Optimal lifetime consumption and investment under drawdown constraint*

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Abstract

We consider the infinite horizon optimal consumption-investment problem under the drawdown constraint, i.e. the wealth process never falls below a fixed fraction of its running maximum. We assume that the risky asset is driven by the constant coefficients Black and Scholes model. For a general class of utility functions, we provide the value function in explicit form, and we derive closed-form expressions for the optimal consumption and investment strategy.

Key words: portfolio allocation, drawdown constraint, duality, verification.

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1 Introduction

Since the seminal papers of Merton [16, 17], there has been an extensive literature on the problem of optimal consumption and investment decision in financial markets subject to imperfections. The case of incomplete markets was first considered by Cox and Huang [4] and Karatzas, Lehoczky and Shreve [12]. Cvitanić and Karatzas [5] considered the case where the agent portfolio is restricted to take values in some given closed convex set. He and Pagès [11] and El Karoui and Jeanblanc [9] extended the Merton model to allow for the presence of labor income. Constantinides and Magill [3], Davis and Norman [7], and Shreve and Soner [21] considered the case where the risky asset is subject to proportional transaction costs. Ben Tahar, Soner and Touzi [2] considered the case where the sales of the risky asset are subject taxes on the capital gains.

In this paper, we study the infinite horizon optimal consumption and investment problem when the wealth never falls below a fixed fraction of its current maximum. This is the so-called drawdown constraint. Fund managers do offer this type of guarantee in order to satisfy the aversion to deception of the investors.

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The drawdown constraint on the wealth accumulation of the fund manager was first considered by Grossman and Zhou [10] for an agent maximizing the long term growth rate of the expected power utility of final wealth, with no intermediate consumption. Their main result is that the optimal investment in the risky asset is an explicit constant proportion of the difference between the current wealth and the imposed fixed fraction of its running maximum. Klass and Nowicki [14] show that the strategy proposed in Grossman and Zhou [10] does not retain its optimal long term growth property when generalized to the discrete time setting. Nevertheless, Cvitanic and Karatzas [6] developed a beautiful martingale approach to the Grossman and Zhou [10] problem which makes the analysis much simpler and allows for more general class of price processes. Their main observation is that strategies based on investment in proportions of the distance between the current wealth and its drawdown constraint, are always admissible. Besides, El Karoui and Meziou [8] recently characterized the optimal portfolio obtained by Cvitanic and Karatzas [6] in terms of Azema-Yor martingales, opening the door to the study of non linear drawdown constraints. A general criticism that one may formulate about the long term growth rate criterion is that it only provides the asymptotic optimal behavior of the fund manager. In other words, there is no penalization for using an arbitrary strategy as long as it coincides with the Grossman and Zhou [10] optimal strategy after some given fixed point in time.

In this paper, we consider the classical Merton criterion, which consists in maximizing the infinite horizon utility of consumption, for a fund manager subject to the drawdown constraint. This problem was considered recently by Roche [19] in the context of the power utility function. Following the initial Merton approach, Roche [19] was able to guess a solution of the dynamic programming equation, and provided some numerical results which highlight some interesting consequences of the drawdown constraint on the optimal consumption-investment strategy. The homogeneity of the power utility is the key-property in order to guess the candidate solution. Notice that Roche [19] does not provide any argument to verify that his candidate solution is indeed the value function of the optimal consumption-investment problem.

In contrast with Roche [19], the analysis of our paper allows for a general class of utility functions whose asymptotic elasticity (see [15]) is bounded by some level depending on the drawdown level, and satisfying some condition related to the relative risk aversion. For any utility function in this class, we derive an explicit expression for the value function of the fund manager, together with the optimal consumption and investment strategy. The key-idea in order to guess the candidate solution is to pass from the dynamic programming equation to the partial differential equation (PDE) satisfied by the dual indirect utility function. The latter PDE being linear inside the state space domain, one can easily account for the Neumann condition related to the drawdown constraint, and derive an explicit candidate solution for any utility function. In order to prove that the thus derived candidate solution is indeed the value function of our optimal consumption-investment problem, we use a verification argument which requires a convenient transversality condition. The verification argument is the main technical step where the above mentioned restrictions on the utility functions are required.

The solution derived in this paper agrees with that of Roche [19] in the zero interest rate and power utility case. However, for positive interest rates, we follow Cvitanic and Karatzas [6] by defining the drawdown constraint in terms of the discounted wealth.

The paper is organized as follows. Section 2 is devoted to the formulation of the problem. The main result of the paper is provided in Section 3. Section 4 presents the formal argument that we used in order to guess our candidate solution. The rigorous proof of our main result is reported in Section 5.

2 Problem formulation

Throughout this paper, we consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ endowed with a Brownian motion $W = \{W_t,\, t \geq 0\}$ with values in $\mathbb{R}$, and we denote by $\mathbb{F} := \{\mathcal{F}_t,\, t \geq 0\}$.

The financial market consists of a non-risky asset, with process normalized to unity, and one risky asset with price process defined by the Black and Scholes model :

$$dS_t \;=\; \sigma S_t \left(dW_t + \lambda dt\right) ,$$

where $\sigma > 0$ is the volatility parameter, and $\lambda \in \mathbb{R}$ is a constant risk premium.

The normalization of the non-risky asset to unity is as usual a reduction of the model obtained by taking this asset as a numéraire. Hence, all amounts are evaluated in terms of their discounted values.

For any continuous process $\{M_t, t \geq 0\}$, we shall denote by

$$M_t^* \;:=\; \sup_{0\leq r\leq t} M_r\,, \qquad t \geq 0\,,$$

the corresponding running maximum process, and we recall that

$$M^* \quad \text{is non-decreasing and} \quad \int_0^\infty \left(M_t^* - M_t\right) dM_t^* \;=\; 0\,. \tag{2.1}$$

2.1 Consumption-portfolio strategies and the drawdown constraint

We next introduce the set of consumption-investment strategies whose induced wealth process X satisfies the drawdown constraint

$$X_t \;\geq\; \alpha X_t^* \quad \text{for every} \quad t \geq 0\,, \;\; \text{a.s.}\,, \tag{2.2}$$

where α is some given parameter in the interval $[0, 1)$.

A portfolio strategy is an $\mathbb{F}-$adapted process $\theta = \{\theta_t,\, t \geq 0\}$, with values in $\mathbb{R}$, satisfying the integrability condition

$$\int_0^T |\theta_t|^2 dt \;<\; \infty \quad \text{a.s.} \qquad \text{for all} \quad T > 0\,. \tag{2.3}$$

A consumption strategy is an $\mathbb{F}-$adapted process $C = \{C_t,\, t \geq 0\}$, with values in $\mathbb{R}_+$, satisfying

$$\int_0^T C_t dt \;<\; \infty \quad \text{a.s.} \qquad \text{for all} \quad T > 0\,. \tag{2.4}$$

Here, θ_t and C_t denote respectively the amount invested in the risky asset and the consumption rate at time t. By the self-financing condition, the wealth process induced by such a pair (C, θ) is defined by

$$X_t^{x,C,\theta} = x - \int_0^t C_r dr + \int_0^t \sigma\theta_r \left(dW_r + \lambda dr\right) \quad t \geq 0\,, \tag{2.5}$$

where x is some given initial capital. We shall denote by $\mathcal{A}_\alpha(x)$ the collection of all such consumption-investment strategies whose corresponding wealth process satisfies the drawdown constraint (2.2).

Remark 2.1 For a given initial wealth x and an admissible consumption-investment strategy $(C, \theta) \in \mathcal{A}_\alpha(x)$, let $X := X^{x,C,\theta}$ and $\tau := \inf\{t > 0 \ : \ X_t = \alpha X_t^*\}$.
• Denoting by $\mathbb{P}^0$ the probability measure under which the process $\{W_t^\lambda := W_t + \lambda t,\ t \geq 0\}$ is a Brownian motion, we see that, for $t \geq 0$, $\mathbb{E}^{\mathbb{P}^0}\left[\int_\tau^{\tau+t} C_r dr | \mathcal{F}_\tau\right] = \mathbb{E}^{\mathbb{P}^0}\left[\alpha X_\tau^* - X_{\tau+t} | \mathcal{F}_\tau\right] \leq 0$ on $\{\tau < \infty\}$. This shows that $\mathbb{E}\left[\int_\tau^\infty C_r dr\right] = 0$.
• Then $X_{\tau+t} = X_\tau + \int_\tau^{\tau+t} \sigma\theta_r dW_r^\lambda$ on $\{\tau < \infty\}$, and in order for the drawdown constraint to be satisfied, it is necessary that $\int_\tau^\infty |\theta_r|^2 dr = 0$.

2.2 A subset of admissible strategies

In order to ensure that the drawdown constraint is satisfied, one may define the consumption and the investment decisions in terms of proportions of the difference $X_t - \alpha X_t^*$:

$$C_t = c_t\left[X_t - \alpha X_t^*\right] \quad \text{and} \quad \theta_t = \pi_t\left[X_t - \alpha X_t^*\right]\,, \tag{2.6}$$

for an $\mathbb{F}-$adapted pair process (c, π) with values in $\mathbb{R}_+ \times \mathbb{R}$. We shall denote in this subsection by $\{X_\alpha^{x,c,\pi}(t),\ t \geq 0\}$ the corresponding wealth process with initial capital x, where the time variable appears in parenthesis, in order to highlight the dependence on α.

Under the self-financing condition, the dynamics of this process is given by

$$dX_\alpha^{x,c,\pi}(t) = \left(X_\alpha^{x,c,\pi}(t) - \alpha\left\{X_\alpha^{x,c,\pi}\right\}^*(t)\right)\left(\pi_t \frac{dS_t}{S_t} - c_t dt\right)\,, \quad t \geq 0\,. \tag{2.7}$$

The following argument reported from Cvitanić and Karatzas [6] shows that for any $\alpha \in [0, 1)$, and for any $\mathbb{F}-$adapted processes (c, π) with values in $\mathbb{R}_+ \times \mathbb{R}$ satisfying

$$\int_0^T c_t dt + \int_0^T |\pi_t|^2 dt < \infty \quad \text{for any} \quad T > 0\,, \tag{2.8}$$

the stochastic differential equation (2.7) has a unique solution satisfying the drawdown condition (2.2), which turns out to be explicit.

First, in the absence of the drawdown constraint, i.e. $\alpha = 0$, the stochastic differential equation (2.7) is well-known to have the following unique solution

$$X_0^{x,c,\pi}(t) = x \exp\left[\int_0^t \left(-c_r + \lambda\sigma\pi_r - \frac{1}{2}|\sigma\pi_r|^2\right) dr + \int_0^t \sigma\pi_r dW_r\right] \ t \geq 0\,,$$

for every initial capital $x > 0$ and every consumption-investment strategy (c, π) satisfying (2.8).

Now, the key ingredient for the construction of a solution to (2.7) is to introduce the process

$$\tilde{X}_\alpha^{x,c,\pi}(t) := \left[X_\alpha^{x,c,\pi}(t) - \alpha \left\{X_\alpha^{x,c,\pi}\right\}^*(t)\right] \left[\left\{X_\alpha^{x,c,\pi}\right\}^*(t)\right]^{\frac{\alpha}{1-\alpha}}, \qquad t \geq 0. \tag{2.9}$$

By Itô's Lemma together with (2.1), it follows that

$$\begin{aligned} d\tilde{X}_\alpha^{x,c,\pi}(t) &= \left[\left\{X_\alpha^{x,c,\pi}\right\}^*(t)\right]^{\frac{\alpha}{1-\alpha}} \left(\frac{\alpha}{1-\alpha}\left[\frac{X_\alpha^{x,c,\pi}(t)}{\left\{X_\alpha^{x,c,\pi}\right\}^*(t)} - 1\right] d\left\{X_\alpha^{x,c,\pi}\right\}^*(t) + dX_\alpha^{x,c,\pi}(t)\right) \\ &= \tilde{X}_\alpha^{x,c,\pi}(t) \left[(\lambda\sigma\pi_t - c_t)\, dt + \sigma\pi_t dW_t\right]. \end{aligned} \tag{2.10}$$

Since the dynamics of $\tilde{X}_\alpha^{x,c,\pi}$ are independent of α, we derive

$$\tilde{X}_\alpha^{x,c,\pi} = \tilde{X}_0^{x(\alpha),c,\pi} = X_0^{x(\alpha),c,\pi} \text{ with } x(\alpha) := \tilde{X}_\alpha^{x,c,\pi}(0) = (1-\alpha)x^{1/(1-\alpha)}. \tag{2.11}$$

We next deduce from (2.9) that, for every $r \leq t$,

$$X_0^{x(\alpha),c,\pi}(r) \leq (1-\alpha)\left\{X_\alpha^{x,c,\pi}\right\}^*(r)^{1/(1-\alpha)} \leq (1-\alpha)\left\{X_\alpha^{x,c,\pi}\right\}^*(t)^{1/(1-\alpha)}. \tag{2.12}$$

At a point of maximum r^* of the process $X_\alpha^{x,c,\pi}$ on $[0, t]$, the previous inequality becomes an equality so that finally

$$\left\{X_0^{x(\alpha),c,\pi}\right\}^*(t) = (1-\alpha)\left\{X_\alpha^{x,c,\pi}\right\}^*(t)^{1/(1-\alpha)}. \tag{2.13}$$

Combining (2.9), (2.11) and (2.13) finally leads to

$$X_\alpha^{x,c,\pi} = \left[X_0^{x(\alpha),c,\pi} + \frac{\alpha}{1-\alpha}\left\{X_0^{x(\alpha),c,\pi}\right\}^*\right]\left(\frac{\left\{X_0^{x(\alpha),c,\pi}\right\}^*}{1-\alpha}\right)^{-\alpha}. \tag{2.14}$$

Since (c, π) satisfies (2.8), $X_0^{x(\alpha),c,\pi}$ is well defined and the above argument shows that the right hand side of (2.14) is the unique solution of (2.7), as one can check by an immediate application of Itô's lemma. Remark also from (2.10) that $\tilde{X}_\alpha^{x,c,\pi}$ is positive so that the solution of (2.7) necessarily satisfies the drawdown condition (2.2).

