) at the local maximum over \( \beta_j \) by \( s \) along the path constructed in Lemma 6.5 connecting \( \beta_j \) with \( \lambda = 0 \) and \( \lambda = \infty \). This deformation decreases the length of the short arc. Afterwards we increase the length of the short arc along the path \( \beta_k \) of the root in the short arc. As a result the whole path stays in \( \mathcal{M}_{\mp} \) and reduces the spectral genus to zero. This completes the proof of Theorem 3.

7. Proof of Theorem 4

Proof of Theorem 4 (i): First we show that \( \mathcal{M}_{\text{rot}}^0 (g = 0 \text{ or } g = 1) \) is open in \( \mathcal{M}_{\mp}^0 \). We use the implicit function theorem to prove that \( \mathcal{M}^g \) is at \((a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^g \) a real submanifold of \( \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \times S^1 \times S^1 \) of dimension \( g + 1 \), the same dimension as \( \mathcal{M}_{\text{rot}} \). Due to Lemma 5.3, \( \mathcal{M}^g \) is at \((a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^g \) a submanifold. Due to Proposition 4.1, \( a \) and \( b \) have a common root only in case \( g = 1 \) and \((a, b, \lambda_1, \lambda_2) \) belongs to the image of (1.6). On the image of (1.6), the polynomial \( a \) is equal to \( a = -(\lambda + 1)^2/16 \) and the coefficients \( \alpha \) and \( H \) in Proposition 4.1 are local coordinates of \( M \). The inequality in (1.1) does not allow simple roots of \( a \) on \( S^1 \), which is equivalent to \( \alpha \geq 2 \). Therefore \( \mathcal{M}^1 \) is at \((a, b, \lambda_1, \lambda_2) \in \mathcal{M}_{\text{rot}}^1 \) a two-dimensional manifold with image of (1.6) as boundary. This implies that \( \mathcal{M}_{\text{rot}}^0 \) and \( \mathcal{M}_{\text{rot}}^1 \) are open in \( \mathcal{M}_{\mp}^0 \) and \( \mathcal{M}_{\mp}^1 \).
For sequences in \( \mathcal{M}^0_{\text{rot}} \) \((g = 0 \text{ or } g = 1)\) which converge in \( \mathcal{M}^0_+ \) the corresponding sequences of parameters \( H \) or \( (H, \alpha) \) have to converge. Therefore \( \mathcal{M}^0_{\text{rot}} \) is closed in \( \mathcal{M}^0_+ \).

**Proof of Theorem 4 (ii):** For \((a, b, \lambda_1, \lambda_2) \in \mathcal{M}^1_{\text{rot}} \), the short arc and the long arc contain both one root of \( b \). Due to Proposition 4.1, \( \tilde{a} \) has at least one pair of double roots away from \( S^1 \). If \( \tilde{a} \) has several pairs of double roots we choose one and remove all others. Hence we may assume \((\tilde{a}, \tilde{b}, \lambda_1, \lambda_2) \in \mathcal{M}^3 \) with \( \tilde{a} \) having a pair of double roots \( \beta, \beta^{-1} \in \mathbb{C} \setminus S^1 \). Since \( b \) has only roots at \( \lambda = \pm 1 \) the corresponding \( \tilde{b} \) has simple roots at \( \beta \) and \( \beta^{-1} \). Now we follow the path described in Theorem 3. We start with step 2 and open with Lemma 5.3 two double roots of \( \tilde{a} \) at \( \beta \) and \( \beta^{-1} \) and eventually a third double root at \( \lambda = -1 \) by a small deformation with decreasing mean curvature. Due to Lemma 6.5, the single root in the short arc is connected along a path with \( \mu \in S^1 \) to \((\lambda = 0, \lambda = \infty)\). We move the roots \( \beta \) and \( \beta^{-1} \) of \( \tilde{b} \) along a path described in Lemma 6.4 to the long arc and then separate them by a small deformation described in step 6 of the proof of Theorem 3. We end up with three roots of \( \tilde{b} \) in the long arc and one root of \( \tilde{b} \) in the short arc. All roots of the long arc are connected with the paths described in Lemma 6.5 with pairs of finite branch points. With the deformations described in Lemma 6.2 we successively shorten these paths from \( S^1 \) to the finite branch points until all three of them are very small. Finally we end up with spectral data of genus zero with three double roots of \( \tilde{a} \) on \( S^1 \). Hence the end point of the deformation in Theorem 3 does not belong to \( \mathcal{M}^0_{\text{rot}} \) which are classified in Proposition 4.1. This completes the proof of Theorem 4.

8. **Mean convex Alexandrov embeddings in \( S^3 \)**

In this section we consider mean convex Alexandrov embeddings \( f : M \to S^3 \) of 2-manifolds \( M \). Some statements apply only to mean convex Alexandrov embeddings with constant mean curvature, but we do not use the special properties of \( \text{cmc} \) cylinders of finite type.

We consider mean convex Alexandrov embedded cylinders in \( S^3 \). In the literature we only found the notion of Alexandrov embeddings for compact domains on the one hand, and the concept of properly Alexandrov embedded immersions from open manifolds into open Riemannian manifolds on the other hand. Since we are interested in immersions of open manifolds into the compact Riemannian manifold \( S^3 \), we introduced in Definition 1.2 the notion of mean convex Alexandrov embeddings for our setting.

We endow the space of cylinders with the topology of uniform convergence on compact sets. Let \( \tilde{f} : M \to S^3 \) be an immersed cylinder close to an Alexandrov embedded cylinder \( f : M \to S^3 \) on a bounded set \( V \). First we will extend the restriction of \( \tilde{f} \) to \( V \) to an open bounded three manifold \( W \) with \( V \subset \partial W \). For this purpose we introduce in subsection 8.2 the notion of local mean convex Alexandrov embedding \( \tilde{f} : V \to S^3 \).

Secondly we give in subsection 8.3 sufficient conditions to glue local mean convex Alexandrov embeddings \((V_p)_{p \in M} \) which cover a cylinder \( \tilde{f} : M \to S^3 \). In this way \( \tilde{f} : M \to S^3 \) will extend to a global Alexandrov embedded cylinder.

For this purpose we need two bounds. First we need a collar with depth uniformly bounded from below and secondly a chord-arc bound. The first bound uses a maximum principle at infinity. If both principal curvatures of a mean convex Alexandrov embedded \( \text{cmc} \) surface are uniformly bounded, then we prove in Proposition 8.1 that the cut locus function is bounded from below by a positive number. We postpone the proof of the chord-arc bound for mean convex Alexandrov embedded \( \text{cmc} \)-cylinders of finite type to the Appendix A.

8.1. **Geometry of Alexandrov embeddings.** In the setting of Definition 1.2, a fixed orientation of \( S^3 \) induces on \( N \) and \( M = \partial N \) an orientation. Conversely, if \( M \) is endowed with
an orientation, then there exists a unique normal, which points inward to the side of $M$ in $S^3$, which induces on the boundary $M$ the given orientation of $M$. In this sense the orientation of $M$ determines the inner normal of $N$.

For each point $p \in M$ of a hypersurface of a Riemannian manifold $N$ there exists a unique arc-length parameterised geodesic $\gamma(p, \cdot)$ emanating from $p = \gamma(p, 0)$ and going in the direction of the inward normal at $p$. Such geodesics are called \textbf{inward $M$-geodesics} [16].

Let $\gamma(p, \cdot)$ be an inward $M$-geodesic. Points $q \in N$ in the ambient manifold that are ‘close to one side’ of $M$ can thus be uniquely parameterised by $(p, t)$ where $p \in M$ and $q = \gamma(p, t)$ for some inward $M$-geodesic $\gamma(p, \cdot)$ and some $t \in \mathbb{R}^+$. The value of $t$ is the geodesic distance of $q$ to $M$. Extending the geodesic further into $N$ it might eventually encounter a point past which $\gamma(p, t)$ has distance smaller than $t$ to $M$. Such a point is called a cut point. The cut locus of $M$ in $N$ consists of the set of cut points along all inward $M$-geodesics. We define the \textbf{cut locus function} as the geodesic distance of the cut point to $M$:

$$c : M \rightarrow \mathbb{R}^+, \quad p \mapsto c(p), \quad \text{such that } \gamma(p, c(p)) \text{ is the cut point.} \quad (8.1)$$

If we want to stress the dependence on $f$ we decorate $\gamma$ and $c$ with index $f$. A known fact from Riemannian Geometry ([16, Lemma 2.1]) asserts that a cut point is either the first focal point on an inward $M$-geodesic, or is the intersection point of two shortest inward $M$-geodesics of equal length. For mean convex surfaces in $S^3$ the first focal point has distance not greater than $\pi/2$ to $M$. Therefore the cut locus function is bounded by $\pi/2$.

For a mean convex Alexandrov embedding $f : M \rightarrow S^3$, the inward $M$-geodesics give us a parametrisation of the 3-manifold $N$ with boundary $M = \partial N$, which we call \textbf{generalised cylinder coordinates}:

$$\gamma_f : \{(p, t) \in M \times \mathbb{R} \mid 0 \leq t < c_f(p)\} \rightarrow N. \quad (8.2)$$

These coordinates define a diffeomorphism onto the complement of the cut locus. The cut locus is homeomorphic to the quotient space $M/\sim_f$ with the following equivalence relation on $M$:

$$p \sim_f q \iff c_f(p) = c_f(q) \quad \text{and} \quad \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q)) \iff \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q)).$$

For each $p \in M$ we denote the corresponding equivalence class by

$$[p]_f = \{q \in M \mid \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q))\}. \quad (8.3)$$

The proofs of the next two results are deferred to the two appendices A and B.

\textbf{Proposition 8.1. (Lower bound on the cut locus function)} Let $f : N \rightarrow S^3$ be a mean convex Alexandrov embedding with constant mean curvature, and principal curvatures bounded by $\kappa_{\text{max}} > 0$. Then the cut locus function is bounded from below by $\arctan(\kappa_{\text{max}}^{-1})$.

