Recurrence of a symmetric Random Walk on $\mathbb{Z}^2$ in Random Medium

Julien Brémont

Université of Rennes 1, 2000

Abstract

We first consider a symmetric random walk in $\mathbb{Z}^d$ ($d \geq 2$) with transition to the closest neighbours and in a stationary random medium defined by an ergodic dynamical system. Following an article of G.Lawler [2], we show the existence of an invariant measure and the validity of the Central Limit Theorem. We will deduce the same result for a random walk $\mathbb{Z}^2$ with symmetric transitions, not limited to the closest neighbours. In this model, we will then prove the recurrence of the random walk by applying a criterion established by JP.Conze [1] for stationary random walks.

1 Description of the model.

Let $(\Omega, \mathcal{F}, \mu, (T_x)_{x\in\mathbb{Z}^d})$ be a dynamical system, that is a probability space $(\Omega, \mathcal{F}, \mu)$ and a $\mathbb{Z}^d$-action. We assume that it is ergodic, that is a measurable function invariant by T_x, $\forall x \in \mathbb{Z}^d$ is constant $\mu - ae$. The set Ω will be interpreted as the space of the environments.

Let d random variables $p_1, \cdots, p_d$ be defined on Ω and such that $p_1 + \cdots + p_d = 1$ and for some constant α, $p_i \geq \alpha > 0$, for $1 \leq i \leq d$. Set then $\pi(\omega) = (p_1(T_x\omega), \cdots, p_d(T_x\omega))_{x\in\mathbb{Z}^d}$ and $\pi_i(\omega, x) = p_i(T_x\omega)$. For fixed ω, define the Markov chain $(\xi_{\pi(\omega)}(n))_{n\geq 0}$ on $\mathbb{Z}^d$ by $\xi_{\pi(\omega)}(0) = 0$ and by the following transition laws (writing $(e_1, \cdots, e_d)$ for the canonical basis of $\mathbb{R}^d$) :

$$\forall x \in \mathbb{Z}^d,\ p(x, x+z, \omega) := \frac{1}{2}\pi_i(\omega, x) \text{ if } z = \pm e_i,\ i = 1, \cdots, d. \tag{1}$$

We write $P_{\pi(\omega)}$ for the operator associated to $(\xi_{\pi(\omega)}(n))_{n\geq 0}$. Let $\Lambda := \{\pm e_1, \cdots, \pm e_d\}$ and P'_ω be the induced measure on the space of jumps $\Lambda^{\mathbb{N}}$. We aim at studying the behaviour of this chain with P'_ω-probability 1 for a fixed ω, $\omega - ae$. Still fixing ω, let $(\omega_n)_{n\geq 0}$, with $\omega_n := T_{\xi_{\pi(\omega)}(n)}\omega$, be the sequence of the environments seen from a particle : after n steps, the particle seem to be in 0 in the medium ω_n. collecting all the chains $(\xi_n(\omega))$, we observe that (ω_n) is a Markov chain on Ω with transition operator P defined by :

$$Pf(\omega) = \sum_{i=1}^{d} \frac{1}{2}\pi_i(\omega, 0)\left[f(T_{e_i}\omega) + f(T_{-e_i}\omega)\right].$$

In order to build all the trajectories of the Markov chains $(\xi_{\pi(\omega)}(n))$, we consider another Markov chain on $(\Omega \times \Lambda)$ defined by $x_k = (\omega_k, z_k)$ where $z_k = \xi_{\pi(\omega)}(k+1) - \xi_{\pi(\omega)}(k)$. We obtain $\xi_{\pi(\omega)}(n) = \sum_{k=0}^{n-1} z_k$. The corresponding transition operator $\tilde{P}$ is :

$$\tilde{P}f(\omega, z) = \sum_{i=1}^{d} \frac{1}{2}\pi_i(T_z\omega, 0)\left[f(T_z\omega, e_i) + f(T_z\omega, -e_i)\right].$$

In the sequel for fixed ω we write E_k for the expectation starting in $k \in \mathbb{Z}^d$ for the chain $(\xi_{\pi(\omega)}(n))_{n\geq 0}$. The dependence in ω will always be implicit. We also write $\Omega'' = \prod_{k=0}^{+\infty} (\Omega \times \Lambda)$ for the space of trajectories of the chain (x_k) and P''_ω for the measure on this space corresponding to the initial distribution $\{\omega, z(0,\omega)\}$, where ω is fixed and $z(0,\omega)$ has distribution :

$$P''_\omega\{z(0,\omega) = z\} = \frac{1}{2}\pi_i(\omega, 0), \text{ if } z = \pm e_i,\ i = 1, \cdots, d.$$

2 Invariant measure and Central Limit Theorem.

We will study all the chains $(\xi_{\pi(\omega)}(n))_{n\geq 0}$ simultaneously by considering (x_n). We search an initial distribution on $(\Omega \times \Lambda)$ and $\tilde{P}$-invariant. From what follows, it is enough to find a measure on Ω which is P-invariant. To get results with probability 1 with respect to μ, we search for a mesure equivalent to μ. In fact it is enough to require absolute continuity. Writing $\varphi(\omega)$ for the density, it has to verify :

$$\varphi \geq 0, \int \varphi \, d\mu = 1 \text{ and } \varphi(\omega) = P^*\varphi(\omega) = \sum_{i=1}^{d} \frac{1}{2}\left[\pi_i(T_{-e_i}\omega, 0)\varphi(T_{-e_i}\omega) + \pi_i(T_{e_i}\omega, 0)\varphi(T_{e_i}\omega)\right]. \tag{2}$$

The following results are detailed in [3] for a general model of random walks on $\mathbb{Z}^d$.

