On the isotropic constant of random polytopes

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Abstract

Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $N > n$ consider N independent random points $x_1, \dots, x_N$ uniformly distributed in K. We prove that, with probability greater than $1 - C_1 \exp(-cn)$ if $N \geq c_1 n$ and greater than $1 - C_1 \exp(-cn/\log n)$ if $n < N < c_1 n$, the random polytopes $K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}$ and $S_N := \mathrm{conv}\{x_1, \dots, x_N\}$ have isotropic constant bounded by an absolute constant $C > 0$.

1 Introduction

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and there is a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the Euclidean unit sphere S_2^{n-1}. It is not hard to see that for every convex body K in $\mathbb{R}^n$ there exists an affine transformation T of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, one may define the isotropic constant L_K as an invariant of the affine class of K. One can check that the isotropic position of K minimizes the quantity

$$\frac{1}{|T(K)|^{1+\frac{2}{n}}} \int_{T(K)} \|x\|_2^2 dx \tag{1.2}$$

over all non-degenerate affine transformations T of $\mathbb{R}^n$. In particular,

$$nL_K^2 \leq \frac{1}{|K|^{1+\frac{2}{n}}} \int_K \|x\|_2^2 dx. \tag{1.3}$$

It is conjectured that there exists an absolute constant $C > 0$ such that $L_K \leq C$ for every $n \in \mathbb{N}$ and every convex body K in $\mathbb{R}^n$. The best known general estimate is currently due to Klartag [13] who proved that $L_K \leq c\sqrt[4]{n}$; Bourgain had proved in [6] that $L_K \leq c\sqrt[4]{n}\log n$. The conjecture is related to the slicing problem, which asks if there exists an absolute constant $c > 0$ such that every convex body with volume 1 has a hyperplane section whose volume exceeds c. The connection comes from the fact that

$$c_1 \leq L_K \cdot |K \cap \theta^{\perp}| \leq c_2 \tag{1.4}$$

for every $\theta \in S^{n-1}$ and every isotropic convex body K, where $c_1, c_2 > 0$ are absolute constants. We refer to the article [15] of Milman and Pajor for background information about isotropic convex bodies.

The purpose of this note is to establish a positive answer to the problem for some classes of random convex bodies. The study of this question was initiated by Klartag and Kozma in [14] with the case of Gaussian random polytopes. They proved that if $N > n$ and if $G_1, \ldots, G_N$ are independent standard Gaussian random vectors in $\mathbb{R}^n$, then the isotropic constant of the random polytopes

$$K_N := \mathrm{conv}\{\pm G_1, \ldots, \pm G_N\} \quad \text{and} \quad S_N := \mathrm{conv}\{G_1, \ldots, G_N\} \tag{1.5}$$

is bounded by an absolute constant $C > 0$ with probability greater than $1 - Ce^{-cn}$. The argument of [14] works for other classes of random polytopes with vertices which have independent coordinates (for example, if the vertices are uniformly distributed in the cube $Q_n := [-1/2, 1/2]^n$ or in the discrete cube $E_2^n := \{-1, 1\}^n$). Alonso–Gutiérrez (see [1]) has recently obtained a positive answer in the situation where K_N or S_N is spanned by N random points uniformly distributed on the Euclidean sphere S_2^{n-1}. We study the following problem:

Question 1.1 *Let K be a convex body in $\mathbb{R}^n$. For every $N > n$ consider N independent random points $x_1, \ldots, x_N$ uniformly distributed in K and define the random polytopes*

$$K_N := \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\} \quad \text{and} \quad S_N := \mathrm{conv}\{x_1, \ldots, x_N\}. \tag{1.6}$$

Is it true that, with probability tending to 1 as $n \to \infty$, one has $L_{K_N} \leq CL_K$ and $L_{S_N} \leq CL_K$ where $C > 0$ is a constant independent from K, n and N?

