Planar random walk in a stratified quasi-periodic environment

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Abstract

Completing former works [4, 5, 6], we study the recurrence of an inhomogeneous Markov chain in the plane, when the environment is horizontally stratified and the heterogeneity of quasi-periodic type.

1 Introduction

This article investigates the question of the recurrence of a class of inhomogeneous Markov chains in the plane, assuming the environment invariant under horizontal translations. This type of random walks were first considered by Matheron and de Marsily [20] around 1980, with a motivation coming from hydrology and the modelization of pollutants diffusion in a porous and stratified ground. In 2003, a discrete version was introduced by Campanino and Petritis in [7].

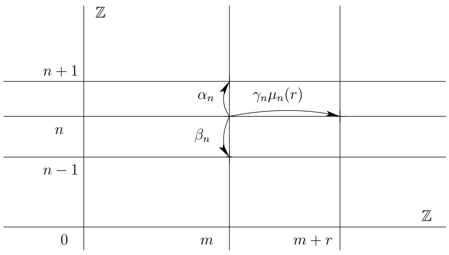
As in [4, 5, 6], we consider an extension of the latter, restricting here to the plane and simplifying a little the hypotheses. We shall define a Markov chain $(S_k)_{k\geq 0}$ in $\mathbb{Z}^2$, starting at the origin, such that the transition laws are constant on each stratum $\mathbb{Z}\times\{n\}$, $n\in\mathbb{Z}$. The first and second coordinates will be respectively called "horizontal" and "vertical". For each (vertical) $n\in\mathbb{Z}$, let positive reals $\alpha_n,\beta_n,\gamma_n$, with $\alpha_n+\beta_n+\gamma_n=1$, and a probability measure μ_n so that:

Hypothesis 1.1.
$\exists\eta>0$, $\forall n\in\mathbb{Z}$, $\min\{\alpha_n,\beta_n,\gamma_n\}\geq\eta$, $Supp(\mu_n)\subset\mathbb{Z}\cap]-1/\eta,1/\eta[$, $\mu_n(0)\leq 1-\eta$.

The transition laws of $(S_k)_{k\geq 0}$ are defined, for all $(m,n)\in\mathbb{Z}^2$ and $r\in\mathbb{Z}$, by:

$$(m,n)\xrightarrow{\alpha_n}(m,n+1),\quad (m,n)\xrightarrow{\beta_n}(m,n-1),\quad (m,n)\xrightarrow{\gamma_n\mu_n(r)}(m+r,n).$$

Here is the corresponding picture:



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The family of transition laws is called the environment, here identified to $((\alpha_n, \beta_n, \gamma_n, \mu_n))_{n\in\mathbb{Z}}$. Introduce the local horizontal drift $\varepsilon_n := \sum_{r\in\mathbb{Z}} r\mu_n(r)$ at each vertical n, i.e. the expectation of μ_n. The special case when $\alpha_n = \beta_n$, $n \in \mathbb{Z}$, is called the "vertically flat model".

With respect to a fully inhomogeneous random walk in the plane, the horizontal stratification of the environment brings the notable simplification that the vertical component of $(S_k)_{k\geq 0}$, in restriction to vertical jumps, is a one-dimensional Markov chain. We call it the "vertical random walk". With $\alpha'_n = \alpha_n/(\alpha_n + \beta_n)$ and $\beta'_n = \beta_n/(\alpha_n + \beta_n)$, its transition laws on $\mathbb{Z}$ are:

$$n \xrightarrow{\alpha'_n} n+1, \quad n \xrightarrow{\beta'_n} n-1.$$

From this, the model inherits some "product structure". For instance, for $(S_k)_{k\geq 0}$ to be recurrent itself, the vertical random walk has first to be. The conditions for this are known for a long time in the context of birth and death processes (cf Karlin and McGregor [16]). Placing in this case, the study of the recurrence of (S_k) essentially reduces to the analysis of the horizontal displacement.

Let us discuss former results concerning the recurrence/transience of such a Markov chain, first for the vertically flat model. The vertical random walk is then simple random walk on $\mathbb{Z}$, which is well-known to be recurrent. The main object of study has been the Campanino-Petritis model [7], corresponding to $\alpha_n = \beta_n = \gamma_n = 1/3$ and $\mu_n = \delta_{x_n}$, for some $(x_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$, where δ_x is Dirac measure at x. Recurrence is shown in [7] when $x_n = (-1)^n$ and transience when $x_n = 1_{n\geq 1} - 1_{n\leq 0}$ or if (x_n) are typical realizations of independent and identically distributed (*i.i.d.*) random variables with law $(\delta_1 + \delta_{-1})/2$. For random (x_n), transience results were obtained by Guillotin-Plantard and Le Ny [15], when the (x_n) are independent, but with different marginals, and also by Pène [21], under some hypotheses of stationarity and decorrelation. Devulder and Pène [12] showed transience for a more general vertically flat model, where $\alpha_n = \beta_n$, the (γ_n) are *i.i.d.* non-constant and $\mu_n = \delta_{x_n}$, for an arbitrary $(x_n) \in \{\pm 1\}^{\mathbb{Z}}$. In [8], for their initial model, Campanino and Petritis studied the case of a random perturbation of a periodic (x_n).

In [4], for the general vertically flat case, a complete recurrence criterion was given. The asymptotic behaviour of the random walk is governed by the sums $(\gamma_{-m}\varepsilon_{-m}/\alpha_{-m} + \cdots + \gamma_{n-1}\varepsilon_{n-1}/\alpha_{n-1})_{-m\leq 0\leq n}$, associated with some horizontal flow defined by the environment and transverse to the vertical layer $[-m, n)$. The central role is played by a two-variable function $\Phi(-m, n)$, introduced below, measuring the "horizontal dispersion" of the previous flow between vertical levels $-m$ and n. The quantity deciding for the recurrence/transience of $(S_k)_{k\geq 0}$ computes some "capacity of dispersion to infinity" of the environment. The abstract form of the criterion in [4] is directly extracted from a Poisson kernel in a half-plane. It seems to be related to some notion of curvature at infinity of the level lines of the function $\Phi(-m, n)$. Several examples were next presented in [4]. Roughly, a growth condition such as $(\log n)^{1+\delta}$ on the one-sided sums $(\gamma_0\varepsilon_0/\alpha_0 + \cdots + \gamma_{n-1}\varepsilon_{n-1}/\alpha_{n-1})$ is sufficient for transience, confirming the natural prevalence of transience results in the litterature on this model.

Extending [4], for the general model where α_n need not equal β_n, a full characterization of the recurrence regime was shown in [5]. With some naturally generalized $\Phi(-m, n)$ (cf Definition 4.7 below), the form of the criterion is the same, emphasizing the fact that the environment defines a new metrization of $\mathbb{Z}^2$. The model appears notably more general than the vertically flat one. Several examples were next given in [5]. However, an empirical observation is that the methods employed for obtaining the structural results of [4, 5] are of very different nature than that used to treat examples. The analysis is in fact naturally divided in two parts, the second one never entering the mechanism of the random walk itself and consisting in studying fine properties of certain ergodic sums. It is a source of interesting and difficult problems, for example closely related to temporal limit theorems and generalizations (cf Dolgopyat-Sarig [13]). In [6], for the general model, the particular case when the transition laws are independent was studied in detail, with a precise quantification of the non-surprising fact that the transience regime largely prevails in the set of parameters.

The purpose of the present article is to complete [4, 5, 6], by extending the applications of [4, 5]. We study for both the vertically flat and the general model the case when the transition laws are described by functions defined above an irrational rotation on the one-dimensional torus.

2 Preliminaries

Let $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ be the one-dimensional torus. Unless otherwise stated, functions are defined on $\mathbb{T}$, with arguments understood modulo one. We write $\|x\|$ for the distance of a real x to $\mathbb{Z}$.

We first recall classical facts about continued fractions. On this topic, one may consult Khinchin's book [18]. Any irrational $0 < \theta < 1$ admits an infinite continued fraction expansion:

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [0, a_1, a_2, \cdots],$$

where the partial quotients $(a_i)_{i\geq 1}$ are integers ≥ 1, obtained by successive iterations of the Gauss map $x \longmapsto \{1/x\}$, starting from θ. The convergents $(p_n/q_n)_{n\geq 1}$ of θ are obtained as the successive truncations $[0, a_1, a_2, \cdots, a_n] = p_n/q_n$, for $n \geq 1$, of this continued fraction. The numerators (p_n) and denominators (q_n) check the same recurrence relation:

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 0,$$

with initial data $p_0 = 0, p_{-1} = 1$ and $q_0 = 1, q_{-1} = 0$. Classically (cf [18], chap. 1):

$$\frac{1}{2q_{n+1}} \leq \frac{1}{q_n + q_{n+1}} \leq \|q_n\theta\| \leq \frac{1}{q_{n+1}}. \tag{1}$$

Fixing $\theta \notin \mathbb{Q}$, we consider the rotation $Tx = x + \theta \mod 1$ on $\mathbb{T}$ and write $T^n f$ for $f \circ T^n$, for any $f : \mathbb{T} \to \mathbb{R}$. We also use cocycle notations, for $x \in \mathbb{T}$:

$$f_n(x) = \begin{cases} f(x) + \cdots + f(T^{n-1}x), & n \geq 1, \\ 0, & n = 0, \\ -f(T^n x) - \cdots - f(T^{-1}x), & n \leq -1. \end{cases}$$

An important property is that $f_{n+p}(x) = f_n(x) + T^n f_p(x)$, for any $x \in \mathbb{T}$ and $n, p \in \mathbb{Z}$.

A function $f : \mathbb{T} \longmapsto \mathbb{R}$ with bounded variation will be said BV, with total variation written as $V(f)$. When f is BV, with $\int_{\mathbb{T}} f(x)\, dx = 0$, the Denjoy-Koksma inequality says that:

$$|f_{q_n}(x)| \leq V(f), \qquad n \geq 1, x \in \mathbb{T}. \tag{2}$$

Let us now recall known facts concerning Ostrowski's expansions (cf Beck [3], p. 23). Every integer $q_m \leq n < q_{m+1}$ can be represented as:

$$n = \sum_{0\leq k\leq m} b_k q_k, \tag{3}$$

with $0 \le b_0 \le a_1 - 1$, $0 \le b_j \le a_{j+1}$, $1 \le j < m$, and $1 \le b_m \le a_{m+1}$. Indeed, $n = b_m q_m + r$, for some $0 \le r < q_m$ and $1 \le b_m \le a_{m+1}$. Iterating this process furnishes the decomposition (3). Setting $A_{-1} = 0$ and $A_k = \sum_{0 \le j \le k} b_j q_j$, for $0 \le k < m$, by (3), we have for any function f:

$$f_n(x) = \sum_{k=0}^{m} f_{b_k q_k}(x + A_{k-1}\theta).$$

When f is BV and centered, the Denjoy-Koksma inequality (2) furnishes the upper-bound:

$$|f_n(x)| \le \sum_{0 \le k \le m} \|f_{q_k}\|_\infty b_k \le V(f) \sum_{0 \le k \le m} b_k, \quad x \in \mathbb{T}. \tag{4}$$

Set $\mathbb{N} = \{0, 1, \cdots\}$. For $g : \mathbb{N} \to \mathbb{R}_+$ increasing to $+\infty$ and $x \ge g(0)$, let $g^{-1}(x)$ be the unique integer $n \ge 0$ such that $g(n) \le x < g(n+1)$. By definition:

$$g(g^{-1}(x)) \le x < g(g^{-1}(x) + 1). \tag{5}$$

Also, $g^{-1}(g(n)) = n$, for large $n \in \mathbb{N}$. Finally, for $f, g : \mathbb{N} \to \mathbb{R}_+$, we write $g \preceq f$ if there exists a constant $C > 0$ so that $g(n) \le C f(n)$, for large $n \in \mathbb{N}$. We write $f \asymp g$ if $g \preceq f$ and $f \preceq g$.

3 The quasi-periodic vertically flat model

We first consider the vertically flat model, i.e. $\alpha_n = \beta_n = (1-\gamma_n)/2$, $n \in \mathbb{Z}$. As a preliminary remark, we discuss the case when the sequence $(\varepsilon_n \gamma_n/(1-\gamma_n))_{n\in\mathbb{Z}}$ is periodic.

Proposition 3.1.
For the vertically flat model, let $(\varepsilon_n \gamma_n/(1-\gamma_n))_{n\in\mathbb{Z}}$ be periodic with period $N \ge 1$. Then the random walk is either recurrent or transient, according to whether:

$$\sum_{0 \le n < N} \varepsilon_n \gamma_n/(1-\gamma_n) = 0 \text{ or } \neq 0.$$

This follows from [4], respectively Prop 1.4 *i)* and Corollary 1.3 *i)*. This extends the case of the Campanino-Petritis model in [7], when $\mu_n = \delta_{x_n}$ with $x_n = \varepsilon_n = (-1)^n$, $\alpha_n = \beta_n = \gamma_n = 1/3$.