Lemma 2.1
If φ satisfies (2), then :
1) $\mu\{\omega,\ \varphi(\omega) > 0\} = 1$ and φ is unique in $L^1(\mu)$.
2) The chain (ω_k) is stationary and ergodic for the initial distribution $\varphi\, d\mu$ on Ω.

In the same way, equipping Λ with counting measure, we define on $(\Omega \times \Lambda)$ the density $\tilde{\varphi}(\omega, z) = \frac{1}{2}\pi_z(\omega, 0)$, if $z \in \Lambda$. We have the following result :

Lemma 2.2
If φ satisfies (2), then for the initial distribution of density $\tilde{\varphi}(\omega, z)$ on $(\Omega \times \Lambda)$, the Markov chain (x_k) is stationary and ergodic.

2.1 Construction of an invariant measure on Ω for the chain (ω_n).

In this section, we propose to show that there exists a random variable φ such that condition (2) is realized. Reformulated in a general framework, the result in the following :

Theorem 2.3
Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $(T_1, \cdots, T_d)$ be elements of $Aut(\Omega, \mathcal{F}, \mu)$. We set $G = \sigma(T_1, \cdots, T_d)$. We assume that G is abelian and that its action on $(\Omega, \mathcal{F}, \mu)$ is ergodic. Let $p_1, \cdots, p_d$ be random variables on $(\Omega, \mathcal{F}, \mu)$ such there exists a constant $\alpha > 0$ with :

$$p_1 + \cdots + p_d = 1 \text{ and } p_i \geq \alpha.$$

We write P for the following operator :

$$Pf(\omega) = \sum_{i=1}^{d} \frac{1}{2} p_i(\omega)[f(T_i\omega) + f(T_i^{-1}\omega)].$$

Then there exists a unique P-invariant probability measure on $(\Omega, \mathcal{F}, \mu)$ equivalent to μ.

Unicity follows from lemma (2.1). We will deduce the existence from the following proposition (2.4). As G is abelian and finitely generated, we assume that G has the form $(T_x)_{x\in\mathbb{Z}^d}$ with $T_xT_y = T_{x+y}$ and $T_0 = Id$, which leads us back to the previous model.
We begin with introducing a few notations. Let $S_\alpha = \{v = (v_1, \cdots, v_d) \in \mathbb{R}^d,\ v_1+\cdots+v_d = 1,\ v_i \geq \alpha > 0\}$ and C_α be the compact metric space of the applications from $\mathbb{Z}^d$ into S_α. If $\pi \in C_\alpha$, we write $(\xi_\pi(j))_{j\geq 0}$ for the random walk on $\mathbb{Z}^d$ starting from 0 and whose transition laws are defined with π as in (1). We write P_π for the corresponding transition operator.

Denote by Π the map from Ω to C_α which to any ω associates $\pi(\omega)$. The chain (ω_n) induces a chain on C_α with transition operator $\mathcal{L}$:

$$\mathcal{L}g(\pi) = \sum_{i=1}^{d} \frac{1}{2}\pi_i(0)\left[g(\tau_{e_i}\pi) + g(\tau_{-e_i}\pi)\right], \tag{3}$$

where for $x \in \mathbb{Z}^d$ and $\pi \in C_\alpha$, $\tau_x\pi(y) = \pi(y+x)$. We first observe that $(C_\alpha, \mathcal{B}, \nu, (\tau_x)_{x\in Z^d})$ is an ergodic dynamical system where $\mathcal{B}$ is the Borel $\sigma-$algebra on C_α and ν is the image by Π of the measure μ. If θ is an $L-$invariant density probability on C_α, we will obtain an invariant density on Ω by simply setting $\varphi = \theta \circ \Pi$.

To build this density θ, we will use approximation. This way we introduce for $n \geq 1$ the set T_n of equivalence classes on $\mathbb{Z}^d$ for the relation : $(z_1, \cdots, z_d) \sim (z'_1, \cdots, z'_d)$ if and only if $z_i - z'_i \in 2n\mathbb{Z}$ for $1 \leq i \leq d$. We write $C_{n,\alpha}$ for the set of applications from T_n in S_α. If $\pi \in C_{n,\alpha}$, π can be seen as a periodic medium. If $\pi \in C_{n,\alpha}$, we define $R_{n,\pi}$ by :

$$R_{n,\pi}g(x) = \sum_{j=0}^{+\infty}\left(1 - \frac{1}{n^2}\right)^j (P_\pi^j g)(x).$$

For $g : T_n \longrightarrow \mathbb{R}$, we ontroduce the usual norms L^p and L^∞ defined by :

$$||g||_2 = \left(\frac{1}{|T_n|}\sum_{x\in T_n}|g(x)|^p\right)^{\frac{1}{p}} \text{ and } ||g||_\infty = \sup_{x\in T_n}|g(x)|. \tag{4}$$

The main result that we will use is the following :

