Nonlinear Schrödinger Equations and Quantum Fluids

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Linear Schrödinger Equation

$$\boxed{i\partial_t\psi + \Delta\psi = 0}$$

Fourier transform :

$$\psi(t,x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} \psi_0(y)dy.$$

Proposition

$\forall t \in \mathbb{R}$,

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

$$E(t) = \|\nabla\psi(t)\|_{L^2}^2 = E_0,$$

$$\|\psi(t)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{d/2}} \|\psi_0\|_{L^1}.$$

Semilinear Schrödinger Equation

$$\boxed{i\partial_t\psi + \Delta\psi = \lambda|\psi|^\alpha\psi}$$

with $\alpha > 0$, $\lambda \in \mathbb{R}^*$ (huge influence of the sign of λ on the behavior of the solution ψ).

Proposition

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

$$E(t) = \|\nabla\psi(t)\|_{L^2}^2 + \frac{\lambda}{\alpha+2}\int_{\mathbb{R}^d}|\psi(t,x)|^{\alpha+2} = E_0.$$

Strichartz estimates *:* $\forall t \in \mathbb{R}$,

$$\|\psi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|\psi_0\|_{L^2},$$

*where (q,r) is an **admissible pair**, ie* $\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right)$.

Logarithmic Schrödinger Equation

$$\boxed{i\partial_t \psi + \Delta\psi = \lambda\psi \log(|\psi|^2)}$$

with $\lambda \in \mathbb{R}^*$.

Proposition

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

$$E(t) = \|\nabla\psi(t)\|_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |\psi(t,x)|^2 \log|\psi(t,x)|^2 dx = E_0.$$

Remark

No Strichartz-like estimates currently known.

Logarithmic Schrödinger Equation

Proposition

Stationnary Solutions

- $\lambda < 0$: *Existence of standing waves*

$$\forall \omega \in \mathbb{R}, \ u(x,t) = e^{i\omega t} e^{\frac{\omega+d}{2}} e^{-\frac{1}{2}|x|^2}.$$

- $\lambda > 0$: *No stationnary solutions, every solution vanishes to 0 :*

$$\|\psi(t,.)\|_{L^\infty(\mathbb{R}^d)} \to 0 \quad \textit{when } t \to 0.$$

Nonlinear behavior *: Note that up to a scaling in time, every solution ψ disperses as a Gaussian function.*

Logarithmic Schrödinger-Langevin Equation

Bohmian mechanics approach of quantum mechanics :

$$\boxed{i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{1}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)} \tag{1}$$

- well-posedness ?
- local/global existence ? long time behavior ?
- stationnary solutions ? stability ?
- physics : $\mu > 0$. Influence of $\lambda > 0$ and $\lambda < 0$?
- numerics ?

Splitting Method

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{1}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)$$

We solve :

- $\partial_t\psi = -\frac{1}{2}i\Delta\psi$ by FFT,
- $\partial_t\psi = -i\lambda\psi\log(|\psi|^2+\varepsilon)$ by the explicit solution

$$\psi(t+\Delta t,.) = \psi(t,.)e^{-i\lambda\Delta t\log(|\psi(t,.)|^2+\varepsilon)},$$

- and $\partial_t\psi = -\frac{1}{2}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)$ by an explicit solution

$$\psi(t+\Delta t,.) = a(t,.)e^{i\theta(t,.)e^{-\mu\Delta t}},$$

where we decompose $\psi(t,.) = a(t,.)e^{i\theta(t,.)}$.

$\lambda < 0$ case

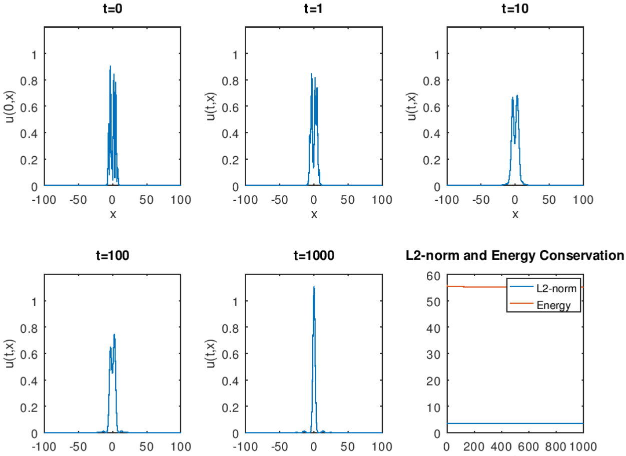

Figure – Solution of equation (8) with initial datum ψ_0 in the focusing case ($\lambda = -0.1$, $\mu = 1$).