Turning to quasi-periodic situations, we shall generalize [4], Prop. 1.5, giving in particular a better understanding of the Campanino-Petritis model in this quasi-periodic context.

Theorem 3.2.
Let $\theta = [0, a_1, a_2, \cdots] \notin \mathbb{Q}$, with $\sum_{n\ge1} \log(1+a_n)/(a_1 + \cdots + a_n) = +\infty$, and $Tx = x + \theta \mod 1$ on $\mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ be piecewise K-Lipschitz, with zero mean. Under Hypothesis 1.1, let $\alpha_n = \beta_n$ and $\varepsilon_n \gamma_n/(1-\gamma_n) = f(n\theta)$, $n \in \mathbb{Z}$. Then the random walk is recurrent.

Remark. — This is shown in [4], Prop. 1.5, for $f = (1_{[0,1/2)} - 1_{[1/2,1)})/2$, corresponding to the Campanino-Petritis model with $\mu_n = \delta_{x_n}$ and $x_n = 1_{[0,1/2)}(n\theta) - 1_{[1/2,1)}(n\theta)$. As a consequence of the above theorem, the random walk is recurrent when $x_n = 1_{[0,1/2)}(x+n\theta) - 1_{[1/2,1)}(x+n\theta)$, $n \in \mathbb{Z}$, for any $x \in \mathbb{T}$, taking $f(.+x)$. As noticed in [5], Prop. 7.1, the condition on θ is generic in measure, since $\sum_{n\ge1} 1/(a_1 + \cdots + a_n) = +\infty$, for a.-e. θ, cf Khinchin [17].

The other direction is in general more delicate, since requiring lower bounds on the ergodic sums. We just give an example.

Proposition 3.3.
Let $\theta = [0, a_1, a_2, \cdots] \notin \mathbb{Q}$, with a_1 odd and a_n even for $n \geq 2$. We suppose that for some $\delta > 1$, $a_{n+1} \geq (a_n)^\delta$, for large n. Let $f = 1_{[0,1/2)} - 1_{[1/2,1)}$. Under Hypothesis 1.1, let $\alpha_n = \beta_n$ and $\varepsilon_n \gamma_n/(1-\gamma_n) = f(x+n\theta)$, $n \in \mathbb{Z}$, for some $x \in \mathbb{T}$. Then, for Lebesgue almost-every $x \in \mathbb{T}$, the random walk is transient.

Remark. — In the last proposition, one may take for example the angle $\theta \in (0,1)$ defined by the partial quotients $a_n = 2^{2^{n-1}-1}$, $n \geq 1$.

3.1 Proof of Theorem 3.2

Fix $\theta = [0, a_1, a_2, \cdots, \cdots] \notin \mathbb{Q}$, $Tx = x + \theta \mod 1$ on $\mathbb{T}$ and f, as in the statement of the theorem. Using cocycle notations $(f_n(x))_{n \in \mathbb{Z}}$, introduce for $n \geq 1$ the following positive functions $\varphi(n)$ and $\varphi_+(n)$ such that:

$$\varphi^2(n) = n^2 + \sum_{-n \leq k < \ell \leq n} (f_\ell(x) - f_k(x))^2 \text{ and } \varphi_+^2(n) = n^2 + \sum_{-n \leq k < \ell \leq n, k\ell \geq 0} (f_\ell(x) - f_k(x))^2.$$

The dependence on x of $\varphi(n)$ and $\varphi_+(n)$ is implicit. Obviously, $n \leq \varphi_+(n) \leq \varphi(n)$. The next lemma gives some control in the other direction.

Lemma 3.4.
There exists a constant $C_0 > 0$, uniform on $x \in \mathbb{T}$, such that for all $n \geq 1$ and $1 \leq m \leq 4a_{n+1}$:

$$\varphi^2(mq_n) \leq 2\varphi_+^2(mq_n) + C_0 m^4 q_n^2.$$

Proof of the lemma:
Step 1. In the sequel, we simplify $f_n(x)$ into f_n, $n \in \mathbb{Z}$. Setting $A = \sum_{-n \leq k \leq -1, 1 \leq \ell \leq n} (f_\ell - f_k)^2$, we have $\varphi^2(n) = \varphi_+^2(n) + A$. Then:

$$A = n \sum_{1 \leq \ell \leq n} f_\ell^2 + n \sum_{-n \leq k \leq -1} f_k^2 - 2 \sum_{-n \leq k \leq -1} f_k \sum_{1 \leq \ell \leq n} f_\ell. \tag{6}$$

We next have:

$$-2 \sum_{-n \leq k \leq -1} f_k \sum_{1 \leq \ell \leq n} f_\ell = \left(\sum_{1 \leq \ell \leq n} (f_\ell - f_{-\ell}) \right)^2 - \left(\sum_{1 \leq \ell \leq n} f_\ell \right)^2 - \left(\sum_{1 \leq \ell \leq n} f_{-\ell} \right)^2. \tag{7}$$

Now, classically:

$$\begin{aligned}
\sum_{1 \leq k < \ell \leq n} (f_\ell - f_k)^2 &= \sum_{2 \leq \ell \leq n} (\ell - 1) f_\ell^2 + \sum_{1 \leq \ell \leq n-1} (n - \ell) f_\ell^2 - 2 \sum_{1 \leq k < \ell \leq n} f_k f_\ell \\
&= \sum_{1 \leq \ell \leq n} (\ell - 1) f_\ell^2 + \sum_{1 \leq \ell \leq n} (n - \ell) f_\ell^2 - 2 \sum_{1 \leq k < \ell \leq n} f_k f_\ell \\
&= n \sum_{1 \leq \ell \leq n} f_\ell^2 - \left(\sum_{1 \leq \ell \leq n} f_\ell \right)^2.
\end{aligned} \tag{8}$$

Proceeding symmetrically for the other part of A, we obtain from (6), (7) and (8):

$$A = \sum_{1\le k<\ell\le n}(f_\ell - f_k)^2 + \sum_{-n\le k<\ell\le -1}(f_\ell - f_k)^2 + \left(\sum_{1\le \ell\le n}(f_\ell - f_{-\ell})\right)^2.$$

Consequently:

$$\varphi^2(n) \le 2\varphi_+^2(n) + \left(\sum_{1\le \ell\le n}(f_\ell - f_{-\ell})\right)^2.$$

Step 2. Let $n \ge 1$ and $1 \le m \le 4a_{n+1}$. Setting $B = \sum_{1\le\ell\le mq_n}(f_\ell - f_{-\ell})$, we have:

$$B = \sum_{0\le u<m}\sum_{1\le\ell\le q_n}(f_{uq_n+\ell} - f_{-uq_n-\ell}) = \sum_{0\le u<m}\sum_{1\le\ell\le q_n}(f_{uq_n} - f_{-uq_n} + T^{uq_n}f_\ell - T^{-uq_n}f_{-\ell}).$$

Using Denjoy-Koksma's inequality (2), for any integer $0 \le u < m$, we have $|f_{uq_n}(x)| \le uV(f)$, idem for $f_{-uq_n}(x)$. As a result:

$$B = O(m^2 q_n) + \sum_{0\le u<m}\sum_{1\le\ell\le q_n}(T^{uq_n}f_\ell - T^{-uq_n}f_{-\ell}).$$

Fixing any integer $0 \le u < m$, we have :

$$\sum_{1\le\ell\le q_n}(T^{uq_n}f_\ell - T^{-uq_n}f_{-\ell}) = \sum_{k=0}^{q_n-1}(q_n-k)f(x+uq_n\theta+k\theta) + \sum_{k=1}^{q_n}(q_n+1-k)f(x-uq_n\theta-k\theta).$$

Using the Denjoy-Koksma inequality (2) for the (q_n+1)-term in the second sum on the right hand side and making a change of variable in the first one, we get:

$$\begin{aligned}\sum_{1\le\ell\le q_n}(T^{uq_n}f_\ell - T^{-uq_n}f_{-\ell}) &= \sum_{k=1}^{q_n}k(f(x+uq_n\theta+(q_n-k)\theta) - f(x-uq_n\theta-k\theta)) + O(q_n)\\ &= \sum_{k=1}^{q_n}k(f(x_+^u-k\theta) - f(x_-^u-k\theta)) + O(q_n),\end{aligned}$$

when setting $x_+^u = x + (u+1)q_n\theta$ and $x_-^u = x - uq_n\theta$. By (1) and the hypothesis $m \le 4a_{n+1}$:

$$\|x_+^u - x_-^u\| \le (2u+1)\|q_n\theta\| \le (8a_{n+1}+1)/q_{n+1} \le 9/q_n.$$

Denote by $[x_-^u, x_+^u]$ the short interval on $\mathbb{T}$ between x_-^u and x_+^u and by D the number of discontinuities of f. Recall also that there is exactly one $k\theta \mod 1$, $1 \le k \le q_n$, in each interval $[\ell/q_n, (\ell+1)/q_n)$, $0 \le \ell < q_n$, on $\mathbb{T}$, since $k\theta = kp_n/q_n + k(\theta - p_n/q_n)$ and $k\|\theta - p_n/q_n\| \le k/(q_nq_{n+1}) < 1/q_n$, for $1 \le k \le q_n$. As a result, for a given discontinuity of f, there at most 10 values of $1 \leq k \leq q_n$ such that $[x_-^u, x_+^u] - k\theta$ contains this discontinuity. Hence, using that f is K-Lipschitz on intervals containing no discontinuity, an upper-bound for the last sum is:

$$D \times 10q_n \times 2\|f\|_\infty + q_n^2 \times K \times \frac{9}{q_n} = O(q_n),$$

As a result, $B = O(m^2 q_n) + O(mq_n) = O(m^2 q_n)$, which ends the proof of the lemma.

□

We turn to the proof of Theorem 3.2. We assume that $\alpha_n = \beta_n$ and that μ_n and thus ε_n are such that $\varepsilon_n\gamma_n/(1-\gamma_n) = f(x+n\theta)$, $n \in \mathbb{Z}$, for some $x \in \mathbb{T}$. The statement of the theorem corresponds to $x = 0$. Introduce the following definition, due to Feller [14]:

Definition 3.5.
A non-decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies dominated variation, if there exists a constant $C > 0$ so that for large $x > 0$, $g(2x) \leq Cg(x)$. Hence, iterating, for all $K > 0$, there exists C_K so that for large $x > 0$, $g(Kx) \leq C_K g(x)$.

In [4], setting $R_k^\ell = \sum_{k \leq i \leq \ell} \varepsilon_i\gamma_i/(1-\gamma_i)$, for integers $k \leq \ell$, we considered the two functions:

$$\Phi^2(n) = n^2 + \sum_{-n \leq k \leq \ell \leq n} (R_k^\ell)^2, \ \Phi_+^2(n) = n^2 + \sum_{-n \leq k \leq \ell \leq n, k\ell > 0} (R_k^\ell)^2,$$

The following results were then established:

Theorem 3.6. *([4], Lemma 6.1, Theorem 1.2. and Corollary 1.3 i))*
1) The functions Φ^{-1} and Φ_+^{-1} satisfy dominated variation with a constant $C = C(\eta)$ depending only on η, where η is introduced in Hypothesis 1.1.

2) The random walk is recurrent if and only if $\sum_{n\geq 1} n^{-2}(\Phi^{-1}(n))^2/\Phi_+^{-1}(n) = +\infty$.

3) The condition $\sum_{n\geq 1} 1/\Phi_+(n) < +\infty$ is sufficient for transience.

Using that f is bounded and that $\big||R_k^\ell| - |f_\ell(x) - f_k(x)|\big| \leq \|f\|_\infty$, $k \leq \ell$, it is immediate that for some constant $C > 1$ depending only on η (hence uniform on $x \in \mathbb{T}$), for all $n \geq 1$:

$$\Phi(n)/C \leq \varphi(n) \leq C\Phi(n) \text{ and } \Phi_+(n)/C \leq \varphi_+(n) \leq C\Phi_+(n).$$

The first inequalities furnish for example that for large $x > 0$, $\Phi^{-1}(x/C) \leq \varphi^{-1}(x) \leq \Phi^{-1}(Cx)$. Hence the previous theorem implies the following result.

Corollary 3.7.
1) The functions φ^{-1} and φ_+^{-1} satisfy dominated variation, i.e. for any $K > 0$, there exists a constant $C_K > 0$, independent of $x \in \mathbb{T}$, so that for large $y > 0$:

$$\varphi^{-1}(Ky) \leq C_K\varphi^{-1}(y) \text{ and } \varphi_+^{-1}(Ky) \leq C_K\varphi_+^{-1}(y). \tag{9}$$

2) The random walk is recurrent if and only if $\sum_{n\geq 1} n^{-2}(\varphi^{-1}(n))^2/(\varphi_+^{-1}(n)) = +\infty$.