Proposition 2.4
There exists a constant $C > 0$ such that for all n, all $\pi \in C_{n,\alpha}$ and all $g : T_n \longrightarrow \mathbb{R}$, we have :

$$||R_{n,\pi}g||_\infty \leq Cn^2||g||_d. \tag{5}$$

Proof of the proposition :
Set first $D_n = \{(z_1, \cdots, z_d) \in Z^d, |z_1| + \cdots + |z_d| \leq n\}$ and fix $\pi \in C_{n,\alpha}$. Before considering $R_{n,\pi}$, we will look at Q, defined on the space of functions $f : D_n \longrightarrow [0, +\infty)$ by :

$$Qf(x) = E_x\left[\sum_{j=0}^{\tau} f(\xi_\pi(j))\right],$$

where τ is the exit time of D_n. In a first step, we will show the existence of a constant C' such that for any function $f : D_n \longrightarrow [0, +\infty)$, we have :

$$||Qf||_{\infty,\text{int }(D_n)} \leq C'n^2||f||_{d,\text{int }(D_n)}, \tag{6}$$

where we define in the same way as in (4) the norms on int (D_n). We set $D = D_n$ and taking $f : D \longrightarrow [0, +\infty)$ such that $f = 0$ on ∂D, we aim at proving that :

$$||Qf||_{\infty,D} \leq C'n^2||f||_{d,D}.$$

This way, if $u : D \longrightarrow \mathbb{R}$ and if $x \in \text{int } (D)$, we define for $1 \leq i \leq d$ the operator Δ_i by :

$$\Delta_i u(x) = u(x + e_i) + u(x - e_i) - 2u(x).$$

We will say that u is *concave* if $\Delta_i u(x) \leq 0$ for $1 \leq i \leq d$ and $x \in \text{int } (D)$. Setting :

$$Mu := (-1)^d \prod_{i=1}^{d} \Delta_i u,$$

we have the following lemma :

Lemma 2.5
Let $f : D \longrightarrow [0, +\infty)$, $f = 0$ on ∂D, then there exists a concave u from D to $[0, +\infty)$ such that :
(i) $u = 0$ on ∂D.
(ii) $(-1)^d Mu = f^d$ on int (D).
(iii) There exists a constant C'', independent on n, such that $||u||_\infty \leq C'' n^2 ||f||_d$.

Proof of the lemma :
Let $\mathcal{A} = \{h$ concave on D with $h = 0$ on ∂D and $(-1)^d Mh \geq f^d$ on int $(D)\}$. First $\mathcal{A}$ is non empty. Indeed, set $k : D \longrightarrow [0, +\infty)$ define by $k(x) = n(n+1) - |x|(|x|+1)$ with $|x| = |x_1| + \cdots + |x_d|$, if $x = (x_1, \cdots, x_d)$. One checks that if $x \in$ int (D) :

$$\Delta_i k(x) = -2 + 2|x| - |x + e_i| - |x - e_i| \leq -2.$$

Therefore $(-1)^d Mk \geq 2^d$. We then choose a function of the form βk, with β large enough. We then observe that if we set $u(x) = \inf_{h \in \mathcal{A}} h(x)$, then $u \in \mathcal{A}$. Let us show that u is a solution. If there exists $x \in$ int (D) such that $(-1)^d Mu(x) > (f(x))^d$, we then have :

$$\prod_{i=1}^{d} [2u(x) - u(x + e_i) - u(x - e_i)] > f(x)^d.$$

Choose then $\gamma < u(x)$ such that :

$$\prod_{i=1}^{d} [2\gamma - u(x + e_i) - u(x - e_i)] = f(x)^d.$$

Setting $v(y) = u(y)$ if $y \neq x$ and $v(x) = \gamma$, we get $v \in \mathcal{A}$, which contradicts the minimality of u. Let us show now that u verifies the point (iii) of the lemma.

If $x \in$ int (D), we set $I(x) = \{(a_1, \cdots, a_d) \in \mathbb{R}^d,\ u(x + e_i) - u(x) \leq a_i \leq u(x) - u(x - e_i)\}$. We have meas $(I(x)) = f(x)^d$. Set $\bar{u} = ||u||_\infty$ and let $\bar{x} \in$ int (D) be such that $u(\bar{x}) = \bar{u}$. Assume that $\bar{u} > 0$.

Set $A = \{a \in \mathbb{R}^d,\ |a_1| + \cdots + |a_d| \leq \frac{\bar{u}}{4n}\}$ and choose $a \in A$. If one chooses a scalar b such that $b \geq \frac{3}{2}\bar{u}$, then for all $x \in D$, we will get $a.x + b > \bar{u}$. Therefore, if a is fixed, there is a smallest b such that $a.x + b \geq u(x)$, $x \in D$. Obviously for some $x_0 \in D$ we will have $a.x_0 + b = u(x_0)$. However one checks that $x_0 \in$ int (D) since we have :

$$a.x_0 + b = a.\bar{x} + b + a.(x_0 - \bar{x}) \geq \bar{u} - \frac{1}{2}\bar{u} = \frac{1}{2}\bar{u} > 0.$$

We then deduce that $a \in I(x_0)$. Indeed, if it wasn't the case, we would get for example for some i : $a_i > u(x_0) - u(x_0 - e_i) \geq a.x_0 - a.(x_0 - e_i) = a_i$.