Hence, for any pair (c, π) of $\mathbb{F}-$adapted processes, with values in $\mathbb{R}_+ \times \mathbb{R}$, and satisfying (2.8), the pair process (C, θ) defined by (2.6) is an admissible consumption-investment strategy in $\mathcal{A}_\alpha(x)$.

2.3 The optimal consumption-investment problem

The previous paragraph shows in particular that, for any initial capital x, the set $\mathcal{A}_\alpha(x)$ contains non-trivial consumption-investment strategies.

We now formulate the optimal consumption-investment problem which will be the focus of this paper. Throughout this paper, we consider a utility function

$$U : \mathbb{R}_+ \to \mathbb{R} \qquad C^2, \text{ concave, satisfying } U'(0+) = \infty \text{ and } U'(\infty) = 0. \tag{2.15}$$

More conditions on U will be needed for our main result, see subsection 3.3 below. For a given initial capital $x > 0$, the optimal consumption-investment problem under drawdown constraint is defined by :

$$u_0^\alpha := \sup_{(C,\theta)\in\mathcal{A}_\alpha(x)} J_0^\alpha(C,\theta) \quad \text{where} \quad J_0^\alpha(C,\theta) := \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t)\, dt\right], \tag{2.16}$$

where $\beta > 0$ is the subjective discount factor which expresses the preference of the agent for the present. For $\alpha = 0$, u_0^0 reduces to the classical Merton optimal consumption-investment problem. We shall use the dynamic programming approach in order to derive an explicit solution of the problem u_0^α. We then need to introduce the dynamic version of this problem :

$$u^\alpha(x,z) := \sup_{(C,\theta)\in\mathcal{A}_\alpha(x,z)} J^\alpha(C,\theta) \quad \text{where} \quad J^\alpha(C,\theta) := \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t)\, dt\right], \tag{2.17}$$

the pair (x,z), with $x \le z$, stands for the initial condition of the state processes (X,Z) defined, for $t \ge 0$, by

$$Z_t^{x,z,C,\theta} := z \vee \left\{X^{x,C,\theta}\right\}_t^* \quad \text{and} \quad X_t^{x,C,\theta} = x - \int_0^t C_r dr + \int_0^t \sigma\theta_r\left(dW_r + \lambda dr\right), \tag{2.18}$$

and $\mathcal{A}_\alpha(x,z)$ is the collection of all $\mathbb{F}-$adapted processes (C,θ) satisfying (2.3)-(2.4) together with the drawdown constraint

$$X_t^{x,C,\theta} \ \ge\ \alpha Z_t^{x,z,C,\theta} \ \text{a.s.}, \qquad t \ge 0. \tag{2.19}$$

Clearly, avoiding the trivial case $x = z = 0$, this restricts the pair of initial condition (x,z) to the closure $\overline{\mathbf{D}}_\alpha$ in $(0,\infty)\times(0,\infty)$ of the domain

$$\mathbf{D}_\alpha := \{(x,z) \,:\, 0 < \alpha z < x \le z\}. \tag{2.20}$$

By the same argument as in Remark 2.1,

$$J^\alpha(C,\theta) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} U(C_t) dt + \frac{U(0)}{\beta} e^{-\beta\tau}\right] \tag{2.21}$$

where

$$\tau := \inf\left\{t > 0 \,:\, X_t^{x,C,\theta} = \alpha Z_t^{x,z,C,\theta}\right\}.$$

In particular, this implies that

$$u^\alpha(x,z) = U(0)/\beta \quad \text{for} \quad (x,z) \in \overline{\mathbf{D}}_\alpha \setminus \mathbf{D}_\alpha. \tag{2.22}$$

We conclude this subsection by stating the following concavity property of the value function u^α, as observed in [19]. This argument can be skipped by the reader as it is not needed for the proof of our main result.

Lemma 2.1 *For any $z > 0$, the function $u^\alpha(.,z)$ is concave.*

Proof. Let $\nu \in [0,1]$ and a triplet (x, x', z) satisfying $(x,z) \in \overline{\mathbf{D}}_\alpha$ and $(x',z) \in \overline{\mathbf{D}}_\alpha$. Take $(C,\theta) \in \mathcal{A}_\alpha(x,z)$ and $(C',\theta') \in \mathcal{A}_\alpha(x',z)$. For any $t \geq 0$, we have

$$\begin{aligned}
\nu X_t^{x,C,\theta} + (1-\nu) X^{x',C',\theta'} &\geq \nu\, \alpha z \vee \left\{X^{x,C,\theta}\right\}_t^* + (1-\nu)\, \alpha z \vee \left\{X^{x',C',\theta'}\right\}_t^* \\
&\geq \alpha\, z \vee \left\{\nu X^{x,C,\theta} + (1-\nu) X^{x',C',\theta'}\right\}_t^* ,
\end{aligned}$$

so that, from the linearity of equation (2.5), we deduce

$$\left(\nu C + (1-\nu)C', \nu\theta + (1-\nu)\theta'\right) \in \mathcal{A}_\alpha\left(\nu x + (1-\nu)x', z\right) .$$

Now, since J^α defined in (2.17) inherits the concavity of U, we get

$$\nu J^\alpha(C,\theta) + (1-\nu)J^\alpha(C',\theta') \leq J^\alpha(\nu C + (1-\nu)C', \nu\theta + (1-\nu)\theta') \leq u^\alpha(\nu x + (1-\nu)x', z),$$

and taking the maximum over (C,θ) and (C',θ') concludes the proof. □

3 The main results

3.1 The corresponding dynamic programming equation

The optimal consumption-investment problem (2.17) is in the class of stochastic control problems studied in Barles, Daher and Romano [1]. The dynamic programming equation is related to the second order operator

$$\mathcal{L}u \ := \ \beta u - \sup_{C\geq 0, \theta\in\mathbb{R}} \left[U(C) + (\theta\sigma\lambda - C)u_x + \frac{\theta^2\sigma^2}{2} u_{xx}\right] . \qquad (3.1)$$

Defining the Legendre-Fenchel transform

$$V(y) \ := \ \sup_{x\geq 0} \ (U(x) - xy) \qquad (3.2)$$

and, recalling the concavity property of u^α stated in Lemma 2.1, the above dynamic programming equation simplifies to

$$\mathcal{L}u \ = \ \beta u - V(u_x) + \frac{\lambda^2}{2}\frac{u_x^2}{u_{xx}} \quad \text{whenever} \quad u \text{ is strictly concave.} \qquad (3.3)$$

with maximizers in (3.1) given by

$$\hat{C} \ = \ -V'(u_x) \ = \ \left(U'\right)^{-1}(u_x) \quad \text{and} \quad \hat{\theta} \ := \ -\frac{\lambda}{\sigma}\frac{u_x}{u_{xx}} . \qquad (3.4)$$

Under some convenient smoothness conditions, we expect the value function u^α to solve the following dynamic programming equation

$$\begin{aligned}
\mathcal{L}u^\alpha(x,z) &= 0, \quad \text{for} \quad (x,z) \in \mathbf{D}_\alpha ; & (3.5)\\
u^\alpha(\alpha z, z) &= 0, \quad \text{for} \quad z \geq 0 ; & (3.6)\\
u_z^\alpha(z,z) &= 0, \quad \text{for} \quad z > 0 . & (3.7)
\end{aligned}$$

We refer to [1] for the rigorous derivation of this dynamic programming equation in the viscosity sense. Since we will be using a verification argument in this paper, we only need to start from this partial differential equation, and "guess" a candidate solution for it.

3.2 The Fenchel-Legendre dual functions

The key-ingredient in order to derive the explicit solution in this paper is to introduce the Legendre-Fenchel transforms of the value function u^α with fixed z :

$$v^\alpha(y,z) \;\; := \;\; \sup_{x\geq 0} \left(u^\alpha(x,z) - xy\right) . \tag{3.8}$$

Since the value function u^α is concave in its first variable, it can indeed be recovered from v^α by the duality relation

$$u^\alpha(x,z) \;\; = \;\; \inf_{y\in\mathbb{R}} \left(v^\alpha(y,z) + xy\right) . \tag{3.9}$$

In the absence of drawdown constraint, the functions u^0 and v^0 are independent of the z variable and the dual function v^0 can be obtained explicitly in terms of the density of the risk-neutral measure. This can be seen by the following formal PDE argument: assuming that u^0 is smooth and satisfies the Inada conditions $(u^0)'(0+) = +\infty$, $(u^0)'(\infty) = 0$, it follows that

$$v^0(y) \;=\; u^0\left([(u^0)']^{-1}(y)\right) - y[(u^0)']^{-1}(y) \quad \text{for} \quad y\geq 0\,, \tag{3.10}$$

and $v^0(y) = \infty$ for $y < 0$. Substituting in the dynamic programming equation (3.5), it follows that v^0 solves on $(0,\infty)$ the linear parabolic partial differential equation

$$\mathcal{L}^* v(y) \;:=\; \beta v(y) - \beta y v_y(y) - \frac{\lambda^2}{2}\, y^2 v_{yy}(y) \;\;=\;\; V(y)\,. \tag{3.11}$$

Under a convenient transversality condition, this provides

$$v^0(y) \;=\; \mathbb{E}\left[\int_0^\infty e^{-\beta t} V\left(e^{\beta t} Y_t\right) dt\right] \quad \text{where} \quad Y_t \;:=\; y\exp\left(-\lambda W_t - \frac{1}{2}\,\lambda^2 t\right) . \tag{3.12}$$

In the particular case of a power utility function, this relation allows to derive explicitly v^0 and u^0 as detailed at the beginning of section 3.5. This result is well-known in the financial mathematics literature, and can be proved rigourously by probabilistic arguments, see e.g. [13].

In this complete market setting, it is remarkable that the Fenchel transform v^0 solves a linear PDE. This is the key-observation in order to guess a candidate solution for the optimal consumption-investment problem under drawdown constraint.

3.3 Assumptions

In this subsection, we collect the assumptions needed for our main result. Our first condition concerns the parameter

$$\gamma \;\;:=\;\; \frac{2\beta}{\lambda^2}\,.$$

Assumption 3.1 $\quad \dfrac{\gamma}{1+\gamma} < 1-\alpha.$

Observe that this condition is automatically satisfied when $\alpha = 0$. Under this condition, we may introduce the positive parameter

$$\delta := \frac{\gamma}{1-\alpha(1+\gamma)} \quad \text{so that} \quad \frac{\gamma}{1+\gamma} = (1-\alpha)\frac{\delta}{1+\delta}, \tag{3.13}$$

and we may express Assumption 3.1 in the equivalent form

$$\delta > 0. \tag{3.14}$$

Our next condition concerns the so-called asymptotic elasticity of the utility function U

$$\mathrm{AE}(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)},$$

as introduced by [15, 20].

Assumption 3.2 $\quad \mathrm{AE}(U) < \dfrac{\delta}{1+\delta}$.

In view of (3.14), Assumption 3.2 is stronger than the usual reasonable asymptotic elasticity condition. From Lemma 6.5 in [15], we deduce the existence of a constant K_0 such that

$$U(x) \le K_0\left(1+\frac{x^{\bar{p}}}{\bar{p}}\right), \quad x \ge 0, \quad \text{where} \quad p := \mathrm{AE}(U). \tag{3.15}$$

Furthermore, since U and V satisfy the relation

$$U(x) = V\left([-V']^{-1}(x)\right) + x\,[-V']^{-1}(x), \qquad x \ge 0,$$

where both terms on the right hand side are positive, it follows from (3.15) together with the fact that $U'(\infty) = 0$ that is

$$\limsup_{y\to 0} -V'(y)y^{\frac{1}{1-p}} < \infty \quad \text{and} \quad \limsup_{y\to 0} V(y)y^{\frac{p}{1-p}} < \infty.$$

In particular, this ensures the following integrability properties

$$\int_0^1 -V'(s)s^{\delta}ds < \infty, \quad \text{and} \quad \int_0^1 V(s)s^{\delta-1}ds < \infty. \tag{3.16}$$

Our final assumption on the utility function is

Assumption 3.3 $\quad \inf_{y>0}\left\{\dfrac{1}{yV''(y)}\displaystyle\int_0^y \frac{-V'(s)}{s}\left(\frac{s}{y}\right)^{1+\delta} ds\right\} > 0.$

Remark 3.1 Let Assumptions 3.1 and 3.2 hold. Then, Assumption 3.3 is satisfied whenever the relative risk aversion of U is uniformly bounded from below. Indeed, if there exist $C' > 0$ such that $-xU''(x) \ge C'U'(x)$ for any $x > 0$, then $C'yV''(y) \le -V'(y)$, for any $y > 0$, and the monotonicity of V' leads to Assumption 3.3.