3) The condition $\sum_{n\geq 1} 1/\varphi_+(n) < +\infty$ is sufficient for transience.

Let us reprove a concrete version of dominated variation of φ^{-1} and φ_+^{-1} in the following lemma. Concerning for example φ_+, we essentially show that $n \longmapsto \varphi_+^2(n)/n$, $n > 0$, is non-decreasing. For $a < b$ in $\mathbb{Z}$ and $x \in \mathbb{T}$, let:

$$\psi(a,b) = \sum_{a \leq k < \ell \leq b} (f_\ell(x) - f_k(x))^2, \tag{10}$$

where the dependence on x is implicit on the left hand side.

Lemma 3.8.
Let integers $a < b < c$ and $x \in \mathbb{T}$. Then:

$$\frac{\psi(a,c)}{c-a} \geq \frac{\psi(a,b-1)}{b-a} + \frac{\psi(b+1,c)}{c-b}.$$

Also, for large n, uniformly in $x \in \mathbb{T}$:

$$\varphi_+(2n) \geq 2^{1/4}\varphi_+(n). \tag{11}$$

Proof of the lemma:
We write f_k in place of $f_k(x)$. Decompose $\psi(a,c) = \psi(a,b) + \psi(b,c) + \sum_{a \leq k < b < \ell \leq c}(f_\ell - f_k)^2$ and then expend:

$$\sum_{a \leq k < b < \ell \leq c} (f_\ell - f_k)^2 \quad = \quad (b-a)\sum_{b < \ell \leq c} f_\ell^2 + (c-b)\sum_{a \leq k < b} f_k^2 - 2\sum_{b < \ell \leq c} f_\ell \sum_{a \leq k < b} f_k.$$

As before, $\psi(b+1,c) = (c-b)\sum_{b<\ell\leq c} f_\ell^2 - (\sum_{b<\ell\leq c} f_\ell)^2$ and $\psi(a,b-1) = (b-a)\sum_{a\leq k<b} f_k^2 - (\sum_{a\leq k<b} f_k)^2$. When substituting:

$$\begin{aligned}\sum_{a \leq k < b < \ell \leq c} (f_\ell - f_k)^2 \quad &= \quad \frac{b-a}{c-b}\left(\psi(b+1,c) + (\sum_{b<\ell\leq c} f_\ell)^2\right) \\ &+ \quad \frac{c-b}{b-a}\left(\psi(a,b-1) + (\sum_{a\leq k<b} f_k)^2\right) - 2\sum_{b<\ell\leq c} f_\ell \sum_{a\leq k<b} f_k.\end{aligned}$$

As a consequence:

$$\begin{aligned}\psi(a,c) \quad &\geq \quad \frac{c-a}{c-b}\psi(b+1,c) + \frac{c-a}{b-a}\psi(a,b-1) \\ &+ \quad \left(\sqrt{\frac{b-a}{c-b}}\sum_{b<\ell\leq c} f_\ell - \sqrt{\frac{c-b}{b-a}}\sum_{a\leq k<b} f_k\right)^2,\end{aligned} \tag{12}$$

giving the result. Concerning the last part of the lemma, for any integer $n \geq 1$, we have:

$$\psi(0,2n) \geq \frac{2n}{n+1}\psi(0,n), \;\; \psi(-2n,0) \geq \frac{2n}{n+1}\psi(-n,0).$$

Since $\varphi_+^2(n) = n^2 + \psi(-n,0) + \psi(0,n)$, we get (11).

$\square$

Remark. — As a complement, let us observe that:

$$\frac{c-a}{c-b}\psi(b+1,c) + \frac{c-a}{b-a}\psi(a,b-1) \geq (\sqrt{\psi(a,b-1)} + \sqrt{\psi(b+1,c)})^2, \tag{13}$$

as this is equivalent to the true relation:

$$\frac{c-b}{b-a}\psi(a,b-1) + \frac{b-a}{c-b}\psi(b+1,c) \geq 2\sqrt{\psi(a,b-1)}\sqrt{\psi(b+1,c)}.$$

Using (13) in (12), this thus implies some reverse triangular inequality:

$$\sqrt{\psi(a,c)} \geq \sqrt{\psi(a,b-1)} + \sqrt{\psi(b+1,c)}.$$

We start the proof of Theorem 3.2. A corollary of Lemma 3.4 is that there exists a constant $C_0 > 0$, independent of $x \in \mathbb{T}$, such that for all $n \geq 1$ and $1 \leq m \leq 4a_{n+1}$:

$$\varphi(mq_n) \leq C_0(\varphi_+(mq_n) + m^2 q_n). \tag{14}$$

Let now $n \geq 1$ and $\ell \geq 0$ be such that $2^\ell \leq 4a_{n+1}$. We make the following discussion:

- *Case 1.* If $\varphi_+(2^\ell q_n) \geq 2^{2\ell} q_n$, then we obtain, using that φ^{-1} is non-decreasing, next (14) and (9) (as well $\varphi^{-1}(\varphi(m)) = m$, for any large integer $m > 0$) at the end:

$$\begin{aligned}\varphi^{-1}(\varphi_+(2^\ell q_n)) \geq \varphi^{-1}((\varphi_+(2^\ell q_n) + 2^{2\ell} q_n)/2) &\geq \varphi^{-1}(\varphi(2^\ell q_n)/(2C_0)) \\ &\geq \frac{\varphi^{-1}(\varphi(2^\ell q_n))}{C_{2C_0}} = \frac{2^\ell q_n}{C_{2C_0}}.\end{aligned}$$

By Ostrowski's expansion (3), for $|k| \leq 2^\ell q_n \leq 4a_{n+1}q_n \leq 4q_{n+1}$, we get that $|f_k| \leq 4 \times V(f)(a_1 + \cdots + a_{n+1})$. Hence, for some $C > 0$, $\varphi_+(2^\ell q_n) \leq C 2^\ell q_n (a_1 + \cdots + a_{n+1})$. Thus, with $C_1 = 1/(CC_{2C_0})$, for large n:

$$\frac{(\varphi^{-1}(\varphi_+(2^\ell q_n)))^2}{2^\ell q_n \varphi_+(2^\ell q_n)} \geq \frac{1}{C_{2C_0}} \frac{2^\ell q_n}{\varphi_+(2^\ell q_n)} \geq \frac{C_1}{a_1 + \cdots + a_{n+1}}. \tag{15}$$

- *Case 2.* Suppose that $\varphi_+(2^\ell q_n) < 2^{2\ell} q_n$. As $\varphi_+(2^\ell q_n) \geq 2^\ell q_n$, there exists $0 \leq \ell' \leq \ell$ so that $2^{2\ell'} q_n \leq \varphi_+(2^\ell q_n) < 2^{2(\ell'+1)} q_n$. We get in this case, using (14) and (9):

$$\begin{aligned}\varphi^{-1}(\varphi_+(2^\ell q_n)) &\geq \varphi^{-1}(2^{2\ell'} q_n) \geq \varphi^{-1}((\varphi_+(2^{\ell'} q_n) + 2^{2\ell'} q_n)/5) \geq \varphi^{-1}(\varphi(2^{\ell'} q_n)/(5C_0)) \\ &\geq \frac{\varphi^{-1}(\varphi(2^{\ell'} q_n))}{C_{5C_0}} = \frac{2^{\ell'} q_n}{C_{5C_0}}.\end{aligned}$$

As a result, we can write in this case, redefining $C_1 := \min\{C_1, 1/(16C_{5C_0})\} > 0$:

$$\frac{(\varphi^{-1}(\varphi_+(2^\ell q_n)))^2}{2^\ell q_n \varphi_+(2^\ell q_n)} \geq \frac{1}{C_{5C_0}} \frac{2^{2\ell'} q_n^2}{2^\ell q_n \varphi_+(2^\ell q_n)} \geq \frac{1}{4C_{5C_0} 2^\ell} \geq \frac{C_1}{a_1 + \cdots + a_{n+1}}.$$

Hence, (15) is true for any $\ell \geq 0$ such that $2^\ell \leq 4a_{n+1}$. This now gives, for large $n > 0$:

$$\begin{aligned}
\sum_{\varphi_+(q_n) \leq k < \varphi_+(4a_{n+1}q_n)} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} &\geq \sum_{0 \leq \ell \leq 1 + \log_2 a_{n+1}} \sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \\
&\geq \sum_{0 \leq \ell \leq 1 + \log_2 a_{n+1}} \frac{(\varphi^{-1}(\varphi_+(2^\ell q_n)))^2}{\varphi_+^{-1}(\varphi_+(2^{\ell+1} q_n))} \sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2} \\
&\geq \sum_{0 \leq \ell \leq 1 + \log_2 a_{n+1}} \frac{(\varphi^{-1}(\varphi_+(2^\ell q_n)))^2}{2^{\ell+1} q_n} \sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2}.
\end{aligned}$$

Using now relation (15), we arrive at:

$$\sum_{\varphi_+(q_n) \leq k < \varphi_+(4a_{n+1}q_n)} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq \frac{C_1}{2(a_1 + \cdots + a_{n+1})} \sum_{0 \leq \ell \leq 1 + \log_2 a_{n+1}} \varphi_+(2^\ell q_n) \sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2}.$$

Using that $1/k^2 \geq 1/k - 1/(k+1)$, we obtain:

$$\sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2} \geq \frac{1}{\varphi_+(2^\ell q_n) + 1} - \frac{1}{\varphi_+(2^{\ell+1} q_n) - 1}.$$

Now, for large n (uniformly in $\ell \geq 0$), applying (11), we have $\varphi_+(2^{\ell+1} q_n) \geq 2^{1/4} \varphi_+(2^\ell q_n)$. As a result, for large n, we obtain:

$$\sum_{\varphi_+(2^\ell q_n) \leq k < \varphi_+(2^{\ell+1} q_n)} \frac{1}{k^2} \geq \frac{1 - 2^{-1/4}}{2} \frac{1}{\varphi_+(2^\ell q_n)}.$$

This thus furnishes, for large $n > 0$:

$$\sum_{\varphi_+(q_n) \leq k < \varphi_+(4a_{n+1}q_n)} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq \frac{C_1(1 - 2^{-1/4})}{4(a_1 + \cdots + a_{n+1})} (1 + \log_2 a_{n+1}).$$

Finally, notice that $4a_{n+1} q_n \leq q_{n+5}$, for $n \geq 0$. Hence, for large $N > 0$:

$$\sum_{k \geq \varphi_+(q_N)} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq \frac{1}{5} \sum_{n \geq N} \sum_{\varphi_+(q_n) \leq k < \varphi_+(q_{n+5})} \frac{1}{k^2} \frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq \frac{C_1(1 - 2^{-1/4})}{5} \sum_{n \geq N} \frac{(1 + \log_2 a_{n+1})}{4(a_1 + \cdots + a_{n+1})}.$$

By hypothesis, the series on the right-hand side diverges. From Corollary 3.7, we conclude that the random walk is recurrent.

□

3.2 Proof of Proposition 3.3

Let us place in the context of this proposition, namely $\alpha_n = \beta_n$ and $\varepsilon_n \gamma_n/(1-\gamma_n) = f(x+n\theta)$, $n \in \mathbb{Z}$, for some $x \in \mathbb{T}$. Here $f = 1_{[0,1/2)} - 1_{[1/2,1)}$ and the angle $\theta = [0, a_1, a_2, \cdots]$ is such that a_1 is odd and a_n is even for $n \geq 2$, together with $a_{n+1} \geq (a_n)^\delta$, for large n, for some fixed $\delta > 1$.

From the relation $q_{n+1} = a_{n+1} q_n + q_{n-1}$, $n \geq 0$, and $q_0 = 1$, $q_{-1} = 0$, we recursively obtain that q_n is odd, $n \geq 1$. This implies that for all $n \geq 1$ and $x \in \mathbb{T}$, $|f_{q_n}(x)| \geq 1$, since the number of $(T^k x)_{0 \leq k < q_n}$ that fall in the intervals $[0,1/2)$ and $[1/2,1)$ are different.