We give an affirmative answer in the case of 1-unconditional convex bodies. That is, we make the additional assumptions that K is centrally symmetric and that, after a linear transformation, the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for $\|\cdot\|_K$: for every choice of real numbers $t_1, \ldots, t_n$ and every choice of signs $\varepsilon_j = \pm 1$,

$$\left\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\right\|_K = \left\|t_1 e_1 + \cdots + t_n e_n\right\|_K. \tag{1.7}$$

Then, it is easily checked that one can bring K to the isotropic position by a diagonal operator. It is also not hard to prove that the isotropic constant of K satisfies $L_K \simeq 1$. The upper bound follows from the Loomis–Whitney inequality; see also [4] where the inequality $2L_K^2 \leq 1$ is proved. On the other hand, recall that for every convex body K in $\mathbb{R}^n$ one has $L_K \geq L_{B_2^n} \geq c$, where $c > 0$ is an absolute constant (see [15]). The precise formulation of our result is the following.

Theorem 1.2 *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $N > n$ consider N independent random points $x_1, \dots, x_N$ uniformly distributed in K. Then, with probability greater than $1 - C_1 \exp(-cn)$ if $N \geq c_1 n$ and greater than $1 - C_1 \exp(-cn/\log n)$ if $n < N < c_1 n$, the random polytopes $K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}$ and $S_N := \mathrm{conv}\{x_1, \dots, x_N\}$ have isotropic constant bounded by an absolute constant $C > 0$.*

The main result is proved in Section 2. Our method is based on the approach of [14] and on precise results of Bobkov and Nazarov from [5] about the ψ_2-behavior of linear functionals on isotropic 1-unconditional convex bodies. We conclude with remarks and comments in Section 3.

Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_p$ the norm of ℓ_p^n, $1 \leq p \leq \infty$, and write B_p^n for the unit ball and S_p^{n-1} for the unit sphere of ℓ_p^n. Volume is denoted by $| \cdot |$. The homothet of B_p^n of volume 1 is denoted by $\overline{B}_p^n$. The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

2 Proof of the theorem

It was mentioned in the Introduction that if D is a convex body in $\mathbb{R}^n$ then $|D|^{2/n} n L_D^2 \leq \frac{1}{|D|} \int_D \|x\|_2^2 dx$. Our starting point will be a stronger estimate for L_D in terms of the ℓ_1^n-norm (see [15, Paragraph 3.6]):

Lemma 2.1 *Let D be a convex body in $\mathbb{R}^n$. Then,*

$$|D|^{1/n} n L_D \leq c \, \frac{1}{|D|} \int_D \|x\|_1 \, dx, \tag{2.1}$$

where $c > 0$ is an absolute constant.

In view of Lemma 2.1, in order to prove that $K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}$ (or $S_N := \mathrm{conv}\{x_1, \dots, x_N\}$) has bounded isotropic constant with probability close to 1, it suffices to give a lower bound for the volume radius $|K_N|^{1/n}$ (or $|S_N|^{1/n}$ respectively) and an upper bound for the expected value of $\| \cdot \|_1$ on K_N (or S_N respectively). Observe that the problem is affinely invariant, and hence, we may assume that K is an isotropic convex body.

2.1 Lower bound for the volume radius

Since $K_N \supseteq S_N$ for every choice of points $x_1, \dots, x_N \in K$, it is enough to give a lower bound for $|S_N|^{1/n}$. This is a consequence of the following observations:

Fact 1. It was proved in [10, Lemma 3.3] (see also [12, Lemma 2.5]) that if K is a convex body in $\mathbb{R}^n$ with volume 1 and if $\overline{B}_2^n$ is a ball in $\mathbb{R}^n$ with volume 1, then

$$\mathrm{Prob}(|S_N| \geq \rho) \geq \mathrm{Prob}(|[\overline{B}_2^n]_N| \geq \rho) \tag{2.2}$$

for every $\rho > 0$. This reduces the problem to the case $K = \overline{B}_2^n$.

Fact 2. It was proved in [11] that there exist $c_1 > 1$ and $c_2 > 0$ such that if $N \geq c_1 n$ and $x_1, \ldots, x_N$ are independent random points uniformly distributed in $\overline{B}_2^n$, then

$$S_N := \mathrm{conv}\{x_1, \ldots, x_N\} \supseteq c_2 \min\left\{\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, 1\right\} \overline{B}_2^n \tag{2.3}$$

with probability greater than $1 - \exp(-n)$. Actually, the argument from [11] shows that, for every $\delta > 0$, if $N \geq (1+\delta)n$ then (2.3) holds true with for a random K_N with $c_2 = c_2(\delta)$; see [1, Lemma 3.1].