\textbf{Proposition 8.2. (Chord-arc bound)} Let $f : N \rightarrow S^3$ be a mean convex Alexandrov embedding with second fundamental form $h$ with respect to the inner normal $N$. If $c > 0$ is a lower bound on the cut locus function $c_f$ (8.1) and $C'$ a bound on the covariant derivative of $h$:

$$||\nabla_X h(Y, Z)|| \leq C' \cdot |X| \cdot |Y| \cdot |Z| \quad \text{for all } p \in M \text{ and } X, Y, Z \in T_pM, \quad (8.4)$$

then there exists a constant $C > 0$ depending only on $c$ and $C'$ such that

$$\text{dist}_N(p, q) \leq \text{dist}_M(p, q) \leq C \text{dist}_N(p, q) \quad \text{for all } p, q \in M. \quad (8.5)$$
8.2. Local mean convex Alexandrov embeddings. In this section we localize the concept of a mean convex Alexandrov embedding. We consider in the following open subsets $V$ of $M$ and open bounded 3-dimensional manifolds $W$ with boundary $V$. More precisely, we denote by $\partial W = V$ the boundary as defined within the concept of manifolds with boundary. It is in general a subset of the topological boundary.

**Definition 8.3.** We call the restriction $f|_V$ of $f : M \to \mathbb{S}^3$ to an open subset $V \subset M$ a local mean convex Alexandrov embedding if $f|_V$ extends as an immersion to an open 3-manifold $W$ with $V = \partial W$ such that the following hold:

(i) The mean curvature of $V$ in $\mathbb{S}^3$ with respect to the inward normal is non-negative everywhere.

(ii) All inward $V$-geodesics exist in $W$ until they reach the cut locus (8.1) (for $t \leq \frac{\pi}{2}$).

(iii) $W = \{\gamma_t(p, t) \mid p \in V \text{ and } 0 < t \leq c_t(p)\}$. 

If $f : M \to \mathbb{S}^3$ is a mean convex Alexandrov embedding which satisfies the chord-arc bound (8.5), then all open subsets $V \subset M$ such that for all $p \in V$ the classes $[p]_t \subset V$ are examples of local mean convex Alexandrov embeddings. To see that we only have to show that

$$W = \{\gamma_t(p, t) \in N \mid p \in V \text{ and } 0 < t \leq c_t(p)\}$$

is open in $N$. Since $V$ is open, $W$ is open at all points away from the cut locus. Consider a cut point $c_t(p)$ and a sequence of $(q_n) \in N$ converging to $c_t(p)$. We prove that $(q_n) \in W$ for $n$ large enough. Let $q_n = \gamma_t(p_n, t) \in N$ with $p_n \in M$. By the chord-arc bound the set $[p]_t$ is bounded in $M$, and there is a subsequence of $p_n$ converging to an element of $[p]_t \subset V$. This proves that $p_n \in V$ for $n$ large enough and $q_n \in W$. Hence $W$ is an open neighborhood around cut points. Since $W$ is an open subset of $N$ with $V = \partial W$, $f|_V$ extends naturally as an immersion to $W$ by restriction of the extension of $f$ to $W$.

We shall prove that ‘mean convex Alexandrov embeddedness’ is an open condition, which will allow us to study deformation families of mean convex Alexandrov embeddings. The main tool is a general perturbation technique of Alexandrov embeddings, which we call **collar perturbation.** We consider local perturbations $\tilde{f}$ of a given smooth immersion $f : M \to \mathbb{S}^3$, which are ‘small’ with respect to the $C^1$-topology on the space of immersions from $M$ into $\mathbb{S}^3$.

**Lemma 8.4. (Collar perturbation).** For given $c > 0$ and $C' > 0$ there exist $\epsilon > 0$ and $R > 0$ with the following property: Let $f : M \to \mathbb{S}^3$ be a mean convex Alexandrov embedding with cut locus function $c_t$ (8.1) bounded from below by $c$ and second fundamental form obeying (8.4), and let $\tilde{f} : M \to \mathbb{S}^3$ be an immersion with non negative mean curvature and principal curvatures bounded by $\kappa_{\text{max}} = \cot(c)$. Furthermore, let both immersions $f$ and $\tilde{f}$ obey

$$(8.6) \quad \text{dist}(f(q), \tilde{f}(q)) < \epsilon \quad \text{and} \quad \|f'(q) - \tilde{f}'(q)\| < \epsilon \quad \text{for all } q \in B(p, R).$$

Here $p \in M$ is some point and $B(p, R) \subset M$ a ball with respect to the metric induced by $\tilde{f}$. Then the restriction $\tilde{f}|_V$ of $\tilde{f}$ to an open neighbourhood $V \subset M$ of $p$ extends to a local mean convex Alexandrov embedding with $V \subset \partial W$.

**Proof.** The generalized cylinder coordinates (8.2) define a diffeomorphism $\gamma_t$ of $M \times [0, c]$ onto an open subset of $N$, which is a collar. Any lower bound on the cut locus function (8.1) is also a lower bound on the distance to the first focal point on the inward $M$-geodesics. Since $f$ is a mean convex Alexandrov embedding, the absolute values of the negative principal curvatures are smaller than the positive principal curvatures. Consequently, due to the formula (A.1), the distances to the first focal points on the outward $M$-geodesics are not smaller than the distances to the first focal points on the inward $M$-geodesics. Hence the normal variation defines an immersion of $M \times (-c, c)$ into $\mathbb{S}^3$. In particular, the induced metric makes $M \times (-c, c)$ into a
Riemannian manifold with constant sectional curvature equal to one. For all elements of this manifold the cylinder coordinates, that is the distances to \( M \approx M \times \{0\} \) and the nearest point in \( M \) are uniquely defined. Hence we can glue \( m \times (-c, c) \) along \( \gamma_\epsilon(M \times \{0, c\}) \) to \( N \), and obtain a larger 3-manifold \( \hat{N} \supset N \) without boundary, such that the generalised cylinder coordinates (8.2) extend to a diffeomorphism \( \hat{\gamma}_\epsilon : (-c, c) \times M \to \hat{N} \) and the immersion \( \hat{f} : N \to \mathbb{S}^3 \) extends to an immersion \( \hat{f} : \hat{N} \to \mathbb{S}^3 \).

An immersion \( \hat{f}|_{B(p, R)} \) which satisfies the inequality (8.6) is a normal exponential graph over \( f|_{B(p, R)} \). Hence \( \hat{f}|_{B(p, R)} \) is an embedded submanifold in \( \hat{N} \cap \hat{\gamma}_\epsilon((-\epsilon, \epsilon) \times M) \). We denote this graph by \( O \). We shall identify \( \hat{f}|_{B(p, R)} \) with the immersion \( \hat{f}|_O \). Due to (8.6) \( \hat{f} \) induces on \( O \) a Riemannian metric denoted by \( \text{dist}_O \) and \( \hat{f}|_{B(p, R)} \) extends as an immersion to \( \hat{N} \) by this identification with \( \hat{f} \).

For all \( q \in O \) let \( t \mapsto \gamma_\epsilon(q, t) \) denote the inward \( O \)-geodesics in \( \hat{N} \). These inward \( O \)-geodesics exist for \( 0 \leq t \leq c - \epsilon \). Let \( U \subset \hat{N} \) denote the union

\[
U = \{ \gamma_\epsilon(q, t) \mid q \in O \text{ and } 0 \leq t \leq c - \epsilon \} \cup \{ \gamma_\epsilon(q, t) \mid q \in M \text{ and } \epsilon < t \leq \epsilon(t)(q) \}.
\]

For sufficiently small \( \epsilon/c \) this subset \( U \) of \( \hat{N} \) is a connected manifold with boundary \( O \). Let \( \text{dist}_U \) denote the distance function of this Riemannian manifold. Due to Proposition 8.2 the immersion \( f \) obeys a chord-arc bound with constant \( C \) depending only on \( c \) and \( C' \). Due to the second inequality of (8.6) the Riemannian metrics induced by \( f \) and \( \hat{f} \) are bounded in terms of each other with a constant depending only on \( \epsilon \). For sufficiently small \( \epsilon \) we obtain the bound

\[
\text{dist}_O(q, q') \leq C''(\text{dist}_U(q, q') + 2\epsilon)
\]

with \( C'' \) depending only on \( c \) and \( C' \). When \( \text{dist}_U(q, q') \geq 2\epsilon \), this inequality gives a chord-arc bound. Now assume that \( \text{dist}_U(q, q') \leq 2\epsilon \). Since the curvature is uniformly bounded on the cylinder, the surface is locally a bounded graph on a totally geodesic disc of radius \( \delta > 0 \). Hence for \( \epsilon > 0 \), the gradient of this graph is uniformly bounded and

\[
\text{dist}_O(q, q') \leq 2\text{dist}_U(q, q') \quad \text{for all } q, q' \in O \text{ with } \text{dist}_U(q, q') \leq 2\epsilon.
\]

These two bounds imply the chord-arc bound

\[
\text{dist}_O(q, q') \leq \bar{C} \text{dist}_U(q, q') \quad \text{for all } q, q' \in O
\]

with \( \bar{C} = 2C'' \geq 2 \). Let \( R = 3\bar{C}(\pi + \epsilon) \) and \( O'' = B(p, \text{dist}_U(q, q') \leq \pi) \subset O' \subset B(p, 2\bar{R}) \subset O \).

