On the Rate of Convergence in the Quantum Central Limit Theorem

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on work with

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The Rate of Convergence in the Central Limit Theorem

\[ X = \sum_{i=1}^{N} X_i \]

\(X_i\) : “weakly correlated”

Central Limit Theorem:

\[
\mathbb{P}[X \leq x] = F(x) \xrightarrow{N \to \infty} G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} dy \ e^{-\frac{(y-\mu)^2}{2\sigma^2}}
\]

\[ \mu = \langle X \rangle, \quad \sigma^2 = \langle (X - \mu)^2 \rangle \]
The Rate of Convergence in the Central Limit Theorem

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Berry—Esseen:

\[ \sup_{x} \left| F(x) - G(x) \right| \leq \frac{C}{\sqrt{N}} \]

\( \mu = \langle X \rangle, \quad \sigma^2 = \langle (X - \mu)^2 \rangle \)
\[ \hat{X} = \sum_{i \in \Lambda} \hat{X}_i = \sum_{k} x_k |k\rangle \langle k| \quad \text{local} \]

\[ \hat{q} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\|\hat{A}\|\|\hat{B}\|} \leq N^z e^{-L/\xi} \]

**Quantum Central Limit Theorem:**

\[ \sum_{x_k \leq x} \langle k | \hat{q} | k \rangle = F(x) \xrightarrow{N \to \infty} G(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{x} dy \ e^{-\frac{(y-\mu)^2}{2\sigma^2}} \]

Goderis, Vets (1989); Hartmann, Mahler, Hess (2004)

**Berry—Esseen:**

\[ \sup_x |F(x) - G(x)| \leq C \frac{\log^{2d}(N)}{\sqrt{N}} \]


\[ \mu = \langle \hat{X} \rangle, \quad \sigma^2 = \langle (\hat{X} - \mu)^2 \rangle \]
The Rate of Convergence in the Quantum Central Limit Theorem

\[ \hat{X} = \sum_{i \in \Lambda} \hat{X}_i = \sum_{k} x_k |k\rangle \langle k| \quad \text{local} \]

\[ \hat{\mathcal{Q}} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\| \hat{A} \| \| \hat{B} \|} \leq N^2 e^{-L/\xi} \]

Quantum Central Limit Theorem:

\[ \sum_{x_k \leq x} \langle k| \hat{\mathcal{Q}} |k\rangle = F(x) \]

Berry—Esseen:

\[ \sup_x |F(x) - G(x)| \leq \frac{C}{\sqrt{N}} \]

\[ \mu = \langle \hat{X} \rangle, \quad \sigma^2 = \langle (\hat{X} - \mu)^2 \rangle \]

relation to density of states: \( \hat{\mathcal{Q}} \propto 1 \)

\[ \left| \left\{ k : E - \Delta E < E_k \leq E \right\} \right| \]

\[ \propto F(E) - F(E - \Delta E) \]

main ingredient (also for (quantum) central limit):

\[
\sup_x |F(x) - G(x)| \leq \frac{1}{T} + \int_0^T dt \frac{\phi(t) - e^{-t^2/2}}{|t|}
\]

Esseen (1945)
main ingredient (also for (quantum) central limit):

\[
\sup_x |F(x) - G(x)| \leq \frac{1}{T} + \int_0^T dt \frac{|\phi(t) - e^{-t^2/2}|}{|t|}
\]

Esseen (1945)

need to bound \( |\phi(t) - e^{-t^2/2}| \)

\( \phi(t) = \langle e^{i\hat{X}t} \rangle \): characteristic function
main ingredient (also for (quantum) central limit):

\[
\sup_x |F(x) - G(x)| \leq \frac{1}{T} + \int_0^T dt \frac{\left| \phi(t) - e^{-t^2/2} \right|}{|t|}
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Esseen (1945)

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\( \phi(t) = \langle e^{i\hat{X}t} \rangle \): characteristic function

\( \hat{X} = \hat{H} \):

- pure state: Loschmidt echo, return probability
- \( \hat{e} = \frac{1}{2^N} \): Fourier transform of d.o.s
The Rate of Convergence in the Quantum Central Limit Theorem: Proof Idea

main ingredient (also for (quantum) central limit):

$$\sup_{x} |F(x) - G(x)| \leq \frac{1}{T} + \int_{0}^{T} dt \frac{|\phi(t) - e^{-t^2/2}|}{|t|}$$

Esseen (1945)

need to bound $|\phi(t) - e^{-t^2/2}|$

$\phi(t) = \langle e^{i\hat{X}t} \rangle$: characteristic function

$\hat{X} = \hat{H}$:
- pure state: Loschmidt echo
- return probability
- dynamical “phase transitions"