Combining the above we have the first part of the next Proposition:

Proposition 2.2 *Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$ and let $x_1, \ldots, x_N$ be independent random points uniformly distributed in K.*

(i) *If $N \geq c_1 n$ then, with probability greater than $1 - \exp(-n)$ we have*

$$|K_N|^{1/n} \geq |S_N|^{1/n} \geq c_2 \min\left\{\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, 1\right\}, \tag{2.4}$$

where $c_1 > 1$ and $c_2 > 0$ are absolute constants.

(ii) *If $n < N < c_1 n$ then (2.4) holds true with probability greater than $1 - \exp(-cn/\log n)$, where $c > 0$ is an absolute constant.*

Part (ii) (the case $n < N < c_1 n$) has to be treated separately. We first consider the symmetric random polytope K_N. Because of Fact 1, we may assume that $K = \overline{B}_2^n$ and, by monotonicity, it is enough to prove that with probability close to one $K_n = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ has the appropriate volume. We write

$$|K_n| = \frac{2^n}{n!} \prod_{k=1}^{n} d(x_k, \mathrm{span}\{x_1, \ldots, x_{k-1}\}), \tag{2.5}$$

where $\mathrm{span}(\emptyset) = \{0\}$ and $d(z, A)$ is the Euclidean distance from z to A. As in [14], we observe that the random variables $Y_k := d(x_k, \mathrm{span}\{x_1, \ldots, x_{k-1}\})$ are independent. Using the fact that the radius of $\overline{B}_2^n$ is of the order of $\sqrt{n}$ and taking into account rotational invariance, we see that there exists an absolute constant $c_2 > 0$ such that

$$\mathrm{Prob}(Y_k \leq c_2 t\sqrt{n}) \leq \mathrm{Prob}(d(x, E_{k-1}) \leq t) \tag{2.6}$$

for every $t > 0$, where x is uniformly distributed in B_2^n and $E_k = \mathrm{span}\{e_1, \ldots, e_k\}$.

A similar question is studied in [2] (where x is uniformly distributed on S^{n-1}, but the proof and the estimates for $x \in B_2^n$ are similar). We will use [2, Theorem 4.3]: assume that $3 \leq k \leq n-3$ and set $\lambda = k/n$. If $\frac{1}{n} \leq \frac{\sin^2 \varepsilon}{1-\lambda} \leq n$ and $\frac{1}{n} \leq \frac{\cos^2 \varepsilon}{\lambda} \leq n$, then

$$c_1 \frac{e^{-\alpha_n u}}{\sqrt{u}} \leq \mathrm{Prob}(\rho(x, E_k) \leq \varepsilon) \leq c_2 \frac{e^{-\alpha_n u}}{\sqrt{u}}, \tag{2.7}$$

where ρ is the geodesic distance, $\alpha_n > 0$ and $\alpha_n \to 1$, $c_1, c_2 > 0$ are absolute constants and $u = \frac{n}{2}\left[(1-\lambda)\log\frac{1-\lambda}{\sin^2\varepsilon} + \lambda\log\frac{\lambda}{\cos^2\varepsilon}\right]$.

We apply this fact as follows: assume that $\lambda = \frac{k}{n} \leq 1 - \frac{1}{\log n}$. We define ε_k by the equation $\sin^2\varepsilon_k = (1-\lambda)/4$. Then,

$$u_k = \frac{n}{2}\left[(1-\lambda)\log 4 + \lambda\log\frac{4\lambda}{3+\lambda}\right] = \frac{n}{2}\left[\log 4 + \lambda\log\frac{\lambda}{3+\lambda}\right]. \tag{2.8}$$

Consider the function $H : [0,1] \to \mathbb{R}$ defined by $H(\lambda) = \log 4 + \lambda\log\frac{\lambda}{3+\lambda} - \delta(1-\lambda)$, where $\delta = \log 2 - 3/8 > 0$. Then, $H'(\lambda) < 0$ on $[0,1]$ and $H(1) = 0$. Therefore,

$$u_k \geq \frac{\delta(1-\lambda)n}{2} \geq \frac{\delta n}{2\log n}. \tag{2.9}$$

Since ρ and d are comparable, it follows that

$$\mathrm{Prob}(Y_k \leq c_3\sqrt{n-k}) \leq \exp(-cn/\log n) \tag{2.10}$$

for all $k \leq k_0 := \lfloor n - \frac{n}{\log n}\rfloor$. For $k > k_0$ we define ε_k by the equation $\sin^2\varepsilon_k = (1-\lambda)/n$; then, it is easy to check that $u_k \geq cn\log n$.