Let $2/3 < \beta < 1$ and define $m_k = (a_{k+1})^\beta$, $k \geq 1$. Introduce :

$$A_k = \{x \in \mathbb{T}, f_{m q_k}(x) = m f_{q_k}(x),\ 0 \leq m \leq m_k\}.$$

Write $\mathcal{L}_\mathbb{T}$ for Lebesgue measure on $\mathbb{T}$ and denote by $[x, x+q_k\theta]$ the short interval on $\mathbb{T}$ determined by x and $x + q_k\theta$. If $T^r[x, x+q_k\theta]$ does not contain either 0 or 1/2, for $0 \leq r < m_k q_k$, then $x \in A_k$, because $f(T^r x) = f(T^{r+q_k} x)$, for $0 \leq r < m_k q_k$. Hence, if $x \in \mathbb{T}\backslash A_k$, there exist $0 \leq r < m_k q_k$ such that either 0 or 1/2 belongs to $T^r[x, x+q_k\theta]$. As a result, $\mathbb{T}\backslash A_k \subset \cup_{0 \leq r < m_k q_k, y \in \{0,1/2\}}([y - q_k\theta, y] - r\theta)$. Using (1):

$$\mathcal{L}_\mathbb{T}(\mathbb{T}\backslash A_k) \leq \frac{2 m_k q_k}{q_{k+1}} \leq \frac{2 m_k}{a_{k+1}} \leq 2(a_{k+1})^{-(1-\beta)}.$$

Since (a_k) grows at least geometrically, $\sum_k a_k^{-(1-\beta)} < +\infty$. By the first lemma of Borel-Cantelli, we deduce that for Lebesgue a.-e. x, $x \in A_k$ for large k.

Let $N_k = a_1 + \cdots + a_k$ and observe that $N_k \sim_{k \to +\infty} a_k$, since for large k, $a_k \geq (a_{k-1})^\delta$, with $\delta > 1$. As $m_k = (a_{k+1})^\beta$, we now choose $\beta < 1$ close enough to 1, so that $m_k \geq 1000 N_k$, for large k. Let now $m_k \geq m \geq 100 N_k$. For $0 \leq \ell, \ell' \leq m q_k$, we make the Euclidean divisions of ℓ, ℓ' by q_k : $\ell = a q_k + b$ and $\ell' = a' q_k + b'$, $0 \leq b, b' < q_k$ and where $0 \leq a, a' \leq m_k$. Then, almost-surely for large k, since $x \in A_k$, we have $f_{a q_k}(x) = a f_{q_k}(x)$, $f_{a' q_k}(x) = a' f_{q_k}(x)$. Thus:

$$f_\ell(x) - f_{\ell'}(x) = (a - a') f_{q_k}(x) + T^{a q_k} f_b(x) - T^{a' q_k} f_{b'}(x).$$

Using the upper-bound (4), coming from Ostrowski's expansion (3), for $\|f_b\|_\infty$ and $\|f_{b'}\|_\infty$, we have (since $V(f) = 2$), $|T^{a q_k} f_b(x)| \leq 2 N_k$ and $|T^{a' q_k} f_{b'}(x)| \leq 2 N_k$. Hence, a.-e., for large k, using the fact that $|f_{q_k}(x)| \geq 1$:

$$|f_\ell(x) - f_{\ell'}(x)| \geq |a - a'||f_{q_k}(x)| - |T^{a q_k} f_b(x) - T^{a' q_k} f_{b'}(x)| \geq |a - a'| - 4 N_k. \tag{16}$$

Consequently, a.-e., for $m_k \geq m \geq 100 N_k$, by (16):

$$\begin{aligned}
\varphi_+^2(m q_k) \geq \sum_{0 \leq \ell' \leq \ell \leq m q_k} (f_\ell - f_{\ell'})^2 \quad &\geq \sum_{0 \leq a' < m/4; m/2 < a < m; 0 \leq b, b' < q_k} (f_{a q_k + b}(x) - f_{a' q_k + b'}(x))^2 \\
&\geq \sum_{0 \leq a' < m/4; m/2 < a < m; 0 \leq b, b' < q_k} (m/4 - 4 N_k)^2 \\
&\geq (m/4)^2 q_k^2 (m/5)^2 \geq m^4 q_k^2 / 400.
\end{aligned} \tag{17}$$

In order to conclude the argument we shall apply Corollary 3.7, 3), and show the convergence of $\sum_{n \geq 1} 1/\varphi_+(n)$. Let us write for a.-e. x and large $K > 0$:

$$\begin{aligned}
\sum_{n\geq q_K}\frac{1}{\varphi_+(n)} &\leq \sum_{k\geq K}\sum_{1\leq m\leq a_{k+1}}\sum_{mq_k\leq n<(m+1)q_k}\frac{1}{\varphi_+(n)} \leq \sum_{k\geq K}\sum_{1\leq m\leq a_{k+1}}\frac{q_k}{\varphi_+(mq_k)}\\
&\leq \sum_{k\geq K}\left[\sum_{1\leq m\leq 100N_k} + \sum_{100N_k<m\leq m_k} + \sum_{m_k<m\leq a_{k+1}}\right]\left(\frac{q_k}{\varphi_+(mq_k)}\right)\\
&= \sum_{k\geq K}\left[\quad U_k \quad + \quad V_k \quad + \quad W_k \quad\right].
\end{aligned}$$

1) Considering $\sum_{k\geq K} U_k$, we fix $k \geq K$ and $1 \leq m \leq 100N_k$. Using first the function ψ introduced in (10), before Lemma 3.8 and next this lemma, we have:

$$\begin{aligned}
\varphi_+^2(mq_k) &= (mq_k)^2 + \psi(-mq_k, 0) + \psi(0, mq_k)\\
&\geq (mq_k)^2 + \frac{mq_k}{m_{k-1}q_{k-1}+1}\left(\psi(-m_{k-1}q_{k-1}, 0) + \psi(0, m_{k-1}q_{k-1})\right)\\
&\geq \frac{mq_k}{m_{k-1}q_{k-1}+1}\left((m_{k-1}q_{k-1})^2 + \psi(-m_{k-1}q_{k-1}, 0) + \psi(0, m_{k-1}q_{k-1})\right)\\
&\geq \frac{mq_k}{m_{k-1}q_{k-1}+1}\varphi_+^2(m_{k-1}q_{k-1}). \qquad (18)
\end{aligned}$$

As a result, using now (17), for some (next generic) constant $c > 0$, a.-e. for large $k > 0$:

$$\varphi_+(mq_k) \geq c\sqrt{mq_k/(m_{k-1}q_{k-1})}\varphi_+(m_{k-1}q_{k-1}) \geq c\sqrt{mq_k/(m_{k-1}q_{k-1})}m_{k-1}^2 q_{k-1}.$$

As a result, using at the end that $a_k q_{k-1} \geq q_k/2$ and recalling that $c > 0$ is generic:

$$\varphi_+(mq_k) \geq c\sqrt{ma_k}m_{k-1}^{3/2}q_{k-1} \geq c\sqrt{m}a_k^{1/2+3\beta/2}q_{k-1} \geq \left(c\sqrt{m}a_k^{3\beta/2-1/2}\right)q_k. \qquad (19)$$

We obtain, a.-e., for large $K > 0$, via (19), still for a generic $c > 0$, using at the end that $N_k \sim_{k\to+\infty} a_k$, $\beta > 2/3$ and that (a_k) grows at least geometrically:

$$\begin{aligned}
\sum_{k\geq K} U_k &\leq c\sum_{k\geq K}\sum_{1\leq m\leq 100N_k}\frac{1}{\sqrt{m}a_k^{3\beta/2-1/2}}\\
&\leq c\sum_{k\geq K}\frac{\sqrt{N_k}}{a_k^{3\beta/2-1/2}} \leq c\sum_{k\geq K}\frac{\sqrt{a_k}}{a_k^{3\beta/2-1/2}} \leq c\sum_{k\geq 1}a_k^{1-3\beta/2} < +\infty.
\end{aligned}$$

2) Considering now $\sum_{k\geq K} V_k$, let $k \geq K$ and $100N_k < m \leq m_k$. Using (17), we have a.-e. for large $K > 0$ that $\varphi_+(mq_k) \geq m^2 q_k/20$. So, for a generic constant $c > 0$:

$$\sum_{k\geq K} V_k \leq c\sum_{k\geq K}\sum_{100N_k<m\leq m_k}\frac{1}{m^2} \leq c\sum_{k\geq K}\frac{1}{N_k} \leq c\sum_{k\geq K}a_k^{-1} < +\infty.$$

3) For $\sum_{k\geq K} W_k$, let $k \geq K$ and $m_k < m \leq a_{k+1}$. Again, using (18), Lemma 3.8 and finally (17), for m_k, we obtain, a.-e., for large $K > 0$, for some generic constant $c > 0$:

$$\varphi_+(mq_k) \geq c\sqrt{m/m_k}\varphi_+(m_kq_k) \geq c\sqrt{m/m_k}m_k^2q_k \geq c\sqrt{m}m_k^{3/2}q_k.$$

As a consequence, we can write, for some generic $c > 0$, using at the end that $\beta > 2/3 > 1/3$:

$$\sum_{k\geq K} W_k \leq c \sum_{k\geq K} \sum_{m_k<m\leq a_{k+1}} \frac{1}{\sqrt{m}m_k^{3/2}} \leq c \sum_{k\geq K} \frac{\sqrt{a_{k+1}}}{m_k^{3/2}} \leq c \sum_{k\geq K} a_{k+1}^{1/2-3\beta/2} < +\infty.$$

This completes the proof of the proposition.

□

4 The quasi-periodic general case

Let again $Tx = x + \theta \mod 1$ be an irrational rotation on $\mathbb{T}$. The basic assumption in this section will be that for some BV functions $f, g : \mathbb{T} \to \mathbb{R}$, with f centered, for some $x \in \mathbb{T}$:

$$\beta_n/\alpha_n = e^{f(T^{n-1}x)}, \gamma_n\varepsilon_n/\alpha_n = g(T^nx),\ n \in \mathbb{Z}.$$

Implicitly, Hypothesis 1.1 will always be realized, uniformly in $x \in \mathbb{T}$. Introduce some definitions.

Definition 4.1.
Fixing $x \in \mathbb{T}$, let $\rho_n = \rho_n(x) = e^{f_n(x)}$, $n \in \mathbb{Z}$. In other words:

$$\rho_n = \begin{cases} \frac{\beta_1}{\alpha_1}\cdots\frac{\beta_n}{\alpha_n}, & n \geq 1, \\ 1, & n = 0, \\ \frac{\alpha_{n+1}}{\beta_{n+1}}\cdots\frac{\alpha_0}{\beta_0}, & n \leq -1. \end{cases}$$

For $n \geq 0$, let:

$$v_+(n) = \sum_{0\leq k\leq n} \rho_k \quad \text{and} \quad v_-(n) = (q_0/p_0) \sum_{-n-1\leq k\leq -1} \rho_k.$$

In the same way, introduce for $n \geq 0$:

$$w_+(n) = \sum_{0\leq k\leq n} 1/\rho_k \quad \text{and} \quad w_-(n) = (p_0/q_0) \sum_{-n-1\leq k\leq -1} 1/\rho_k.$$

As already mentioned in the Introduction, for the random walk to be recurrent, the vertical random walk has first to be. The necessary and sufficient condition for this (cf [16]) is:

$$\lim_{n\to+\infty} v_+(n) = +\infty \quad \text{and} \quad \lim_{n\to+\infty} v_-(n) = +\infty.$$

Since f is BV and centered, the Denjoy-Koksma inequality (2) says that $|f_{\pm q_n}(x)| \leq V(f)$, for any $x \in \mathbb{T}$. As a result, $\rho_n = e^{f_n(x)}$ does not go to zero, neither as $n \to +\infty$, nor as $n \to -\infty$, implying that the two previous conditions hold.

Some quasi-invariant measures on $\mathbb{T}$ will play a role. In the sequel, we consider the space of Borel probability measures on $\mathbb{T}$, equipped with its usual (metrizable) weak-$*$ topology (using continuous $w : \mathbb{T} \to \mathbb{R}$, as test functions), for which this space is compact. We denote by $T\nu$ the image by T of a Borel probability measure ν on $\mathbb{T}$. By definition, $\int w\, dT\nu = \int Tw\, d\nu$, for any bounded measurable $w : \mathbb{T} \to \mathbb{R}$. Let us recall the following folklore result.

Theorem 4.2. *([9], Prop. 5.8, or [1], Prop. 1.1.)*
Let $h : \mathbb{T} \to \mathbb{R}$ be BV and centered. There exists a unique Borel probability measure ν_h on $\mathbb{T}$ such that $dT\nu_h = e^{T^{-1}h} d\nu_h$. This measure has no atom.