$\hat{\rho} = \frac{1}{2^N}$: Fourier transform of d.o.s
The Rate of Convergence in the Quantum Central Limit Theorem: Proof Idea

need to bound \[ |\phi(t) - e^{-t^2/2}| \]

set up differential equation for char. function and bound its derivative

\[
\langle \hat{X} e^{it\hat{X}} \rangle = \left( i\langle \hat{X}, \mathcal{X} \rangle t + g(t) \right) \varphi(t) + h(t),
\]

where \( g(t) = g_1(t) + g_2(t) + g_3(t) \), \( h(t) = h_1(t) + h_2(t) + h_3(t) \),

\[
g_1(t) = -i(\langle \hat{X}, \hat{z}_1 \rangle - \langle \hat{X}, \langle \hat{z}_1 \rangle \rangle)t,
\]

\[
g_2(t) = \langle \hat{X}, \hat{\xi}_1(t) \rangle + i\langle \hat{X}, \hat{z}_1 \rangle t - i\langle \hat{X}, \mathcal{X} \rangle t,
\]

\[
g_3(t) = \langle \hat{X}, \hat{\xi}_1(t) \rangle \langle \hat{n}_2(t) \rangle + \sum_{n=3}^k \langle \hat{X}, \hat{\xi}_{n-1}(t) \rangle \langle \hat{n}(t) + 1 \rangle,
\]

\[
h_1(t) = \sum_{n=1}^k \left( \langle \hat{X}, \hat{\xi}_{n-1}(t) e^{iz_n t} \rangle - \langle \hat{X}, \hat{\xi}_{n-1}(t) \rangle \langle e^{iz_n t} \rangle \right),
\]

\[
h_2(t) = \sum_{n=2}^k \langle \hat{X}, \hat{\xi}_{n-1}(t) \rangle \langle (\hat{n}(t) - \langle \hat{n}(t) \rangle) e^{i\hat{X} t} \rangle,
\]

\[
h_3(t) = \langle \hat{X}, \hat{\xi}_k(t) e^{iz_k t} \rangle + \sum_{n=0}^{k-1} \langle \hat{X}, \hat{\xi}_n(t) \hat{r}(t) e^{iz_{n+1} t} \rangle + \sum_{n=2}^k \langle \hat{X}, \hat{\xi}_{n-1}(t) \rangle \langle \hat{s}(t) \rangle,
\]

\[
\hat{r}(t) = e^{i(\hat{z}_n - \hat{z}_{n+1}) t} \left( e^{-i(\hat{z}_n - \hat{z}_{n+1}) t} e^{i\hat{z}_n t} e^{-i\hat{z}_{n+1} t} - \hat{R}_{n+1}(t) \right) = e^{i(\hat{z}_n - \hat{z}_{n+1})} \left( \hat{Z}_{R,n+1}(t) - \hat{R}_{n+1}(t) \right)
\]

\[
\hat{s}(t) = e^{-i(\hat{X} + \hat{Z}_n) t} e^{-i\hat{X} t} e^{i\hat{Z}_n t} - \hat{S}_n(t) \rangle e^{-i\hat{Z}_n t} e^{i\hat{X} t} =: \left( \hat{Z}_{S,n}(t) - \hat{S}_n(t) \right) e^{-i\hat{Z}_n t} e^{i\hat{X} t}.
\]

The Rate of Convergence in the Quantum Central Limit Theorem: Application

\[
\hat{H} = \sum_{i \in \Lambda} \hat{H}_i = \sum_{k} E_k |k\rangle \langle k| \text{ local}
\]

\[
\hat{\mathcal{Q}}_T : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\| \hat{A} \| \| \hat{B} \|} \leq N^z e^{-L/\xi}
\]

d = 1 : Araki (1969)
d > 1, \ T > T_c : Kliesch, Gogolin, Kastoryano, Riera, Eisert (2014)

canonical state \( \hat{\mathcal{Q}}_T = e^{-\hat{H}/T}/Z \)
\[ \hat{H} = \sum_{i \in \Lambda} \hat{H}_i = \sum_k E_k |k\rangle \langle k| \text{ local} \]

\[ \hat{\rho}_T : \frac{|\langle \hat{A}\hat{B}\rangle - \langle \hat{A}\rangle \langle \hat{B}\rangle|}{||\hat{A}|| ||\hat{B}||} \leq N^z e^{-L/\xi} \]

canonical state \( \hat{\rho}_T = e^{-\hat{H}/T}/Z \)

with energy density \( u(T) = \frac{\text{tr}[\hat{H}\hat{\rho}_T]}{N} \) \( (= \frac{\mu}{N}) \)

specific heat capacity \( c(T) = \frac{\partial}{\partial T} u(T) \) \( (= \frac{\sigma^2}{N T^2}) \)
The Rate of Convergence in the Quantum Central Limit Theorem: Application

$$\hat{H} = \sum_{i \in \Lambda} \hat{H}_i = \sum_k E_k \ket{k}\bra{k} \text{ local}$$

$$\hat{Q}_T: \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\|\hat{A}\| \|\hat{B}\|} \leq N^z e^{-L/\xi}$$