Finally we have $A \subset \bigcup_{x \in \text{ int } (D)} I(x)$ and then :

$$\text{mes } (A) \leq \sum_{x \in \text{ int } D} \text{mes } (I(x)).$$

There exists then a constant $c > 0$ such that :

$$c\frac{\bar{u}^d}{n^d} \leq \sum_{x \in \text{ int } (D)} f(x)^d,$$

that is a constant $C'' > 0$ such that :

$$\bar{u} \leq C'' n^2 ||f||_d.$$

□

End of the proof of the proposition :
We will first establish the inequality (6). Set $Y(j) = \xi_\pi(j \wedge \tau)$ where τ is the exit time of D. For $j \geq 1$, using the concave function u of lemma (2.5), we get for $x \in D$:

$$\begin{aligned} E_x[u(Y(j)) - u(Y(j-1))] &= E_x\left[1_{\{j-1<\tau\}}\frac{1}{2}\sum_{i=1}^{d}\pi_i(Y(j-1))\Delta_i(Y(j-1))\right] \\ &\leq -\frac{d\alpha}{2}E_x\left[1_{\{j-1<\tau\}}|Mu(Y(j-1))|^{\frac{1}{d}}\right] \\ &\leq -\frac{d\alpha}{2}E_x\left[1_{\{j-1<\tau\}}f(Y(j-1))\right], \end{aligned}$$

that is :

$$E_x\left[u(Y(j)) - u(Y(j-1)) + \frac{d\alpha}{2}1_{\{j-1<\tau\}}f(Y(j-1))\right] \leq 0.$$

Summing on j, we obtain :

$$E_x\left[u(\xi_\pi(j\wedge\tau)) - u(x) + \frac{d\alpha}{2}\sum_{k=0}^{(j-1)\wedge\tau} f(\xi_\pi(k))\right] \leq 0.$$

Let j tend to $+\infty$ we arrive at :

$$E_x\left[\sum_{k=0}^{\tau} f(\xi_\pi(k))\right] \leq \frac{2}{d\alpha}u(x),$$

which shows inequality (6). We will now prove (5) by taking $g : T_n \longrightarrow \mathbb{R}^+$. Shifting n, one can assume that (6) is verified on D and not only on int (D). There exists then a generic constant c such that (using the periodicity of g) :

$$||Qg||_{\infty,T_n} \leq ||Qg||_{\infty,D_{2n}} \leq cn^2||g||_{d,D_{2n}} \leq cn^2||g||_{d,T_n}.$$

For fixed $x \in \mathbb{Z}^d$, we define the stopping times $(\tau_i^x)_{i\geq 0}$ in the following way : $\tau_0^x = 0$, then τ_1^x is the first time when the random walk (ξ_π) the boundary of the hypercube with center in x whose vertices have coordinates $(x \pm ne_1, \cdots, x \pm ne_d)$ and then $\tau_{k+1}^x = \tau_1^{(\xi_\pi)(\tau_k^x)}$. Setting $R = R_{n,\pi}$, we then get :

$$\begin{aligned} Rg(x) &= \sum_{j=0}^{+\infty}\left(1-\frac{1}{n^2}\right)^j E_x[g(\xi_\pi(j))] = E_x\left[\sum_{k=0}^{+\infty}\sum_{j=\tau_k^x}^{\tau_{k+1}^x-1} g(\xi_\pi(j))\left(1-\frac{1}{n^2}\right)^j\right] \\ &\leq E_x\left[\sum_{k=0}^{+\infty}\left(1-\frac{1}{n^2}\right)^{\tau_k^x}\sum_{j=\tau_k^x}^{\tau_{k+1}^x-1} g(\xi_\pi(j))\right] \\ &\leq \sum_{k=0}^{+\infty}E_x\left[\left(1-\frac{1}{n^2}\right)^{\tau_k^x}E_{\xi_\pi(\tau_k^x)}\left[\sum_{j=0}^{\tau_1^{(\xi_\pi)(\tau_k^x)}} g(\xi_\pi(j))\right]\right]. \end{aligned}$$

Finally :

$$\begin{aligned} Rg(x) &\leq ||Qg||_\infty \sum_{k=0}^{+\infty} E_x\left[e^{-\frac{\tau_k^x}{n^2}}\right] \leq ||Qg||_\infty \sum_{k=0}^{+\infty} \left|\left|E_{(.)}\left[e^{-\frac{\tau_1^{(.)}}{n^2}}\right]\right|\right|_\infty^k \\ &= ||Qg||_\infty \left(1-\left|\left|E_{(.)}\left[e^{-\frac{\tau_1^{(.)}}{n^2}}\right]\right|\right|_\infty\right)^{-1}. \end{aligned}$$

To conclude the proof of the proposition, we will give for n large enough, a minoration uniform in x of $P_x[\tau \geq \varepsilon n^2]$, where $\tau = \tau_1^x$ and ε will be fixed later. If $x = (x_1, \cdots, x_d)$, we write $\tau_{X_1}, \cdots, \tau_{X_d}$ for the exit times of the intervals $[-n+x_i, n+x_i]$ for the coordinate processes $(\langle e_i, \xi_\pi \rangle)$, $1 \leq i \leq d$. We have :

$$\begin{aligned} P_x[\tau \geq \varepsilon n^2] &\geq P_x[\tau_{X_i} \geq \varepsilon n^2, 1 \leq i \leq d] \\ &\geq 1 - \sum_{i=1}^{d} P_{x_i}[\tau_{X_i} < \varepsilon n^2]. \end{aligned}$$

However we observe (for example when considering the times when these processes jump) that for fixed i :

$$P_{x_i}[\tau_{X_i} < \varepsilon n^2] \leq P_0[\tau_{\frac{1}{2},\frac{1}{2}} < \varepsilon n^2],$$

denoting by $\tau_{\frac{1}{2},\frac{1}{2}}$ the exit time of the $[-n, n]$ for the standard random walk on $\mathbb{Z}$ (with jumps ± 1, with probability $1/2$). As the previous walk satisfies a functional CLT, taking ε small enough, we deduce the result.