3.4 Explicit solution under drawdown constraint

According to (3.16), under Assumptions 3.1 and 3.2, the function

$$g(\zeta) \;:=\; \frac{\delta}{\beta(1+\delta)} \left(\int_0^\zeta \frac{-V'(s)}{s} \left(\frac{s}{\zeta}\right)^{1+\delta} ds + \int_\zeta^\infty \frac{-V'(s)}{s}\, ds \right), \quad \zeta > 0\,, \tag{3.17}$$

is a well defined positive C^1 function from $(0,\infty)$ to $(0,\infty)$, with negative derivative

$$g'(\zeta) \;=\; -\frac{\delta}{\beta\zeta} \int_0^\zeta \frac{-V'(s)}{s} \left(\frac{s}{\zeta}\right)^{1+\delta} ds \;<\; 0\,, \qquad \zeta > 0\,. \tag{3.18}$$

We denote $\varphi := g^{-1}$ its inverse which is a C^1 decreasing positive function from $(0,\infty)$ to $(0,\infty)$ defined implicitly by the relation

$$z \;:=\; \frac{\delta}{\beta(1+\delta)} \left(\int_0^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds + \int_{\varphi(z)}^\infty \frac{-V'(s)}{s}\, ds \right), \quad z > 0\,. \tag{3.19}$$

We now introduce the function

$$\begin{aligned} h(y,z) \;&:=\; \alpha z + \frac{\gamma}{\beta(1+\gamma)} \left(\frac{\varphi(z)}{y}\right)^{1+\gamma} \int_0^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds \\ &\quad + \frac{\gamma}{\beta(1+\gamma)} \left\{ \int_{\varphi(z)}^y \frac{-V'(s)}{s} \left(\frac{s}{y}\right)^{1+\gamma} ds + \int_y^\infty \frac{-V'(s)}{s} ds \right\}, \quad y \geq \varphi(z)\,. \end{aligned} \tag{3.20}$$

Lemma 3.1 *Let Assumptions 3.1 and 3.2 hold. For any $z > 0$, the function $h(.,z)$ is invertible and its inverse denoted $f(.,z)$ is a strictly decreasing C^1 function from $(\alpha z, z]$ to $[\varphi(z),\infty)$ whose derivative satisfies*

$$-\frac{f_x(x,z)}{f(x,z)} \;=\; \left((\gamma+1)(x-\alpha z) + \frac{\gamma}{\beta} \int_{f(x,z)}^\infty \frac{V'(s)}{s} ds \right)^{-1}, \qquad (x,z) \in \mathbf{D}_\alpha\,. \tag{3.21}$$

Proof. Fix $z > 0$. The function $h(.,z)$ is C^1 on $(\varphi(z),\infty)$ and

$$h_y(y,z) \;=\; -\frac{\gamma}{\beta y} \left\{ \left(\frac{\varphi(z)}{y}\right)^{1+\gamma} \int_0^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds + \int_{\varphi(z)}^y \frac{-V'(s)}{s} \left(\frac{s}{y}\right)^{1+\gamma} ds \right\}$$

which is strictly negative. Therefore, since $h(\varphi(z),z) = z$ and $h(\infty,z) = \alpha z$, h is invertible and its inverse $f(.,z)$ is a strictly decreasing C^1 function from $(\alpha z, z]$ to $[\varphi(z),\infty)$. Simple computation then leads to (3.21). □

We now introduce our candidate feedback solutions for the consumption-investment problem:

$$\hat{C}(x,z) \;:=\; -[V' \circ f](x,z) \quad \text{and} \quad \hat{\theta}(x,z) \;:=\; \frac{\lambda}{\sigma} \left\{ (\gamma+1)(x-\alpha z) - \frac{\gamma}{\beta} \int_{f(x,z)}^\infty \frac{-V'(s)}{s} ds \right\} \tag{3.22}$$

for $(x,z) \in \mathbf{D}_\alpha$, and $\hat{C}(x,z) = \hat{\theta}(x,z) = 0$ on $\overline{\mathbf{D}}_\alpha \setminus \mathbf{D}_\alpha$.

Lemma 3.2 *Let Assumptions 3.1, 3.2 and 3.3 hold. Then, the functions $\hat{C}$ and $\hat{\theta}$ are Lipschitz on $\overline{\mathbf{D}}_\alpha$.*

The proof of this lemma requires precise regularity properties of the function f and is reported in Section 5.2. Given an initial condition $(x,z) \in \overline{\mathbf{D}}_\alpha$, we consider the stochastic differential equation

$$d\hat{X}_t \;=\; -\hat{C}(\hat{X}_t, \hat{Z}_t)dt + \hat{\theta}(\hat{X}_t, \hat{Z}_t)\sigma\left(dW_t + \lambda dt\right), \tag{3.23}$$

where we used the previous notation

$$\hat{Z}_t \;:=\; z \vee \hat{X}^*_t\,, \qquad t \geq 0\,.$$

Lemma 3.3 *Let Assumptions 3.1, 3.2 and 3.3 hold. Then the stochastic differential equation* (3.23) *has a unique strong solution $(\hat{X}, \hat{Z})$ for any initial condition $(x,z) \in \overline{\mathbf{D}}_\alpha$. Moreover the pair process*

$$(C^*, \theta^*) \;:=\; \left(\hat{C}, \hat{\theta}\right)(\hat{X}_t, \hat{Z}_t) \in \mathcal{A}_\alpha(x,z)$$

so that $\hat{X}_t$ satisfies the drawdown constraint (2.19).

Proof. We first extend continuously $\hat{C}$ and $\hat{\theta}$ to $\{(x,z) : x \leq z\}$ by setting them equal to zero, so that they remain Lipschitz, see Lemma 3.2. We shall denote by $K > 0$ a common Lipschitz constant. For a fixed z, we consider the map $G(t,\mathbf{x}) := \hat{C}\left(\mathbf{x}(t), z \vee \mathbf{x}^*(t)\right)$ defined on $\mathbb{R}_+ \times C^0(\mathbb{R}_+)$. Since $\hat{C}$ is Lipschitz, We directly estimate that

$$|G(t,\mathbf{x}) - G(t,\mathbf{y})| \;\leq\; K\left\{|\mathbf{x}(t) - \mathbf{y}(t)| + |z \vee \mathbf{x}^*(t) - z \vee \mathbf{y}^*(t)|\right\} \;\leq\; 2K\,|\mathbf{x} - \mathbf{y}|^*_t\,,$$

for $t \geq 0$ and $\mathbf{x}, \mathbf{y} \in C^0(\mathbb{R}_+)$. This proves that G is a functional Lipschitz function in the sense of Protter [18]. By a similar calculation, we also show that the diffusion coefficient of the stochastic differential equation (3.23) is also functional Lipschitz. The existence and uniqueness of a strong solution to (3.23) follows from Theorem 7 p197 in [18].

Finally, the functions $\hat{c}$ and $\hat{\pi}$ defined by

$$\hat{c}(x,z) \;:=\; \frac{\hat{C}(x,z)}{x - \alpha z} \quad \text{and} \quad \hat{\pi}(x,z) \;:=\; \frac{\hat{\theta}(x,z)}{x - \alpha z}\,, \qquad (x,z) \in \mathbf{D}_\alpha\,, \tag{3.24}$$

are bounded since $\hat{C}$ and $\hat{\theta}$ are Lipschitz functions satisfying, for any $z > 0$, $\hat{C}(\alpha z, z) = \hat{\theta}(\alpha z, z) = 0$. The functions $\hat{c}$ and $\hat{\pi}$ can be arbitrary extended to $\overline{\mathbf{D}}_\alpha$ so that the processes $\hat{c}(X_t, Z_t)$ and $\hat{\pi}(X_t, Z_t)$ are well defined and bounded for $(X_t, Z_t) \in \overline{\mathbf{D}}_\alpha$. Following the same argument as in Section 2.2, this implies in particular that $(C^*, \theta^*) \in \mathcal{A}_\alpha(x,z)$. □

We are now ready for the statement of our main result.

Theorem 3.1 *Let Assumptions 3.1, 3.2, and 3.3 hold. Then, $u^\alpha = U(0)/\beta$ on $\overline{\mathbf{D}}_\alpha \setminus \mathbf{D}_\alpha$,*

$$u^\alpha(x,z) \;=\; f(x,z)\left(\frac{\gamma+1}{\gamma}\,(x - \alpha z) \;+\; \frac{1}{\beta}\int_{f(x,z)}^{\infty} \frac{V(s)}{s^2}ds\right)\,,\;\; (x,z) \in \mathbf{D}_\alpha\,, \tag{3.25}$$

and the consumption-investment strategy (C^*, θ^*) is an optimal solution of the problem u^α.

Moreover, u^α is a $C^0\left(\overline{\mathbf{D}}_\alpha\right) \cap C^{2,1}\left(\mathbf{D}_\alpha\right)$ function, and the corresponding dual function v^α defined in (3.8) *is given by*

$$v^\alpha(y,z) \;=\; \begin{cases} y\left(-\alpha z + \dfrac{1}{\gamma} h(y,z) + \dfrac{1}{\beta}\displaystyle\int_y^\infty \frac{V(s)}{s^2} ds\right) & \text{for} \quad y \geq \varphi(z) \\ v^\alpha\left(\varphi(z), z\right) + z\left(\varphi(z) - y\right) & \text{for} \quad y \leq \varphi(z) \end{cases}$$

The proof of this result is reported in Section 5, and relies on a verification argument which requires to guess the explicit form of the theorem. The construction of the candidate explicit solution is provided for completeness in Section 4.

3.5 The power utility case

In the absence of drawdown constraint, the value function associated to a power utility function and its Fenchel transform are well-known to be explicit. The main result of this section is that, under the drawdown constraint, the Fenchel transform of the value function associated to a power utility function is completely explicit, and the expressions of the optimal strategy and the value function are considerably simplified.

A power utility function is characterized by its asymptotic elasticity $p \in (0,1)$ and is given by

$$U_p(x) \;:=\; \frac{x^p}{p}, \qquad x > 0,$$

Its Fenchel transform satisfies

$$V_p(y) \;=\; \frac{y^{-q}}{q}, \quad y > 0, \quad \text{with} \quad \frac{1}{p} - \frac{1}{q} = 1.$$

We first recall briefly the solution of the Merton problem in the absence of the drawdown constraint. From section 3.2, under a convenient transversality condition, the Fenchel transform v_p^0 of the value function u_p^0 is given by (3.12). One immediately checks that, under the so called Merton condition

$$\frac{\gamma}{1+\gamma} > p, \tag{3.26}$$

the Fenchel tranform v_p^0 is given by

$$v_p^0(y) \;=\; \frac{(1-p)^3}{\beta p}\left(1 - \frac{1+\gamma}{\gamma} p\right)^{-1} y^{\frac{p}{p-1}} \;<\; \infty, \qquad y > 0,$$

and the value function u_p^0 is obtained by direct calculation from (3.9),

$$u_p^0(x) \;=\; \left[\frac{\beta}{(1-p)^2}\left(1 - \frac{1+\gamma}{\gamma} p\right)\right]^{p-1} \frac{x^p}{p}, \qquad x > 0.$$

The optimal consumption-investment strategy is identified as the maximizer in the dynamic programming equation (3.1), and given by $\hat{C}(x) = c_0^* x$ and $\hat{\theta}(x) = \pi_0^* x$, where

$$c_0^* \;:=\; \frac{\beta}{(1-p)^2}\left(1 - \frac{1+\gamma}{\gamma} p\right), \qquad \pi_0^* \;:=\; \frac{\lambda}{\sigma(1-p)}. \tag{3.27}$$

We now turn to the solution of the optimal consumption-investment problem under drawdown constraint. Let

$$b_\alpha \;:=\; \frac{\beta}{(1-p)^2}\left(1-\frac{1+\delta}{\delta}p\right). \tag{3.28}$$

Observe that the optimal consumption rate in the Merton problem without drawdown constraint is $c_0^* = b_0$, since $\delta = \gamma$ whenever $\alpha = 0$. Notice also from (3.13) that Assumption 3.2 which rewrites

$$b_\alpha > 0\,, \quad \text{i.e.} \quad (1-\alpha)p \;<\; \frac{\gamma}{1+\gamma}\,,$$

is weaker than the Merton condition (3.26), and reduces to it when $\alpha = 0$. Since the relative risk aversion of the power utility function U_p is a positive constant, Assumption 3.3 is always satisfied under Assumptions 3.1 and 3.2, see Remark 3.1.