$d = 1$: Araki (1969)

$d > 1$, $T > T_c$: Kliesch, Gogolin, Kastoryano, Riera, Eisert (2014)

canonical state $\hat{\varrho}_T = e^{-\hat{H}/T}/Z$

$\hat{\varrho}$: state on microcanonical subspace

$$M_\delta = \{ \ket{k} : |E_k - Nu(T)| \leq \delta \sqrt{N} \}, \quad \frac{\log^{2d}(N)}{\sqrt{N}} \lesssim \delta \lesssim 1$$

quantum Berry–Esseen

$$S(\hat{\varrho} \| \hat{\varrho}_T) \lesssim \log(|M_\delta|) - S(\hat{\varrho}) + \log^{2d}(N)$$
Why Do Systems Thermalize?

\[ \hat{\rho}_T = e^{-\hat{H}/T}/Z \]
Why Do Systems Thermalize?

lack of knowledge, ignorance

Jaynes’ principle

\[ \hat{\varrho}_T = e^{-\hat{H}/T}/Z \]
part of a large (closed) system $\hat{\mathcal{O}}_C = \text{tr}_C [\hat{\mathcal{O}}]$
part of a large (closed) system $\hat{\rho}_C = \text{tr}_C[\hat{\rho}]$

$\approx e^{-\hat{H}_C/T}/Z$
part of a large (closed) system \( \hat{\mathcal{O}}_C = \text{tr}_C[\hat{\mathcal{O}}] \)
\[ \approx \text{tr}_C[\text{e}^{-\hat{H}/T}/Z] \]
part of a large (closed) system \( \hat{\rho}_C \approx \text{tr}_C [e^{-\hat{H}/T}/Z] \)

- in contact with heat bath

\[ \hat{\rho}_C (0) \otimes \hat{\rho}_B \]
part of a large (closed) system \( \hat{\mathcal{C}} \approx \text{tr}_C \left[ e^{-\hat{H} / T} / Z \right] \)

in contact with heat bath, unitary evolution

\[
e^{-it\hat{H}} (\hat{\mathcal{C}}(0) \otimes \hat{\mathcal{B}}) e^{it\hat{H}}
\]
part of a large (closed) system \( \hat{\mathcal{C}} \approx \text{tr}_C \left[ e^{-\hat{H}/T} / Z \right] \)

in contact with heat bath, unitary evolution

\[
\text{tr}_C \left[ e^{-it\hat{H}} (\hat{\mathcal{C}}(0) \otimes \hat{\mathcal{B}}) e^{it\hat{H}} \right] = \hat{\mathcal{C}}(t)
\]
part of a large (closed) system \( \hat{\mathcal{C}} \approx \text{tr}_{\mathcal{C}} \left[ e^{-\hat{H}/T} / Z \right] \)

in contact with heat bath, unitary evolution

\[
\text{tr}_{\mathcal{C}} \left[ e^{-it\hat{H}} (\hat{\mathcal{C}}(0) \otimes \hat{\mathcal{B}}) e^{it\hat{H}} \right] = \hat{\mathcal{C}}(t) \\
\xrightarrow{t \to \infty} \text{tr}_{\mathcal{C}} \left[ e^{-\hat{H}/T} / Z \right]
\]
part of a large (closed) system $\hat{\mathcal{C}} \approx \text{tr}_\mathcal{C}[e^{-\hat{H}/T}/Z]$

quantum quench

$$\text{tr}_\mathcal{C}[e^{-it\hat{H}}(\hat{\mathcal{C}}(0) \otimes \hat{\mathcal{B}})e^{it\hat{H}}] = \hat{\mathcal{C}}(t)$$

$$\xrightarrow{t \to \infty} \text{tr}_\mathcal{C}[e^{-\hat{H}/T}/Z]$$
part of a large (closed) system $\hat{\mathcal{C}} \approx \text{tr}_\mathcal{C} \left[ e^{-\hat{H}/T}/Z \right]$

- **Canonical Typicality**

- **Entanglement and the foundations of statistical mechanics**

- **Thermalization in Nature and on a Quantum Computer**

- **Thermalization and Canonical Typicality in Translation-Invariant Quantum Lattice Systems**

- **Equivalence of Statistical Mechanical Ensembles for Non-Critical Quantum Systems**
  Brandão, Cramer, arxiv:1502.03263

quantum quench $\hat{\mathcal{C}}(t) \xrightarrow{t \to \infty} \text{tr}_\mathcal{C} \left[ e^{-\hat{H}/T}/Z \right]$