□

Proof of theorem (2.3) *:*
As we have already observed, it is enough to build a density θ on C_α, invariant by $\mathcal{L}$ (cf (3)). As $(C_\alpha, \mathcal{B}, \nu, (\tau_x)_{x \in Z^2})$ is an ergodic dynamical system and as C_α is a compact metric space, there exists a generic point π in C_α, that is such that :

$$\nu_n := \frac{1}{(2n)^d} \sum_{x \in \mathbb{Z}^d, |x_i| \leq n} \delta_{\tau_x \pi} \rightharpoonup \nu.$$

Consider now for all n the element π_n of $C_{n,\alpha}$ that we define on the hypercube with vertices $(\pm n, \cdots, \pm n)$ as the restriction of π to this hypercube. We then introduce the measure μ_n in the same way as ν_n by :

$$\mu_n := \frac{1}{(2n)^d} \sum_{x \in \mathbb{Z}^d, |x_i| \leq n} \delta_{\tau_x \pi_n}.$$

We directly see that if A is a borel set of C_α depending on a finite number of coordinates, then : $\mu_n(A) - \nu_n(A) \longrightarrow 0$. If O is an open set, there exists another open set $O' \subset O$ depending on finitely many coordinates and such that : $\nu(O') \geq \nu(O) - \varepsilon$. Then :

$$\nu(O') \leq \liminf \nu_n(O') \leq \liminf \mu_n(O') \leq \liminf \mu_n(O).$$

Therefore : $\nu(O) \leq \liminf \mu_n(O)$ and consequently : $\mu_n \rightharpoonup \nu$.

Let now Φ_n be a density (withrespect to counting measure on T_n) of the measure invariant by P_{π_n}, that is such that :

$$P_{\pi_n}^* \Phi_n = \Phi_n \text{ and } ||\Phi_n||_1 = 1,\ \Phi_n \geq 0.$$

Write R_n for the operator R_{n,π_n}. From proposition (2.4), we have $||R_n||_{L^d, L^\infty} \leq Cn^2$. Consequently we have $||R_n^*||_{L^1, L^{\frac{d}{d-1}}} \leq Cn^2$. As $R_n^* \Phi_n = n^2 \Phi_n$, we deduce :

$$||\Phi_n||_{\frac{d}{d-1}} \leq C.$$

Let λ_n be the probability measure on C_α supported by the orbit of $(\tau_x \pi_n)_{x\in T_n}$ and defined by :

$$\lambda_n(\tau_x\pi_n) = \frac{1}{(2n)^d}\Phi_n(x).$$

The measure λ_n is $\mathcal{L}$-invariante, as if g is any bounded measurable function on C_α, setting $\tilde{g}(x) = g(\tau_x\pi_n)$, we obtain :

$$\begin{aligned}\int \mathcal{L}g(\pi)\ d\lambda_n(\pi) &= \frac{1}{(2n)^d}\sum_{x\in T_n}\Phi_n(x)(P_{\pi_n}\tilde{g})(x) \\ &= \frac{1}{(2n)^d}\sum_{x\in T_n}\Phi_n(x)g(\tau_x\pi_n) = \int g(\pi)\ d\lambda_n(\pi).\end{aligned}$$

Using now the fact that C_α is a compact metric space, we can assume, up to an extraction, that $\lambda_n \rightharpoonup \lambda$. The measure λ is also $\mathcal{L}$-invariant. Let us show that it is absolutely continuous with respect to ν. Let g be continuous and positive on C_α. We have :

$$\begin{aligned}\int g\ d\lambda_n &= \int g\ \Phi_n\ d\nu_n = \int g^{\frac{1}{d}+\frac{d-1}{d}}\ \Phi_n\ d\nu_n \\ &\leq \left(\int g\ d\nu_n\right)^{\frac{1}{d}}\left(\int g\ \Phi_n^{\frac{d}{d-1}}\ d\nu_n\right)^{\frac{d-1}{d}} \\ &\leq ||g||_\infty^{\frac{d-1}{d}}\ C\left(\int g\ d\nu_n\right)^{\frac{1}{d}}.\end{aligned}$$

Consequently we get :

$$\int g\ d\lambda \leq ||g||_\infty^{\frac{d-1}{d}}\ C\left(\int g\ d\nu\right)^{\frac{1}{d}}.$$

If E is a borel set such that $\nu(E)=0$, there exists an open set O containing E such that $\nu(O)\leq\varepsilon$. As $\nu(O) = \sup_{gC^\circ, g\leq 1_O}\nu(g)$, we get $\lambda(O)\leq C\varepsilon^{\frac{1}{d}}$ and then $\lambda \ll \nu$, which concludes the demonstration.