With this choice of ε_k it is clear that, with probability greater than $1 - \exp(-cn/\log n)$, we have

$$|K_n| \geq \frac{2^n}{n!}\prod_{k=1}^{k_0}(c_3\sqrt{n-k}) \times \prod_{k=k_0+1}^{n}\frac{c_4}{\sqrt{n}} \geq \left(\frac{c}{\sqrt{n}}\right)^n. \tag{2.11}$$

This extends the estimate (2.4) of Proposition 2.2 to the range $n \leq N < c_1 n$ (in the symmetric case) with a slightly worse probability estimate.

For the random polytope S_N we follow [14]: we may assume that $N = n+1$. We define $y_i = x_i - x_1$, $i = 1, \dots, n+1$ and consider the symmetric random polytope $K'_{n+1} = \mathrm{conv}\{\pm y_2, \dots, \pm y_{n+1}\}$. By the Rogers–Shephard inequality we have

$$|S_{n+1}| = |\mathrm{conv}\{0, y_2, \dots, y_{n+1}\}| \geq 4^{-n}|K'_{n+1}|, \tag{2.12}$$

and hence, it remains to estimate $|K'_{n+1}|$ from below. Consider the linear map F defined by $F(x_i) = x_i - x_1$, $2 \leq i \leq n+1$. With probability one, $x_2, \dots, x_{n+1}$ are linearly independent, and $K'_{n+1} = F(D_n)$, where $D_n = \mathrm{conv}\{\pm x_2, \dots, \pm x_{n+1}\}$. Therefore,

$$|K'_{n+1}| = |\det F| \cdot |D_n|. \tag{2.13}$$

Let $v \in \mathbb{R}^n$ be such that $\langle v, x_i\rangle = 1$, $2 \leq i \leq n+1$. Since $\|x_i\|_2 \leq c\sqrt{n}$ for all i, we have $\|v\|_2 \geq c_1/\sqrt{n}$ by the Cauchy–Schwarz inequality. Observe that $F(x) = x - \langle x, v\rangle x_1$ for every $x \in \mathbb{R}^n$; therefore, $\det F = 1 - \langle v, x_1\rangle$. This implies that

(2.14)

$$\mathrm{Prob}(|\det F| < 2^{-n}) = \mathbb{E}_v\left[\mathrm{Prob}(|\langle v, x\rangle - 1| < 2^{-n})\right] \leq \left|\left\{x : |\langle x, \theta_v\rangle| \leq \frac{1}{\|v\|_2 2^n}\right\}\right|,$$

where $\theta_v = v/\|v\|_2$, because the centered strip has maximal volume among all strips of width 2^{-n} which are perpendicular to θ_v. Since $\|v\|_2 \geq c/\sqrt{n}$, we easily check that the last quantity in (2.14) is bounded by $\sqrt{n}\exp(-cn)$. We have already seen that, with probability greater than $1-\exp(-cn/\log n)$, the volume of D_n is larger than $(c/\sqrt{n})^n$. Since we also have $|\det F| \geq 2^{-n}$, the proof is complete.

2.2 Upper bound for the expectation of $\|\cdot\|_1$

Let (Ω, μ) be a probability space and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing convex function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The Orlicz space $L_\phi(\mu)$ is the space of all measurable functions f on Ω for which $\int_\Omega \phi(|f|/t)d\mu < \infty$ for some $t > 0$, equipped with the norm $\|f\|_\phi = \inf\{t > 0 : \int_\Omega \phi(|f|/t)d\mu \leq 1\}$. We will only need the functions $\psi_\alpha(t) = e^{t^\alpha} - 1$. In particular,

$$\|f\|_{\psi_2} = \inf\left\{t > 0 : \int e^{(f(x)/t)^2} d\mu(x) \leq 2\right\}. \tag{2.15}$$

We will make use of the following Bernstein type inequality (see [8]):

Lemma 2.3 *Let $g_1, \ldots, g_m$ be independent random variables with $\mathbb{E}\, g_j = 0$ on some probability space (Ω, μ). Assume that $\|g_j\|_{\psi_2} \leq A$ for all $j \leq m$ and some constant $A > 0$. Then,*

$$\mathrm{Prob}\left\{\left|\sum_{j=1}^m g_j\right| > \alpha m\right\} \leq 2\exp(-\alpha^2 m/8A^2) \tag{2.16}$$

for every $\alpha > 0$.

Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. The ψ_2 behavior of linear functionals $x \mapsto \langle x, \theta\rangle$ on K is described by the following result of Bobkov and Nazarov from [5].

Lemma 2.4 *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $\theta \in \mathbb{R}^n$,*

$$\|\langle\cdot, \theta\rangle\|_{\psi_2} \leq c\sqrt{n}\|\theta\|_\infty, \tag{2.17}$$

where $c > 0$ is an absolute constant.

Now, let $y_1, \ldots, y_n$ be independent random points uniformly distributed in K. We fix $\theta \in \mathbb{R}^n$ with $\|\theta\|_\infty = 1$ and a choice of signs $\varepsilon_j = \pm 1$, and apply Lemma 2.3 to the random variables $g_j(y_1, \ldots, y_n) = \langle \varepsilon_j y_j, \theta\rangle$ on $\Omega = K^n$. From Lemma 2.4 (with $m = n$) we see that

$$\mathrm{Prob}\left\{|\langle \varepsilon_1 y_1 + \cdots + \varepsilon_n y_n, \theta\rangle| > \alpha n\right\} \leq 2\exp(-c\alpha^2) \tag{2.18}$$

for every $\alpha > 0$. Consider a 1/2-net $\mathcal{N}$ for S^n_∞ with cardinality $|\mathcal{N}| \leq 5^n$. Choosing $\alpha = C\sqrt{n}\sqrt{\log(2N/n)}$ where $C > 0$ is a large enough absolute constant, we see that, with probability greater than $1 - \exp(-c_1 n \log(2N/n))$ we have

$$|\langle \varepsilon_1 y_1 + \cdots + \varepsilon_n y_n, \theta\rangle| \leq C n^{3/2}\sqrt{\log(2N/n)} \tag{2.19}$$

for every $\theta \in \mathcal{N}$ and every choice of signs $\varepsilon_j = \pm 1$. Using a standard successive approximation argument, and taking into account all 2^n possible choices of signs $\varepsilon_j = \pm 1$, we get that, with probability greater than $1 - \exp(-c_2 n \log(2N/n))$,

$$\max_{\varepsilon_j = \pm 1} \|\varepsilon_1 y_1 + \cdots + \varepsilon_n y_n\|_1 \leq C n^{3/2}\sqrt{\log(2N/n)}. \tag{2.20}$$

Now, let $N \geq n$ and let $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Since the number of subsets $\{y_1, \ldots, y_n\}$ of $\{\pm x_1, \ldots, \pm x_N\}$ is bounded by $(2eN/n)^n$, we immediately get the following.

Proposition 2.5 *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Fix $N > n$ and let $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Then, with probability greater than $1 - \exp(-cn \log(2N/n))$ we have*

$$\max_{\varepsilon_j = \pm 1} \|\varepsilon_1 x_{i_1} + \cdots + \varepsilon_n x_{i_n}\|_1 \leq C n^{3/2}\sqrt{\log(2N/n)} \tag{2.21}$$

for all $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, N\}$.

Observe that, with probability equal to 1, all the facets of K_N or S_N are simplices. Also, if $F = \mathrm{conv}\{y_1, \ldots, y_n\}$ is a facet of K_N then we must have $y_j = \varepsilon_j x_{i_j}$ and $i_j \neq i_s$ for all $1 \leq j \neq s \leq n$. In other words, x_i and $-x_i$ cannot belong to the same facet of K_N.

We first consider the case of the symmetric random polytope K_N. The next lemma reduces the computation of the expectation of $\|x\|_1$ on K_N to a similar problem on the facets of K_N (the idea comes from [14]).