- **Time-dependence of correlation functions following a quantum quench**

- **Relaxation in a Completely Integrable Many-Body Quantum System**

- **Effect of suddenly turning on interactions in the Luttinger model**

- **Quenching, Relaxation, and a Central Limit Theorem for Quantum Lattice Systems**

- **Thermalization and its mechanism for generic isolated quantum systems**

- **Foundation of Statistical Mechanics under Experimentally Realistic Conditions**

- **Quantum mechanical evolution towards thermal equilibrium**
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- **Canonical Typicality**
  Goldstein, Lebowitz, Tumulka, Zahn

- **Entanglement and the foundation**

- **Thermalization in Nature and…**

- **Thermalization and Canonical**
  Mueller, Adlam, Masanes, Wiebe,

- **Equivalence of Statistical Mech…**

for random \( |\psi\rangle \in \mathcal{H}_R \subset \mathcal{H}_C \otimes \mathcal{H}_B \)
with high probability \( \hat{\mathcal{C}} \approx \text{tr}_C \left[ \mathbb{1}_R / d_R \right] \)
...thermal?

quantum quench \( \hat{\mathcal{C}}(t) \xrightarrow{t \to \infty} \text{tr}_C \left[ e^{-\hat{H}/T} / Z \right] \)

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part of a large (closed) system $\hat{\rho}_C \approx \text{tr}_C \left[ e^{-\hat{H}/T}/Z \right]$ for random $|\psi\rangle \in \mathcal{H}_R \subset \mathcal{H}_C \otimes \mathcal{H}_B$ with high probability $\hat{\rho}_C \approx \text{tr}_C \left[ \mathbb{1}_R/d_R \right]$ ...thermal?

quantum quench $\hat{\rho}_C(t) \xrightarrow{t \to \infty} \text{tr}_C \left[ e^{-\hat{H}/T}/Z \right]$ for random $|\psi\rangle \in \mathcal{H}_R \subset \mathcal{H}_C \otimes \mathcal{H}_B$ with high probability $\hat{\rho}_C \approx \text{tr}_C \left[ \mathbb{1}_R/d_R \right]$ ...thermal?

Integrable

no thermalization instead: generalized Gibbs ensemble

“equilibrium state”, close to it for most times ...thermal? time scale?

Non-integrable
\[ \hat{H} = \sum_{ij} \left( \hat{b}_i^\dagger A_{ij} \hat{b}_j + \hat{b}_i B_{ij} \hat{b}_j + \text{h.c.} \right) \text{ local, t.i.} \]

\[ \hat{\rho}(0) \in \mathcal{H}_C \otimes \mathcal{H}_B \text{ sufficiently clustering} \]

(not necessarily Gaussian)

A Quantum Central Limit Theorem for Non-Equilibrium Systems:  
Exact Local Relaxation of Correlated States  
\( \hat{H} = \sum_{ij} (\hat{b}_i^\dagger A_{ij} \hat{b}_j + \hat{b}_i B_{ij} \hat{b}_j + \text{h.c.}) \) local, t.i.

\( \hat{\varrho}(0) \in \mathcal{H}_C \otimes \mathcal{H}_B \) sufficiently clustering (not necessarily Gaussian)

\[
\| \hat{\varrho}_C(t) - \hat{G}(t) \|_{\text{tr}} \leq \epsilon \quad \text{for all} \quad t \in [t_1(\epsilon, N), t_2(\epsilon, N)]
\]

\( \hat{G}(t) \): Gaussian with same second moments as \( \hat{\varrho}_C(t) \)

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*A Quantum Central Limit Theorem for Non-Equilibrium Systems: Exact Local Relaxation of Correlated States*

\[ \hat{H} = \sum_{ij} \left( \hat{b}_i^\dagger A_{ij} \hat{b}_j + \hat{b}_i B_{ij} \hat{b}_j + \text{h.c.} \right) \text{ local, t.i.} \]

\[ \hat{\rho}(0) \in \mathcal{H}_C \otimes \mathcal{H}_B \text{ sufficiently clustering} \]

\[ \text{(not necessarily Gaussian)} \]

\[ \| \hat{\rho}_C(t) - \hat{G}(t) \|_{\text{tr}} \leq \epsilon \text{ for all } t \in [t_1(\epsilon, N), t_2(\epsilon, N)] \]