All cut locus functions of mean convex Alexandrov embeddings are uniformly bounded from above by \( \pi \), since otherwise a sphere with negative principal curvatures would touch \( M \) inside of \( N \) contradicting Hopf's maximum principle. The distance of two points of \( O' \), whose inward \( O \)-geodesics intersect at distances not larger than \( \frac{\pi}{2} \), is not larger than \( \bar{C}\pi \). For all \( q \in O'' \), we have

\[
\{ q' \in O \mid \exists t \in [0, \frac{\pi}{2}] \text{ with } \text{dist}_U(\gamma_\epsilon(q, t), \epsilon(t)) \leq t \} \subset \{ q' \in O \mid \text{dist}_U(q, q') \leq \pi \} \subset O'.
\]

Therefore, for all \( q \in O'' \) the cut locus function \( c_\epsilon \) is well defined. For all such \( q \in O'' \) let \( [q]_\epsilon \) denote the set

\[
[q]_\epsilon = \{ q' \in O \mid \text{dist}_N(\gamma_\epsilon(q, c_\epsilon(q)), q') = c_\epsilon(q) \}.
\]

For any closed subset \( A \subset O \) the set \( \{ q \in O'' \mid [q]_\epsilon \cap A \neq \emptyset \} \) is a closed subset of \( O'' \). Hence \( V = \{ q \in O'' \mid [q]_\epsilon \subset O'' \} \) is an open subset of \( O \). Furthermore \( W = \{ \gamma_\epsilon(q, t) \mid q \in V \text{ and } 0 \leq t \leq c_\epsilon(q) \} \) is a subset of \( \hat{N} \) with boundary \( V \). By construction \( \hat{f}|_V \) is a local mean convex Alexandrov embedding \( [q]_\epsilon \subset V \). By choice of \( R \), \( V \) is an open neighbourhood of \( p \) in \( O \).
8.3. From local to global mean convex Alexandrov embeddings. In this section we consider an immersion \( \tilde{f} : M \to S^3 \) which is covered by a family of local mean convex Alexandrov embedded neighborhoods \( \tilde{f}_p : V_p \to S^3 \) of \( p \in M \). We explain how to extend \( \tilde{f} : M \to S^3 \) to the 3-manifold \( \tilde{N} = \bigcup_{p \in M} W_p \), with \( \partial \tilde{N} = M \). The set of local immersions \( \tilde{f}_p \) are constructed by local collar perturbations. We will apply a theorem of Myers-Steenrod [29] in our construction.

**Proposition 8.5.** For given \( c > 0 \) and \( C' > 0 \) there exist \( \epsilon > 0 \) and \( R > 0 \) with the following property: Let \( f : M \to S^3 \) be an immersion with non negative mean curvature and principal curvatures bounded by \( \kappa_{\text{max}} = \cot(c) \) such that for all \( p \in M \) there exists a mean convex Alexandrov embedding \( \tilde{f}_p : M \to S^3 \) with cut locus function (8.1) bounded from below by \( c \) and second fundamental form obeying (8.4). If \( \tilde{f} \) and \( \tilde{f}_p \) obey (8.6) on \( B(p, R) \subset M = \partial N_p \) for all \( p \in M \), then \( \tilde{f} \) extends to a mean convex Alexandrov embedding \( \tilde{f} : \tilde{N} \to S^3 \) with \( \partial \tilde{N} = M \).

**Proof.** Let \( C \) and \( \tilde{C} \) denote the constants as in Lemma 8.4. We apply Lemma 8.4 to all \( p \in M \) with the same \( R = 3\tilde{C}(\pi + \epsilon) \) and decorate the corresponding objects with an index \( p \). The balls \( B(p, R) \) are embedded as Riemannian submanifolds \( O_p \subset N_p \). We obtain a covering of \( M \) by open subsets \( V_p = \{ q \in O''_p \mid [q]_{\tilde{f}} \subset O''_p \} \) of the balls \( O''_p = B(p, \frac{R}{3}) \) with respect to the metric induced by \( \tilde{f} \). Furthermore, the restrictions \( \tilde{f}_{|V_p} \) of \( \tilde{f} \) to the members \( V_p \) of this covering extend to local mean convex Alexandrov embeddings \( \tilde{f}_p : W_p \to S^3 \) with open subsets \( W_p \subset U_p \).

We shall glue the manifolds \( (W_p)_{p \in M} \) to obtain a 3-manifold \( \tilde{N} \) with boundary \( M \). It suffices to consider \( p, q \in M \) with \( \text{dist}_M(p, q) < \epsilon \). The intersection \( O''_p \cap O''_q \) is a connected subset of the Riemannian manifold \( M \) with the metric induced by \( \tilde{f} \). The immersions \( \tilde{f}_p \) and \( \tilde{f}_q \) define on \( U_p \subset \tilde{N}_p \) and \( U_q \subset \tilde{N}_q \) chord-arc distance \( \text{dist}_{U_p}(p', q') \) and \( \text{dist}_{U_q}(p', q') \). Now define \( R(t) = \tilde{C}(3(\pi + \epsilon) - t) \), and set the related shrunken open sets \( O_p(t), O_q(t) \) and \( O_{pq}(t) = O_p(t) \cap O_q(t) \). Note that \( O_p \cup O_q \subset O_{pq}(t) \). Denote

\[
A_p(t) = \{(p', q') \in \tilde{O}_p(t) \times \tilde{O}_q(t) \mid \text{dist}_{U_p}(p', q') = \text{dist}_{S^3}(\tilde{f}(p'), \tilde{f}(q')) = t\},
\]

\[
A_q(t) = \{(p', q') \in \tilde{O}_q(t) \times \tilde{O}_q(t) \mid \text{dist}_{U_q}(p', q') = \text{dist}_{S^3}(\tilde{f}(p'), \tilde{f}(q')) = t\}.
\]

If \( (p', q') \in A_p(t) \) then \( p' \) and \( q' \) are connected by a geodesic segment lying in \( U_p \) which is mapped by \( \tilde{f}_p \) to the unique shortest geodesic in \( S^3 \) from \( \tilde{f}(p') \) to \( \tilde{f}(q') \). By Lemma 8.4 the points in \( O_p \) obey in \( U_p \) a chord-arc bound with constant \( \tilde{C} \). If the geodesic segment in \( U_p \) from \( p' \) to \( q' \) meets a point \( p'' \in M \), then \( p'' \in O_{pq}(\text{dist}_{U_p}(p', p'')) \). The chord-arc bound implies that

\[
\text{dist}_{O_p}(p', p'') \leq \tilde{C} \text{dist}_{U_p}(p', p'') \text{ and } \text{dist}_{O_p}(p'', q') \leq \tilde{C} \text{dist}_{U_p}(p'', q') \text{ and }
\]

\[
\text{dist}_{O_p}(p, p'') \leq \text{dist}_{O_p}(p, q') + \text{dist}_{O_p}(q', p'') \leq R(\text{dist}_{U_p}(p', p'')).
\]

By definition of the sets \( A_p(t) \) this implies that \( (p', p'') \in A_p(\text{dist}_{U_p}(p', p'')) \) and \( (p'', q') \in A_p(\text{dist}_{U_p}(p'', q')) \).

**Claim 1.** \( A_p(t) = A_q(t) \) for all \( t \in [0, \pi] \).

We shall prove this claim later, and first show that it implies that \( c_{\tilde{f}_p} \) and \( c_{\tilde{f}_q} \) coincide on \( V_p \cap V_q \).

Let us assume on the contrary \( c_{\tilde{f}_p}(p') < c_{\tilde{f}_q}(p') \) for \( p' \in V_p \cap V_q \). The domain \( U_p \) contains a ball of radius \( c_{\tilde{f}_p}(p') \) centered at \( \gamma_{\tilde{f}_p}(p', c_{\tilde{f}_p}(p')) \) (analogously with \( U_q \) and \( c_{\tilde{f}_q}(p') \)).

Then there exists \( p'' \neq q'' \in [p']_{\tilde{f}_p} \subset O''_p \cap O''_q \). Then \( p' \) and \( q' \) are connected by a segment of a geodesic in \( U_p \), which meets the boundary only at the end points \( p' \) and \( q' \) by strict convexity of a geodesic ball centered at the cut point. Therefore there does not exist a point \( p'' \in O_p \) with \( \text{dist}_{U_p}(p', q') = \text{dist}_{U_p}(p', p'') + \text{dist}_{U_p}(p'', q') \). Due to the claim 1 there does not exist \( p'' \in O_q \) with \( \text{dist}_{U_q}(p', q') = \text{dist}_{U_q}(p', p'') + \text{dist}_{U_q}(p'', q') \). Therefore the shortest path in \( U_q \) connecting \( p' \) and \( q' \) is also a segment of a geodesic, which meets the boundary only at the end point \( p' \).
and \( q' \). Furthermore, both immersions \( \hat{f}_p \) and \( \hat{f}_q \) map these segments onto the same geodesic in \( S^3 \) connecting \( \hat{f}(p') \) and \( \hat{f}(q') \). Since \( p' \) and \( q' \) both belong to \([p']_{\tilde{I}_p}\) there exists a unique geodesic 2-sphere in \( S^3 \), which intersects the image of \( \tilde{f} \) orthogonally at \( f(p') \) and at \( \tilde{f}(q') \). Hence there exists in \( U_p \) a geodesic 2-sphere, which intersects \( O_q \) orthogonally at \( p' \) and \( q' \). The pre-image of this 2-sphere in \( U_q \) intersects \( O_q \) orthogonally at \( p' \) and \( q' \), and contains a segment of a geodesic connecting \( p' \) and \( q' \). This segment of a geodesic is in \( U_q \), because it is contained in a geodesic ball of larger radius and centered at the cut point of \( p' \) in \( U_q \). This implies that the inward \( O_q \)-geodesics at \( p' \) and \( q' \) also is contained in the larger geodesic ball of \( U_q \) and they meet each other at distance \( c_{\tilde{f}_p}(p') = c_{\tilde{f}_q}(q') \) in contradiction to \( c_{\tilde{f}_p}(p') < c_{\tilde{f}_q}(q') \). Interchanging \( p \) and \( q \) we get the other inequality, and thus both cut locus functions \( c_{\tilde{f}_p} \) and \( c_{\tilde{f}_q} \) coincide on \( V_p \cap V_q \).