The main observation for the particular case of a power utility function, is that the function φ, defined as the inverse of g given by (3.17) is fully explicit:

$$\varphi(z) \;=\; U_p'(b_\alpha z) = (b_\alpha z)^{p-1}\,, \qquad z > 0\,.$$

Furthermore, the value function u_p^α inherits the homogeneity property from the power utility function U_p, so that

$$u_p^\alpha(x,z) \;=\; z^p\, u_p^\alpha\left(\frac{x}{z},1\right), \qquad (x,z)\in \mathbf{D}_\alpha\,. \tag{3.29}$$

Therefore, the function $\hat{C}$ defined in (3.4) satisfies

$$\hat{C}(x,z) \;=\; -V_p'\left(z^{p-1}\,\nabla_x u_p^\alpha\left(\frac{x}{z},1\right)\right) \;=\; -z\,V_p'\left(\nabla_x u_p^\alpha\left(\frac{x}{z},1\right)\right) \;=\; z\,\hat{C}\left(\frac{x}{z},1\right),$$

for $(x,z)\in\mathbf{D}_\alpha$, where $\nabla_x u_p^\alpha$ denotes the derivative of u_p^α with respect to its first component. As a consequence, the function $(x,z)\mapsto -[V_p'\circ f](x,z)/(x-\alpha z)$ reduces to a function of the single variable x/z. Direct calculation reveals that this function is the inverse of the function F defined by

$$F(\xi) \;:=\; \alpha + \frac{b_\alpha}{\xi}\left(\frac{1-\frac{b_0}{\xi}}{1-\frac{(1-\alpha)b_0}{b_\alpha}}\right)^{\frac{\lambda^2}{2(1-p)^2}b_0^{-1}} \tag{3.30}$$

which is a C^1 function from $[b_0^+, b_\alpha/(1-\alpha)]$ to $[\alpha,1]$. By passing to the limit in (3.30), we observe that

$$F(\xi) \;=\; \alpha + \frac{b_\alpha}{\xi}\exp\left[\frac{1}{\alpha\gamma}\left(1-\alpha-\frac{b_\alpha}{\xi}\right)\right] \quad \text{whenever} \quad b_0 \;=\; 0\,. \tag{3.31}$$

Indeed, under Assumptions 3.1 and 3.2, F is strictly increasing so that its inverse F^{-1} is well defined and a strictly increasing continuous function from $[\alpha,1]$ to $[b_0^+, b_\alpha/(1-\alpha)]$. The functions $\hat{c}$ and $\hat{\pi}$ defined in (3.24) are now given by

$$\hat{c}_p(x,z) \;:=\; F^{-1}\left(\frac{x}{z}\right) \quad \text{and} \quad \hat{\pi}_p(x,z) \;:=\; \frac{\lambda}{\sigma}(\gamma+1) - \frac{2}{\sigma\lambda}(1-p)F^{-1}\left(\frac{x}{z}\right), \quad (x,z)\in\overline{\mathbf{D}}_\alpha\,.$$

As in lemma 3.3, under Assumptions 3.1 and 3.2, the stochastic differential equation

$$d\hat{X}_t \;=\; \left(\hat{X}_t - \alpha\hat{Z}_t\right)\left[-\hat{c}_p\left(\hat{X}_t,\hat{Z}_t\right)dt + \hat{\pi}_p\left(\hat{X}_t,\hat{Z}_t\right)\sigma\left(dW_t+\lambda dt\right)\right],$$

has a unique strong solution $(\hat{X},\hat{Z})$ for any initial condition $(x,z)\in\overline{\mathbf{D}}_\alpha$ and the pair process

$$\left(C_p^*,\theta_p^*\right) \;:=\; (\hat{X}-\alpha\hat{Z})\left(\hat{c}_p(\hat{X},\hat{Z}),\hat{\pi}_p(\hat{X},\hat{Z})\right) \;\in\; \mathcal{A}_\alpha(x,z)\,.$$

For completeness, we restate Theorem 3.1 in the context of a power utility function

Theorem 3.2 *Let $U=U_p$, Assumptions 3.1 and 3.2 hold. Then $u_p^\alpha=0$ on $\overline{\mathbf{D}}_\alpha\setminus\mathbf{D}_\alpha$,*

$$u_p^\alpha(x,z) \;:=\; \left(\frac{\gamma+1}{\gamma}+\frac{(1-p)^2}{\beta p}F^{-1}\left(\frac{x}{z}\right)\right)\left[F^{-1}\left(\frac{x}{z}\right)\right]^{p-1}(x-\alpha z)^p\,, \qquad (x,z)\in\mathbf{D}_\alpha\,,$$

and the consumption-investment strategy $\left(C_p^*,\theta_p^*\right)$ is an optimal solution of the problem u^α.

Furthermore, u_p^α is a $C^0\left(\overline{\mathbf{D}}_\alpha\right)\cap C^{2,1}\left(\mathbf{D}_\alpha\right)$ function, and the corresponding dual function v_p^α is given by

$$v_p^\alpha(y,z) \;=\; \begin{cases} -\alpha zy - \dfrac{\alpha(b_\alpha z)^p}{b_\alpha\left(\gamma-(1+\gamma)p\right)}\left(\dfrac{(b_\alpha z)^{p-1}}{y}\right)^\gamma + \dfrac{1-p}{p\,b_0}\,y^{-\frac{p}{1-p}} & \text{for} \quad y\geq(b_\alpha z)^{p-1} \\ v^\alpha\left((b_\alpha z)^{p-1},z\right)+z\left((b_\alpha z)^{p-1}-y\right) & \text{for} \quad y\leq(b_\alpha z)^{p-1} \end{cases}$$

The above solution agrees with the candidate solution derived by [19] in the case of possibly positive interest rates. Therefore, Theorem 3.2 confirms that the candidate solution derived by [19] is indeed the solution of the optimal consumption-investment problem.

3.6 Properties of the solution

In this subsection, we analyse the behavior of an agent maximizing its lifetime power utility of consumption under the drawdown constraint (2.19). The particular case of a power utility function enables us to compare our solution to the well-known benchmark Merton solution in the absence of drawdown constraint. Remark furthermore that, since the value functions u_p^α and the consumption-investment strategy (C_p,θ_p) inherit the homogeneity properties of U_p and V_p, all the evaluations and comparisons can be realized in terms of fraction of wealth x/z. The results presented here are similar to the ones observed by Roche [19] and are reported here for completeness.

Considering a particular set of parameters $\{p,\sigma,\lambda,\beta\}=\{0.2,1,3,3\}$ satisfying the Merton condition (3.26), we report the value functions and optimal consumption-investment strategies associated to different values of α satisfying Assumption 3.1. Of course, the results observed when α reaches zero coincide with the benchmark Merton one. Because these three functions equal zero whenever the drawdown constraint binds, the reader can easily identify in each of the figures the slopes associated to the different values of α.

We first observe in Figure 1 that the amount of wealth invested in the risky asset decreases with α. Nevertheless, when the drawdown constraint nearly binds, the marginal investment strategy does not depend on α. But, as the fraction of wealth increases, the agent is more reluctant to investment in the risky asset as α increases. Finally, when the wealth process

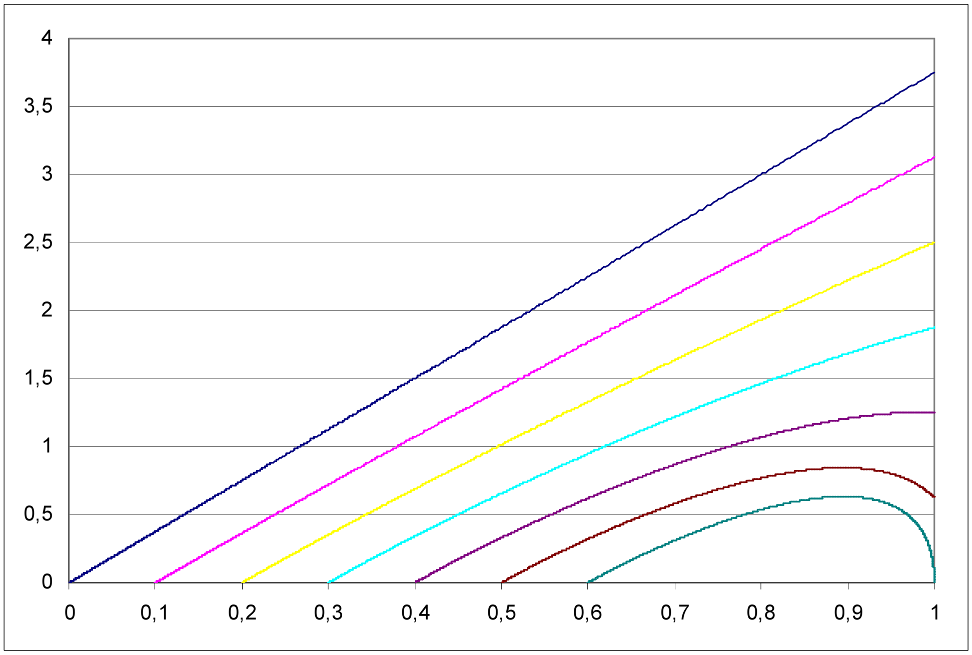

Figure 1: Investment θ_p versus the fraction of wealth x/z for $\alpha = 0$ to 0.6

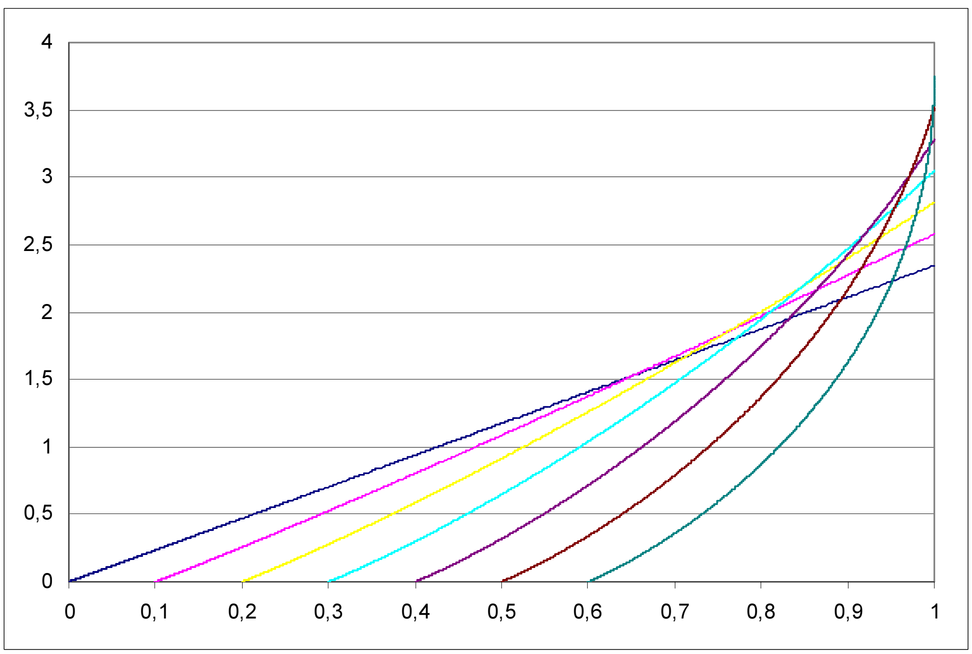

Figure 2: Consumption C_p versus the fraction of wealth x/z for $\alpha = 0$ to 0.6

approaches its maximum, the amount invested in the risky asset even decreases for α high enough. Conversely, the consumption of the agent reported in Figure 2 is decreasing in α when the proportion of wealth is close to the drawdown constraint but increases with α whenever the wealth process approaches its current maximum.

The key intuition behind those observations is the anticipation of the agent to the possibility that the drawdown constraint may be binding in the future. Therefore its aversion to risk increases and this explains why its investment and consumption strategy decrease with α. The particular behavior of the optimal strategy of the agent when its wealth approaches its current maximum relies in the ratcheting feature of the drawdown constraint. The agent anticipates that reaching its current maximum of wealth will increase the floor imposed by the drawdown constraint, and therefore chooses to consume instead of investing in the risky asset. Remark that, considering an agent maximizing the long term growth rate of expected utility of its final wealth, the optimal investment strategy derived by Grossman and Zhou [10] is conversely always linearly increasing with the fraction of wealth.

Finally Figure 3 shows the dependence of the value function u^{α} in terms of α. Since the set of possible consumption-investment strategies decreases with α, u_{α} is decreasing in α. This effect, due to the drawdown constraint, decreases with the proximity of the wealth to its current maximum.