The proof of existence, relying on non-atomicity, is incomplete in [9] and too abstract in [1]. Inspired by [9] (Prop. 4.2), let us reprove existence and atomicity in an elementary way. Fixing f as before Definition 4.1 and using the latter, introduce the notation :

$$A(n, g, x) = \frac{\sum_{k=0}^{n} g(T^k x)/\rho_k(x)}{\sum_{k=0}^{n} 1/\rho_k(x)}.$$

Lemma 4.3.
i) (Partial reproof of Theorem 4.2) Let $(x_n)_{n\geq 1}$ in $\mathbb{T}$ and (N_n) be an increasing sequence of integers. For $x \in \mathbb{T}$, write $\rho_n(x) = e^{f_n(x)}$, $n \in \mathbb{Z}$. Then any cluster point μ, for the weak-$*$ topology, of the sequence of probability measures on $\mathbb{T}$:

$$\left(\frac{\sum_{k=0}^{N_n} \delta_{T^k x_n}/\rho_k(x_n)}{\sum_{k=0}^{N_n} 1/\rho_k(x_n)} \right)_{n\geq 1} \tag{20}$$

is non-atomic and verifies $dT\mu = e^{T^{-1}f} d\mu$. The solution of this last equation is unique.

ii) Let $g : \mathbb{T} \to \mathbb{R}$ be BV. Then $A(n, g, x) \to_{n\to+\infty} \int_{\mathbb{T}} g d\nu_f$, uniformly in x.

Proof of the lemma:
As a preliminary remark, for any (x_n) and (N_n), we have, as $n \to +\infty$:

$$\sum_{k=0}^{N_n} 1/\rho_k(x_n) \to +\infty \text{ and } (1/\rho_{N_n}(x_n))/(\sum_{k=0}^{N_n} 1/\rho_k(x_n)) \to 0. \tag{21}$$

The first point comes from the observation that, independently of x_n, $\rho_{q_\ell}(x_n) \geq e^{-V(f)}$, for any $\ell \geq 1$, as follows from the Denjoy-Koksma inequality (2). Next, the second point can be equivalently rewritten as:

$$\sum_{k=0}^{N_n} e^{-f_k(x_n)+f_{N_n}(x_n)} = \sum_{k=0}^{N_n} e^{T^k f_{N_n-k}(x_n)} = \sum_{k=0}^{N_n} e^{T^{N_n-k} f_k(x_n)} \to +\infty,$$

for the same reason.

i) Call $(\mu_n)_{n\geq 1}$ the sequence in (20) and consider a cluster point μ of it for the weak-$*$ topology. For simplicity, we keep the same notations (x_n), (N_n) and assume that (μ_n) converges to μ. Let $a \in \mathbb{T}$. For any $\delta > 0$, we have $\mu((a-\delta, a+\delta)) \leq \liminf \mu_n((a-\delta, a+\delta))$. It is therefore enough to show that $\mu_n((a-\delta, a+\delta))$ is arbitrary small for large n, for a well-chosen $\delta > 0$.

Fix an integer $K \geq 1$ and take $\delta > 0$ so that the intervals $(a-\delta, a+\delta) - k\theta$, $0 \leq k \leq 2q_K$, on $\mathbb{T}$ are disjoint. Then:

$$\mu_n((a-\delta, a+\delta)) = \frac{\sum_{k=0}^{N_n} 1_{(a-\delta,a+\delta)-k\theta}(x_n)/\rho_k(x_n)}{\sum_{k=0}^{N_n} 1/\rho_k(x_n)}.$$

Let $0 \leq \tau_{1,n} < \cdots < \tau_{L_n,n} \leq N_n$, for some $L_n \geq 0$, be the subsequence of $0 \leq k \leq N_n$ such that $x_n \in (a-\delta, a+\delta) - k\theta$. If $L_n = 0$, we have $\mu_n((a-\delta, a+\delta)) = 0$. When $L_n \geq 2$, by hypothesis on δ, we have $\tau_{k,n} + q_K < \tau_{k+1,n}$, for $1 \leq k < L_n$, and $\tau_{L_n-1,n} + 2q_K < \tau_{L_n,n}$. Using the Denjoy-Koksma inequality (2), giving $1/\rho_{k\pm q_l}(x_n) \geq e^{-V(f)}/\rho_k(x_n)$, we obtain, when $L_n \geq 2$:

$$\begin{aligned}\sum_{k=0}^{N_n} \frac{1}{\rho_k(x_n)} &\geq \sum_{1\leq k\leq L_n-1} \sum_{1\leq \ell\leq K} \frac{1}{\rho_{\tau_{k,n}+q_\ell}(x_n)} + \sum_{1\leq \ell\leq K} \frac{1}{\rho_{\tau_{L_n,n}-q_\ell}(x_n)} \\ &\geq \sum_{1\leq k\leq L_n} (1/\rho_{\tau_{k,n}}(x_n)) \sum_{1\leq \ell\leq K} e^{-V(f)} = Ke^{-V(f)} \sum_{1\leq k\leq L_n} (1/\rho_{\tau_{k,n}}(x_n)).\end{aligned}$$

When $L_n = 1$, noticing that either $\tau_{L_n,n} > q_K$ or $\tau_{L_n,n} + q_K < N_n$, as soon as n is large enough, the same reasoning holds and we obtain the same equality. Hence, always:

$$\mu_n((a-\delta, a+\delta)) \leq \frac{\sum_{1\leq k\leq L_n} 1/\rho_{\tau_{k,n}}(x_n)}{Ke^{-V(f)} \sum_{1\leq k\leq L_n} (1/\rho_{\tau_{k,n}}(x_n))} = \frac{e^{V(f)}}{K}.$$

This can be made arbitrary small, when choosing K large enough. Hence μ is non-atomic.

Next, for any continuous $h : \mathbb{T} \to \mathbb{R}$, $A(N_n, h, x_n) \to \int_{\mathbb{T}} h d\mu$. Since μ is non-atomic, this holds for any h continuous except at countably many points and in particular if h is BV. Since f is BV, e^{-f} is also BV. Thus for any continuous h, $A(N_n, e^{-f}Th, x_n) \to \int_{\mathbb{T}} e^{-f}Thd\mu$. It now follows from (21), that for any continuous h:

$$\int_{\mathbb{T}} e^{-f}Thd\mu = \int_{\mathbb{T}} hd\mu,$$

giving $dT\mu = e^{T^{-1}f}d\mu$. Thus μ solves the equation $dT\mu = e^{T^{-1}f}d\mu$. For unicity of the solution, see [9] (Theorem 5.6). This solution is hence non-atomic. This completes the proof of this point.

ii) If the result is not true, using that $|A(n, g, x)| \leq \|g\|_\infty$, there exists $a \in \mathbb{R}$, (x_n) in $\mathbb{T}$ and $N_n \to +\infty$, such that $A(N_n, g, x_n) \to a \neq \int_{\mathbb{T}} gd\nu_f$. By compacity of the weak-$*$ topology, for some subsequence $(N_{\zeta(n)})$ of (N_n) and $(x_{\zeta(n)})$ of (x_n), the sequence of measures:

$$\frac{\sum_{k=0}^{N_{\zeta(n)}} \delta_{T^k x_{\zeta(n)}}/\rho_k(x_{\zeta(n)})}{\sum_{k=0}^{N_{\zeta(n)}} 1/\rho_k(x_{\zeta(n)})}, n \geq 1,$$

converges to some probability measure μ on $\mathbb{T}$. By *i)*, $dT\mu = e^{T^{-1}f}d\mu$ and so $\mu = \nu_f$, by unicity. As ν_f is non-atomic, $A(N_{\zeta(n)}, g, x_{\zeta(n)}) \to \int_{\mathbb{T}} gd\nu_f$, contradicting $A(N_n, g, x_n) \to a \neq \int_{\mathbb{T}} gd\nu_f$. This ends the proof of the lemma.

□

It is well-known (cf [9], Theorem 3.6) that ν_h is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\mathbb{T}}$ if and only if $h = \log u - \log Tu$, for some $\mathcal{L}_{\mathbb{T}}$-integrable $u > 0$. Otherwise it is singular. When ν_h is as in Theorem 4.2, we notice for the sequel the following relation, for any bounded measurable $w : \mathbb{T} \to \mathbb{R}$:

$$\int_{\mathbb{T}} w \; d\nu_h = \int_{\mathbb{T}} e^{-h}Tw \; d\nu_h. \tag{22}$$

We shall show in this section:

Theorem 4.4.
Let $\theta \notin \mathbb{Q}$ and $Tx = x + \theta \mod 1$ on $\mathbb{T}$. Let BV functions $f, g : \mathbb{T} \to \mathbb{R}$, with f centered. Assume that $\beta_n/\alpha_n = e^{f(T^{n-1}x)}$ and $\gamma_n\varepsilon_n/\alpha_n = g(T^n x), n \in \mathbb{Z}$, for some $x \in \mathbb{T}$.

i) Suppose that $\int_{\mathbb{T}} gd\nu_f \neq 0$. Then for all $x \in \mathbb{T}$, the random walk is transient.

ii) Assume that $g = h - e^{-f}Th$, for some bounded h. Introduce:
1) $f = u - Tu$, with $e^u \in L^1(\mathcal{L}_{\mathbb{T}})$.
2) $f(x_0 + y) = f(x_0 - y)$, for some $x_0 \in \mathbb{T}$ and $\mathcal{L}_{\mathbb{T}}$-a.-e. $y \in \mathbb{T}$.
Then, under either condition 1) or 2), for $\mathcal{L}_{\mathbb{T}}$-a.-e. x, the random walk is recurrent.

Remark. — As soon as f is not identically zero (i.e. $\nu_f \neq \mathcal{L}_{\mathbb{T}}$), it is possible that $\int_{\mathbb{T}} gd\nu_f \neq 0$, while $\int_{\mathbb{T}} g(x)dx = 0$. Indeed, there exist an interval I and a real t such that $\nu_f(I) \neq \nu_f(I + t)$, so $g = 1_I - 1_{I+t}$ convenes. In item *ii*) of the theorem, the condition $g = h - e^{-f}Th$, for some bounded h, implies $\int_{\mathbb{T}} gd\nu_f = 0$. Reciprocally, when $\int_{\mathbb{T}} gd\nu_f = 0$ and supposing a Diophantine condition on θ together with a regularity condition on both f and g, one can find a bounded h so that $g = h - e^{-f}Th$. For instance, one has the following statement:

Lemma 4.5.
Introduce the Diophantine type of θ:

$$\eta(\theta) = \sup\{r \in \mathbb{R}_+^*,\ \liminf_{q \to +\infty} q^r \|q\theta\| = 0\} \geq 1. \tag{23}$$

Let $m > \eta(\theta)$ be an integer. Assume that $f \in C^{2m}(\mathbb{T}, \mathbb{R})$ is centered and $g \in C^m(\mathbb{T}, \mathbb{R})$ verifies $\int_{\mathbb{T}} g\ d\nu_f = 0$. Then $g = h - e^{-f}Th$, for some continuous h on $\mathbb{T}$.

Proof of the lemma:
By Arnold [2], cf also Conze-Marco [11] (Thm 2.1), since $f^{(m)}$ is C^m and $m > \eta(\theta)$, one has $f^{(m)} = v - Tv$, for some continuous v. By successive integrations, we have $f = u - Tu$, with u of class C^m and zero mean. Hence $e^{-f} = e^{Tu}/e^u$ and so, using [9] Prop. 4.3, ν_f is the measure with density γe^u with respect to $\mathcal{L}_{\mathbb{T}}$, for some constant $\gamma > 0$. The hypothesis on g is thus $\int (ge^u)(x)dx = 0$. As ge^u is of class C^m, using one more time [2], we have $ge^u = H - TH$, for a continuous H. Finally, $h = e^{-u}H$ is bounded, as continuous on $\mathbb{T}$, and satisfies $g = h - e^{-f}Th$. $\square$

In the context of Theorem 4.4 *ii*), when $\int_{\mathbb{T}} gd\nu_f = 0$, for instance when $g = h - e^{-f}Th$ with h bounded (and even simply when $g = 0$), transience requires some strongly dissymmetric behaviour between $v_+(n)$ and $w_+(n)$ (cf (49), below, a quantified version of the condition $\lim_{n\to+\infty} w_+(n)/v_+(n) = +\infty$) or between $v_-(n)$ and $w_-(n)$, as $n \to +\infty$. We build an example in the next proposition. Condition 1) of Theorem 4.4 *ii*), for example satisfied for $f = 1_{[0,1/2)} - 1_{[1/2,1)}$ with $x_0 = 1/4$, prevents this dissymmetry to occur.

Proposition 4.6.
There exist $\theta \notin \mathbb{Q}$ (setting then $Tx = x + \theta \mod 1$ on $\mathbb{T}$) and a BV centered $f : \mathbb{T} \to \mathbb{R}$, verifying $f = u - Tu$ with $u \geq 0$, so that for any bounded $g : \mathbb{T} \to \mathbb{R}$, when $\beta_n/\alpha_n = e^{f(T^{n-1}x)}$ and $\gamma_n\varepsilon_n/\alpha_n = g(T^n x), n \in \mathbb{Z}$, for some $x \in \mathbb{T}$, then for $\mathcal{L}_{\mathbb{T}}$-a.-e. x, the random walk is transient.