We claim that the open set \( V_{pq} = V_p \cap V_q \) extends in two different but isometric extensions \( Z_p \subset W_p \) and \( Z_q \subset W_q \) as local mean convex Alexandrov embeddings with

\[
\begin{align*}
Z_p &= \{ \gamma_{\tilde{f}_p}(p', t) \in W_p \mid p' \in V_{pq} \text{ and } 0 < t \leq c_{\tilde{f}_p}(p') \}, \\
Z_q &= \{ \gamma_{\tilde{f}_q}(q', t) \in W_q \mid q' \in V_{pq} \text{ and } 0 < t \leq c_{\tilde{f}_q}(q') \}.
\end{align*}
\]

We identify \( Z_p \) with \( Z_q \) by a diffeomorphism \( \psi : Z_p \to Z_q \) such that its restriction \( \psi|_{V_{pq}} \) to \( V_{pq} \) is the identity and \( \tilde{f}_p|_{Z_p} = \tilde{f}_q|_{Z_q} \circ \psi \). A distance preserving bijection \( \psi : Z_p \to Z_q \) between two Riemannian manifold is a diffeomorphism by a theorem of Myers-Steenrod [29] (see also [30], Theorem 18, p 147).

We define \( \psi \) by \( \psi \circ \gamma_{\tilde{f}_p}(p', t) = \gamma_{\tilde{f}_q}(p', t) \) for \( p' \in V_{pq} \). This map is well-defined if \( c_{\tilde{f}_p}(p') = c_{\tilde{f}_q}(p') \) for all \( p' \in V_{pq} \). This indeed implies that for \( p' \in V_{pq} \) the equivalence classes \([p']_{\tilde{I}_p}\) and \([p']_{\tilde{I}_q}\)

\[ (8.3) \]

of the local mean convex Alexandrov embeddings \( \tilde{f}_p : W_p \to S^3 \) and \( \tilde{f}_q : W_q \to S^3 \) coincide. Hence we identify the class \([p']_{\tilde{I}} \in V_{pq}\) and then \( \psi \) is a bijection. By construction the exponential coordinates obey \( \tilde{f}_p|_{Z_p} = \tilde{f}_q|_{Z_q} \circ \psi \). Since the immersions \( \tilde{f}_p \) and \( \tilde{f}_q \) induce the metrics of \( Z_p \) and \( Z_q \), the map \( \psi \) preserves distances.

The union of mean convex Alexandrov embeddings extend to a manifold \( N \). It remains to show that \( N \) is complete with respect to the Riemannian metric induced by \( \tilde{f} \). For all \( r < c \), the submanifolds

\[
\{ \gamma_{\tilde{f}_p}(q, t) \in N_p \mid q \in M \text{ and } -r \leq t \leq c_{\tilde{f}_p}(q) \} \subset N_p
\]

are complete with respect to the Riemannian metric induced by \( \tilde{f}_p \). By construction, every point of the Riemannian manifold \( N \) with the metric induced by \( \tilde{f} \) is the center of an \( \epsilon \)-ball contained in one of these complete submanifolds of \( \hat{N}_p \). Therefore \( N \) is complete.

**Proof of Claim 1.** We consider the set \( B \) of all \( t_0 \in [0, \pi] \) such that \( A_p(t) = A_q(t) \) holds for all \( t \in [0, t_0] \). We prove that \( B \) is both closed and open in \([0, \pi] \), and thus \( B = [0, \pi] \). Due to the lower bound \( c \) of the cut locus function, \( B \) contains the set \([0, c - \epsilon] \).

Set \( t_0 = \sup B \), and suppose \( (p', q') \in A_p(t_0) \). We need to consider two cases. In the first case we assume that the unique geodesic in \( U_p \) from \( p' \) to \( q' \) goes through a point \( p'' \in O_p \setminus \{p', q'\} \). This implies that

\[
(p', p'') \in A_p(\text{dist}_{U_p}(p', p'')) = A_q(\text{dist}_{U_p}(p', p'')) \text{ and } (p'', q') \in A_p(\text{dist}_{U_p}(p'', q')) = A_q(\text{dist}_{U_p}(p'', q')).
\]

Both geodesics in \( U_q \) which connect \( p' \) to \( p'' \), and \( p'' \) to \( q' \), are mapped by \( \tilde{f}_q \) onto the unique geodesic in \( S^3 \) connecting \( \tilde{f}(p') \) to \( \tilde{f}(q') \). This implies that \( p' \) and \( q' \) are connected in \( U_q \) by a smooth geodesic, and hence \( (p', q') \in A_q(t_0) \). This proves \( A_p(t_0) \subset A_q(t_0) \), and analogously also the other inclusion, and therefore \( A_p(t_0) = A_q(t_0) \).
For the second case we may assume that there is no point \( p'' \in O_p \setminus \{ p', q' \} \) on the geodesic in \( U_p \) from \( p' \) to \( q' \). The mean convexity of the surface implies that \( (p', q') \) is not a local minimum of the function \( \text{dist}_{U_p} \) on \( O_p \times O_p \) (see Lemma B.1). Then there exists a sequence \( (p_n, q_n) \) which converges to \( (p', q') \) such that \( \text{dist}_{U_p}(p_n, q_n) < t_0 \). The continuity of \( \text{dist}_{U_p} \) and \( \text{dist}_{U_q} \) imply that

\[
\text{dist}_{U_q}(p', q') = \lim_{n \to \infty} \text{dist}_{U_q}(p_n, q_n) = \lim_{n \to \infty} \text{dist}_{U_p}(p_n, q_n) = \text{dist}_{U_p}(p', q') .
\]

Hence \( (p', q') \in A_q(t_0) \), and thus \( A_p(t_0) \subset A_q(t_0) \). The other inclusion is obtained by switching the roles of \( p \) and \( q \). Therefore \( A_p(t_0) = A_q(t_0) \) in both cases, which shows \( t_0 \in B \), and proves that \( B \) is closed.

We now show that \( B \) is open. If the maximum \( t_0 \) is smaller than \( \pi \), then there exists a sequence \( t_n \in (t_0, \pi] \) which converges to \( t_0 \) such that \( A_p(t_n) \neq A_q(t_n) \). By passing to a subsequence we may assume without loss of generality that there exists \( (p_n, q_n) \in A_p(t_n) \) but \( (p_n, q_n) \notin A_q(t_n) \). A subsequence of \( (p_n, q_n) \) converges to \( (p', q') \in A_p(t_0) = A_q(t_0) \). Therefore \( p' \) and \( q' \) are connected by smooth geodesic segments in both \( U_p \) and \( U_q \). Both of these geodesic segments are mapped by \( \tilde{f}_p \) and \( \tilde{f}_q \) to the unique shortest geodesic in \( \mathbb{S}^3 \) from \( \tilde{f}(p') \) to \( \tilde{f}(q') \). Furthermore, both these geodesic segments meet the boundaries \( O_p \) and \( O_q \) in the same points. The balls of radii \( c - \epsilon \) around each of such boundary points in \( U_p \) and \( U_q \) are isometric. In the complement of these balls both geodesic segments have positive distances to the boundaries \( O_p \) and \( O_q \).

Hence two tubular neighbourhoods of these geodesic segments in \( U_p \) and \( U_q \) are isometric as well.

For large \( n \) the geodesic segments in \( U_p \) connecting \( p_n \) with \( q_n \) belong to the tubular neighbourhood in \( U_p \). They are isometric to geodesic segments in \( U_q \) connecting \( p_n \) with \( q_n \). This implies \( (p_n, q_n) \notin A_q(t_0) \) for large \( n \). This contradicts the assumption that \( t_0 < \pi \). Hence we have that \( B = [0, \pi] \), and the claim is proven.

9. Spectral data of mean convex Alexandrov embedded cylinders

Now we apply the analysis of section 8 to finite type cylinders \( f \in \mathcal{A} \). First we show that the isospectral action in Definition 3.5 preserves mean convex Alexandrov embedded cylinders:

**Proposition 9.1.** Let \( (\xi, \lambda_1, \lambda_2) \) correspond to a mean convex Alexandrov embedded cylinder \( f : \mathbb{C}/\tau \mathbb{Z} \to \mathbb{S}^3 \). If \( \xi \) has no roots on \( \lambda \in \mathbb{C}^* \), then \( (\pi(t)\xi, \lambda_1, \lambda_2) \) corresponds for all \( t \in \mathbb{C}^\theta \) to a mean convex Alexandrov embedded cylinder (compare [15, Proposition 8.1 and 8.2]).

(i) If \( a(\lambda) \) has only simple roots, then \( \{ \pi(t)\xi \mid t \in \mathbb{C}^\theta \} = \mathcal{I}(a) \) and \( (\tilde{\xi}, \lambda_1, \lambda_2) \) corresponds for any \( \tilde{\xi} \in \mathcal{I}(a) \) to a mean convex Alexandrov embedded cylinder.

(ii) If \( \xi \in \mathcal{I}(p^2a) = \mathcal{I}(a) \), then \( \mathcal{I}(a) \) is the closure of \( \{ \pi(t)\xi \mid t \in \mathbb{C}^\theta \} \) and \( (\tilde{\xi}, \lambda_1, \lambda_2) \) corresponds for any \( \tilde{\xi} \in \mathcal{I}(a) \) to a mean convex Alexandrov embedded cylinder.

**Proof.** Due to Proposition 3.10, all \( (\tilde{\xi}, \lambda_1, \lambda_2) \) with \( \tilde{\xi} \in \mathcal{I}(a) \) correspond to CMC cylinders \( \tilde{f} : \mathbb{C}/\tau \mathbb{Z} \to \mathbb{S}^3 \). All these CMC cylinders \( \tilde{f} \) have due to Proposition 3.9 uniform bounded principal curvatures and uniform bounded covariant derivatives of the second fundamental form (8.4). For the mean convex Alexandrov embedded cylinders amongst them, Proposition 8.1 implies a lower bound on the cut locus function 8.1 and Proposition 8.2 a chord-arc bound (8.5).