**\( \hat{G}(t) \): Gaussian with same second moments as \( \hat{\rho}_C(t) \)**

maximum entropy state

equilibration, non-thermal: Tegmark, Yeh (1994)
characteristic function
(FT of Wigner function, Bochner’s theorem)

$$\chi_{\hat{\phi}_C(t)}(\bm{\beta}) = \text{tr}_C[\hat{\phi}_C(t) \hat{D}(\bm{\beta})]$$

$$\hat{D}(\bm{\beta}) = \prod_{i \in C} e^{\beta_i \hat{b}_i^\dagger - \beta_i^* \hat{b}_i}$$
characteristic function
(FT of Wigner function, Bochner’s theorem)

\[ \chi_{\hat{\rho}_C(t)}(\beta) = \text{tr}_C [\hat{\rho}_C(t) \hat{D}(\beta)] = \text{tr} [\hat{\rho}(0) \hat{D}(\alpha(t, \beta))] \]

\[ \hat{D}(\beta) = \prod_{i \in C} e^{\beta_i \hat{b}_i^\dagger - \beta_i^* \hat{b}_i} \]

\[ \alpha_i = \sum_{j \in C} \beta_j (e^{itA})_{ij} \]
for which states $\hat{\varrho}$, $\hat{\tau}$ (and which $I$) is

$$\|\hat{\varrho}_C - \hat{\tau}_C\|_{tr} \leq \epsilon$$

Local Closeness – A Lemma

Equivalence of Statistical Mechanical Ensembles for Non-Critical Quantum Systems
Brandão, Cramer, arxiv:1502.03263
for which states $\hat{\rho}$, $\hat{\tau}$ (and which $l$) is

$$\|\hat{\rho}_C - \hat{\tau}_C\|_{\text{tr}} \leq \epsilon$$

non-t.i.: $\mathbb{E}[ ]$
\[
\hat{T} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\|\hat{A}\| \|\hat{B}\|} \leq N^{\gamma} e^{-L/\xi}
\]

for which states \( \hat{\varrho} \) (and which \( l \)) is

\[
\| \hat{\varrho}_C - \hat{T}_C \|_{\text{tr}} \leq \epsilon
\]
\( \hat{\tau} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle |}{\| \hat{A} \| \| \hat{B} \|} \leq N^z e^{-L/\xi} \)

for which states \( \hat{\rho} \)
(and which \( I \) is

\[ \| \hat{\rho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon \]

for those with

\[ \frac{S(\hat{\rho} \| \hat{\tau})}{\epsilon^2} + l^d \leq \frac{(\epsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)} \]
Equivalence of Statistical Mechanical Ensembles for Non-Critical Quantum Systems

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\[ \hat{\tau} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\|\hat{A}\|\|\hat{B}\|} \leq N^2 e^{-L/\xi} \]

for which states \( \hat{\varrho} \)
(and which \( I \)) is

\[ \|\hat{\varrho}_C - \hat{\tau}_C\|_{tr} \leq \epsilon \]

for those with

\[ \frac{S(\hat{\varrho}||\hat{\tau})}{\epsilon^2} + I^d \leq \frac{1}{\ln(N)} \left( \frac{\epsilon^2 N}{d+1} \right) \]

- **Quantum Substate Theorem** Jain, Radhakrishnan, Sen (2009); Jain, Nayak (2011)

\[ S^{2\sqrt{\epsilon}}(\hat{\varrho}||\hat{\tau}) \leq S^{2\sqrt{\epsilon}}_{\max}(\hat{\varrho}||\hat{\tau}) \leq \frac{S(\hat{\varrho}||\hat{\tau})+1}{\epsilon} + \log(\frac{1}{1-\epsilon}) \]
\[ \hat{\tau} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\| \hat{A} \| \| \hat{B} \|} \leq N^2 e^{-L/\xi} \]

for which states \( \hat{\varrho} \) (and which \( l \) is

\[ \| \hat{\varrho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon \]

for those with

\[ \frac{S(\hat{\varrho} \| \hat{\tau})}{\epsilon^2} + l^d \leq \frac{(\epsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)} \]

- Quantum Substate Theorem Jain, Radhakrishnan, Sen (2009); Jain, Nayak (2011)
- Lemma Datta, Renner (2009); Brandão, Plenio (2010); Brandão, Horodecki (2012)

\[ S_{\text{max}}(\hat{\varrho} \| \hat{\tau}) \leq \lambda \]

\[ \kappa = 2^\lambda \| \hat{\tau} - \hat{\pi} \|_{tr} \quad \rightarrow \quad S_{\text{max}}^{\sqrt{8\kappa}}(\hat{\varrho} \| \hat{\tau}) \leq \lambda + \log(\frac{1}{1-\kappa}) \]
\( \hat{\tau} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\| \hat{A} \| \| \hat{B} \|} \leq N^z e^{-L/\xi} \)

for which states \( \hat{\rho} \)