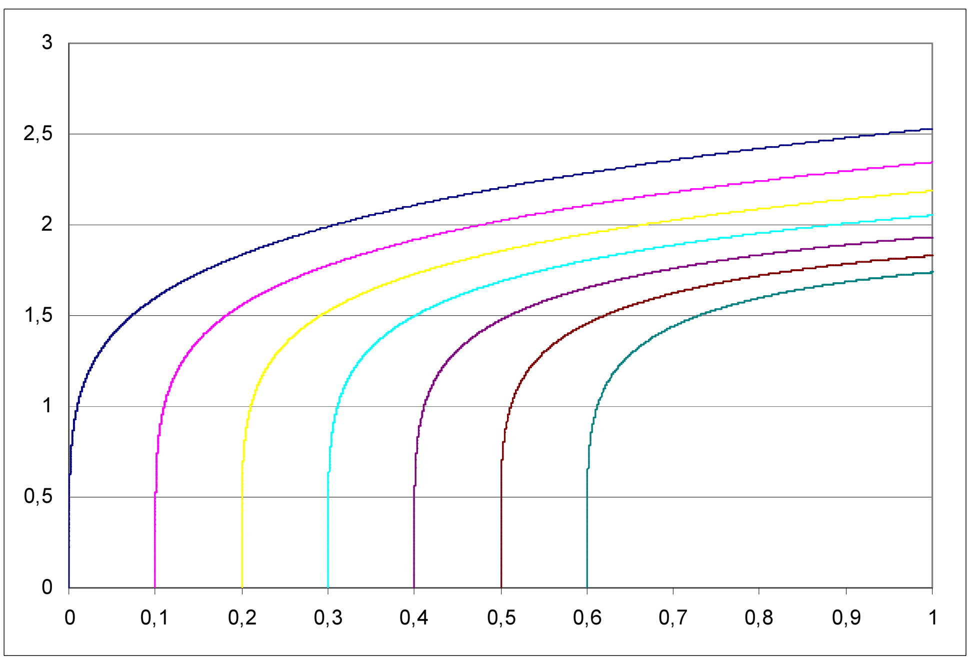

Figure 3: Value function u_p^{α} versus the fraction of wealth x/z for $\alpha = 0$ to 0.6

4 Guessing a candidate solution for the dual function

In this section, we show with a formal argument how the dual function v^α can be guessed. We shall assume throughout that, for any $z > 0$,

$$u^\alpha(.,z) \qquad \text{is a smooth increasing function.} \tag{4.1}$$

From the discussion of Section 3.1, the dynamic programming equation for the value function u^α is

$$\mathcal{L}u^\alpha \;:=\; \beta u - V(u_x) + \frac{\lambda^2}{2}\frac{u_x^2}{u_{xx}} \;.=\; 0\,, \qquad (x,z) \in \mathbf{D}_\alpha\,; \tag{4.2}$$

$$u^\alpha(\alpha z, z) \;=\; U(0)/\beta\,, \qquad z \geq 0\,; \tag{4.3}$$

$$u_z^\alpha(z,z) \;=\; 0\,, \qquad z > 0\,. \tag{4.4}$$

<u>Step 1:</u> *The PDE satisfied by v^α.*

We first introduce the functions

$$\varphi(z) \;:=\; u_x^\alpha(z,z) \quad \text{and} \quad \psi(z) \;:=\; u_x^\alpha(\alpha z, z)\,, \quad z > 0\,.$$

For any $z > 0$, by the concavity property of $u^\alpha(.,z)$, see Lemma 2.1, we have $\varphi(z) \leq \psi(z)$. From the definition of the dual function v^α, we have

$$v^\alpha(y,z) \;=\; u^\alpha\left(\hat{x}(y,z),z\right) - \hat{x}(y,z)y \quad \text{if} \quad u_x^\alpha\left(\hat{x}(y,z),z\right) = y \in [\varphi(z),\psi(z)]\;, \tag{4.5}$$

$$v^\alpha(y,z) \;=\; u^\alpha(z,z) - yz \quad \text{if} \quad y \leq \varphi(z)\,, \tag{4.6}$$

$$v^\alpha(y,z) \;=\; u^\alpha(\alpha z,z) - \alpha yz \;=\; U(0)/\beta - \alpha yz \quad \text{if} \quad y \geq \psi(z)\,, \tag{4.7}$$

where the last equality follows from (4.3). In the situation of (4.5) where $y \in [\varphi(z), \psi(z)]$, we obtain by a direct change of variable in (4.2) that

$$\mathcal{L}^* v^\alpha(y,z) \;=\; V(y) \quad \text{for} \quad \varphi(z) < y < \psi(z)\,, \tag{4.8}$$

where $\mathcal{L}^*$ is the linear operator defined in (3.11). We also observe that the Neumann boundary condition (4.4) is converted into

$$v_z^\alpha(y,z) \;=\; \varphi(z) - y \quad \text{for} \quad y \leq \varphi(z)\,. \tag{4.9}$$

<u>Step 2:</u> *From the Neumann condition to a Dirichlet condition.*

Let introduce the function

$$w^\alpha(y,z) \;:=\; v_z^\alpha(y,z) \quad \text{for} \quad z > 0 \text{ and } \varphi(z) < y < \psi(z)\,. \tag{4.10}$$

Since $\mathcal{L}^*$ is a linear operator, it follows that w^α satisfies

$$\mathcal{L}^* w^\alpha \;=\; \beta w^\alpha - \beta y w_y^\alpha - \frac{\lambda^2}{2} y^2 w_{yy}^\alpha \;=\; 0\,, \qquad \varphi(z) < y < \psi(z)\,. \tag{4.11}$$

Condition (4.7) and the Neumann condition (4.9) on v^α provide the following Dirichlet conditions on w^α,

$$w^\alpha\left(\varphi(z),z\right) \;=\; 0 \;\text{ and }\; w^\alpha\left(\psi(z),z\right) \;=\; -\alpha\psi(z)\,, \qquad z>0\,. \tag{4.12}$$

For every fixed $z>0$, the system (4.11)-(4.12) has a unique C^2 solution $w^\alpha(.,z)$ given by

$$w^\alpha(y,z) \;=\; -\alpha y\left(1-\left(\frac{\varphi(z)}{\psi(z)}\right)^{1+\gamma}\right)^{-1}\left(1-\left(\frac{\varphi(z)}{y}\right)^{1+\gamma}\right), \qquad \varphi(z)<y<\psi(z) \tag{4.13}$$

<u>*Step 3:*</u> *Infinite marginal utility when the drawdown constraint nearly binds.*

Since we will be using a verification argument, we just need to find a solution to the dynamic programming equation (4.2)-(4.3)-(4.4). We then seek for a candidate solution satisfying

$$\psi(z) \;=\; u_x^\alpha(\alpha z,z) \;=\; +\infty\,, \qquad z>0.$$

From the economic viewpoint, this means that the marginal indirect utility is infinite when the wealth process approaches the drawdown constraint. This is understandable as the amounts of consumption and investment reduce to zero for the remaining lifetime whenever the drawdown constraint binds, i.e. $X_t=\alpha Z_t$, see Remark 2.1. So, any small departure from this constraint is very important for the investor as investment on the financial market and consumption are again possible. In this case, (4.13) reduces to

$$w^\alpha(y,z) \;=\; -\alpha y\left(1-\left(\frac{\varphi(z)}{y}\right)^{1+\gamma}\right), \qquad \varphi(z)<y\,. \tag{4.14}$$

<u>*Step 4:*</u> *Derivation of a generic form for v_y^α.*

Integrating (4.14) with respect to z leads to

$$v^\alpha(y,z) = -\alpha yz + \alpha y\int_{z_0}^{z}\left(\frac{\varphi(s)}{y}\right)^{1+\gamma}ds + \phi(y)\,, \qquad \varphi(z)<y\,,$$

where z_0 and $\phi(.)$ are still to be determined. Differentiating now with respect to y, we get

$$v_y^\alpha(y,z) = -\alpha z - \alpha\gamma\int_{z_0}^{z}\left(\frac{\varphi(s)}{y}\right)^{1+\gamma}ds + \phi'(y)\,, \qquad \varphi(z)<y\,, \tag{4.15}$$

with the two boundary conditions $v_y^\alpha(\varphi(z),z)=-z$ and $v_y^\alpha(\infty,z)=-\alpha z$ given respectively by (4.6) and (4.7). In order to determine ϕ', we observe from (4.8), that ϕ satisfies an ordinary differential equation which provides, after differentiation with respect to y,

$$(\gamma+2)\phi'''(y)+y\phi''(y) \;=\; -\frac{\gamma}{\beta}\frac{V'(y)}{y}\,, \qquad \varphi(z)<y\,.$$

We deduce

$$\phi''(y) \;=\; -\frac{\gamma}{\beta y}\int_{y_0}^{y}\frac{V'(s)}{s}\left(\frac{s}{y}\right)^{1+\gamma}ds\,, \qquad \varphi(z)<y\,,$$

with y_0 a constant to be determined. Integrating with respect to y, we obtain the expression of ϕ' up to a constant which is fixed by the boundary condition $\phi'(\infty) = 0$ given by $v^\alpha_y(\infty, z) = -\alpha z$. Reporting this expression in (4.15), we finally get

$$v^\alpha_y(y,z) = -\alpha z - \alpha\gamma \int_{z_0}^{z} \left(\frac{\varphi(s)}{y}\right)^{1+\gamma} ds + \frac{\gamma}{\beta(1+\gamma)} \int_{y_0}^{y} \frac{V'(s)}{s} \left(\frac{s \wedge y}{y}\right)^{1+\gamma} ds \qquad (4.16)$$

for $\varphi(z) < y$, with the boundary condition $v^\alpha_y(\varphi(z), z) = -z$.

<u>Step 5:</u> *Implicit obtention of the marginal utility $\varphi(z)$.*

The function $\varphi(z)$ will be implicitly given by the boundary condition $v^\alpha_y(\varphi(z), z) = -\alpha z$. Rewriting the boundary condition according to (4.16) and differentiating with respect to z, we compute

$$\varphi'(z)\, v^\alpha_{yy}(\varphi(z), z) \;=\; -\frac{\gamma}{\delta}, \qquad z > 0\,. \qquad (4.17)$$

Assuming that φ is invertible and denoting g its inverse, we notice that (4.17) rewrites as an ordinary differential equation satisfied by g

$$(1+\delta)g(\zeta) + \zeta\, g'(\zeta) = \frac{\delta}{\beta} \int_{\zeta}^{\infty} \frac{-V'(s)}{s} ds\,, \qquad \zeta > 0\,,$$

whose solution is explicitly given by

$$g(\zeta) = \frac{\delta}{\beta(1+\delta)} \left(\int_{\zeta_0}^{\zeta} \frac{-V'(s)}{s} \left(\frac{s}{\zeta}\right)^{1+\delta} ds + \int_{\zeta}^{\infty} \frac{-V'(s)}{s} ds \right)\,,\ \zeta > 0\,, \qquad (4.18)$$

with ζ_0 a constant to be determined. From (3.13), $\delta/(1+\delta) > 0$ and since we require g to be a positive function, ζ_0 must be 0 or ∞ depending on the sign of δ. Nevertheless, in both cases, direct computation shows that g' and then φ' are negative. Since we require the dual function v^α to be convex, equation (4.17) imposes $\delta > 0$ which corresponds to assumption 3.1. Therefore $\zeta_0 = 0$ and g coincides with (3.17) which is well-defined under Assumption 3.2, see (3.16). Therefore the function $\varphi(z)$ is implicitly defined by the relation

$$z = \frac{\delta}{\beta(1+\delta)} \left(\int_{0}^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\zeta}\right)^{1+\delta} ds + \int_{\varphi(z)}^{\infty} \frac{-V'(s)}{s} ds \right)\,,\ z > 0\,. \qquad (4.19)$$

<u>Step 6:</u> *Deducing the dual function v^α.*

Now, combining (4.16), (4.19) and the boundary condition $v^\alpha_y(\varphi(z), z) = -z$, we compute

$$\alpha(1+\gamma) \int_{z_0}^{z} \left(\frac{\varphi(s)}{\varphi(z)}\right)^{1+\gamma} ds - \frac{1}{\beta} \int_{y_0}^{\varphi(z)} \frac{V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\gamma} ds = -\frac{1}{\beta} \int_{0}^{\varphi(z)} \frac{V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds\,,$$

for $z > 0$, which reported in (4.16), leads to

$$\begin{aligned} v^\alpha_y(y,z) \;=\; & -\alpha z - \frac{\gamma}{\beta(1+\gamma)} \left(\frac{\varphi(z)}{y}\right)^{1+\gamma} \int_{0}^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds \\ & - \frac{\gamma}{\beta(\gamma+1)} \left(\int_{\varphi(z)}^{y} \frac{-V'(s)}{s} \left(\frac{s}{y}\right)^{1+\gamma} ds + \int_{y}^{\infty} \frac{-V'(s)}{s} ds \right)\,, \qquad \varphi(z) < y\,. \end{aligned}$$

Starting from this expression of v_y^α, the ordinary differential equation (4.8) directly leads to the expression of v^α announced in Theorem 3.1. In order to deduce the value function u^α, we simply need, for any $z>0$, to invert the function $v_y^\alpha(.,z)$, which corresponds to inverting the function $h(.,z)$ defined in (3.20).

Remark 4.1 In the particular case of the power utility function, u_p^α inherits the homogeneity property of U_p so that $\varphi(z)=\varphi(1)z^{p-1}$. Therefore, we can skip step 5 and $\varphi(1)$ is explicitly determined by the boundary condition $v_y^\alpha(\varphi(1),1)=-1$.

5 The verification argument

This section is devoted to the proof of Lemma 3.2 and Theorem 3.1.

5.1 A general version of the verification theorem

We recall the definition of the operator $\mathcal{L}$:

$$\mathcal{L}u \;=\; \beta u - \sup_{C\geq 0,\theta\in\mathbb{R}} \left\{U\left(C\right)+\mathcal{L}^{C,\theta}u\right\} \quad \text{where} \quad \mathcal{L}^{C,\theta}u \;:=\; \frac{1}{2}\theta^2\sigma^2 u_{xx}+(\theta\sigma\lambda-C)u_x\,.$$

We first derive a general verification theorem adapted to our maximization under drawdown constraint problem.