4.1 Preliminaries

As in [5], we introduce functions $\Phi_{str}(n)$, $\Phi(n)$ and $\Phi_+(n)$ describing the average horizontal macrodispersion of the environment. The last two respectively correspond to $\Phi_u(n)$ and $\Phi_{u,+}(n)$ in [5], Definition 2.3, with $d = 1$, $u = 1 \in \mathbb{R}_+$ and $\varepsilon_s = m_s$, with the notations of [5].

Definition 4.7.
i) The structure function, depending only on the vertical, is defined for $n \geq 0$ by:

$$\Phi_{str}(n) = \left(n \sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} \frac{1}{\rho_k} \right)^{1/2}.$$

2) For $m, n \geq 0$, introduce:

$$\Phi(-m,n) = \left(\sum_{-v_-^{-1}(m) \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left[\frac{1}{\rho_k^2} + \frac{1}{\rho_\ell^2} + \left(\sum_{s=k}^{\ell} \frac{\gamma_s \varepsilon_s}{\alpha_s \rho_s} \right)^2 \right] \right)^{1/2}.$$

For $n \geq 0$, set $\Phi(n) = \Phi(-n,n)$ and $\Phi_+(n) = \sqrt{\Phi^2(-n,0) + \Phi^2(0,n)}$.

As in [6], we rectify a misleading point appearing in [5], Definition 2.3 1), where the term "standard Lebesgue measure" on the half Euclidean ball $S_+^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1,\ x_1 \geq 0\}$ has to be understood as "uniform probability measure". The following result is extracted from [5], Theorem 2.4, Proposition 2.5 1) and Lemma 6.11.

Theorem 4.8.
i) The random walk is recurrent if and only if $\displaystyle\sum_{n \geq 1} \frac{1}{n^2} \frac{(\Phi^{-1}(n))^2}{\Phi_+^{-1}(n)} = +\infty$.
ii) The condition $\sum_{n\geq 1} 1/\Phi(n) < +\infty$ is sufficient for the transience of the random walk. It is equivalent to transience whenever $\Phi \preceq \Phi_+$.

As a general fact, it is rather directly verified that $\Phi_{str} \preceq \Phi_+ \preceq \Phi$. Other general results, fully detailed in [6], section 3.1, are:

$$\Phi_+(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq 0 \text{ or } 0 \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{\gamma_s \varepsilon_s}{\alpha_s \rho_s} \right)^2 \right)^{1/2}, \tag{24}$$

as well as:

$$\Phi(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{\gamma_s \varepsilon_s}{\alpha_s \rho_s} \right)^2 \right)^{1/2}. \tag{25}$$

An essential point (recalled in detail in [6], end of Section 3.1) is that the inverse functions Φ_{str}^{-1}, Φ_+^{-1} and Φ^{-1} check dominated variation in the sense of Definition 3.5.

4.2 Proof of Theorem 4.4 *i)*

We turn to the proof of Theorem 4.4 *i)*, fixing BV functions f and g, with f centered and $\int_{\mathbb{T}} g d\nu_f \neq 0$. Let $x \in \mathbb{T}$ and $\rho_n = \rho_n(x) = e^{f_n(x)}$, $n \in \mathbb{Z}$, as before. Observe first that (cf [6], section 3.1, 2), for the first inequality):

$$\Phi_{str}(n) \preceq \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \left(\frac{\rho_k}{\rho_\ell} + \frac{\rho_\ell}{\rho_k} \right) \right)^{1/2} \preceq \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2},$$

considering, for the second inequality, only in the last inside sum the terms for $s = k$ and $s = \ell$. Introduce now the following function Ψ, essentially corresponding to Φ when $g = 1$ ("essentially", because, by (25) and the previous inequality, the definition can be simplified when $g = 1$):

$$\Psi(n) = \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}. \tag{26}$$

Notice now that (cf again [6], section 3.1, 2), for the first line), using at the end that g is bounded:

$$\begin{aligned} \Phi(n) &\asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{\gamma_s \varepsilon_s}{\alpha_s \rho_s} \right)^2 \right)^{1/2} \\ &= \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2} \preceq \Psi(n). \end{aligned}$$

We now prove the reverse inequality. Using Lemma 4.3 *ii*), let first $M \ge 1$ be such that for $n \ge M$ and all $x \in \mathbb{T}$:

$$\left| \frac{\sum_{k=0}^{n} g(T^k x)/\rho_k(x)}{\sum_{k=0}^{n} 1/\rho_k(x)} - \int_{\mathbb{T}} g d\nu_f \right| \le \left| \int_{\mathbb{T}} g d\nu_f \right| /2.$$

This gives the inequality $|\sum_{s=k}^{\ell} g(T^s x)/\rho_s(x)| \ge (1/2) |\int_{\mathbb{T}} g d\nu_f| \times \sum_{s=k}^{\ell} 1/\rho_s(x)$, whenever $\ell - k > M$. Consequently:

$$\begin{aligned} \Phi(n) &\ge \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k > M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2} \\ &\ge \frac{1}{2} \left| \int_{\mathbb{T}} g d\nu_f \right| \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k > M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}. \end{aligned} \tag{27}$$

Observe now that:

$$\left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k \le M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2} \le \|g\|_\infty \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k \le M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

As M is fixed and $\eta \leq \rho_k(y)/\rho_{k+1}(y) \leq 1/\eta$, for any $k \in \mathbb{Z}$ and $y \in \mathbb{T}$, where η comes from Hypothesis 1.1, we get that for some constant $C_M > 0$, whenever $0 \leq \ell - k \leq M$:

$$1/C_M \leq \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \leq C_M.$$

Therefore, for some constant $C > 0$, depending on M:

$$\begin{aligned} \left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq v_+^{-1}(n), \ell - k \leq M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} &\leq C \left(\sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} 1 \right)^{1/2} \\ &\preceq \sqrt{v_-^{-1}(n) + v_+^{-1}(n)}. \end{aligned} \tag{28}$$

We now show that $v_+^{-1}(n)/\Psi^2(n) \to 0$. Indeed, by (26):

$$\begin{aligned} \Psi^2(n) \geq \sum_{0 \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 &\succeq \sum_{0 \leq k \leq \ell \leq v_+^{-1}(n)} \left(\frac{\rho_k}{\rho_\ell} + \frac{\rho_\ell}{\rho_k} \right) \\ &\succeq \sum_{0 \leq k \leq v_+^{-1}(n)} \rho_k \sum_{0 \leq \ell \leq v_+^{-1}(n)} 1/\rho_\ell \succeq (v_+^{-1}(n))^2, \end{aligned}$$

by the Cauchy-Schwarz inequality in the final step. Thus $\Psi^2(n)/v_+^{-1}(n) \succeq v_+^{-1}(n) \to +\infty$, as $n \to +\infty$. In the same way, $\Psi^2(n)/v_-^{-1}(n) \to +\infty$, as $n \to +\infty$. By (28):

$$\left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq v_+^{-1}(n), \ell - k \leq M} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} = o(\Psi(n)).$$

From (27) and (28), we deduce that $\Psi \preceq \Phi$ and thus finally $\Psi \asymp \Phi$. The same argumentation shows that $\Phi_+ \asymp \Psi_+$, where Ψ_+ "essentially" corresponds to $g = 1$ and is defined by:

$$\Psi_+(n) = \left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq v_+^{-1}(n), k\ell \geq 0} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}. \tag{29}$$

By Theorem 4.8 *i*), we are therefore left to proving:

$$\sum_{n \geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_+^{-1}(n)} < +\infty. \tag{30}$$

We shall give two proofs of (30). The first one is just a reinterpretation. Consider another random walk, defined by (changing only g and keeping the same BV function f and $x \in \mathbb{T}$):

$$\mu_n = \delta_{+1},\ \varepsilon_n = 1,\ \alpha_n = \gamma_n,\ \beta_n/\alpha_n = e^{f(T^{n-1}x)},\ n \in \mathbb{Z}.$$

Since $\gamma_n \varepsilon_n / \alpha_n = 1$, this case corresponds to $g = 1$. As previously indicated (before (26)), the functions $\tilde{\Phi}$ and $\tilde{\Phi}_+$ for this new random walk verify $\tilde{\Phi} \asymp \Psi$ and $\tilde{\Phi}_+ \asymp \Psi_+$. By Theorem 4.8 *i)*, condition (30) is therefore the transience criterion of this second random walk. The latter being obviously transient (a.-s., the horizontal coordinate tends monotonically to $+\infty$), the condition in (30) is verified and this ends the first proof.

One may be interested in showing directly the convergence of the series in (30). We now furnish the argument. This may help in the future to manipulate the recurrence criterion for studying other examples. Introduce the following functions:

$$\Psi_{++}(n) = \left(\sum_{0 \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}, \tag{31}$$

as well as:

$$\Psi_{+-}(n) = \left(\sum_{-v_-^{-1}(n) \leq k \leq \ell \leq 0} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

For the sequel, notice that Ψ^{-1}, Ψ_+^{-1}, Ψ_{++}^{-1} and Ψ_{+-}^{-1} all verify dominated variation in the sense of Definition 3.5. For the first two, this follows directly from $\tilde{\Phi} \asymp \Psi$ and $\tilde{\Phi}_+ \asymp \Psi_+$, and for the last two, one just has to follow once more the same lines, see [6] section 3.1. We shall next use repeatedly properties like $(a+b)/2 \leq \max(a,b) \leq a+b$, $(\sqrt{a}+\sqrt{b})/2 \leq \sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, etc, for $a, b \geq 0$. From the definition of Ψ_+ given in (29), we have:

$$\Psi_+(n) \asymp \Psi_{++}(n) + \Psi_{+-}(n).$$

Hence, using dominated variation, $\Psi_+^{-1}(n) \asymp \min\{\Psi_{++}^{-1}(n), \Psi_{+-}^{-1}(n)\}$. This thus furnishes:

$$\frac{1}{\Psi_+^{-1}(n)} \asymp \frac{1}{\Psi_{++}^{-1}(n)} + \frac{1}{\Psi_{+-}^{-1}(n)}.$$

In order to prove (30), we thus have to show the two convergences:

$$\sum_{n \geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_{++}^{-1}(n)} < +\infty \text{ and } \sum_{n \geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_{+-}^{-1}(n)} < +\infty. \tag{32}$$

We establish the first one, the case of the second one being similar.

First, using Hypothesis 1.1, for some $c > 1$ depending only on η, we have $\sum_{0 \leq k \leq n+1} \rho_k \leq c \sum_{0 \leq k \leq n} \rho_k$, for all $n \geq 0$. Hence, for large $n > 0$, uniformly on $x \in \mathbb{T}$, using (5):

$$n \geq \sum_{0 \leq \ell \leq v_+^{-1}(n)} \rho_\ell \geq \sum_{0 \leq \ell \leq v_+^{-1}(n)+1} \rho_\ell / c \geq n/c. \tag{33}$$

Thus, for large $n > 0$ (uniformly on $x \in \mathbb{T}$):

$$\sum_{v_+^{-1}(n/c^2) \leq \ell \leq v_+^{-1}(n)} \rho_\ell \asymp n, \tag{34}$$

We now have, using (26):

$$\Psi(n) \geq \left(\sum_{-v_-^{-1}(n) \leq k \leq 0 \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}. \tag{35}$$

The variables k and ℓ are now independent on the right hand side. Let us define:

$$\zeta(n) = \sqrt{n} \left(\sum_{0 \leq \ell \leq v_+^{-1}(n)} \rho_\ell \left(\sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

Notice that ζ^{-1} verifies dominated variation in the sense of Definition 3.5, like the inverse of any function of the forme $n^\gamma h(n)$, with $\gamma > 0$ and h non-decreasing. Obviously, by (33):

$$\zeta(n) \preceq n \sum_{s=0}^{v_+^{-1}(n)} \frac{1}{\rho_s}. \tag{36}$$

Analogously to (33), we have $\sum_{-v_-^{-1}(n) \leq k \leq 0} \rho_k \asymp n$, so we get, by (35) and (34):

$$\begin{aligned} \Psi(n) \succeq \zeta(n) &\geq \sqrt{n} \left(\sum_{v_+^{-1}(n/c^2) \leq \ell \leq v_+^{-1}(n)} \rho_\ell \left(\sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} \\ &\succeq n \sum_{s=0}^{v_+^{-1}(n/c^2)} \frac{1}{\rho_s} \geq (n/c^2) \sum_{s=0}^{v_+^{-1}(n/c^2)} \frac{1}{\rho_s}. \end{aligned} \tag{37}$$

Set $F(n) = \sum_{k=0}^{v_+^{-1}(n)} 1/\rho_k$ and $G(n) = nF(n)$, where G^{-1} thus verifies dominated variation (cf Definition 3.5). The first inequality in (37) gives $\Psi^{-1}(n) \preceq \zeta^{-1}(n)$. Moreover, the last inequalities in (37) also provide, together with (36) (and using dominated variation):

$$\zeta^{-1}(n) \asymp G^{-1}(n).$$

In order to establish the first part of (32), it is thus sufficient to show that:

$$\sum_{n \geq 1} \left(\frac{G^{-1}(n)}{n} \right)^2 \frac{1}{\Psi_{++}^{-1}(n)} < +\infty. \tag{38}$$