The continuity and the commutativity of the isospectral action in Definition 3.5 [14, Section 4]

\[
\pi(z + t)\xi = \pi(z)\pi(t)\xi = \pi(t)\pi(z)\xi
\]
and the compactness of $\mathcal{I}(a)$ implies that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that
\[
\|\pi(z)\pi(t)\tilde{\xi} - \pi(z)\tilde{\xi}\| = \|\pi(t)\pi(z)\tilde{\xi} - \pi(z)\tilde{\xi}\| \leq \sup_{\tilde{\xi} \in \mathcal{I}(a)} \|\pi(t)\xi - \xi\| \leq \varepsilon \quad \text{for all } \tilde{\xi} \in \mathcal{I}(a) \text{ and } |t| < \delta.
\]

Hence Proposition 8.5 implies that there exists a $\delta > 0$, such that for all $t \in B(0, \delta)$ the CMC cylinders corresponding to $(\pi(t)\tilde{\xi}, \lambda_1, \lambda_2)$ are mean convex Alexandrov embedded, if $(\tilde{\xi}, \lambda_1, \lambda_2)$ corresponds to a mean convex Alexandrov embedded CMC cylinder. Therefore the set of all $t \in \mathbb{C}^2$ such that $(\pi(t)\tilde{\xi}, \lambda_1, \lambda_2)$ corresponds to a mean convex Alexandrov embedding is $\mathbb{C}^2$, and the whole orbit of the isospectral action corresponds to mean Alexandrov embeddings (compare [15, Section 8]). This proves (i).

Since $\xi \in \mathcal{I}(p^2a)$ vanishes at all roots of $p$ on $S^1$, $p$ does not vanish on $S^1$ and $\deg p$ is even. In the case where $\hat{a} = p^2a$ and $\deg p = 2$ we parameterize in [14, Section 6] $\mathcal{I}(\hat{a})$ by pairs $(L, \xi)$ of lines $L \in \mathbb{CP}^1$ together with $\xi \in \mathcal{I}(a)$. The elements $\xi \in \mathcal{I}(\hat{a})$ without roots correspond to pairs such that $L^\perp$ is not an eigenline of the value of $\xi$ at a root of $p$ [14, Proposition 6.6]. Such $\xi$ form a dense orbit in $\mathcal{I}(\hat{a})$. By induction in $\deg p$ we conclude for general $p$ without roots on $S^1$ that $\{\pi(t)\xi \mid t \in \mathbb{C}^{2+\deg p}\}$ is dense in $\mathcal{I}(\hat{a})$, if $\xi \in \mathcal{I}(\hat{a})$ has no roots. The first assertion together with Proposition 8.5 implies (ii).

Due to Proposition 3.4 (iii), all $f \in \mathcal{A}$ correspond to a potential $\xi$ without roots. Proposition 9.1 shows that for $f \in \mathcal{A}_{\text{Ac}}$ all elements of the corresponding isospectral sets are mean convex Alexandrov embedded. Hence for all $f \in \mathcal{A}_{\text{Ac}}$ there exists $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g_+$ with $\mathcal{A}(a, b, \lambda_1, \lambda_2) \subset \mathcal{A}_{\text{Ac}}$. Now we show that continuous deformations preserve such spectral data (compare [15, Proposition 2.7 and 2.8]):

**Proposition 9.2.** For all $g \in \mathbb{N} \cup \{0\}$ the space $\mathcal{M}^g_{\text{Ac}}$ is an open and closed subset of $\mathcal{M}^g_+$.

**Proof.** We shall use Proposition 8.5, and the fact that the map $A$ (3.9) is open and proper, to conclude that $\mathcal{M}^g_{\text{Ac}}$ is open and closed in $\mathcal{M}^g_+$. Let $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g_+$ and let $K \subset \mathcal{M}^g$ be a compact neighbourhood of $(a, \lambda_1, \lambda_2) \in \mathcal{M}^g$. Due to Proposition 3.9 the CMC immersions of all $(\tilde{\xi}, \lambda_1, \lambda_2) \in A^{-1}[K]$ have curvatures bounded by some $\kappa_{\text{max}}$ and covariant derivatives of the second fundamental form bounded by some $C' > 0$ (8.4). For the mean convex Alexandrov embedded cylinders amongst them Proposition 8.1 implies a lower bound on the cut locus function, and Proposition 8.2 a chord-arc bound (8.5).

For $(a, b, \lambda_1, \lambda_2) \in \mathcal{M}^g_{\text{Ac}}$ all $(\xi, \lambda_1, \lambda_2) \in A^{-1}[\{(a, \lambda_1, \lambda_2)\}]$ have in $A^{-1}[K]$ an open neighbourhood, whose CMC immersions $\tilde{\xi}$ obey on $B(0, R) \subset \mathbb{C}$ the estimate (8.6) with the constants $\varepsilon > 0$ and $R > 0$ of Proposition 8.5. The union $U$ of these open neighbourhoods is an open neighbourhood of the compact subset $A^{-1}[\{(a, \lambda_1, \lambda_2)\}]$ in $A^{-1}[K]$. Proposition 8.5 implies that $V = \{(\alpha, \beta, \lambda_1, \lambda_2) \in \mathcal{M}^g_+ \mid \mathcal{I}(\alpha) \subset U\}$ is a subset of $\mathcal{M}^g_{\text{Ac}}$. We claim that $V$ is a neighbourhood of $(a, b, \lambda_1, \lambda_2)$. The complement of $U$ in $A^{-1}[K]$ is a closed subset. Due to Lemma 3.8 the map $A$ is proper. Therefore $A^{-1}[K]$ and all closed subsets are compact. The complement of $U$ is mapped by $A$ onto a compact subset of $K$, which does not contain $(a, \lambda_1, \lambda_2)$. The complement of this compact subset is an open neighbourhood $O \subset V$ of $(a, \lambda_1, \lambda_2)$. This shows the claim and implies that $\mathcal{M}^g_{\text{Ac}}$ is open in $\mathcal{M}^g_+$.

Now we show that $\mathcal{M}^g_{\text{Ac}}$ is closed in $\mathcal{M}^g_+$. Let $(a_n, b_n, \lambda_{1,n}, \lambda_{2,n})$ be a sequence in $\mathcal{M}^g_{\text{Ac}}$ converging in $\mathcal{M}^g_+$ to $(a, b, \lambda_1, \lambda_2)$. We have to show that any $(\xi, \lambda_1, \lambda_2) \in A^{-1}[\{(a, \lambda_1, \lambda_2)\}]$ corresponds to a mean convex Alexandrov cylinder. By Lemma 3.8 the map $A$ is open. Therefore every neighbourhood of $(\xi, \lambda_1, \lambda_2)$ contains elements of $A^{-1}[\{(a_n, \lambda_{1,n}, \lambda_{2,n})\}]$. Therefore the CMC cylinder corresponding to $(\xi, \lambda_1, \lambda_2)$ obeys the condition of Proposition 8.5 and is mean convex Alexandrov embedded. 

\[ \square \]
Appendix A. Proof of the lower bound of cut locus function

Proof of Proposition 8.1. For the hypersurface $M_t = \cup_{p \in M} \gamma(p, t)$, the mean curvature

\begin{equation}
H(t) = \frac{1}{2} \left( \cot (\arctan(\kappa_1^{-1}) - t) + \cot (\arctan(\kappa_2^{-1}) - t) \right)
\end{equation}

is positive for all $t \in (0, t_{\text{ foc}})$, and strictly increasing since

\[ H'(t) = \frac{1}{2} \left( \sin^{-2}(\arctan(\kappa_1^{-1}) - t) + \sin^{-2}(\arctan(\kappa_2^{-1}) - t) \right) > 0. \]

Let $c_t$ denote the cut locus function (8.1). If there exists a point $p \in M$ for which $c_t(p) < \arctan(\kappa_{\max}^{-1}) \leq t_{\text{ foc}}$, then two inward $M$-geodesics $\gamma(p, \cdot), \gamma(q, \cdot)$ through $p, q \in M$ respectively, have to intersect at a distance of $c_t(p)$ from $M$, and thus $c_t(p) = c_t(q)$. Hence, if there exists a point $p \in M$ with $c_t(p) < \arctan(\kappa_{\max}^{-1})$ then $M_t$ intersects itself for a value of $t < \arctan(\kappa_{\max}^{-1})$ over two points $p, q \in M$. Let

\[ c_0 = \inf\{t \mid M_t \text{ intersects over two points of } M\}. \]

Since over all points $p \in M$ the mean curvature of $M_t$ is positive for all $0 < t < \arctan(\kappa_{\max}^{-1})$ with respect to the inner normals, the surfaces $M_t$ cannot touch each other from different sides over two points for $0 < t < \arctan(\kappa_{\max}^{-1})$. Now let $(p_k)_{k \in \mathbb{N}}$ be a sequence in $M$ with

\[ \lim_{k \to \infty} c_t(p_k) = c_0 = \inf \{c_t(p) \mid p \in M\}. \]