(and which \( l \) is

\[ \| \hat{\rho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon \ ? \]

for those with

\[ \frac{S(\hat{\rho} \| \hat{\tau})}{\epsilon^2} + ld \lesssim \left( \frac{\epsilon^2 N}{\ln(N)} \right)^{d+1} \]

- **Quantum Substate Theorem** Jain, Radhakrishnan, Sen (2009); Jain, Nayak (2011)
- **Lemma** Datta, Renner (2009); Brandão, Plenio (2010); Brandão, Horodecki (2012)

\[ \| \hat{\tau}_{C_1 \ldots C_M} - \hat{\tau}_{C_1} \otimes \cdots \otimes \hat{\tau}_{C_1} \| \leq \sum_{j=2}^{M} \text{cov}(\hat{A}_1 \cdots \hat{A}_{j-1}, \hat{A}_j) \]
\[ \hat{\tau} : \frac{|\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle|}{\| \hat{A} \| \| \hat{B} \|} \leq N^z e^{-L/\xi} \]

for which states \( \hat{\rho} \)
(and which \( l \)) is

\[ \| \hat{\rho}^C - \hat{\tau}^C \|_{tr} \leq \epsilon \]

for those with

\[ \frac{S(\hat{\rho} \| \hat{\tau})}{\epsilon^2} + l^d \leq \frac{(\epsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)} \]

- Quantum Substate Theorem  
  Jain, Radhakrishnan, Sen (2009); Jain, Nayak (2011)
- Lemma  
  Datta, Renner (2009); Brandão, Plenio (2010); Brandão, Horodecki (2012)
- Pinsker’s inequality  
  \[ \| \hat{\rho} - \hat{\tau} \|_{tr} \leq \ln(4) S(\hat{\rho} \| \hat{\tau}) \]
- Super-additivity  
  \[ \sum_{j=1}^{M} S(\hat{\rho}_{C_j} \| \hat{\tau}_{C_j}) \leq S(\hat{\rho} \| \hat{\tau}_{C_1} \otimes \cdots \otimes \hat{\tau}_{C_M}) \]

Equivalence of Statistical Mechanical Ensembles for Non-Critical Quantum Systems
Brandão, Cramer, arxiv:1502.03263
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$ for which states $\hat{\varrho}$ (and which $I$) is $\| \hat{\varrho}_C - \hat{\tau}_C \|_{\text{tr}} \leq \varepsilon$?
canonical state \( \hat{\tau} = e^{-\hat{H}/T}/Z \)
for which states \( \hat{\rho} \) (and which \( I \)) is
\[ \| \hat{\rho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon \] ?
which states \( \hat{\rho} \) are locally thermal?
Local Equivalence of Micro- and Macrocanonical Ensembles

canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$ for which states $\hat{\varrho}$ (and which $I$) is

$$\|\hat{\varrho}_C - \hat{\tau}_C\|_{\text{tr}} \leq \epsilon$$

which states $\hat{\varrho}$ are locally thermal?
for microcanonical states $\hat{\varrho} = \frac{1_{M_\delta}}{|M_\delta|}$
this question goes back to Boltzmann and Gibbs

previous work:

- Thermodynamical functions
  [Lebowitz, Lieb (1969); Lima (1971/72); Touchette (2009)]
- States [Mueller, Adlam, Masanes, Wiebe (2013)]
- Popescu, Short, Winter (2005); Riera, Gogolin, Eisert (2011)
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$
for which states $\hat{\rho}$ (and which $I$) is

$$\| \hat{\rho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon$$

which states $\hat{\rho}$ are locally thermal?

for microcanonical states $\hat{\rho} = \frac{I_{M\delta}}{|M\delta|}$
this question goes back to Boltzmann and Gibbs

here:

- Finite size, explicit bounds
- Not necessarily translational invariant
- More general than microcanonical
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$
for which states $\hat{\varrho}$ (and which $I$) is
\[ \| \hat{\varrho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon \]?
which states $\hat{\varrho}$ are locally thermal?

microcanonical states $\hat{\varrho} = \frac{\mathbb{1}_{M_\delta}}{|M_\delta|}$

with
\[ M_\delta = \{|k\rangle : |E_k - Nu(T)| \leq \delta \sqrt{N} \}, \quad \frac{\log^{2d}(N)}{\sqrt{N}} \lesssim \delta \lesssim 1 \]

and $I$ such that $I^d \lesssim \frac{(\epsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)}$
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$
for which states $\hat{\varrho}$ (and which $I$) is
$$\|\hat{\varrho}_C - \hat{\tau}_C\|_{tr} \leq \epsilon$$?
which states $\hat{\varrho}$ are locally thermal?
microcanonical states $\hat{\varrho} = \frac{\mathbb{1}_{M_\delta}}{|M_\delta|}$
with
$$M_\delta = \{ |k\rangle : |E_k - Nu(T)| \leq \delta \sqrt{N} \}, \quad \frac{\log^{2d}(N)}{\sqrt{N}} \lesssim \delta \lesssim 1$$
and $I$ such that $I^d \lesssim \frac{(\epsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)}$
$\delta = 0 :$ Eigenstate Thermalization
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$

for which states $\hat{\varrho}$ (and which $l$) is

$$\| \hat{\varrho}_C - \hat{\tau}_C \|_{tr} \leq \epsilon$$

which states $\hat{\varrho}$ are locally thermal?