Theorem 5.1 *Let ψ be a $C^0\left(\overline{\mathbf{D}}_\alpha\right)\cap C^{2,1}\left(\mathbf{D}_\alpha\right)$ function.*
(i) *If ψ satisfies $\mathcal{L}\psi\geq 0$ and $-\psi_z(z,z)\geq 0$, then $\psi\geq u^\alpha$.*
(ii) *Assume in addition that*
(a) *$\mathcal{L}\psi=0$, $\psi(\alpha z,z)=U(0)/\beta$ and $-\psi_z(z,z)=0$;*
(b) *there exist $K>0$ and $0<p_0<\delta/(1+\delta)$ such that*

$$\psi(x,z) \;\leq\; K\left(1+z^{\alpha p_0}(x-\alpha z)^{(1-\alpha)p_0}\right), \qquad (x,z)\in\overline{\mathbf{D}}_\alpha\,;$$

(c) *$\mathcal{L}\psi \;=\; \beta\psi-U(\tilde{C})+\mathcal{L}^{\tilde{C},\tilde{\theta}}\psi$ where $\tilde{C}(x,z)=(x-\alpha z)\tilde{c}(x,z)$, $\tilde{\theta}(x,z)=(x-\alpha z)\tilde{\pi}(x,z)$, and the stochastic differential equation*

$$d\tilde{X}_t \;\;=\;\; -\tilde{C}(\tilde{X}_t,\tilde{Z}_t)dt+\sigma\,\tilde{\theta}(\tilde{X}_t,\tilde{Z}_t)\,(dW_t+\lambda dt) \quad t\geq 0\,,$$

has a unique strong solution $(\tilde{X},\tilde{Z})$ for any initial condition $(\tilde{X}_0,\tilde{Z}_0)=(x,z)\in\overline{\mathbf{D}}_\alpha$ satisfying

$$\int_0^T \tilde{c}(\tilde{X}_t,\tilde{Z}_t)dt<\infty \;\; a.s. \quad \text{and} \quad ||\tilde{\pi}(\tilde{X}_.,\tilde{Z}_.)||_\infty<\infty\,.$$

Then $\psi \;=\; u^\alpha$.

Proof. We first observe that $\mathcal{L}\psi\geq 0$ implies

$$\beta\psi \;\geq\; V(\psi_x) \;\;\geq\;\; U(0)\,, \tag{5.1}$$

since V is a decreasing function and $V(\infty) = U(0)$. For $(x,z) \in \overline{\mathbf{D}}_\alpha \backslash \mathbf{D}_\alpha$, we have $u^\alpha(x,z) = U(0)/\beta$, and therefore the statement of the theorem is trivial. From now on, we fix a pair $(x,z) \in \mathbf{D}_\alpha$.

(i) Let (C,θ) be an arbitrary admissible consumption-investment strategy in $\mathcal{A}_\alpha(x,z)$, set $(X,Z) := \left(X^{x,C,\theta}, Z^{x,z,C,\theta}\right)$ the solution of (2.18) with initial condition $(X_0,Z_0) = (x,z)$, and define the sequence of stopping times

$$\tau_n \;:=\; \inf\left\{t > 0 \;:\; X_t - \alpha Z_t < n^{-1}\right\}.$$

By Itô's formula, we obtain

$$\begin{aligned} e^{-\beta T\wedge\tau_n}\psi\left(X_{T\wedge\tau_n}, Z_{T\wedge\tau_n}\right) \;&=\; \psi(x,z) + M_T + \int_0^{T\wedge\tau_n} e^{-\beta t}\psi_z(X_t,Z_t)dZ_t \\ &\quad + \int_0^{T\wedge\tau_n} e^{-\beta t}\left[\mathcal{L}^{C_t,\theta_t}\psi - \beta\psi\right](X_t,Z_t)dt \end{aligned}$$

where

$$M_T \;:=\; \int_0^{T\wedge\tau_n} e^{-\beta t}\theta_t\sigma\psi_x(X_t,Z_t)dW_t\,, \qquad T \geq 0\,.$$

Since $-\psi_z(z,z) \geq 0$, Z is an increasing process and $dZ_t = 0$ whenever $X_t < Z_t$, it follows that the integral term with respect to Z is non-negative. Using in addition the fact that $\mathcal{L}\psi \geq 0$, we get

$$\psi(x,z) \;\geq\; e^{-\beta T\wedge\tau_n}\psi\left(X_{T\wedge\tau_n}, Z_{T\wedge\tau_n}\right) + \int_0^{T\wedge\tau_n} e^{-\beta t}U(C_t)dt - M_T\,. \tag{5.2}$$

Recall that ψ_x is continuous on $\mathbf{D}_\alpha$. Then, it follows from the definition of τ_n that the stopped process $\psi_x(X,Z)$ is a.s. continuous on $[0, T\wedge\tau_n]$. Since $\int_0^T \theta_t^2 dt < \infty$, this implies that M is a local martingale. By the lower bound (5.1) on ψ, it follows from (5.2) that M is uniformly bounded from below. Then M is a supermartingale. Taking expected values in (5.2), and using again the lower bound (5.1) on ψ, this implies that

$$\psi(x,z) \;\geq\; \mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-\beta t}U(C_t)dt + \frac{U(0)}{\beta}\,e^{-\beta T\wedge\tau_n}\right].$$

By the monotone convergence theorem together with Remark 2.21, this implies that

$$\psi(x,z) \;\geq\; \mathbb{E}\left[\int_0^{\tau_\infty} e^{-\beta t}U(C_t)dt + \frac{U(0)}{\beta}\,e^{-\beta\tau_\infty}\right] \;=\; \mathbb{E}\left[\int_0^{\infty} e^{-\beta t}U(C_t)dt\right],$$

which proves that $\psi(x,z) \geq u^\alpha(x,z)$ by the arbitrariness of $(C,\theta) \in \mathcal{A}_\alpha(x,z)$.

(ii) For simplicity, we denote $(\tilde{C}_t, \tilde{\theta}_t, \tilde{c}_t, \tilde{\pi}_t) := (\tilde{C}, \tilde{\theta}, \tilde{c}, \tilde{\pi})(\tilde{X}_t, \tilde{Z}_t)$, for any $t \geq 0$. By the same argument as in (2.10), we have

$$(\tilde{X}_t - \alpha\tilde{Z}_t)\tilde{Z}_t^{\alpha/(1-\alpha)} \;=\; \exp\left\{-\int_0^t \sigma\tilde{\pi}_r dW_r - \int_0^t (\tilde{c}_r + \left(-\lambda\sigma\tilde{\pi}_r + \frac{(\sigma\tilde{\pi}_r)^2}{2}\right)dr\right\}. \tag{5.3}$$

In particular, this implies that the sequence of stopping times

$$\tilde{\tau}_n \;:=\; \inf\left\{t > 0 \;:\; \tilde{X}_t - \alpha \tilde{Z}_t < n^{-1} \;\text{ or }\; \tilde{Z}_t > n\right\} \quad\longrightarrow\quad \infty \text{ a.s.}$$

Since we have $\beta\psi - U(\tilde{C}) - \mathcal{L}^{\tilde{C},\tilde{\theta}}\psi = 0$, it follows from Itô's lemma that

$$\psi(x,z) \;=\; e^{-\beta T\wedge\tilde{\tau}_n}\psi\left(\tilde{X}_{T\wedge\tilde{\tau}_n}, \tilde{Z}_{T\wedge\tilde{\tau}_n}\right) + \int_0^{T\wedge\tilde{\tau}_n} e^{-\beta t}U(\tilde{C}_t)dt - \tilde{M}_T \tag{5.4}$$

where

$$\tilde{M}_T \;:=\; \int_0^{T\wedge\tilde{\tau}_n} e^{-\beta t}\sigma[\tilde{\theta}\psi_x](\tilde{X}_t, \tilde{Z}_t)dW_t\,, \qquad T \geq 0\,.$$

Since ψ_x is continuous on $\mathbf{D}_\alpha$, and the stopped process $(\tilde{X}, \tilde{Z})$ takes values in a compact subset of $\mathbf{D}_\alpha$, it follows that the process $\psi_x(\tilde{X}, \tilde{Z})$ is uniformly bounded on $[0, \tilde{\tau}_n]$. Using the boundedness of the process $\tilde{\pi}$, we deduce that $\tilde{M}$ is a martingale, and

$$\psi(x,z) \;=\; \mathbb{E}\left[e^{-\beta T\wedge\tilde{\tau}_n}\psi\left(\tilde{X}_{T\wedge\tilde{\tau}_n}, \tilde{Z}_{T\wedge\tilde{\tau}_n}\right)\right] + \mathbb{E}\left[\int_0^{T\wedge\tilde{\tau}_n} e^{-\beta t}U(\tilde{C}_t)dt\right]. \tag{5.5}$$

We introduce the notation $p_\alpha := (1-\alpha)p_0$ where p_0 is defined in (ii-b) and recall from (3.13) that $p_\alpha < \gamma/(1+\gamma)$. From (5.3) together with condition (ii-b) of the theorem, we have

$$e^{-\beta t}\,\psi(\tilde{X}_t, \tilde{Z}_t) \leq K\left(1 + N_t\,\exp\left\{-\int_0^t \beta + p_\alpha\left(\tilde{c}_r - \lambda\sigma\tilde{\pi}_r + (1-p_\alpha)\frac{(\sigma\tilde{\pi}_r)^2}{2}\right)dr\right\}\right)$$

for any $t > 0$, where N is the Doléans-Dade exponential of $\int_0^t \sigma p_\alpha \tilde{\pi}_s dW_s$. We next compute that

$$\begin{aligned}
\eta_s \;&:=\; \beta + p_\alpha\left(\tilde{c}_s - \lambda\sigma\tilde{\pi}_s + (1-p_\alpha)\frac{(\sigma\tilde{\pi}_s)^2}{2}\right)\\
&\geq\; \frac{\lambda^2}{2}\left\{\gamma + p_\alpha\left((1-p_\alpha)\left(\frac{\sigma\tilde{\pi}_s}{\lambda} - \frac{1}{1-p_\alpha}\right)^2 - \frac{1}{(1-p_\alpha)}\right)\right\}\\
&\geq\; \frac{\lambda^2}{2}\left\{\gamma - \frac{p_\alpha}{1-p_\alpha}\right\} =: \eta > 0\,,
\end{aligned}$$

since $p_\alpha < \gamma/(1+\gamma)$. Therefore, it follows that

$$\mathbb{E}\left[e^{-\beta T\wedge\tilde{\tau}_n}\psi\left(X_{T\wedge\tilde{\tau}_n}, Z_{T\wedge\tilde{\tau}_n}\right)\right] \;\leq\; K\,\mathbb{E}\left[e^{-\beta T\wedge\tilde{\tau}_n} + e^{-\eta T\wedge\tilde{\tau}_n}N_{T\wedge\tilde{\tau}_n}\right]. \tag{5.6}$$

Furthermore, by the Cauchy-Schwartz inequality, $\mathbb{E}\left[e^{-\eta T\wedge\tilde{\tau}_n}N_{T\wedge\tilde{\tau}_n}\right]$ is bounded from above, for any $\varepsilon > 0$, by

$$\mathbb{E}\left[\exp\left\{(1+\varepsilon^{-1})\left(-\eta T\wedge\tilde{\tau}_n + \varepsilon\int_0^{T\wedge\tilde{\tau}_n}|\sigma p_\alpha\tilde{\pi}_s|^2 ds\right)\right\}\right]^{\varepsilon/(1+\varepsilon)} \mathbb{E}\left[N^\varepsilon_{T\wedge\tilde{\tau}_n}\right]^{1/(1+\varepsilon)}\,,$$

where N^ε is a martingale, the Doléans-Dade exponential of $\int_0^t (1+\varepsilon)p_\alpha\,\sigma\tilde{\pi}_s dW_s$. Since $\tilde{\pi}$ is uniformly bounded, by taking ε small enough, we finally deduce from (5.6) that

$$\mathbb{E}\left[e^{-\beta T\wedge\tilde{\tau}_n}\psi\left(\tilde{X}_{T\wedge\tilde{\tau}_n}, \tilde{Z}_{T\wedge\tilde{\tau}_n}\right)\right] \;\leq\; K\left(\mathbb{E}\left[e^{-\beta T\wedge\tilde{\tau}_n}\right] + \mathbb{E}\left[e^{-\eta T\wedge\tilde{\tau}_n}\right]^{\varepsilon/(1+\varepsilon)}\right).$$

Therefore, sending respectively n and T to infinity in (5.5), the dominated and the monotone convergence theorem provide

$$\psi(x,z) \;=\; \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(\tilde{C}_t) dt\right].$$

In view of (i), this implies that $\psi = u^\alpha$.

5.2 Proof of Theorem 3.1

We now turn to the proof of Theorem 3.1 by verifying that the explicit expression reported in there fulfills the conditions of the verification theorem 5.1. One of these conditions will indeed require the proof of Lemma 3.2. We first need to establish additional properties of the function f.