□

2.2 Central Limit Theorem.

Let φ verify (2). Set $\mathcal{P} = \varphi(\omega)P''_\omega d\mu(\omega)$ for the probability measure on the trajectories of $(x_n)_{n\geq 0}$, which makes the previous one stationary and ergodic (see lemma (2.2)). We will prove the following result :

Theorem 2.6
There exists d constants $b_i > 0$, $1\leq i\leq d$, such that under the law $\mathcal{P}$:

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} z_k \xrightarrow{\mathcal{L}} \mathcal{N}(0,\ diag\ (b_i)).$$

Proof of the theorem :
For fixed ω, consider $(\xi_{\pi(\omega)}(n))_{n\geq 0}$ and recall that $(z_k)_{k\geq 0}$ is defined as the sequence of steps, that is $z_k = \xi_{\pi(\omega)}(k+1) - \xi_{\pi(\omega)}(k)$. Writing z for the function $(\omega, z)\longmapsto z$, we have $z_k = z(x_k)$. Denote by $\mathcal{F}_k$ the $\sigma-$algebra $\sigma\{z_0,\cdots,z_{k-1}\}$. We have the following conditional laws :

$$P''_\omega[z_k = \pm e_i|\mathcal{F}_{k-1}] = \frac{1}{2}\pi_i(\omega_k, 0),\ i = 1,\cdots,d.$$

Consequently, since the model is symmetric, we see that $(\xi_{\pi(\omega)}(n))_{n\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Moreover, with E''_ω designing the expectation under P''_ω :

$$E''_\omega[z_k^i z_k^j|\mathcal{F}_{k-1}] = \begin{cases} 0, \text{ if } i \neq j. \\ \pi_i(\omega_k, 0), \text{ si } i = j. \end{cases}$$

Set now $V_n^i = \sum_{j=0}^{n-1} \pi_i(\omega_j, 0)$. As the measure $\varphi d\mu$ on Ω makes the chain $(\omega_n)_{n\geq 0}$ stationary and ergodic, there exists some constant vector $b = (b_1, \cdots, b_d)$ with $b_i \geq \alpha$, such the following law of large numbers holds :

$$\frac{V_n}{n} \longrightarrow b, P''_\omega - ps, \mu - ae.$$

The hypotheses of Brown's Theorem [4] are then verified $\mu - ae$ and then :

$$\left(\frac{\xi_{\pi(\omega)}([nt])}{\sqrt{n}}\right)_{t\in[0,1]} \rightharpoonup \mathcal{W}(0, \text{ diag } (b_i)), \mu - ae.$$

The result follows from Lebesgue's dominated convergence Theorem.

□

3 Recurrence of a symmetric random walk in $\mathbb{Z}^2$.

3.1 General model.

Let $\mathcal{D}'$ be a finite subset of $\mathbb{Z}^2$, symmetric with respect to 0, not containing 0 for simplicity and not contained in a single line. Write $\mathcal{D}'$ as the disjoint union of two symmetric parts $\mathcal{D}' = \mathcal{D} \cup \text{Sym}(\mathcal{D})$. Let $(\Omega, \mathcal{F}, \mu, (T_x)_{x\in\mathbb{Z}^2})$ be a dynamical system equipped with an action of $\mathbb{Z}^2$ where we suppose that the action of $\mathcal{D}$ on Ω is ergodic. Let $(p_z)_{z\in\mathcal{D}}$ be a collection of random walks defined on Ω, such that $\sum_{z\in\mathcal{D}} p_z = 1$. We assume that there exists a constant α such that $p_z \geq \alpha > 0$ for $z \in \mathcal{D}$. Let $d =$ card (D). We index $\mathcal{D}$, that is $\mathcal{D} = (z(i))_{1\leq i\leq d}$. For fixed ω, consider the Markov chain $(\xi_\omega(n))_{n\geq 0}$ on $\mathbb{Z}^2$ defined by $\xi_\omega(0) = 0$ and by the transition laws :

$$\forall x \in \mathbb{Z}^2, \; p(x, x+z, \omega) := \frac{1}{2}p_z(T_x\omega) \text{ if } z \in \pm\mathcal{D}.$$

To this random walk we associate $(\zeta_\omega(n))_{n\geq 0}$, the random walk on $\mathbb{Z}^d$ in ergodic random medium defined by $\zeta_\omega(0) = 0$ and by the transition laws :

$$\forall x \in \mathbb{Z}^d, \; p(x, x \pm e_i, \omega) := \frac{1}{2}p_{z(i)}\left(T_{\sum_{i=1}^d x(i)z(i)}\omega\right), \text{ for } i = 1, \cdots, d.$$

Writing $(\zeta_\omega^i(n))_{n\geq 0}$, $1 \leq i \leq d$, for the coordinate processes of $(\zeta_\omega(n))_{n\geq 0}$, we have :

$$\xi_\omega(n) = \sum_{i=1}^d \zeta_\omega^i(n) z(i).$$

Consequently, $\frac{1}{\sqrt{n}}\xi_\omega(n) \rightharpoonup \mathcal{N}(0, M)$, where the covariance matrix M is defined by :

$$\begin{aligned} {}^t\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} M \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} &= \sum_{i=1}^d b_i^2(t_1\langle z(i), e_1\rangle + t_2\langle z(i), e_2\rangle)^2 \\ &= \sum_{i=1}^d b_i^2(\langle z(i), t_1e_1 + t_2e_2\rangle)^2, \end{aligned}$$

introducing the collection $(b_i)_{1\leq i\leq d}$ corresponding to $(\zeta_\omega(n))_{n\geq 0}$, as in theorem (2.6). If the previous value is 0, as dim (Vect $\{z(i), i = 1, \cdots, d\}$) = 2, this implies that $t_1e_1 + t_2e_2 = 0$, that is $t_1 = t_2 = 0$. Therefore M is non-degenerated.

Remark. — The case when $\mathcal{D}'$ contains 0 reduces to the previous one. Indeed, if one restricts the built random walk to jumping times, we get the previous walk.

Remark. — If $\mathcal{D}'$ is contained in a line, say the horizontal one for simplicity, the recurrence will follow by projection, that considering a symmetric random walk in an ergodic medium invariant by vertical translation and whose laws conditional to horizontal movement are the ones of the initial random walk.