Lemma 2.6 *Let $F_1, \ldots, F_m$ be the facets of K_N. Then,*

$$\frac{1}{|K_N|}\int_{K_N} \|x\|_1 dx \leq \max_{1 \leq s \leq m} \frac{1}{|F_s|}\int_{F_s} \|u\|_1 du. \tag{2.22}$$

Proof. Following [14, Lemma 2.5], one can check that

$$\frac{1}{|K_N|}\int_{K_N} \|x\|_1 dx = \frac{1}{|K_N|}\sum_{s=1}^{m} \frac{d(0, F_s)}{n+1}\int_{F_s} \|u\|_1 du, \tag{2.23}$$

where $d(0, F_s)$ is the Euclidean distance from 0 to the affine subspace determined by F_s. Since

$$|K_N| = \frac{1}{n}\sum_{s=1}^{m} d(0, F_s)|F_s|, \tag{2.24}$$

the result follows. $\square$

Let $y_1, \dots, y_n \in \mathbb{R}^n$ and define $F = \mathrm{conv}\{y_1, \dots, y_n\}$. Then, $F = T(\Delta^{n-1})$ where $\Delta^{n-1} = \mathrm{conv}\{e_1, \dots, e_n\}$ and $T_{ij} = \langle y_j, e_i \rangle =: y_{ji}$. Assume that $\det T \neq 0$. It follows that

$$\begin{aligned}\frac{1}{|F|}\int_F \|u\|_1 du &= \frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} \|Tu\|_1 du \\ &= \frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} \sum_{i=1}^n \left|\sum_{j=1}^n y_{ji}u_j\right| du \\ &= \sum_{i=1}^n \frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} \left|\sum_{j=1}^n y_{ji}u_j\right| du \\ &\leq \sum_{i=1}^n \left(\frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} \left(\sum_{j=1}^n y_{ji}u_j\right)^2 du\right)^{1/2}.\end{aligned}$$

Using the fact that

$$\frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} u_{j_1}u_{j_2} = \frac{1+\delta_{j_1,j_2}}{n(n+1)}, \tag{2.25}$$

we see that

$$\begin{aligned}\frac{1}{|F|}\int_F \|u\|_1 du &\leq \frac{1}{\sqrt{n(n+1)}}\sum_{i=1}^n \left(\sum_{j=1}^n y_{ji}^2 + \left(\sum_{j=1}^n y_{ji}\right)^2\right)^{1/2} \\ &\leq \frac{1}{n}\sum_{i=1}^n \left[\left(\sum_{j=1}^n y_{ji}^2\right)^{1/2} + \left|\sum_{j=1}^n y_{ji}\right|\right].\end{aligned}$$

It now follows from the classical Khintchine inequality (see [17] for the best constant $\sqrt{2}$) that

$$\frac{1}{|F|}\int_F \|u\|_1 du \leq \frac{\sqrt{2}+1}{n} \max_{\varepsilon_j=\pm 1} \|\varepsilon_1 y_1 + \cdots + \varepsilon_n y_n\|_1 . \tag{2.26}$$

Now, Proposition 2.5 and Lemma 2.6 immediately imply our upper bound:

Proposition 2.7 *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Fix $N > n$ and let $x_1, \dots, x_N$ be independent random points uniformly distributed in K. Then, with probability greater than $1 - \exp(-cn\log(2N/n))$ we have*

$$\frac{1}{|K_N|}\int_{K_N} \|x\|_1 dx \leq C\sqrt{n}\sqrt{\log(2N/n)} \tag{2.27}$$

where $C > 0$ is an absolute constant.

The case of S_N requires some minor modifications. First of all, the role of 0 is played by the vector $w = \frac{1}{N}(x_1+\cdots+x_N)$ which belongs to $S_N := \mathrm{conv}\{x_1,\dots,x_N\}$. The substitute for (2.23) is

$$\frac{1}{|S_N|}\int_{S_N}\|x\|_1dx = \frac{1}{|S_N|}\sum_{s=1}^{m}\frac{d(0,F_s)}{n+1}\int_{F_s}\|u-w\|_1du, \tag{2.28}$$

where $F_1,\dots,F_m$ are the facets of S_N (see [14, Lemma 2.5]). As in Lemma 2.6 (and since $\|u-w\|_1 \le \|w\|_1 + \|u\|_1$ for every $s \le m$ and for every $u \in F_s$) we see that

$$\begin{aligned}\frac{1}{|S_N|}\int_{S_N}\|x\|_1dx &\le \max_{1\le s\le m}\frac{1}{|F_s|}\int_{F_s}\|u-w\|_1du\\ &\le \|w\|_1 + \max_{1\le s\le m}\frac{1}{|F_s|}\int_{F_s}\|u\|_1du.\end{aligned}$$