Then there exists a sequence $\Theta_k$ of isometries of $S^3$ which transform each point $p_k$ into a fixed reference point $p_0 \in S^3$, and the tangent plane of $M$ at $p_k$ into the tangent plane of a fixed geodesic sphere $S^2_{p_0} \subset S^3$ which contains $p_0$. This sequence of isometries transforms neighbourhoods $U_k$ of $p_k \in M$ into normal CMC graphs $\Theta_k[U_k]$ over $B(p_0, r) \subset S^2_{p_0}$. Due to Arzelà-Ascoli, and the a-priori gradient bound from Proposition 4.1 in [13], this bounded sequence of normal CMC graphs over $B(p_0, r) \subset S^2_{p_0}$ has a convergent subsequence. By passing to a subsequence we may achieve that these graphs converge to a normal CMC graph $U$ over $B(p_0, r) \subset S^2_{p_0}$, which is tangent to $S^2_{p_0}$ at $p_0$. For $c_0 < \arctan(\kappa_{\max}^{-1})$, the sets $[p_k]$ contain besides $p_k$ another point $q_k$ for large $k$. Furthermore, the sequence of isometries $\Theta_k$ transforms the sequence of geodesic 2-spheres tangent to $M$ at $q_k$ into a converging sequence of spheres with limit $S^2_{q_0}$. This sphere contains the limit $q_0 = \lim \Theta_k(q_k)$ with distance $\text{dist}(p_0, q_0) \leq 2c_0$. For large $k$ the points $q_k$ have neighbourhoods $V_k$, whose transforms $\Theta_k[V_k]$ are normal CMC graphs over $B(q_0, r) \subset S^2_{q_0}$. By passing again to a subsequence the normal CMC graphs $\Theta_k[V_k]$ converge to a normal CMC graph $V$ tangent to $S^2_{q_0}$ at $q_0$. The transformed inward $M$-geodesics nearby $p_k$ and $q_k$ converge to normal geodesics of these two limiting CMC surfaces $U$ and $V$ in $S^3$. Let $U_c$ and $V_c$ denote the surfaces $U$ and $V$ shifted by $c_0$ along these normal geodesics. If we shift both sequences $\Theta_k[U_k]$ and $\Theta_k[V_k]$ by $c_0$ along the transformed $M$-geodesics, they converge to $U_c$ and $V_c$. Therefore these surfaces $U_c$ and $V_c$ touch each other from different sides at the limit of the transformed cut points $\lim \Theta_k(\gamma_t(p_k, c(p_k))) = \lim \Theta_k(\gamma_t(q_k, c(q_k)))$. Hence the shifted surfaces cannot have positive mean curvature with respect to the inner normal. This implies $c_0 = 0$ and $H = 0$.

This case contradicts the maximum principle at infinity [25, Theorem 7]. In fact, the generalised cylinder coordinates 8.2 define an immersion from the 3-manifold $(x, t) \in M \times [0, 1/2 \arctan(\kappa_{\max}^{-1})]$ with boundary components $M \times \{0\}$ and $M \times \{1/2 \arctan(\kappa_{\max}^{-1})\}$. Since the infimum of the cut locus function is smaller than $1/2 \arctan(\kappa_{\max}^{-1})$, some pieces of $M$ sit inside of this 3-manifold with possible boundaries in the second boundary component. All assumptions of [25, Theorem 7] are fulfilled and these pieces have constant distance to the first boundary. Since constant distance surfaces of minimal surfaces in $S^3$ are not minimal, this is impossible.
Appendix B. Proof of the chord arc bound

Proof of Proposition 8.2: For all \( p, q \in M \) we have \( \text{dist}_N(p, q) \leq \text{dist}_M(p, q) \). In general, these distances do not coincide. We shall construct a path from \( p \) to \( q \) of length at most \( C \text{dist}_N(p, q) \). Due to (Rinow [34], pages 172 and 141) the points \( p \) and \( q \) are joined by a shortest path in \( N \). In case this shortest path touches at some points the boundary [1, Theorem 1.], we decompose it into pieces. The boundary points of a shortest path might have accumulation points. But any point of a shortest path, which is not a boundary point, belongs to a unique geodesic piece in \( N \), which has only two boundary points at both ends. Hence it suffices to construct such a path for two points \( p \) and \( q \), which are connected by a geodesic in \( N \) with only two boundary points at both ends. Let \( \chi_p, \chi_q \in [0, \frac{\pi}{2}] \) denote the angles in \( T_pN \) and \( T_qN \) between the inward geodesic \( \gamma \) connecting \( p \) and \( q \) and the inward normal to \( M \), respectively.

Due to [16, Lemma 2.1] the cut locus function \( c_t(p) \) (8.1) is for all \( p \in M \) not larger than the first focal point \( \gamma(p, t_{foc}) \), where \( t_{foc} = \arctan((\max\{\kappa_1, \kappa_2\})^{-1}) \geq \arctan(\kappa_{\max}^{-1}) \). Hence both principal curvatures are uniformly bounded by \( \kappa_{\max} = \cot(c) \) with \( 0 < c \leq \frac{\pi}{2} \).

Claim: Suppose \( p, q \in M \) are connected in \( N \) by a geodesic of length \( \geq \pi \). Then there exists \( \hat{q} \in M \) with

\[
\text{dist}_N(p, \hat{q}) < \pi \quad \text{and} \quad \text{dist}_N(p, q) = \text{dist}_N(p, \hat{q}) + \text{dist}_N(\hat{q}, q).
\]

To prove this claim, note that all geodesics \( \gamma \) in \( N \) starting at \( p \in M \) which do not meet \( M \) in distances \( d \in (0, \pi) \), meet each other at the antipode of \( p \). The tangent space \( T_pN \) contains a unique half space of initial directions of geodesics starting at \( p \in M \). If the pre-image of \( B(p, \pi) \subset N \) with respect to \( \exp_p \) contains the intersection of \( B(0, \pi) \subset T_pN \) with this half space, then due to Hopf’s maximum principle (see e.g. [12]), \( B(p, \pi) \subset M \) is a geodesic sphere in \( S^3 \) and the claim is obvious. Otherwise there starts at \( p \) a geodesic which touches \( M \) for some \( t \in (0, \pi) \), and the claim follows in this case. This proves the claim.

In the sequel we consider points \( p, q \in M \) connected by a geodesic in \( N \) with \( \text{dist}_N(p, q) < \pi \). In the following discussion we use the construction in the proof of Lemma 8.4 of a larger 3-manifold \( \tilde{N} \supset N \) without boundary, such that the generalised cylinder coordinates (8.2) extend to a diffeomorphism \( \tilde{\gamma}_t : M \times (-c, c) \rightarrow \tilde{N} \) and the immersion \( f : N \rightarrow S^3 \) extends to an immersion \( \tilde{f} : \tilde{N} \rightarrow S^3 \). Along any geodesic in \( \tilde{N} \), which starts at some point \( p \in M \) the distance to \( M \) can only increase with the same speed as the distance to \( p \), if the geodesic is an \( M \)-geodesic. Therefore such geodesics exist in \( \tilde{N} \) at least up to distances smaller than \( c \) from the initial point \( p \in M \). This shows that in \( \tilde{N} \) the absolute value of the second coordinate of \( \tilde{\gamma}_t \) is the distance to \( M \). Let \( (p, q) \) be any pair of points in \( M \), which are connected in \( \tilde{N} \) by a geodesic. We distinguish between the following three cases:

\begin{enumerate}
  \item[(i)] The geodesic stays inside \( \{\tilde{\gamma}_t(p, t) \mid (p, t) \in M \times [-\frac{c}{2}, \frac{c}{2}]\} \).
  \item[(ii)] The geodesic stays inside \( N \) with distance \( > \frac{c}{2} \) to \( M \) and \( (\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} \leq \epsilon \).
  \item[(iii)] The geodesic stays inside \( N \) with distance \( > \frac{c}{2} \) to \( M \) and \( (\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} > \epsilon \).
\end{enumerate}

In case (i) the first entries of the cylinder coordinates of the geodesic yields a smooth path in \( M \) from \( p \) to \( q \). Since the curvature is bounded by \( \kappa_{\max} = \cot(c) \) the derivative of the generalised cylinder coordinates at \( (p, t) \in M \times [-\frac{c}{2}, \frac{c}{2}] \) in direction of \( (p', t') \in T_{(p,t)}M \times (-c, c) \) obeys

\[
|d\tilde{\gamma}_t(p, t)(p', t')| \geq (1 - \kappa_{\max} \sin(\frac{c}{2})) |p'| = (1 - \cot(c) \sin(\frac{c}{2})) |p'| \geq \left( 1 - \frac{1}{2\cos(\frac{c}{2})} \right) |p'| \geq \frac{1}{4} |p'|.
\]

The integral along the geodesic of this inequality yields for all such pairs \( (p, q) \)

\[
(B.1) \quad \text{dist}_M(p, q) \leq 4 \text{dist}_N(p, q).
\]

This includes all \( (p, q) \) which are connected in \( N \) by a geodesic with \( \text{dist}_N(p, q) < c \).
**In case (ii)** we shall consider smooth families of geodesics $\gamma$ connecting two smooth paths $s \mapsto p(s)$ and $s \mapsto q(s)$ in $M$ parameterised by a real parameter $s$. For fixed $s$ the geodesic is parameterised by the real parameter $t$. The derivatives with respect to $s$ are denoted by prime and the derivatives with respect to $t$ by dot. For example $p' \in T_{p(s)}M$ and $q' \in T_{q(s)}M$ denotes the tangent vectors along the paths $s \mapsto p(s)$ and $s \mapsto q(s)$. The geodesic $\gamma$ extends in $S^3$ to a closed geodesic. For any $(p', q') \in T_pM \times T_qM$ there exists a Killing field $\mathcal{V}$ on $S^3$, which moves the closed geodesic $\gamma$ in such a way, that the intersection points at $p$ and $q$ moves along $p'$ and $q'$, respectively. Conversely, all Killing fields $\mathcal{V}$ generate a one-dimensional group of isometries of $S^3$. Let $s \mapsto \gamma(s, \cdot)$ denote the corresponding family of geodesics and $s \mapsto (p(s), q(s))$ the corresponding intersection points with $M$.

**Lemma B.1.** There exist $\epsilon, \delta > 0$ and $0 < s_0 < \min\{\frac{\pi}{2}, \frac{3}{2}\}$ depending only on $c$ and $C'$ with the following property: Let $p, q \in M$ be connected by a geodesic in $N$ obeying (ii). Then there exists a non-trivial Killing field $\mathcal{V}$, such that $d : s \mapsto d(s) = \text{dist}_{S^3}(p(s), q(s))$ obeys

\[(B.2) \quad d'(s) \leq 0, \quad d''(s) \leq -\delta \cos\left(\frac{d(s)}{2}\right), \quad |p'(s)| + |q'(s)| \leq 3 \cos\left(\frac{d(s)}{2}\right) \quad \text{for all} \quad s \in [0, s_0].\]

**Proof.** We shall construct a Killing field $\mathcal{V}$ with the desired properties, which rotates $\gamma$ around two antipodes of $\gamma$. The corresponding rotated geodesics $\gamma_\theta(s, \cdot)$ belong to a unique geodesic 2-sphere $S^2 \subset S^3$. The corresponding paths $s \mapsto p(s)$ and $s \mapsto q(s)$ move along the intersection of this 2-sphere $S^2$ with $M$. Hence we can calculate all derivatives on this sphere.