3.2 Recurrence.

We will follow in a particular case the proof of a recurrence criterion established by J-P.Conze [1] on stationary random walks in $\mathbb{Z}^2$. That criterion has been generalized by K.Schmidt [5]. Denoting in the dame way P'_ω the measure induced by $(\xi_\omega(n))_{n\geq}$ on the space of trajectories $(\Omega \times \mathcal{D}')$, we will prove the following result :

Theorem 3.1
The random walk $((\xi_\omega(n))_{n\geq 0}$ is recurrent, $P'_\omega - ae$, $\mu - ae$.

The next lemma is valid in a larger context and can be found in [6].

Lemma 3.2
(i) If (p_n) is a sequence of probability laws on $\mathbb{R}^2$ which converges weakly to some probability p, then for any function f with compact support in $\mathbb{R}^2$:

$$\frac{1}{n}\sum_{k=1}^{n}\left|\int_{\mathbb{R}^2} f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp_k(u) - \int_{\mathbb{R}^2} f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp(u)\right| \longrightarrow 0.$$

(ii) Let (Z_n) be a sequence of random variables on a probability space $(E, \mathcal{A}, \mu)$ with values in $\mathbb{R}^2$ and verifying :

$$\frac{1}{\sqrt{n}} Z_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, A),$$

where the covariance matrix A is non-degenerated. Set :

$$\alpha(n,\varepsilon) = \frac{1}{n\varepsilon^2}\sum_{1\leq k\leq n}\mu(Z_k \in B(\varepsilon\sqrt{n})),$$

and $\alpha(\varepsilon) = \underline{\lim}\ \alpha(n,\varepsilon)$. then $\lim \alpha(\varepsilon) = +\infty$, as $\varepsilon \to 0$.

Proof of the lemma :
(i) Fix $\varepsilon > 0$. Using the equicontinuity of the family of functions $(f(\theta.), \varepsilon \leq \theta \leq 1)$, we get the following convergence :

$$\lim_n \sup_{\varepsilon\leq\theta\leq 1}\left|\int_{\mathbb{R}^2} f(\theta u)\ dp_k(u) - \int_{\mathbb{R}^2} f(\theta u)\ dp(u)\right| = 0.$$

For $\varepsilon > 0$, we have the upper bound :

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left|\int f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp_k(u) - \int f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp(u)\right| \\
\leq\ & 2\varepsilon\|f\|_\infty + \frac{1}{n}\sum_{\varepsilon\leq\frac{k}{n}\leq 1}\left|\int f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp_k(u) - \int f\left(\frac{\sqrt{k}}{\sqrt{n}}u\right)\ dp(u)\right|
\end{aligned}$$

The second dominating term tends to 0, as $n \longrightarrow +\infty$, from the previous remark, which establishes the first point of the lemma.

(ii) Using (i) and a change of coordinates, as A is non-degenerated, we are led to find a lower bound for the probability of euclidian balls of the form $E(\varepsilon\sqrt{n}/\sqrt{k})$, where for any $\rho > 0$ we set $E(\rho) = \{(x,y) : x^2 + y^2 \leq \rho^2\}$. Write $\mathcal{N}$ for the standard normal law in $\mathbb{R}^2$. We have :

$$\mathcal{N}\left(E\left(\varepsilon\sqrt{\frac{n}{k}}\right)\right) = \frac{1}{2\pi}\frac{n}{k}\int\int_{\mathbb{R}^2} 1_{E(\varepsilon)}(x,y)\exp\left(-\frac{1}{2}\frac{n}{k}(x^2+y^2)\right)\, dx\, dy.$$

Using polar coordinates we get :

$$\begin{aligned}\alpha(\varepsilon) &\geq \lim_n \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\frac{1}{k}\int_0^{\varepsilon}\exp\left(-\frac{n}{k}\frac{r^2}{2}\right)\, dr = \lim_n \frac{1}{n\varepsilon^2}\sum_{k=1}^{n}\left(1-\exp\left(-\frac{n}{k}\frac{\varepsilon^2}{2}\right)\right)\, dr \\ &= \frac{1}{\varepsilon^2}\int_0^1\left(1-\exp\left(-\frac{\varepsilon^2}{2r}\right)\right)\, dr = \frac{1}{2}\int_0^{\frac{2}{\varepsilon^2}}\left(1-\exp\left(-\frac{1}{u}\right)\right)\, du \sim \ln\left(\frac{1}{\varepsilon}\right).\end{aligned}$$

This concludes the proof of the lemma. □

Proof of theorem (3.1) *:*
Assume the contrary. Write $(E, T, \mathcal{P})$ for the ergodic dynamical system constituted by $E = (\Omega \times \mathcal{D}')^{\mathbb{N}}$, the shift T and the measure $\mathcal{P}$. For $y \in \mathbb{Z}^2$, we set $||y|| := \max\{|y_1|, |y_2|\}$. If the result is false, from the properties of φ, there exists a subset D_1 of E of measure > 0 formed with points x that never come back to 0. Writing z for the application $(\omega, z) \longmapsto z$, we then have :

$$\forall x \in D_1,\ \forall n \geq 1,\ ||S_n z(x)|| \geq 1, \text{ setting } S_n z(x) = \sum_{k=0}^{n-1} T^k z(x).$$