pure states $\hat{\varrho}$ drawn from the subspace spanned by $M_\delta$:

$$\mathbb{P} \left[ \| \hat{\varrho}_C - \text{(m.c.)}_C \|_{tr} \leq \sqrt{\epsilon} + 2^{l^d} / \sqrt{|M_\delta|} \right] \geq 1 - 2e^{-|M_\delta| \epsilon}$$

Popescu, Short, Winter (2005)
canonical state \( \hat{\tau} = e^{-\hat{H}/T}/Z \) for which states \( \hat{\rho} \) (and which \( I \)) is
\[
\| \hat{\rho}_C - \hat{\tau}_C \|_{\text{tr}} \leq \varepsilon
\]
which states \( \hat{\rho} \) are locally thermal?

pure states \( \hat{\rho} \) drawn from the subspace spanned by \( M_\delta \):
\[
\mathbb{P} \left[ \| \hat{\rho}_C - (\text{m.c.})_C \|_{\text{tr}} \leq \sqrt{\varepsilon} + 2^d / \sqrt{|M_\delta|} \right] \geq 1 - 2e^{-|M_\delta|\varepsilon}
\]
Popescu, Short, Winter (2005)

\[
\geq 1 - 2 \exp \left[ -\varepsilon \exp (S(\hat{\tau}) - \log^{2d}(N)\sqrt{N}) \right] =: p
\]
QBE
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$ for which states $\hat{\varrho}$ (and which $I$) is

$$\|\hat{\varrho}_C - \hat{\tau}_C\|_{tr} \leq \epsilon$$

which states $\hat{\varrho}$ are locally thermal?
pure states $\hat{\varrho}$ drawn from the subspace spanned by $M_\delta$:

$\hat{\tau}, M_\delta, \delta, I$ as before $\rightarrow$ with probability at least $p$

$$\|\hat{\varrho}_C - \hat{\tau}_C\|_{tr} \leq \epsilon + 2^I d \exp \left[ - (S(\hat{\tau}) - \log^{2d}(N)\sqrt{N}) \right]$$

canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$ for which states $\hat{\sigma}$ is
\[ \| \hat{\sigma}_C - \hat{\tau}_C \|_{\text{tr}} \leq \varepsilon \]
which states $\hat{\sigma}$ are locally thermal?

$\hat{\tau}, l$ as before then those

- with small free energy $F_T(\hat{\sigma}) \lesssim F_T(\hat{\tau}) + \frac{T \varepsilon^2 (\varepsilon^2 N)^{\frac{1}{d+1}}}{\ln(N)}$

\[ F_T(\hat{\sigma}) = \text{tr}[\hat{H}\hat{\sigma}] - TS(\hat{\sigma}) \]
canonical state $\hat{\tau} = e^{-\hat{H}/T}/Z$

for which states $\hat{\phi}$ is

$$\|\hat{\phi}_C - \hat{\tau}_C\|_{\text{tr}} \leq \epsilon$$

which states $\hat{\phi}$ are locally thermal?

$\hat{\tau}, M_\delta, \delta, I$ as before then those

- with small free energy $F_T(\hat{\phi}) \leq F_T(\hat{\tau}) + \frac{T\epsilon^2(\epsilon^2N)^{\frac{1}{d+1}}}{\ln(N)}$

or

- in microcanonical subspace with large entropy $S(\hat{\phi}) \geq \log(|M_\delta|) - \frac{\epsilon^2(\epsilon^2N)^{\frac{1}{d+1}}}{\ln(N)}$ (in fact, “almost all” states in this subspace)
\[ \hat{\varrho}(t) = e^{-i t \hat{H}} \hat{\varrho}_0 e^{i t \hat{H}} \]

\[ \hat{H} = \sum_k E_k |k\rangle \langle k| \]

\[ \hat{\omega} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt \, \hat{\varrho}(t) \]
\[ \hat{\varrho}(t) = e^{-i t \hat{H}} \hat{\varrho}_0 e^{i t \hat{H}} \]

\[ \hat{H} = \sum_k E_k |k\rangle \langle k| \]

\[ \hat{\omega} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \! dt \, \hat{\varrho}(t) = \sum_k \langle k| \hat{\varrho}_0 |k\rangle |k\rangle \langle k| \]

\[ \text{non-degen. energy gaps} \]