We parameterise this 2-sphere by the real parameter $s$ of the family $s \mapsto \gamma_\theta(s, \cdot)$ of rotated geodesics, and the real arc length parameter $t$ of these geodesics. We choose the equator as the points corresponding to $t = 0$ with distance $\frac{\pi}{2}$ to the rotation axis. Let $t_p$ and $t_q$ denote the values of this parameter $t$ at the points $p(s)$ and $q(s)$. Hence the distance $\text{dist}_N(p, q)$ is equal to $\text{dist}_N(p, q) = |t_p - t_q|$. The vector fields $\mathcal{V}$ and the geodesic vector field $\mathcal{G}$ along the geodesics $\gamma_\theta(s, \cdot)$ form an orthogonal basis of the tangent spaces of this 2-sphere away from the zeroes of $\mathcal{V}$. The vector fields $\mathcal{V}$ and $\mathcal{G}$ have at $(s, t)$ the scalar products

\[g(\mathcal{V}, \mathcal{V}) = \cos^2(t), \quad g(\mathcal{V}, \mathcal{G}) = 0, \quad g(\mathcal{G}, \mathcal{G}) = 1.\]

Since $\mathcal{G}$ is a geodesic vector field the derivative $\nabla_G \mathcal{G}$ vanishes. Moreover, the normal curvature of the integral curve of $\mathcal{V}$ starting at $(s, t)$ is equal to $\tan(t)$. Therefore at $(s, t)$ we have

\[
\nabla_G \mathcal{V} = \cos^2(t) \tan(t) \mathcal{G} = \cos(t) \sin(t) \mathcal{G}, \quad \nabla_G \mathcal{G} = -\tan(t) \mathcal{V}, \quad \nabla_G \mathcal{G} = 0.
\]

We parameterise a neighbourhood of the geodesic from $p$ to $q$ in such a way that the corresponding vector field $\mathcal{G}$ points inward to $\bar{N}$ at $p$ and outward of $\bar{N}$ at $q$, respectively. The derivatives of $s \mapsto p(s)$ and $s \mapsto q(s)$ are equal to

\[
p' = \mathcal{V}(p) - \mathcal{G}(p) \frac{g(\mathcal{N}(p), \mathcal{V}(p))}{g(\mathcal{N}(p), \mathcal{G}(p))}, \quad q' = \mathcal{V}(q) - \mathcal{G}(q) \frac{g(\mathcal{N}(q), \mathcal{V}(q))}{g(\mathcal{N}(q), \mathcal{G}(q))}.
\]

The lengths $|p'|$ and $|q'|$ depend on the angles $\angle(\mathcal{N}(p), \mathcal{V}(p))$ and $\angle(\mathcal{N}(q), \mathcal{V}(q))$. Since $\mathcal{V}$ is orthogonal to $\mathcal{G}$ and $\angle(\mathcal{N}(p), \mathcal{G}(p)) = \chi_p$ and $\angle(\mathcal{N}(q), \mathcal{G}(q)) = \chi_q$ these angles obey $\angle(\mathcal{N}(p), \mathcal{V}(p)) \in \left[\frac{\pi}{2} - \chi_p, \frac{\pi}{2}\right]$ and $\angle(\mathcal{N}(q), \mathcal{V}(q)) \in \left[\frac{\pi}{2} - \chi_q, \frac{\pi}{2}\right]$. Then we have

\[(B.3) \quad |p'| \leq \frac{|\cos(t_p)|}{\cos(\chi_p)}, \quad |q'| \leq \frac{|\cos(t_q)|}{\cos(\chi_q)}.
\]

\[
d' = \frac{g(\mathcal{N}(p), \mathcal{V}(p))}{g(\mathcal{N}(p), \mathcal{G}(p))} - \frac{g(\mathcal{N}(q), \mathcal{V}(q))}{g(\mathcal{N}(q), \mathcal{G}(q))} = \frac{g(\mathcal{N}(p), \mathcal{V}(p))}{\cos(\chi_p)} - \frac{g(\mathcal{N}(q), \mathcal{V}(q))}{\cos(\chi_q)}.
\]
Along the paths $p$ and $q$ with $X = p'$ and $X = q'$, respectively, we have at $(s, t)$

\[
\nabla_X \frac{g(\mathfrak{M}, \vartheta)}{g(\mathfrak{M}, \hat{\vartheta})} = \frac{g(\nabla_X \mathfrak{M}, \vartheta) + g(\mathfrak{M}, \nabla_X \vartheta)}{g(\mathfrak{M}, \hat{\vartheta})} - \frac{g(\mathfrak{M}, \vartheta)(g(\nabla_X \mathfrak{M}, \hat{\vartheta}) + g(\mathfrak{M}, \nabla_X \hat{\vartheta}))}{(g(\mathfrak{M}, \hat{\vartheta}))^2} = \frac{g(\nabla_X \mathfrak{M}, X) + g(\mathfrak{M}, \nabla_X \vartheta)}{g(\mathfrak{M}, \hat{\vartheta})} - \frac{g(\mathfrak{M}, \vartheta)g(\mathfrak{M}, \nabla_X \hat{\vartheta})}{g(\mathfrak{M}, \hat{\vartheta})^2} = -\frac{h(X, X)}{g(\mathfrak{M}, \hat{\vartheta})} + \cos(t) \sin(t) + 2 \tan(t) \left( \frac{g(\mathfrak{M}, \vartheta)}{g(\mathfrak{M}, \hat{\vartheta})} \right)^2.
\]

Hence the second derivative is equal to

\[
d'' = -\frac{h(p', p')}{\cos(\chi_p)} - \frac{h(q', q')}{\cos(\chi_q)} + \frac{\sin(2t_2) - \sin(2t_q)}{2} + 2 \tan(t_p) \left( \frac{g(\mathfrak{M}(p), \varrho(p))}{g(\mathfrak{M}(p), \hat{\varrho}(p))} \right)^2 - 2 \tan(t_q) \left( \frac{g(\mathfrak{M}(q), \varrho(q))}{g(\mathfrak{M}(q), \hat{\varrho}(q))} \right)^2.
\]

If along the rotation of the geodesic for $s \in [0, s_0]$ the following inequalities are satisfied

(B.4) $-\frac{\pi}{2} \leq t_p \leq 0, \quad 0 \leq t_q \leq \frac{\pi}{2}, \quad \frac{\pi}{2} \leq d = t_q - t_p, \quad \text{and} \quad (\sin^2(\chi_p) + \sin^2(\chi_q))^\frac{1}{2} \leq \frac{1}{2},$

then $\min\{\cos(\chi_p), \cos(\chi_q)\} \geq \frac{\sqrt{2}}{2}$ implies the third inequality of (B.2):

\[
|p'| + |q'| \leq \frac{\cos(t_p)}{\cos(\chi_p)} + \frac{\cos(t_q)}{\cos(\chi_q)} \leq \frac{2}{\sqrt{3}} (\cos(t_p) + \cos(t_q)) = \frac{4}{\sqrt{3}} \cos \left( \frac{d}{2} \right) \cos \left( \frac{t_p + t_q}{2} \right) \leq 3 \cos \left( \frac{d}{2} \right).
\]

Furthermore, the last two terms of $d''$ are bounded by

\[
\left| \tan(t_p) \left( \frac{g(\mathfrak{M}(p), \varrho(p))}{g(\mathfrak{M}(p), \hat{\varrho}(p))} \right)^2 \right| \leq \sin(|t_p|) \cos(t_p) \tan^2(\chi_p) \leq \frac{\sin(2|t_p|)}{2 \cdot 3},
\]

\[
\left| \tan(t_q) \left( \frac{g(\mathfrak{M}(q), \varrho(q))}{g(\mathfrak{M}(q), \hat{\varrho}(q))} \right)^2 \right| \leq \sin(|t_q|) \cos(t_q) \tan^2(\chi_q) \leq \frac{\sin(2|t_q|)}{2 \cdot 3}.
\]

Due to $\sin(2t_q) - \sin(2t_p) = 2 \sin(t_q - t_p) \cos(t_p + t_q)$ and we arrive at

(B.5) \[
d''(s) \leq -\frac{h(p', p')}{\cos(\chi_p)} - \frac{h(q', q')}{\cos(\chi_q)} - \sin(d) \cos(t_p + t_q) \left( 1 - \frac{3}{2} \right).
\]

Now we claim that the second inequality of (B.2) is implied by (B.4) and

(B.6) \[
\delta \leq \frac{1}{9} \sin \left( \frac{\pi}{2} \right) \cos(t_p + t_q), \quad -\frac{h(p', p')}{|p'|^2} \leq \frac{\delta}{2} \quad \text{and} \quad -\frac{h(q', q')}{|q'|^2} \leq \frac{\delta}{2}.
\]

The assumption (B.4) implies $t_p \leq -d + \frac{\pi}{2}$ and $d - \frac{\pi}{2} \leq t_q$. For $d \in \left[ \frac{\pi}{2}, \pi \right)$ we get $\cos(t_p) \leq \sin(d)$ and $\cos(t_q) \leq 1$. For $d \in \left[ \frac{\pi}{2}, \frac{\pi}{2} \right)$ we use $\cos(t_p) \leq 1$ and $\cos(t_q) \leq 1$ and obtain

\[
\frac{1}{9} \sin \left( \frac{\pi}{2} \right) \max \left\{ \frac{|p'|^2}{\cos(\chi_p)}, \frac{|q'|^2}{\cos(\chi_q)} \right\} \leq \frac{3}{8} \sin \left( \frac{\pi}{2} \right) \max \left\{ \frac{|p'|^2}{\cos(\chi_p)}, \frac{|q'|^2}{\cos(\chi_q)} \right\} \leq \frac{\delta}{9} \sin(d).
\]

Together with (B.6) we can estimate the first two terms in (B.5):

\[
-\frac{h(p', p')}{\cos(\chi_p)} \leq \frac{\delta}{2} \frac{|p'|^2}{\cos(\chi_p)} \leq \frac{1}{9} \sin(d) \cos(t_p + t_q) \quad \text{and} \quad -\frac{h(q', q')}{\cos(\chi_q)} \leq \frac{\delta}{2} \frac{|q'|^2}{\cos(\chi_q)} \leq \frac{1}{9} \sin(d) \cos(t_p + t_q).
\]

The third inequality of (B.4) implies $\sin \left( \frac{\pi}{2} \right) \cos \left( \frac{d}{2} \right) \leq 2 \sin \left( \frac{\pi}{2} \right) \cos \left( \frac{d}{2} \right) \leq 2 \sin \left( \frac{d}{2} \right) \cos \left( \frac{d}{2} \right) = \sin(d)$. Thus the second inequality of (B.2) indeed follows with the help of (B.5) from (B.4) and (B.6).