For all $x \in E$, let $(k_n(x))_{n\geq 1}$ be the increasing sequence of passage times of x in D_1 under the action of T. As the system $(E, T, \mathcal{P})$ is ergodic this sequence is well defined $\mathcal{P} - ae$. Consequently, if $k_n(x) > k_{n'}(x)$, as $T^{k_{n'}(x)}x \in D_1$, we get :

$$\left|\left|S_{k_n(x)}z(x) - S_{k_{n'}(x)}z(x)\right|\right| = \left|\left|S_{k_n(x)-k_{n'}(x)}z(T^{k_{n'}(x)}x)\right|\right| \geq 1.$$

Consider the sequence $(R_n(x))_{n\geq 1} = (S_{k_n(x)}z(x))_{n\geq 1}$, of ergodic sums associated to the function z, restricted to the time $k_n(x)$. Using Kac's lemma, we obtain :

$$\lim_n \frac{k_n(x)}{n} = \frac{1}{\mathcal{P}(D_1)},\ \mathcal{P} - ae.$$

From the following lemma (3.3), the sequence (R_n) verifies the bidimensional Central Limit Theorem. Recover the ball $B(\varepsilon\sqrt{N})$ of radius $\varepsilon\sqrt{N}$ centred at the origin (for the norm ewe have just introduced) by 2^{2r} balls $B(c, \rho)$ of radius $\rho = \varepsilon\sqrt{N}2^{-r}$, r being the integer defined by $\frac{1}{2} \leq \rho \leq 1$. Let J be the set of the centers of these balls. Taking the notations of lemma (3.2), we then obtain :

$$\begin{aligned}\alpha(N,\varepsilon) = \frac{1}{N\varepsilon^2}\sum_{1\leq n\leq N}\mathcal{P}(R_n \in B(\varepsilon\sqrt{N})) &= \frac{1}{N\varepsilon^2}\int\sum_{1\leq n\leq N} 1_{B(\varepsilon\sqrt{N})}(R_n(x))\, d\mathcal{P}(x) \\ &\leq \frac{1}{N\varepsilon^2}\int\sum_{c\in J}\left[\sum_{1\leq n\leq N} 1_{B(c,\rho)}(R_n(x))\right]\, d\mathcal{P}(x) \\ &\leq \frac{1}{N\varepsilon^2}\text{Card } J = \frac{2^{2r}}{N\varepsilon^2} = \rho^{-2} \leq 4,\end{aligned}$$

as for every c and every x, there is always only one non-zero term in [ ]. The previous majoration contradicts lemma (3.2).

□

Lemma 3.3
If $(k_n(x))_{n\geq 1}$ *is a strictly increasing sequence of measurable functions with values in* $\mathbb{N}$ *such that* $\lim_n (k_n(x)/n) = a$ *exists* $\mathcal{P} - ae$*, a being a constant* $\in [1, +\infty)$*, then under the law* $\mathcal{P}$ *:*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{k_n(.)} T^k z(.) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{a} M\right).$$

Proof of the lemma :
We check as before that $(z(x_n))_{n\geq 0}$ is a sequence of bounded martingale differences with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n = \sigma\{z(x_0), \cdots, z(x_{n-1})\}$. Let $\varepsilon > 0$ and $\delta > 0$ be fixed. We have :

$$\begin{aligned} \mathcal{P}\left\{\left\|\frac{1}{\sqrt{an}} \sum_{k=0}^{k_n(.)} T^k z(.) - \frac{1}{\sqrt{an}} \sum_{k=0}^{[an]} T^k z(.)\right\| \geq \varepsilon\right\} &\leq \mathcal{P}\left\{|k_n(.) - [an]| \geq \delta n\right\} \\ &+ \mathcal{P}\left\{\sup_{|l-[an]|<\delta n} \frac{1}{\sqrt{an}} \left\|\sum_{j=[an]}^{l} T^k z(.)\right\| \geq \varepsilon\right\}. \end{aligned}$$

It is enough to consider the second term in the right hand side and we denote it by r. By stationary and distinguishing the cases $l \geq [an]$ and $l < [an]$, it is bounded by :

$$\begin{aligned} r &\leq 2\mathcal{P}\left\{\sup_{0\leq l<\delta n} \frac{1}{\sqrt{an}} \left\|\sum_{j=0}^{l} T^k z(.)\right\| \geq \varepsilon\right\} \\ &\leq 2\sum_{i=1,2} \mathcal{P}\left\{\sup_{0\leq l<\delta n} \frac{1}{\sqrt{an}} |Y_{l,i}| \geq \varepsilon\right\}, \end{aligned}$$

considering for $i = 1$ and 2, the coordinate martingales $Y_{n,i} = \langle \sum_{j=0}^{n} T^k z(.), e_i \rangle$. As $(Y_{n,1}^2)$ and $(Y_{n,1}^2)$ are sub-martingales with respect to $(\mathcal{F}_n)$, we obtain :

$$\begin{aligned} r &\leq 2\sum_{i=1,2} \frac{1}{an\varepsilon^2} E_{\mathcal{P}}[Y_{[n\delta],i}^2] \\ &\leq \frac{2}{an\varepsilon^2} \delta n (\sigma_1^2 + \sigma_2^2) = \delta\left(\frac{2(\sigma_1^2 + \sigma_2^2)}{a\varepsilon^2}\right), \end{aligned}$$

setting $\sigma_i^2 = E_{\mathcal{P}}[\langle z(x_1), e_i \rangle^2]$, for $i = 1$ and 2. The last inequality proves the result. □

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