Lemma 5.1 *Let Assumptions 3.1 and 3.2 hold. Then $f \in C^1(\mathbf{D}_\alpha)$ and we have*

$$\frac{f_z(x,z)}{f(x,z)} = \alpha\left(\gamma\left(\frac{\varphi(z)}{f(x,z)}\right)^{\gamma+1} + 1\right)\left((\gamma+1)(x-\alpha z) + \frac{\gamma}{\beta}\int_{f(x,z)}^\infty \frac{V'(s)}{s} ds\right)^{-1} \tag{5.7}$$

for $(x,z) \in \mathbf{D}_\alpha$.

Proof. We recall from lemma 3.1 that, for any $z > 0$, $f(.,z)$ is a decreasing C^1 function on $(\alpha z, z]$ whose derivative is given by (3.21). Furthermore, by construction, we have

$$f[h(y,z),z] \;=\; y\,, \text{ for } y \geq \varphi(z)\,, \quad \text{and} \quad h[f(x,z),z] \;=\; x\,, \text{ for } (x,z) \in \mathbf{D}_\alpha\,. \tag{5.8}$$

Now, from the definition of h, see (3.20), $h \in C^{1,1}(\{(y,z), y \geq \varphi(z)\})$ and we have

$$0 \;\leq\; h_z(y,z) \;=\; \alpha\left(\gamma\left(\frac{\varphi(z)}{y}\right)^{\gamma+1} + 1\right) \;\leq\; \alpha(1+\gamma)\,, \qquad y \geq \varphi(z)\,. \tag{5.9}$$

Therefore, h and f are increasing in z. Hence f is decreasing in x, increasing in z and $\varphi : z \mapsto f(z,z)$ is decreasing. In order to prove that $f \in C^1(D_\alpha)$, we shall prove that f is differentiable in each variable with continuously partial derivatives.

1. In this step, we show that $f \in C^0(\mathbf{D}_\alpha)$, which implies that $f_x \in C^0(\mathbf{D}_\alpha)$ by (3.21). We take $(x,z) \in \mathbf{D}_\alpha$ and study separately the cases where $x < z$ and $x = z$.

- If $x < z$, for l' small enough, $(x, z+l') \in \mathbf{D}_\alpha$ and we deduce from (5.9) that

$$h(f(x,z+l'),z) - x \;=\; h(f(x,z+l'),z) - h(f(x,z+l'),z+l') \;\;\leq\;\; \alpha(1+\gamma)\, l' \underset{l'\to 0}{\longrightarrow} 0\,.$$

Therefore, since $f(x,z+l') \geq \varphi(z)$ from the monotonicity of f, combining (5.8) and the continuity of $f(.,z)$, we obtain

$$f(x,z+l') - f(x,z) \;=\; f(h(f(x,z+l'),z),z) - f(x,z) \underset{l'\to 0}{\longrightarrow} 0\,. \tag{5.10}$$

Moreover, we remark that, for ℓ small enough, $(x+l, z+l') \in \mathbf{D}_\alpha$ and we have

$$f(x+l,z+l') - f(x,z) = f_x(x_l, z+l')\, l + f(x,z+l') - f(x,z) \tag{5.11}$$

for some $x_l \in [x, x+l]$. Now, since f is monotonic in both its variables, we deduce from (3.21) that f and f_x are bounded on any compact subset of $\mathbf{D}_\alpha$ containing (x,z). Therefore, combining (5.10) and (5.11), we deduce that f is continuous at point (x,z).

- If $x = z$, we have, for any l and l' satisfying $(z+l, z+l') \in \mathbf{D}_\alpha$,

$$f(z+l, z+l') \;=\; f_x(z_l, z+l')(l'-l) + \varphi(z+l')\,, \quad \text{for some } z_l \in [z+l, z+l']\,.$$

Therefore similar arguments as above combined with the continuity of φ lead to the continuity of f on $\mathbf{D}_\alpha$.

2. We now prove that f is differentiable with respect to z with continuous partial derivatives. Take $(x,z) \in \mathbf{D}_\alpha$ and l' such that $(x, z+l') \in \mathbf{D}_\alpha$. Combining $f(x,z) \geq \varphi(z+l')$ with (5.8), we deduce

$$\begin{aligned} \frac{1}{l'}\left\{f(x,z+l') - f(x,z)\right\} &= \frac{1}{l'}\left\{f(x,z+l') - f(h(f(x,z), z+l', z+l'))\right\} \\ &= f_x(x_{l'}, z+l')\,\frac{1}{l'}\left\{h(f(x,z),z) - h(f(x,z), z+l')\right\}, \end{aligned}$$

for some $x_{l'} \in [x, x+l']$. Since $f_x \in C^0(\mathbf{D}_\alpha)$ and $h_z(f(x,z), .)$ is continuous, we obtain

$$\frac{1}{h'}\left\{f(x,z+h') - f(x,z)\right\} \underset{h'\to 0}{\longrightarrow} -f_x(x,z)\; h_z(f(x,z), z)\,.$$

Finally, combining (3.21) and (5.9), simple computations lead to (5.7) and f_z inherits the continuity of f on $\mathbf{D}_\alpha$. □

We are now ready for the proof of Lemma 3.2 which states that the functions $\hat{C}$ and $\hat{\theta}$ defined in (3.22) are Lipschitz on $\overline{\mathbf{D}}_\alpha$.

Proof of Lemma 3.2. Remark from lemma 5.1 that $\hat{\theta}$ and $\hat{C}$ are in $C^1(\mathbf{D}_\alpha)$.
1.We first study $\hat{\theta}$ and, since f_x and V' are negative functions, we have

$$\hat{\theta}_x(x,z) \;=\; \frac{\lambda}{\sigma}\left(\gamma + 1 - \frac{\gamma}{\beta}\frac{f_x(x,z)}{f(x,z)}[V'\circ f](x,z)\right) \;\leq\; \frac{\lambda}{\sigma}(\gamma+1)\,, \quad (x,z) \in \mathbf{D}_\alpha\,. \tag{5.12}$$

Notice that, combining the definition of f and (3.21), we get

$$\begin{aligned} \frac{\beta}{\gamma}\frac{f(x,z)}{f_x(x,z)} &= \left(\frac{\varphi(z)}{f(x,z)}\right)^{1+\gamma}\int_0^{\varphi(z)}\frac{V'(s)}{s}\left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds + \int_{\varphi(z)}^{f(x,z)}\frac{V'(s)}{s}\left(\frac{s}{f(x,z)}\right)^{1+\gamma} ds \\ &\leq \int_0^{f(x,z)}\frac{V'(s)}{s}\left(\frac{s}{f(x,z)}\right)^{1+\delta} ds\,, \quad (x,z) \in \mathbf{D}_\alpha\,, \end{aligned} \tag{5.13}$$

since $\varphi(z) \leq f(x,z)$ and $\gamma \leq \delta$. Now, since V' is a negative increasing function, we deduce

$$\frac{f(x,z)}{f_x(x,z)[V'\circ f](x,z)} \geq \frac{\gamma}{\beta}\int_0^{f(x,z)}\frac{1}{s}\left(\frac{s}{f(x,z)}\right)^{1+\delta} ds \;=\; \frac{\gamma}{\beta(1+\delta)} \;>\; 0 \tag{5.14}$$

by Assumption 3.1. Combining this inequality with (5.12), we deduce that the function θ_x is bounded on $\mathbf{D}_\alpha$. Similarly we compute that, for $(x,z) \in \mathbf{D}_\alpha$,

$$\hat{\theta}_z(x,z) \;=\; -\frac{\lambda}{\sigma}\left(\alpha(\gamma+1) + \frac{\gamma}{\beta}\frac{f_z(x,z)}{f(x,z)}[V'\circ f](x,z)\right) \;\geq\; -\frac{\lambda}{\sigma}\,\alpha(\gamma+1)\,,$$

since f_z and $-V'$ are positive functions. Combining (3.21) and (5.7), we compute

$$\frac{f(x,z)}{f_z(x,z)} \;=\; -\frac{1}{\alpha}\left(\gamma\left(\frac{\varphi(z)}{f(x,z)}\right)^{1+\gamma}+1\right)^{-1}\frac{f(x,z)}{f_x(x,z)} \;\geq\; -\frac{1}{\alpha(\gamma+1)}\frac{f(x,z)}{f_x(x,z)}\,, \quad (5.15)$$

for $(x,z)\in\mathbf{D}_\alpha$. We then deduce from (5.14) that $\hat{\theta}_z$ is bounded from above and that $\hat{\theta}$ is a Lipschitz function on $\mathbf{D}_\alpha$. Since, for any $z>0$, $\hat{\theta}(0+,z)=0=\hat{\theta}(0,z)$, the function $\hat{\theta}$ is in fact Lipschitz on $\overline{\mathbf{D}}_\alpha$.

2. We now study $\hat{C}$ whose derivatives are given by

$$\hat{C}_x(x,z) \;=\; -f_x(x,z)[V''\circ f](x,z)\;\geq\;0 \quad\text{and}\quad \hat{C}_z(x,z)\;=\;-f_z(x,z)[V''\circ f](x,z)\;\leq\;0\,,$$

for $(x,z)\in\mathbf{D}_\alpha$. We deduce from (5.13) that

$$\hat{C}_x(x,z)\;\leq\;\frac{\beta}{\gamma}f(x,z)[V''\circ f](x,z)\left(\int_0^{f(x,z)}\frac{-V'(s)}{s}\left(\frac{s}{f(x,z)}\right)^{1+\delta}\right)^{-1}, \quad (x,z)\in\mathbf{D}_\alpha\,, \quad (5.16)$$

so that $\hat{C}_x$ is bounded according to Assumption 3.3. Combining (5.15) and (5.16), we obtain a lower bound on $\hat{C}_z$ and therefore $\hat{C}$ is a Lipschitz function on $\mathbf{D}_\alpha$. □

Before stating the proof of Theorem 3.1, we first isolate two particular properties of the candidate value function denoted $\hat{u}^\alpha$ and defined in Theorem 3.1 by

$$\hat{u}^\alpha(x,z)\;:=\;f(x,z)\left(\frac{\gamma+1}{\gamma}(x-\alpha z)\;+\;\frac{1}{\beta}\int_{f(x,z)}^{\infty}\frac{V(s)}{s^2}ds\right),\quad (x,z)\in\mathbf{D}_\alpha\,, \quad (5.17)$$

and $\hat{u}^\alpha=U(0)/\beta$ on $\overline{\mathbf{D}}_\alpha\setminus\mathbf{D}_\alpha$.

Lemma 5.2 *Let Assumptions 3.1 and 3.2 hold. Then $\hat{u}^\alpha$ is a $C^0\left(\overline{\mathbf{D}}_\alpha\right)\cap C^{2,1}\left(\mathbf{D}_\alpha\right)$ function satisfying*

$$\hat{u}^\alpha_x(x,z)\;=\;f(x,z)\quad\textit{and}\quad\hat{u}^\alpha_z(z,z)\;=\;0\,,\qquad (x,z)\in\mathbf{D}_\alpha\,. \quad (5.18)$$

Proof. Under Assumptions 3.1 and 3.2, $f\in C^1\left(\mathbf{D}_\alpha\right)$, see lemma 5.1. Therefore $\hat{u}^\alpha\in C^1\left(\mathbf{D}_\alpha\right)$ and by direct differentiation in (5.17), it follows from (3.21) that $\hat{u}^\alpha_x=f$. Then $\hat{u}^\alpha$ is a $C^{2,1}\left(\mathbf{D}_\alpha\right)$ function and we compute from (5.7) that

$$\hat{u}^\alpha_z(x,z)\;=\;\alpha f(x,z)\left(\left(\frac{\varphi(z)}{f(x,z)}\right)^{\gamma+1}-1\right),\qquad (x,z)\in\mathbf{D}_\alpha\,, \quad (5.19)$$

which leads to (5.18).

We now prove that $\hat{u}^\alpha\in C^0\left(\overline{\mathbf{D}}_\alpha\right)$. Since V' is a negative function, we derive from (3.21),

$$-\frac{f_x(x,z)}{f(x,z)}\;\geq\;\frac{1}{(\gamma+1)(x-\alpha z)}\,,\qquad (x,z)\in\mathbf{D}_\alpha\,.$$

Integrating this inequality on the interval $[x,z]$, we obtain, up to the composition with the exponential function,

$$f(x,z)\;\;\geq\;\;\varphi(z)[(1-\alpha)z]^{1/(1+\gamma)}(x-\alpha z)^{-1/(1+\gamma)}\,,\qquad (x,z)\in\mathbf{D}_\alpha\,. \quad (5.20)$$

Remark now that, combining (5.17) with the definition of f, we derive, by an integration by part argument,

$$\hat{u}^\alpha(x,z) = \frac{\delta}{\beta}\left(\frac{\varphi(z)}{f(x,z)}\right)^\gamma \int_0^{\varphi(z)} \frac{V(s)}{s}\left(\frac{s}{\varphi(z)}\right)^\delta ds + \frac{\gamma}{\beta}\int_{\varphi(z)}^{f(x,z)} \frac{V(s)}{s}\left(\frac{s}{f(x,z)}\right)^\gamma ds\,, \quad (5.21)$$

for $(x,z) \in \mathbf{D}_\alpha$. Since the function V is decreasing, it is bounded from below by $V(\infty) = U(0)$, which plugged in (5.21) leads to $\hat{u}^\alpha \geq U(0)/\beta$. Fix now $z_0 > 0$, $\epsilon > 0$ and $\mathcal{C}_0$ a compact subset of $\mathbb{R}^+$ containing z_0. Remark that there exists a constant M such that $|V(y) - U(0)| \leq \beta\epsilon/2$ for $y \geq M$.