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \! dt \, \| \hat{\varrho}_C(t) - \hat{\omega}_C \|_\text{tr} \leq 2^{\! |C|} \sqrt{\text{tr}[\hat{\omega}^2]} \]
\[ \hat{\mathcal{C}}(t) = e^{-it\hat{H}} \hat{\mathcal{C}}_0 e^{it\hat{H}} \]

\[ \hat{H} = \sum_k E_k |k\rangle \langle k| \]

\[ \hat{\omega} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \hat{\mathcal{C}}(t) = \sum_k \langle k| \hat{\mathcal{C}}_0 |k\rangle |k\rangle \langle k| \]

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \| \hat{\mathcal{C}}(t) - \hat{\omega}_C \|_{tr} \leq 2^{\frac{|C|}{2}} \sqrt{\text{tr}[\hat{\omega}^2]} \]

fraction of times for which
\[ \| \hat{\mathcal{C}}(t) - \hat{\omega}_C \|_{tr} \leq \epsilon \]

is at least
\[ 1 - 2^{\frac{|C|}{2}} \sqrt{\text{tr}[\hat{\omega}^2]}/\epsilon \]

Linden, Popescu, Short, Winter (2008)
\[ \hat{\varphi}(t) = e^{-it\hat{H}} \hat{\varrho}_0 e^{it\hat{H}} \]

\[ \hat{H} = \sum_k E_k |k\rangle \langle k| \]

\[ \hat{\omega} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \hat{\varphi}(t) = \sum_k \langle k| \hat{\varrho}_0 |k\rangle |k\rangle \langle k| \]

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \| \hat{\varphi}_C(t) - \hat{\omega}_C \|_{\text{tr}} \leq 2^{\|C\|} \sqrt{\text{tr}[\hat{\omega}^2]} \]

- Geometry irrelevant
- Even “global” observables
- Also “local” quenches

fraction of times for which 
\[ \| \hat{\varphi}_C(t) - \hat{\omega}_C \|_{\text{tr}} \leq \epsilon \] is at least \[ 1 - 2^{\|C\|} \sqrt{\text{tr}[\hat{\omega}^2]} / \epsilon \]

Linden, Popescu, Short, Winter (2008)
\( \hat{\varrho}(t) = e^{-i t \hat{H}} \hat{\varrho}_0 e^{i t \hat{H}} \)

\( \hat{H} = \sum_k E_k |k\rangle \langle k| \)

\( \hat{\omega} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \! dt \, \hat{\varrho}(t) = \sum_k \langle k| \hat{\varrho}_0 |k\rangle |k\rangle \langle k| \)

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \! dt \| \hat{\varrho}_C(t) - \hat{\omega}_C \|_{\text{tr}} \leq 2^{|C|} \sqrt{\text{tr}[\hat{\omega}^2]} \]

- Purity?
- Thermal?
- Time scale?

fraction of times for which
\[ \| \hat{\varrho}_C(t) - \hat{\omega}_C \|_{\text{tr}} \leq \epsilon \] is at least
\[ 1 - 2^{|C|} \sqrt{\text{tr}[\hat{\omega}^2]} / \epsilon \]

Linden, Popescu, Short, Winter (2008)
Local Hamiltonian, sufficiently weakly correlated initial state: $\text{tr}[\hat{\omega}^2] \lesssim \frac{\ln^{2d}(N)}{\sqrt{N}}$.
Local Hamiltonian, sufficiently weakly correlated initial state: 
\[ \text{tr}[\hat{\omega}^2] \lesssim \frac{\ln^{2d}(N)}{\sqrt{N}}. \]

integrable: no thermalization (instead generalized Gibbs ensemble)
local Hamiltonian, sufficiently weakly correlated initial state: 
\[ \text{tr}[\hat{\omega}^2] \lesssim \frac{\ln^{2d}(N)}{\sqrt{N}} \]

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most Hamiltonians that are unitarily equivalent to a local Hamiltonian lead to fast thermalization*


*the subsystem spends most of the times in \( [0, N^{\frac{1}{3a}}] \) close to the maximally mixed state
local Hamiltonian, sufficiently weakly correlated initial state: \( \text{tr}[\hat{\omega}^2] \lesssim \frac{\ln 2^d (N)}{\sqrt{N}} \)

integrable: no thermalization (instead generalized Gibbs ensemble)

most Hamiltonians that are unitarily equivalent to a local Hamiltonian lead to fast thermalization*


transl. inv., thermodynamic limit: entropic condition on initial state implies thermalization


QBE non-t.i., finite size

*the subsystem spends most of the times in \([0, N^{\frac{1}{2d} - \frac{1}{2}}]\) close to the maximally mixed state