$\lambda > 0$ case

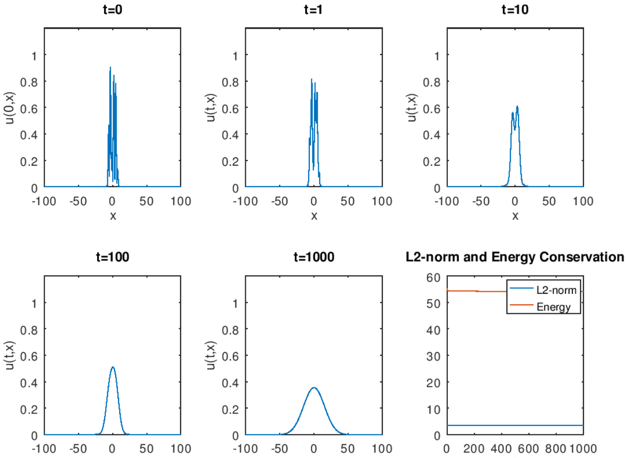

Figure – Solution of equation (8) with initial datum ψ_0 in the defocusing case ($\lambda = 0.1$, $\mu = 1$).

Logarithmic Schrödinger Equations and Quantum Fluids

Madelung transform

$$\boxed{i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = V\psi}$$

Madelung transform $\psi = \sqrt{\rho}e^{iS/\hbar}$, and define $J = \rho\nabla S$, then :

$$\partial_t\rho + \mathrm{div} J = 0, \tag{2}$$

$$\partial_t J + \mathrm{div}\left(\frac{J \otimes J}{\rho}\right) + \rho\nabla V = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{3}$$

Remark

- Quantum Euler Equations without pressure.
- Pathological 3rd order quantum potential $\rho\nabla\left(\Delta\sqrt{\rho}/\sqrt{\rho}\right)$.

Logarithmic Schrödinger-Langevin Equation with potential

In the case of the Schrödinger-langevin equation

$$\boxed{i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{\hbar}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right) + V\psi,}$$

we get the following system :

$$\partial_t\rho + \mathrm{div} J = 0, \tag{4}$$

$$\partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \lambda\nabla\rho + \mu J + \rho\nabla V = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{5}$$

Augmented formulation

Denoting $I = \hbar\nabla\rho$, previous system can be written :

$$\partial_t \rho + \mathrm{div} J = 0, \tag{6}$$

$$\partial_t J + \mathrm{div}\left(\frac{J \otimes J}{\rho}\right) + \lambda\nabla\rho + \mu J + \rho\nabla V = \frac{\hbar}{4}\,\mathrm{div}\left(\nabla I - \frac{1}{\rho} I \cdot I\right), \tag{7}$$

$$\partial_t I + \mathrm{div}\left(\frac{I \otimes J}{\rho}\right) = -\frac{\hbar}{4}\,\mathrm{div}\left({}^t\nabla J - \frac{1}{\rho}\,{}^tJ \cdot {}^tI\right). \tag{8}$$

Remark

This system is over-determined as we have the fourth equation $I = \hbar\nabla\rho$. Note also that it fails to be a classical hyperbolic system.

Preserving structure

Our discretization had to preserve some structural properties of our system, as

- mass conservation

$$\|\rho(t,.)\|_{L^1} = \|\rho_0\|_{L^1},$$

- positivity

$$\rho \geq 0,$$

- over-determination

$$J = \hbar \nabla \rho.$$

Aim

Benefits of such method (vs Spectral method in particular) :

- no hard structural hypothesis,
- should works equally for both linear and non-linear system.

What we could aim about our discretization (vs Splitting Method in particular) :

- long-time accuracy under strict CFL,
- regularity-preserving ($\rho_0 \in H^k \Rightarrow \rho \in H^k$).

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Thanks for your attention.