Now, since φ and V are continuous functions and therefore bounded on compact sets, we deduce from (5.21) the existence of a constant $K > 0$ satisfying

$$\hat{u}^\alpha(x,z) \quad \leq \quad \left(\frac{K}{f(x,z)}\right)^\gamma + \frac{U(0)}{\beta} + \frac{\epsilon}{2}, \quad (x,z) \in \mathbf{D}_\alpha\,,\ z \in \mathcal{C}_0\,.$$

Observe now from (5.20) that there exists $\eta > 0$ such that, for any $(x,z) \in \mathbf{D}_\alpha$ with $z \in \mathcal{C}_0$ and $|x - \alpha z| < \eta$, we have $f(x,z) > K(\epsilon/2)^{-1/\gamma}$ which leads to

$$\frac{U(0)}{\beta} \leq \hat{u}^\alpha(x,z) \quad \leq \quad \frac{U(0)}{\beta} + \epsilon\,.$$

Therefore $\hat{u}^\alpha \in C^0\left(\overline{\mathbf{D}}_\alpha\right)$ and the proof is complete. □

Lemma 5.3 *Let Assumptions 3.1 and 3.2 hold. Then, there exists $K > 0$ such that*

$$\hat{u}^\alpha(x,z) \quad \leq \quad K\left(1 + z^{\alpha p}(x - \alpha z)^{(1-\alpha)p}\right), \qquad (x,z) \in \overline{\mathbf{D}}_\alpha\,.$$

Proof. First remark that this property is straightforward for $(x,z) \in \overline{\mathbf{D}}_\alpha \backslash \mathbf{D}_\alpha$. According to lemma 5.2, we compute

$$\hat{u}^\alpha_x(x,z) \;=\; f(x,z) \;=\; \hat{u}^\alpha(x,z)\left(\frac{\gamma+1}{\gamma}(x - \alpha z) \;+\; \frac{1}{\beta}\int_{f(x,z)}^\infty \frac{V(s)}{s^2}ds\right)^{-1}, \qquad (5.22)$$

for $(x,z) \in \mathbf{D}_\alpha$.

1. We first derive (5.19) for a power utility function U_p and denote $\hat{u}^\alpha_p$ the candidate value function. As detailled in section 3.5, $f(x,z)$ rewrites as $(F^{-1}(x/z)(x - \alpha z))^{p-1}$ on $\mathbf{D}_\alpha$ so that (5.22) leads to

$$\nabla_x \hat{u}^\alpha_p(x,z) \;=\; \hat{u}^\alpha_p(x,z)\left(\frac{\gamma+1}{\gamma}(x - \alpha z) \;+\; \frac{(1-p)^2}{\beta p}F^{-1}\left(\frac{x}{z}\right)(x - \alpha z)\right)^{-1}, \quad (x,z) \in \mathbf{D}_\alpha\,,$$

where $\nabla_x \hat{u}^\alpha_p$ denotes the partial derivative of $\hat{u}^\alpha_p$ with respect to x. Since F^{-1} is an increasing function and $F^{-1}(1) = b_\alpha/(1-\alpha)$ where b_α is defined in (3.28), simple computations combined with (3.13) lead to

$$\frac{\nabla_x \hat{u}^\alpha_p(x,z)}{\hat{u}^\alpha_p(x,z)} \;\geq\; \frac{(1-\alpha)p}{x - \alpha z}\,, \qquad (x,z) \in \mathbf{D}_\alpha\,. \qquad (5.23)$$

Integrating this inequality on the interval $[x,z]$, we obtain, up to the composition with the exponential function

$$\frac{\hat{u}_p^\alpha(z,z)}{\hat{u}_p^\alpha(x,z)} \;\geq\; \left(\frac{(1-\alpha)z}{x-\alpha z}\right)^{(1-\alpha)p}\,, \qquad (x,z)\in\mathbf{D}_\alpha\,. \tag{5.24}$$

Since $\hat{u}_p^\alpha$ inherits the homogeneity property of U_p, $u_p^\alpha(z,z)=u_p^\alpha(1,1)\,z^p$, for any $z>0$, and we deduce from (5.24) the existence of $K>0$ such that

$$\hat{u}_p^\alpha(x,z) \;\leq\; K\,z^p(x-\alpha z)^{(1-\alpha)p}\,, \qquad (x,z)\in\mathbf{D}_\alpha\,. \tag{5.25}$$

2. We next consider the case where the utility function is given by $U_p^0 = K^0(1+U_p)$ where K^0 is the constant defined in (3.15). Observe that U_p^0 satisfies the required Assumptions 3.2 and 3.3. Simple computations show that the corresponding marginal utilities f_p^0 and f_p associated to the candidate value function $\hat{u}_0^\alpha$ and $\hat{u}_p^\alpha$ are related by $f_p^0 = K^0 f_p$. Combining (5.17) and (5.25), we easily derive

$$\hat{u}_0^\alpha(x,z) \;=\; K^0(1+\hat{u}_p^\alpha(x,z)) \;\leq\; K\,K^0(1+\,z^p(x-\alpha z)^{(1-\alpha)p})\,, \qquad (x,z)\in\mathbf{D}_\alpha\,. \tag{5.26}$$

3. We finally consider the general case. We recall from (3.15) that $U\leq U_p^0$ so that their Fenchel transforms satisfy also $V\leq V_p^0$. In this step, we shall prove that $\hat{u}^\alpha\leq\hat{u}_0^\alpha$ which combined with (5.26) concludes the proof.

Set $V^\epsilon := V+\epsilon(V_p^0-V)$, for $0\leq\epsilon\leq 1$, and denote $(V^\epsilon)'$, φ^ϵ, f^ϵ and $\hat{u}^{\alpha,\epsilon}$ the associated functions defined in section 3.4. Observe first that all these functions are differentiable in ϵ. We intend to prove that $\hat{u}^{\alpha,\epsilon}$ is an increasing function of ϵ on $[0,1]$, which implies the required result as $V^0=V$ and $V^1=V_p^0$.

For ease of notation, we denote Υ the operator defined for $(V,f,\varphi)\in C^1(\mathbb{R}^+,\mathbb{R}^+)\times\mathbb{R}^+\times\mathbb{R}^+$ by

$$\Upsilon[V,f,\varphi] \;:=\; \frac{\delta}{\beta}\left(\frac{\varphi}{f}\right)^{1+\gamma}\int_0^\varphi \frac{V(s)}{s^2}\left(\frac{s}{\varphi}\right)^{1+\delta}ds + \frac{\gamma}{\beta}\int_\varphi^f \frac{V(s)}{s^2}\left(\frac{s}{f}\right)^{1+\gamma}ds - \frac{1}{\beta}\int_f^\infty \frac{V(s)}{s^2}ds\,.$$

By an integration by part argument on (3.19), φ^ϵ is implicitly defined, for $\epsilon\in[0,1]$, by

$$\Upsilon[V^\epsilon,\varphi^\epsilon,\varphi^\epsilon](z) \;\;=\;\; \frac{1+\gamma}{\gamma}(1-\alpha)z\,, \quad z>0\,.$$

Denoting ∇_ϵ the differential operator with respect to ϵ, we deduce

$$(1+\delta)\frac{\nabla_\epsilon\varphi^\epsilon}{\varphi^\epsilon}\left(\Upsilon[V^\epsilon,\varphi^\epsilon,\varphi^\epsilon]-\frac{1}{\beta}\int_{\varphi^\epsilon}^\infty\frac{(V^\epsilon)'(s)}{s}ds\right) \;\;=\;\; \Upsilon[\nabla_\epsilon V^\epsilon,\varphi^\epsilon,\varphi^\epsilon]\,. \tag{5.27}$$

Similarly f^ϵ is defined, for $\epsilon\in[0,1]$, by

$$\Upsilon[V^\epsilon,f^\epsilon,\varphi^\epsilon](x,z) = \frac{1+\gamma}{\gamma}(x-\alpha z)\,, \quad (x,z)\in\mathbf{D}_\alpha\,,$$

and differentiation with respect to ϵ combined with (5.27) leads to

$$(1+\gamma)\frac{\nabla_\epsilon f^\epsilon}{f^\epsilon}\left(\Upsilon[V^\epsilon,f^\epsilon,\varphi^\epsilon]+\frac{1}{\beta}\int_{f^\epsilon}^\infty\frac{(V^\epsilon)'(s)}{s}ds\right) = \Upsilon[\nabla_\epsilon V^\epsilon,f^\epsilon,\varphi^\epsilon]-\frac{\delta-\gamma}{1+\delta}\Upsilon[\nabla_\epsilon V^\epsilon,\varphi^\epsilon,\varphi^\epsilon]\,. \tag{5.28}$$

Combining the definition of f^ϵ and (5.17), we rewrite $\hat{u}^{\alpha,\epsilon}$ as

$$\hat{u}^{\alpha,\epsilon} \;=\; \left(\Upsilon[V^\epsilon, f^\epsilon, \varphi^\epsilon] \;+\; \frac{1}{\beta}\int_{f^\epsilon}^{\infty} \frac{V^\epsilon(s)}{s^2} ds\right) f^\epsilon\,, \qquad 0 \le \epsilon \le 1\,.$$

Differentiating this expression with respect to ϵ, we compute from (5.27) and (5.28) that

$$\begin{aligned}
\frac{\nabla_\epsilon \hat{u}^{\alpha,\epsilon}}{f^\epsilon} &= \frac{1}{1+\gamma}\Upsilon[\nabla_\epsilon V^\epsilon, f^\epsilon, \varphi^\epsilon] - \frac{\delta-\gamma}{(1+\gamma)(1+\delta)}\Upsilon[\nabla_\epsilon V^\epsilon, \varphi^\epsilon, \varphi^\epsilon] + \frac{1}{\beta}\int_{f^\epsilon}^{\infty}\frac{\nabla_\epsilon V^\epsilon(s)}{s} ds \\
&= \frac{\delta}{\beta(1+\delta)}\left(\frac{\varphi^\epsilon}{f^\epsilon}\right)^{1+\gamma}\int_0^{\varphi^\epsilon}\frac{\nabla_\epsilon V^\epsilon(s)}{s^2}\left(\frac{s}{\varphi^\epsilon}\right)^{1+\delta} ds + \frac{1}{\beta}\int_{f^\epsilon}^{\infty}\frac{\nabla_\epsilon V^\epsilon(s)}{s} ds \\
&\quad + \frac{\gamma-\delta}{\beta(1+\gamma)(1+\delta)}\int_{\varphi^\epsilon}^{\infty}\frac{\nabla_\epsilon V^\epsilon(s)}{s} ds + \frac{\gamma}{\beta(1+\gamma)}\int_{\varphi^\epsilon}^{f^\epsilon}\frac{\nabla_\epsilon V^\epsilon(s)}{s^2}\left(\frac{s}{\varphi^\epsilon}\right)^{1+\gamma} ds\,,
\end{aligned}$$

for any $\epsilon \in [0,1]$. We now observe that all the above integrals are positive since $\nabla_\epsilon V^\epsilon = V_p^0 - V \ge 0$. Since $\gamma \le \delta$ and $f^\epsilon \ge 0$, this shows that $u^{\alpha,\epsilon}$ is non-decreasing in ε. □

We are now ready for the

Proof of Theorem 3.1. We will simply check that the candidate value function $\hat{u}^\alpha$ defined in (5.17) satisfies the hypothesis of Theorem 5.1. First, from lemma 5.2, $\hat{u}^\alpha \in C^0\left(\overline{\mathbf{D}}_\alpha\right) \cap C^{2,1}\left(\mathbf{D}_\alpha\right)$. Combining (3.21) and (5.18), we easily check that $\hat{u}^\alpha$ satisfies (ii-a) in Theorem 5.1. Remark also that condition (ii-b) in Theorem 5.1 is exactly given by lemma 5.3. By construction, the functions $(\hat{C}, \hat{\theta})$ defined in (3.22) satisfy (3.4) so that $\mathcal{L}\hat{u}^\alpha \;=\; \beta\hat{u}^\alpha - U(\hat{C}) + \mathcal{L}^{\hat{C},\hat{\theta}}\hat{u}^\alpha$. Now, Lemma 3.3 ensures existence and uniqueness of a solution $(\hat{X}, \hat{Z})$ to the SDE (3.23) for any initial condition $(x,z) \in \overline{\mathbf{D}}_\alpha$, and, since $\hat{c}$ and $\hat{\pi}$ defined in (3.24) are bounded functions, $\hat{u}^\alpha$ satisfies (ii-c) in Theorem 5.1. Therefore $\hat{u}^\alpha = u^\alpha$ and simple computations lead to the expression of the dual function of v^α. □

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