We shall show first that there exists a vector field $\vartheta$ obeying at $s = 0$

\[
\delta \leq \frac{1}{18} \sin \left( \frac{\pi}{2} \right) \cos(t_p + t_q), \quad -\frac{h(p', p')}{|p'|^2} \leq \frac{\delta}{4} \quad \text{and} \quad -\frac{h(q', q')}{|q'|^2} \leq \frac{\delta}{4}.
\]
The Killing field \( \vartheta \) is uniquely determined by two choices: firstly, the choice of a geodesic sphere \( S^2 \subset S^3 \), which contains the closed geodesic from \( p \) to \( q \), and secondly, a choice of the zeroes of \( \vartheta \), or equivalently a choice of the coordinates \( t_p \) and \( t_q \) with \( t_q - t_p = d \mod \pi \). We start with \( t_q = 0 \) and set \( t_p = \frac{d}{2} \).

Now we choose the 2-sphere \( S^2 \subset S^3 \). It is uniquely determined either by the unique line in \( T_p M \) or by the unique line in \( T_q M \), which is tangent to \( S^2 \). Since \( f \) is a mean convex Alexandrov embedding and both principal curvatures are uniformly bounded by \( \kappa_{\max} \), the cone angles of the double cones \( \{ X \in T_p M \mid h(X, X) \geq -\frac{1}{4} \delta \| X \|^2 \} \) and \( \{ X \in T_q M \mid h(X, X) \geq -\frac{1}{4} \delta \| X \|^2 \} \) are not smaller than \( \frac{\pi}{2} + O(\delta) \). For sufficiently small \( \epsilon \geq (\sin^2(\chi_p) + \sin^2(\chi_q))^\frac{1}{2} \) the tangent direction in the plane orthogonal to \( \gamma(p) \) in \( T_p N \), and in the plane orthogonal to \( \gamma(q) \) in \( T_q N \) of the corresponding spheres build two double cones with cone angles not smaller than \( \frac{\pi}{2} \). Hence the intersection of both double cones is non-empty and there exists such a 2-sphere.

Secondly we shall show that the inequalities (B.4) and (B.6) are satisfied for \( s \in [0, s_0] \) with some \( s_0 > 0 \). Since the curvature is bounded by \( \cot(c) \) and due to the assumption \( (\sin^2(\chi_p) + \sin^2(\chi_q))^\frac{1}{2} \leq \epsilon \) there exists \( s_0 \) such that \( t_p \) and \( t_q \) do not reach the roots of \( \vartheta \) for \( s \in [0, s_0] \). Since the derivatives of \( \cos(\chi_p) \), \( \cos(\chi_q) \), \( t_p \) and \( t_q \) with respect to \( s \) are uniformly bounded, there exists \( s_0 > 0 \) such that the inequalities (B.4) and the first inequality of (B.6) are satisfied for \( s \in [0, s_0] \). Due to (8.4) also the derivatives of \( h(p', p') \) and \( h(q', q') \) are uniformly bounded. Hence there exists \( s_0 > 0 \) only depending on \( c \) and \( C' \), such that the second and the third inequality of (B.2) are satisfied for \( s \in [0, s_0] \).

Finally we have to satisfy the first inequality of (B.2). At the start point \( s = 0 \) this is always the case for one choice of the sign of \( \vartheta \). Now the second inequality of (B.2) implies the first. \( \square \)

**Continuation of the proof of Proposition 8.2.** For pairs \( (p, q) \) connected in \( N \) by a geodesic obeying (ii) there exists by Lemma B.1 a Killing field \( \vartheta \) and two paths \( s \mapsto p(s) \) and \( s \mapsto q(s) \) along which the length \( d \) is reduced for \( 0 \leq s \leq s_0 \). The inequality

\[ -d'(s) \leq |p'(s)| + |q'(s)| \leq 3 \cos\left(\frac{d(s)}{2}\right) \leq 3 \cos\left(\frac{d(s_0)}{2}\right) \]

implies \( \cos\left(\frac{d(0)}{2}\right) \geq (1 - \frac{3}{2} s_0) \cos\left(\frac{d(s_0)}{2}\right) \). Twice integration of (B.2) implies the following inequality together with the separate inequalities for the numerator and the denominator:

\[ \frac{d(0) - d(s_0)}{\text{dist}_M(p(s_0), p(0)) + \text{dist}_M(q(s_0), q(0))} \geq \frac{\delta s_0^2}{3s_0} \cos\left(\frac{d(0)}{2}\right) \geq \frac{\delta s_0^2}{6} (1 - \frac{3}{2} s_0) . \]

Since \( s_0 \) is bounded by \( \frac{\pi}{2} \) the geodesic connecting \( p(s) \) and \( q(s) \) stays inside \( \{ \gamma_t(p, t) \mid p \in M, -\frac{\pi}{2} \leq t \leq c_t(p) \} \subset \hat{N} \). We divide the geodesic from \( p(s_0) \) to \( q(s_0) \) in \( \hat{N} \) into segments outside of \( N \) and inside of \( N \). In this way an application of Lemma B.1 transforms the geodesic from \( p \) to \( q \) obeying (ii) into two paths \( s \mapsto p(s) \) from \( p(0) = p \) to \( p(s_0) \) and \( s \mapsto q(s) \) from \( q(0) = q \) to \( q(s_0) \) in \( M \) and several geodesic segments in \( \hat{N} \) from \( p(s_0) \) to \( q(s_0) \) obeying either (i), or (ii) or (iii). Due to (B.7) the sum of the lengths of the paths in \( M \) times \( \delta s_0^2 (1 - \frac{3}{2} s_0) \) plus the lengths of the geodesic segments is smaller than the length of the original geodesic.

**In case** (iii) we apply the gradient flow of \( \text{dist}_\hat{N} \). As long as \( p \neq q \) are connected in \( \hat{N} \) by a geodesic, the function \( \text{dist}_\hat{N} \) is smooth. The Riemannian metric on \( M \times M \) identifies the negative of the gradient of \( \text{dist}_\hat{N} \) with a unique vector field \( (p', q') \). We follow this flow as long as \( (\sin^2(\chi_p) + \sin^2(\chi_q))^\frac{1}{2} \geq \epsilon \) holds and the geodesic in \( \hat{N} \) connecting \( p \) and \( q \) stays inside \( \{ \gamma_t(p, t) \mid p \in M, -\frac{\pi}{2} < t \leq c_t(p) \} \). Along the corresponding integral curves \( s \mapsto (p(s), q(s)) \) we
have $|p'| = \sin(\chi_p)$ and $|q'| = \sin(\chi_q)$. Therefore $\text{dist}_N(p(s), q(s))$ decreases with derivative

(B.8) \[ -\frac{d}{ds} \text{dist}_N(p(s), q(s)) = \sin(\chi_p)|p'(s)| + \sin(\chi_q)|q'(s)| \geq \frac{\epsilon}{\sqrt{2}} (|p'(s)| + |q'(s)|). \]

For such $(p, q)$ the vector field $(p', q')$ is smooth with length bounded from above and below. For finite $s = s_0$ either the geodesic shrinks to point and $p$ becomes equal to $q$, or we reach a pair $(p, q)$ connected by a geodesic in $N$ such that either the geodesic touches the boundary of $\{\gamma_t(p, t) \mid p \in M, -\frac{s}{2} \leq t \leq c_t(p)\}$ or $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} \leq \epsilon$ holds. We integrate (B.8) to

(B.9) \[ \text{dist}_M(p(s_0), p(0)) + \text{dist}_M(q(s_0), q(0)) \leq \frac{\sqrt{2}}{\epsilon} \left( \text{dist}_N(p(0), q(0)) - \text{dist}_N(p(s_0), q(s_0)) \right). \]

Again we decompose the final geodesic in geodesic segments obeying either (i) or (ii) or (iii). To sum up the application of the gradient flow transforms the geodesic from $p$ to $q$ obeying (iii) into two paths $s \mapsto p(s)$ from $p(0) = p$ to $p(s_0)$ and $s \mapsto q(s)$ from $q(0) = q$ to $q(s_0)$ in $M$ and several geodesic segments in $N$ from $p(s_0)$ to $q(s_0)$ obeying either (i), or (ii) or (iii). For all other applications of the gradient flow reduces $\text{dist}_N(p, q)$ by a number $\geq \frac{\epsilon^2}{2\sqrt{2}}$. Therefore finitely many applications of Lemma B.1 and the gradient flow transform the geodesic connecting $p$ and $q$ into finitely many paths in $M$, which connect $p$ with $q$ and whose total length is bounded by $C \text{dist}_N(p, q)$ with $C = \max\{4, \frac{12}{\delta(2s_0 - 3|s_0|)}, \frac{\sqrt{2}}{\epsilon}\}$.

References

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