From (2.26) and Proposition 2.5 we get

$$\max_{1\le s\le m}\frac{1}{|F_s|}\int_{F_s}\|u\|_1du \le C\sqrt{n}\sqrt{\log(2N/n)} \tag{2.29}$$

It remains to estimate $\|w\|_1$. But, applying Lemma 2.3 (with $m = N$) to the random variables $g_j(x_1,\dots,x_N) = \langle x_j,\theta\rangle$, where $\theta \in S_\infty^{n-1}$, we see that

(2.30)

$$\mathrm{Prob}\left\{|\langle x_1+\cdots+x_N,\theta\rangle| > C\sqrt{n}\sqrt{\log(2N/n)}N\right\} \le 2\exp(-cN\log(2N/n))$$

and continuing as in §2.2 we can check that

$$\|w\|_1 = \frac{1}{N}\|x_1+\cdots+x_N\|_1 \le C\sqrt{n}\sqrt{\log(2N/n)} \tag{2.31}$$

with probability greater than $1-C\exp(-cN\log(2N/n))$. This leads to the analogue of Proposition 2.7 for S_N.

2.3 Proof of the main result

Lemma 2.1 tells us that

$$|K_N|^{1/n}nL_{K_N} \le c\,\frac{1}{|K_N|}\int_{K_N}\|x\|_1\,dx, \tag{2.32}$$

where $c > 0$ is an absolute constant. Assume first that $N \le \exp(cn)$. Propositions 2.2 and 2.7 show that, with probability greater than $1 - C_1\exp(-cn)$ if $N \ge c_1 n$ and greater than $1 - C_1\exp(-cn/\log n)$ if $n < N < c_1 n$, K_N satisfies

$$\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}\cdot nL_{K_N} \le c\cdot C\sqrt{n}\sqrt{\log(2N/n)}. \tag{2.33}$$

It follows that $L_{K_N} \leq C_1 := c \cdot C$.

It is proved in [9, Section 5] that if $N \geq \exp(cn)$ then, with probability greater than $1 - \exp(-cn)$, one has

$$c_1 K \subseteq S_N \subseteq K_N \subseteq K \subseteq c_2 \overline{B}_1^n. \tag{2.34}$$

The last inclusion is established in [4] for isotropic 1-unconditional convex bodies. Then, $|K_N|^{1/n} \geq |S_N|^{1/n} \geq c_1$ and

$$\frac{1}{|K_N|} \int_{K_N} \|x\|_1 \, dx \leq \frac{1}{|K_N|} \int_{K_N} c_3 n \|x\|_{K_N} \, dx \leq c_3 n. \tag{2.35}$$

Therefore, (2.32) gives $L_{K_N} \leq c_4 := c_3/c_1$ in this case as well.

Similar arguments work for S_N. □

3 Remarks

§3.1. Let K be an isotropic convex body in $\mathbb{R}^n$ with the property $\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C\|\langle \cdot, \theta \rangle\|_2$ for every $\theta \in \mathbb{R}^n$, where $C > 0$ is an absolute constant. This class of ψ_2–bodies includes the balls $\overline{B}_q^n$ of ℓ_q^n, $2 \leq q \leq \infty$ (see [3]). It is also known that ψ_2–bodies have bounded isotropic constant; this was proved by Bourgain in [7]. Starting with (1.3) instead of Lemma 2.1 and using the method of Section 2 one can prove that, with probability greater than $1 - \exp(-cn)$, the isotropic constants of K_N and S_N are bounded by an absolute constant. Actually, the argument is completely parallel to the one of Alonso-Gutiérrez in [1] for the case of random points from S_2^{n-1}. Note that 1-unconditional isotropic convex bodies are not necessarily ψ_2–bodies.