For the analysis of Ψ_{++}, we fix an integer $K > c^2$, where the constant $c > 1$ appears in (33). Define now for any integer $u \geq 0$ the quantity:

$$A_u = \sum_{v_+^{-1}(K^u) < k \leq v_+^{-1}(K^{u+1})} 1/\rho_k. \tag{39}$$

Starting from the definition (31) of Ψ_{++}:

$$\begin{aligned}\Psi_{++}(K^n) &= \left(\sum_{0\le k\le \ell\le v_+^{-1}(K^n)} \rho_k\rho_l\left(\sum_{k\le s\le \ell} 1/\rho_s\right)^2\right)^{1/2}\\ &\preceq \left(\sum_{v_+^{-1}(K^0)<k\le \ell\le v_+^{-1}(K^n)} \rho_k\rho_l\left(\sum_{k\le s\le \ell} 1/\rho_s\right)^2\right)^{1/2}.\end{aligned}$$

Proceeding as for (34), we also have $\sum_{v_+^{-1}(K^u)<k\le v_+^{-1}(K^{u+1})}\rho_k \asymp K^u$. Using this, we continue:

$$\begin{aligned}\Psi_{++}(K^n) &\preceq \left(\sum_{0\le u\le v\le n-1}\ \sum_{\substack{v_+^{-1}(K^u)<k\le v_+^{-1}(K^{u+1})\\ v_+^{-1}(K^v)<\ell\le v_+^{-1}(K^{v+1}),k\le \ell}} \rho_k\rho_l\left(\sum_{v_+^{-1}(K^u)<s\le v_+^{-1}(K^{v+1})} 1/\rho_s\right)^2\right)^{1/2}\\ &\preceq \left(\sum_{0\le u\le v\le n-1} K^uK^v(A_u+\cdots+A_v)^2\right)^{1/2}.\end{aligned}$$

We arrive at:

$$\begin{aligned}\Psi_{++}(K^n) &\preceq \sum_{0\le u\le v<n} K^{u/2}K^{v/2}(A_u+\cdots+A_v)\\ &\preceq \sum_{0\le \ell<n} A_\ell \sum_{0\le u\le \ell} K^{u/2}\sum_{\ell\le v<n} K^{v/2} \preceq K^{n/2}\sum_{0\le \ell<n} A_\ell K^{\ell/2}. \qquad (40)\end{aligned}$$

Let $N\ge 1$ be an integer. We have, by (40):

$$\sum_{n=0}^{N}\frac{\Psi_{++}(K^n)}{K^n} \preceq \sum_{0\le n\le N}\frac{1}{K^{n/2}}\sum_{0\le \ell<n} A_\ell K^{\ell/2} = \sum_{0\le \ell<N} A_\ell \sum_{\ell<n\le N}\frac{K^{\ell/2}}{K^{n/2}} \asymp \sum_{0\le \ell<N} A_\ell \asymp F(K^N), \quad (41)$$

as $F(n)\to+\infty$, as $n\to+\infty$. Now, (41) furnishes, still for $N\ge 1$:

$$\begin{aligned}\sum_{1\le k\le \Psi_{++}(K^N)}\frac{1}{\Psi_{++}^{-1}(k)} &\preceq \sum_{0\le n<N}\ \sum_{\Psi_{++}(K^n)<k\le \Psi_{++}(K^{n+1})}\frac{1}{\Psi_{++}^{-1}(k)}\\ &\preceq \sum_{0\le n<N}\frac{\Psi_{++}(K^{n+1})}{K^n} \preceq F(K^N). \qquad (42)\end{aligned}$$

Let us define for $n\ge 1$:

$$Z(n) = \sum_{1 \le k \le n} 1/\Psi_{++}^{-1}(k).$$

We extend the notation to real $x \ge 1$, by $Z(x) = Z(\lfloor x \rfloor)$, using the floor function. Idem for Ψ_{++}. The last inequality (42) thus says that for $n \ge 1$, taking N so that $K^{N-1} \le n < K^N$:

$$Z(\Psi_{++}(n/K)) \preceq F(n) = \frac{G(n)}{n}.$$

In particular, using (5):

$$Z(\Psi_{++}(G^{-1}(n)/K)) \preceq \frac{G(G^{-1}(n))}{G^{-1}(n)} \preceq \frac{n}{G^{-1}(n)}.$$

Next, if $n \le \Psi_{++}(G^{-1}(n)/K)$, then $Z(n) \preceq n/G^{-1}(n)$. Otherwise, the last inequality gives:

$$\begin{aligned} Z(n) &\le Z(\Psi_{++}(G^{-1}(n)/K)) + \sum_{\Psi_{++}(G^{-1}(n)/K) < k \le n} 1/\Psi_{++}^{-1}(k) \\ &\le \frac{n}{G^{-1}(n)} + \frac{n}{\Psi_{++}^{-1}(\Psi_{++}(G^{-1}(n)/K))} \preceq \frac{n}{G^{-1}(n)}. \end{aligned}$$

Finally, using the last inequality and the definition of $Z(n)$, we show (38):

$$\begin{aligned} \sum_{n \ge 1} \left(\frac{G^{-1}(n)}{n}\right)^2 \frac{1}{\Psi_{++}^{-1}(n)} \preceq \sum_{n \ge 1} \left(\frac{1}{Z(n)}\right)^2 \frac{1}{\Psi_{++}^{-1}(n)} &\le \sum_{n \ge 1} \frac{1}{Z(n-1)Z(n)} \frac{1}{\Psi_{++}^{-1}(n)} \\ &\le \sum_{n \ge 1} \left(\frac{1}{Z(n-1)} - \frac{1}{Z(n)}\right) < +\infty. \end{aligned}$$

This concludes the second proof of (30) and of Theorem 4.4 *i*).

□

Remark. — The proof shows in particular that for the general model, the condition $\Phi_{str}^2 \preceq \Phi$ implies the transience of the random walk. Indeed, in this case $G(n) \le \Phi_{str}^2(n) \preceq \Phi(n)$, so $\Phi^{-1}(n) \preceq G^{-1}(n)$ and we apply (38), giving the first half of (32) (second one is obtained idem).

4.3 Proof of Theorem 4.4 *ii*)

In this section, f and g are BV functions, with f centered. We suppose that $g = h - e^{-f}Th$, for a bounded h. As $T^s g(x) = \gamma_s \varepsilon_s / \alpha_s$ and $T^s(e^{-f}Th)/\rho_s = (T^{s+1}h)/\rho_{s+1}$, we first have:

$$
\begin{aligned}
\left(\sum_{-v_-^{-1}(n)\le k\le \ell\le v_+^{-1}(n)} \rho_k\rho_\ell \left(\sum_{s=k}^{\ell}\frac{\gamma_s\varepsilon_s}{\alpha_s\rho_s}\right)^2\right)^{1/2} &= \left(\sum_{-v_-^{-1}(n)\le k\le \ell\le v_+^{-1}(n)} \rho_k\rho_\ell \left(\sum_{s=k}^{\ell}\frac{T^s g}{\rho_s}\right)^2\right)^{1/2} \\
&\preceq \left(\sum_{-v_-^{-1}(n)\le k\le \ell\le v_+^{-1}(n)} \rho_k\rho_\ell \left(\sum_{s=k}^{\ell}\left(\frac{T^s h}{\rho_s}-\frac{T^{s+1}h}{\rho_{s+1}}\right)\right)^2\right)^{1/2} \\
&\preceq \left(\sum_{-v_-^{-1}(n)\le k\le \ell\le v_+^{-1}(n)} \rho_k\rho_\ell \left(1/\rho_k^2+1\rho_l^2\right)\right)^{1/2} \\
&\asymp \Phi_{str}(n).
\end{aligned}
$$

See for example [6], section 3.1, 2), for details of the last step. Next, we deduce by (25) that $\Phi(n) \asymp \Phi_+(n) \asymp \Phi_{str}(n)$. Using Theorem 4.8 *ii*), the recurrence of the random walk is now equivalent to the divergence of $\sum_{n\ge1} 1/\Phi_{str}(n)$, or, using that $w_+(n+1)\asymp w_+(n)$ and $w_-(n+1)\asymp w_-(n)$ (cf Definition 4.1), of:

$$\sum_{n\ge1}\frac{1}{\sqrt{n(w_+\circ v_+^{-1}(n)+w_-\circ v_-^{-1}(n))}}.$$

Because of the monotonicity of the general term of the previous series, by usual condensation, this is equivalent, for any fixed $K>1$, to showing the divergence of:

$$\sum_{n\ge1}\frac{\sqrt{K^n}}{\sqrt{w_+\circ v_+^{-1}(K^n)+w_-\circ v_-^{-1}(K^n)}}. \tag{43}$$

1) Suppose first that $f=u-Tu$, with $e^u\in L^1(\mathcal{L}_{\mathbb{T}})$. Then, by the Law of Large Numbers (for the second step), a.-e., as $n\to+\infty$:

$$w_+(n)(x)=\sum_{k=0}^{n} e^{-u(x)+T^k u(x)} \sim ne^{-u(x)}\int_{\mathbb{T}} e^{u(y)}dy. \tag{44}$$

In the same way, a.-e., $w_-(n)(x)$ is asymptotically linear. For v_+, using again the Law of Large Numbers, for a positive maybe non-integrable function, there is a.-e. some $\kappa(x)>0$ such that:

$$v_+(n)(x)=e^{u(x)}\sum_{k=0}^{n} e^{-T^k u(x)} \ge n\kappa(x), \text{ for } n\ge 1.$$

The same property is true, a.-e., for $v_-(n)$, as $n\to+\infty$. Hence, a.-e., there is some $c(x)>0$ so that $v_+^{-1}(n)\le c(x)n$ and $v_-^{-1}(n)\le c(x)n$, for large n. We obtain, a.-e., for large $n>0$:

$$w_+\circ v_+^{-1}(n)\le w_+(c(x)n)\preceq n.$$

Idem, $w_-\circ v_-^{-1}(n)\preceq n$. These make the general term in (43) not go to zero, so the series diverges and we conclude that the random walk is recurrent.

2) Suppose that (instead of the L^1-condition) for some $x_0 \in \mathbb{T}$, then $f(x+x_0) = f(x_0 - x)$, for a.-e. $x \in \mathbb{T}$. Using the denominators (q_n) of the convergents of the angle θ, one has:

$$\sum_{0\le k\le q_n} e^{f_k(x)} = \sum_{0\le k\le q_n} e^{f_{q_n-k}(x)} = \sum_{0\le k\le q_n} e^{f_{-k}(x)+T^{-k}f_{q_n}(x)} \asymp \sum_{0\le k\le q_n} e^{f_{-k}(x)}, \tag{45}$$

using the Denjoy-Koksma's inequality (2). As a result, one obtains that $v_+(q_n) \asymp v_-(q_n)$, as $n \to +\infty$, uniformly on $x \in \mathbb{T}$. For the same reason:

$$w_+(q_n) \asymp w_-(q_n), \tag{46}$$

as $n \to +\infty$, also uniformly on $x \in \mathbb{T}$. Recall that $v_+(n+1) \asymp v_+(n)$ and similarly $v_-(n+1) \asymp v_-(n)$, uniformly on $x \in \mathbb{T}$. We can thus fix some large $K > 1$ and an integer $p_0 \ge 1$ such that for all $x \in \mathbb{T}$ and all $n \ge 1$, there exists an integer $p \ge 0$ with $K^p \le v_+(q_n) \le K^{p+p_0}$ and $K^p \le v_-(q_n) \le K^{p+p_0}$. This gives in particular $v_+^{-1}(K^p) \le q_n$ and $v_-^{-1}(K^p) \le q_n$. In (43), the term corresponding to p verifies (uniformly on $x \in \mathbb{T}$):

$$\frac{\sqrt{K^p}}{\sqrt{w_+ \circ v_+^{-1}(K^p) + w_- \circ v_-^{-1}(K^p)}} \ge \frac{\sqrt{K^{-p_0} v_+(q_n)}}{\sqrt{w_+(q_n) + w_-(q_n)}} \asymp \frac{\sqrt{v_+(q_n)}}{\sqrt{w_+(q_n)}}, \tag{47}$$

using (46) for the last step. Next, the set E of $x \in \mathbb{T}$ for which the right-hand side of (47) goes to zero is measurable and T-invariant, as for example:

$$v_+(n)(Tx) = \sum_{0\le k\le q_n} \rho_k(Tx) \asymp \sum_{0\le k\le q_n} \rho_k(x) = v_n(x),$$

since $\log(\alpha/\beta)(x)$ is bounded. Hence this set has Lebesgue measure zero or one, by ergodicity of $(\mathbb{T}, T, \mathcal{L}_{\mathbb{T}})$. If it were one, then for a.-e. y:

$$(v_+(q_n)/w_+(q_n))(x_0+y) \to 0 \text{ and } (v_+(q_n)/w_+(q_n))(x_0-y) \to 0, \tag{48}$$

as $n \to +\infty$. However, using (46) and the symmetry assumption in the final step, we can write:

$$\frac{v_+(q_n)}{w_+(q_n)}(x_0+y) \asymp \frac{v_+(q_n)}{w_-(q_n)}(x_0+y) = \frac{\sum_{0\le k\le q_n} e^{f_k(x_0+y)}}{\sum_{0\le k\le q_n} e^{-f_{-k}(x_0+y)}} \asymp \frac{\sum_{0\le k\le q_n} e^{f_k(x_0+y)}}{\sum_{0\le k\le q_n} e^{f_k(x_0-y)}}.$$

As a result, for a.-e. $y \in \mathbb{T}$:

$$\frac{v_+(q_n)}{w_+(q_n)}(x_0+y) \asymp \frac{w_+(q_n)}{v_+(q_n)}(x_0-y).$$

By (48), the left hand side goes to 0, as $n \to +\infty$, whereas the right hand side goes to $+\infty$. This contradiction shows that E has measure zero. Hence for a.-e. $x \in \mathbb{T}$, the right-hand side of (47) does not go to zero. Hence the series in (43) diverges and we conclude that the random walk is recurrent for a.-e. x. This completes the proof of Theorem 4.4 *ii*).

□

Remark. — In a similar way, but without the symmetry assumption, suppose that $f(x) = u(x) - u(x+y)$, for some BV function u and some parameter $y \in \mathbb{T}$. Let us show that for a.-e. $(x, y) \in \mathbb{T}^2$ the random walk is recurrent. Indeed, in the previous proof, part 2), if ever transience holds for some (x, y), then, by (47) and the convergence of the series in (43):

$$v_+(n)/w_+(n) = \left(\sum_{k=0}^{q_n} e^{u_k(x)-u_k(x+y)}\right) / \left(\sum_{k=0}^{q_n} e^{-u_k(x)+u_k(x+y)}\right) \to 0.$$

Now, the set of $(x, y) \in \mathbb{T}^2$ verifying this property is clearly invariant by the joint action of $T \times Id$ and $Id \times T$ on $\mathbb{T}^2$, which is ergodic. Hence this set has measure 0 or 1. If this is 1, one obtains that for a.-e. (x, y), as $n \to +\infty$:

$$\left(\sum_{k=0}^{q_n} e^{u_k(x)-u_k(y)}\right) / \left(\sum_{k=0}^{q_n} e^{-u_k(x)+u_k(y)}\right) \to 0.$$

This is impossible again, when reversing the roles of x and y. We thus have recurrence for a.-e. $(x, y) \in \mathbb{T}^2$. Rather generally, in Theorem 4.4 *ii*), it would be interesting if the symmetry assumption 2) could be dropped. This raises the question, for $f : \mathbb{T} \to \mathbb{R}$, BV and centered, of understanding the a.-e. behaviour, as $n \to \infty$, of ratios of the form $\sum_{k=0}^{n} e^{f_k(x)} / \sum_{k=0}^{n} e^{-f_k(x)}$.

Also in Theorem 4.4 *ii*), when supposing only $\int_{\mathbb{T}} g d\nu_f = 0$ for g (in place of $g = h - e^{-f} Th$, with h bounded) the analysis requires to find sharp upper-bounds on sums of the form:

$$\sum_{k=0}^{n} e^{-f_k(x)} T^k g(x).$$

4.4 Proof of Proposition 4.6

By Theorem 4.8 *ii*), to prove transience for the random walk, it is enough to show the convergence of $\sum_{n \geq 1} 1/\Phi_{str}(n)$. By definition of Φ_{str}, it is sufficient to establish that:

$$\sum_{n \geq 1} 1/\sqrt{n w_+ \circ v_+^{-1}(n)} < +\infty. \tag{49}$$

We choose f in the form $f = u - Tu$, with $u \geq 0$. Then e^{-u} is integrable, so $v_+(n)$ is a.-e. asymptotically linear, as $n \to +\infty$, by the Law of Large Numbers, as for (44). As a consequence, we obtain that for any $x \in \mathbb{T}$, $v_+^{-1}(n) \asymp n$, as $n \to +\infty$.

Setting $U = e^u$, we have $w_+(n) \sim e^{-u(x)} U_n(x)$, so in order to obtain (49), it is enough to show that for a.-e. x:

$$\sum_{n \geq 1} 1/\sqrt{n U_n} < +\infty. \tag{50}$$

Let us build u (and $f = u - Tu$). Let the rotation angle $\theta \notin \mathbb{Q}$ be defined by the partial quotients $a_m = m^6$, $m \geq 1$. Introduce the function $h_{B,\Delta}(x) = B(1 - |x|/\Delta)_+$, for $\Delta > 0$, $B > 0$, from $\mathbb{R}$ to $\mathbb{R}$. It is a piecewise linear pick function of height B and width Δ, centered at zero.

Let (q_m) be the denominators of the convergents of θ. For $m \geq 1$, set $h^m = h_{B_m,\Delta_m}$, with:

$$\Delta_m = 1/(2m^2 q_m), \qquad B_m = m^2/q_m. \tag{51}$$

As $\Delta_m \leq 1/2$, we periodize h^m with period one and see it as a function on $\mathbb{T}$. We first define $f = \sum_{m\geq 1} f^m$ on $\mathbb{T}$, where:

$$f^m = \sum_{k=0}^{q_m-1} T^{-k}(h^m - T^{q_m} h^m).$$

For large m, the $(k\theta)_{0\leq k<q_m}$ are approximately equally spaced and in the definition of f^m, the $T^{-k}(h^m - T^{q_m} h^m)$, $0 \leq k < q_m$, have disjoint supports. Indeed, for $0 \leq k < q_m$, the support of $T^{-k}(h^m - T^{q_m} h^m)$ is included in $k\theta + [-\Delta_m - \|q_m\theta\|, \Delta_m + \|q_m\theta\|]$. As for $0 \leq k < k' < q_m$, $d_{\mathbb{T}}(k\theta, k'\theta) \geq \|q_{m-1}\theta\| \geq 1/(2q_m)$, by (1), the inequalities $1/(2q_m) \geq 2\Delta_m + 2/q_{m+1} \geq 2\Delta_m + 2\|q_m\theta\|$, valid for large m, prove this claim.

From the definition of f^m, we have $V(f^m) \leq q_m V(h^m - T^{q_m} h^m)$. For $m \geq 1$, f^m is centered. Using (51), (1) and the fact that $h^m - T^{q_m} h^m$ is flat except on three intervals of lengths $\|q_m\theta\|$, with respective slopes $\pm B_m/\Delta_m$, $\pm 2B_m/\Delta_m$, $\pm B_m/\Delta_m$, we get:

$$V(f^m) \leq C q_m (B_m/\Delta_m)\|q_m\theta\| \leq C/m^2.$$

As a result, f is BV and centered. We now check that f is a.-e. equal to some $u - Tu$. For $m \geq 1$, $f^m = u^m - Tu^m$, with:

$$u^m = \sum_{k=0}^{q_m-1} T^{-k} \sum_{\ell=0}^{q_m-1} T^{\ell} h^m = \sum_{|\ell|<q_m} (q_m - |\ell|) T^{\ell} h^m. \tag{52}$$

The Lebesgue measure of the support of u^m is $\leq 4q_m\Delta_m$. As $\sum_{m\geq 1} q_m\Delta_m < +\infty$, by the first lemma of Borel-Cantelli, a.-e. $x \in \mathbb{T}$ belongs to the support of u^m, for only finitely many m. Hence $u = \sum_{m\geq 1} u^m$ is well-defined a.-e. and $f = u - Tu$, a.-e..

As (a_m) is sub-exponential, the Diophantine type $\eta(\theta) \geq 1$ of θ, defined in (23), equals 1. Indeed, fixing $r > 1$, for any integer q, choose m such that $q_m \leq q < q_{m+1}$. Then $q^r\|q\theta\| \geq q_m^r/q_{m+1} \geq q_m^{r-1}(m+1)^{-6} \to +\infty$, as $q \to +\infty$. Hence $\eta(\theta) = 1$. For $x \in \mathbb{T}$ and $r > 0$, let $\tau_r(x) = \min\{n \geq 1, \|T^n x\| < r\}$. By a theorem of Kim and Marmi [19], for a.-e. x:

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1. \tag{53}$$

Recall that $U = e^u$ and let $s_m = \tau_{\Delta_m/2}(x)$, $m \geq 1$. For a.-e. $x \in \mathbb{T}$, decompose now:

$$\sum_{n>s_1} \frac{1}{\sqrt{nU_n}} = \sum_{m\geq 1} \sum_{s_m<n\leq s_{m+1}} \frac{1}{\sqrt{nU_n}} \preceq \sum_{m\geq 1} \sqrt{s_{m+1}}/\sqrt{U_{s_m+1}}. \tag{54}$$

We next have, using (52) and $h^m(T^{s_m}x) \geq B_m/2$:

$$U_{s_m+1} = (e^u)_{s_m+1}(x) \geq e^{u(T^{s_m}x)} \geq e^{u^m(T^{s_m}x)} \geq e^{q_m h^m(T^{s_m}x)} \geq e^{m^2/2}. \tag{55}$$

Also, a.-e., using (53), for large m, $s_m \leq (2/\Delta_m)^2 \preceq (m^2 q_m)^2$. Next, the denominators of the convergents of θ verify $q_m = m^6 q_{m-1} + q_{m-2} \leq (2m)^6 q_{m-1}$, so brutally $q_m \leq (2m)^{6m} \leq e^{cm\log m}$. Hence $s_{m+1} = O(e^{3cm\log m})$. Together with (55), we deduce that the series in (54) is finite. This ends the proof of the proposition.

□

As a final remark, we mention that the sums $(v_+(n))_{n\geq 0}$ can be realized as ergodic sums over a skew-product in infinite measure, a two-step extension of $(\mathbb{T}, T)$. In this spirit we shall show that, when f is not an additive coboundary, then $(\log(v_+(n)/w_+(n)))_{n\geq 1}$ somehow behaves as a classical stationary random walk in $\mathbb{R}$ (cf Conze-Leborgne [10], section 2.5, for example).

Lemma 4.9.
Let $f \in \mathcal{L}^1(\mathbb{T}, \mathbb{R})$, centered, and not a.-e. equal to $u - Tu$, for some measurable u. Define $v_+(n) = v_+(n)(x)$ and $w_+(n) = w_+(n)(x)$, $n \geq 1$, as in Definition 4.1. Then one of the following three situations occur:

1) $v_+(n)/w_+(n) \to +\infty$, a.s., as $n \to +\infty$.
2) $v_+(n)/w_+(n) \to 0$, a.s., as $n \to +\infty$.
3) $\limsup_{n\to+\infty} v_+(n)/w_+(n) = +\infty$, and $\liminf_{n\to+\infty} v_+(n)/w_+(n) = 0$, a.s., as $n \to +\infty$.

Proof of the lemma :
As a first point, $v_+(n) \to +\infty$, a.-e., since, a.-e., the random walk $(f_k)_{k\geq 0}$ is recurrent, as $k \to +\infty$, since f is integrable and centered. We have $v_+(n)(x) \sim e^{f(x)} v_+(n-1)(Tx)$ and $w_+(n)(x) \sim e^{-f(x)} w_+(n-1)(Tx)$, so the set $E_1 = \{x \in \mathbb{T},\ \limsup v_+(n)(x)/w_+(n)(x) < +\infty\}$ is T-invariant. It hence has Lebesgue measure 0 or 1, by ergodicity. If this measure is 1, we can a.-e. define $\psi(x) = \limsup_{n\to+\infty}(v_+(n)(x)/w_+(n)(x))$. By the opening remark:

$$\psi(x) = e^{2f(x)}\psi(Tx).$$

Now, the set $E_2 = \{\psi(x) > 0\}$ is T-invariant and again has measure 0 or 1. If this measure is 1, one has $f = (\log \psi)/2 - (\log T\psi)/2$, contrary to the hypothesis. Hence E_2 has measure 0.

Finally, $\limsup v_+(n)/w_+(n) = +\infty$, a.-s., or $v_+(n)/w_+(n) \to 0$, a.-s., as $n \to +\infty$. Symmetrically, $\limsup w_+(n)/v_+(n) = +\infty$, a.-s., or $w_+(n)/v_+(n) \to 0$, a.-s.. Intersecting the possibilities, we obtain the three cases given in the statement of the lemma.

□

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