§3.2. If $x_1, \ldots, x_N$ are independent random points uniformly distributed in a convex body K of volume 1 in $\mathbb{R}^n$, we define

$$\mathbb{E}(K, N) = \mathbb{E}\,|S_N|^{1/n} = \mathbb{E}|\mathrm{conv}\{x_1, \ldots, x_N\}|^{1/n}. \tag{3.1}$$

In [11] it was proved that if K is an isotropic 1-unconditional convex body in $\mathbb{R}^n$, then, for every $N \geq n + 1$,

$$\mathbb{E}(K, N) \leq C \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, \tag{3.2}$$

where $C > 0$ is an absolute constant. Observe that this is a direct consequence of Proposition 2.7. We have

$$|K_N|^{1/n} n L_{K_N} \leq C\sqrt{n}\sqrt{\log(2N/n)} \tag{3.3}$$

with probability greater than $1 - \exp(-cn)$, so the result follows from the fact that $L_{K_N} \geq c_1$, where $c_1 > 0$ is an absolute constant. This was observed by A. Pajor.

In [10] it was proved that if K is *any* convex body in $\mathbb{R}^n$, then $\mathbb{E}(K, N) \leq C L_K \frac{\log(2N/n)}{\sqrt{n}}$. Using the methods of [10], [11] and the concentration result of G.

Paouris (see [16]) one can prove that for any convex body K in $\mathbb{R}^n$, if $n+1 \leq N \leq ne^{\sqrt{n}}$ then

$$\mathbb{E}\,(K,N) \leq CL_K\, \frac{\sqrt{\log(N/n)}}{\sqrt{n}}, \tag{3.4}$$

where $C>0$ is an absolute constant. This would be a consequence (for the full range of values of the parameter N) of an affirmative answer to Question 1.1.

References

[1] D. Alonso–Gutiérrez, *About the isotropy constant of random convex sets*, Preprint: `http://arxiv.org/abs/0707.1570`.

[2] S. Artstein–Avidan, *Proportional concentration phenomena on the sphere*, Israel J. Math. **132** (2002), 337–358.

[3] F. Barthe, O. Guédon, S. Mendelson and A. Naor, *A probabilistic approach to the geometry of the ℓ_p^n-ball*, Ann. Prob. **33** (2005), 480–513.

[4] S. G. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 53–69.

[5] S. G. Bobkov and F. L. Nazarov, *Large deviations of typical linear functionals on a convex body with unconditional basis*, Stochastic Inequalities and Applications, Progr. Probab. 56, Birkhäuser, Basel, 2003, 3–13.

[6] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1469** (1991), 127–137.

[7] J. Bourgain, *On the isotropy constant for ψ_2-bodies*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 114–121.

[8] J. Bourgain, J. Lindenstrauss and V. D. Milman, *Minkowski sums and symmetrizations*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1317** (1988), 44–74.

[9] A. Giannopoulos and V. D. Milman, *Concentration property on probability spaces*, Adv. in Math. **156** (2000), 77–106.

[10] A. Giannopoulos and A. Tsolomitis, *On the volume radius of a random polytope in a convex body*, Math. Proc. Cambridge Phil. Soc. **134** (2003), 13–21.

[11] A. Giannopoulos, M. Hartzoulaki and A. Tsolomitis, *Random points in isotropic unconditional convex bodies*, J. London Math. Soc. **72** (2005), 779–798.

[12] M. Hartzoulaki and G. Paouris, *Quermassintegrals of a random polytope in a convex body*, Arch. Math. **80** (2003), 430–438.

[13] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Analysis **16** (2006), 1274–1290.

[14] B. Klartag and G. Kozma, *On the hyperplane conjecture for random convex sets*, Preprint: `http://arxiv.org/abs/math.MG/0612517`.

[15] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geom. Aspects of Funct. Analysis (Lindenstrauss-Milman eds.), Lecture Notes in Math. **1376** (1989), 64–104.

[16] G. Paouris, *Concentration of mass in convex bodies*, Geom. Funct. Analysis **16** (2006), 1021–1049.

[17] S. J. Szarek, *On the best constant in the Khinchine inequality*, Studia Math. **58** (1976), 197–208.

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