Random self-similar series over a rotation

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Abstract

We study the law of random self-similar series defined above an irrational rotation on the Circle. This provides a natural class of continuous singular non-Rajchman measures.

1 Introduction

Dynamical setting. Consider a probability space $(\Omega, \mathcal{F}, P)$, with a measurable transformation $T : \Omega \to \Omega$, preserving P. The dynamical system $(\Omega, \mathcal{F}, P, T)$ is supposed to be ergodic.

Given real random variables $b(\omega)$ and $r(\omega) > 0$ on $(\Omega, \mathcal{F})$, define for $\omega \in \Omega$ the real affine map $\varphi_\omega(y) = b(\omega) + r(\omega)y$, $y \in \mathbb{R}$. We assume that $\{\varphi_\omega,\ \omega \in \Omega\} = S$ is countable (with $\forall \varphi \in S$, $P(\varphi_\omega = \varphi) > 0$), $b \in L^1$, $\log r \in L^1$ and $\int_\Omega \log r \, dP < 0$. Setting $r_n(\omega) = r(\omega) \cdots r(T^{n-1}\omega)$, with $r_0(\omega) = 1$, introduce the a.-e. defined random variable :

$$X(\omega) = \sum_{n \geq 0} r_n(\omega) b(T^n \omega).$$

The law, or occupation measure, of X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is denoted by P_X, i.e. $P_X(A) = P(X^{-1}(A))$, $A \in \mathcal{B}(\mathbb{R})$. The "self-similar" relation $X(\omega) = \varphi_\omega(X(T\omega))$, equivalently rewritten in the "coboundary" form $b(\omega) = X(\omega) - r(\omega)X(T\omega)$, will be central. It differs from the usual relations of self-similarity for measures, which require some form of independence, not supposed here. Note that if $b(\omega) = \alpha(\omega) - r(\omega)\alpha(T\omega)$, for some random α, then necessarily $\alpha = X$, a.-e..

Such a setting includes the traditional self-similar measures (cf Varjú [5] for a survey), corresponding to the independent case, i.e. Ω a product space with the left shift T, P a product measure and b, r functions of the first coordinate. Bernoulli convolutions are a famous example, cf the review of Solomyak [4]. The present ergodic extension can be motivated by the case when all affine maps are strict contractions. There is then a self-similar set associated with S and this broader class of measures, supported by that set, may help studying its properties.

A fundamental question concerns the type of P_X with respect to Lebesgue measure *Leb* and, first of all, the purity of the Radon-Nikodym decomposition. The law of pure types of Jessen and Wintner may be applied to some extent (cf Jessen and Wintner [2], Theorem 35, or Elliott [1], Lemma 1.22), but it seems clearer to give a direct proof in the present situation.

Lemma 1.1. *The law P_X is of pure type.*

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Proof of the lemma :
Let $S^{(n)} = S \circ \cdots \circ S$, $n \geq 0$, and $C = \{\varphi \in \cup_{n\geq 1} S^{(n)}$, strict contraction$\}$, countable. Each $\varphi \in C$ having a unique fixed point $fix(\varphi)$, the set $\mathcal{P} = \{fix(\varphi),\ \varphi \in C\}$ is countable.

- If there exists $a \in \mathbb{R}$, $A = \{X = a\}$, with $P(A) > 0$, then ω a.-e. on A, there exists $n \geq 1$ such that $T^n\omega \in A$ and $\varphi_\omega \cdots \varphi_{T^{n-1}\omega} \in C$. As $X(\omega) = X(T^n\omega) = a$, we get $a = \varphi_\omega \cdots \varphi_{T^{n-1}\omega}(a)$, so $a \in \mathcal{P}$. Now, ω a.-e. on Ω, there exists $n \geq 0$ such that $T^n\omega \in A$, thus $X(\omega) \in \{\varphi(c),\ c \in \mathcal{P}, \varphi \in \cup_{n\geq 0} S^{(n)}\} =: \mathcal{Q}$, a countable set. Therefore $P_X(\mathcal{Q}) = 1$ and P_X is purely atomic.

- If P_X is continuous and if there exists $A \in \mathcal{B}(\mathbb{R})$ with $Leb(A) = 0$ and $P_X(A) > 0$, introduce $B = \cup_{\varphi \in \cup_{n\geq 0} S^{(n)}} \varphi^{-1}(A)$. Clearly $Leb(B) = 0$. Since $X(\omega) \in B$ implies $X(T\omega) = \varphi_\omega^{-1}(X(\omega)) \in B$, the set $X^{-1}(B)$ is T-invariant. As $P(X^{-1}(B)) \geq P(X^{-1}(A)) > 0$, ergodicity implies that $P_X(B) = P(X^{-1}(B)) = 1$. Therefore $P_X \perp Leb$. □

Pure atomicity. Let us discuss the continuity of P_X. Clearly, $P_X = \delta_c$ if and only if $\forall \varphi \in S$, $\varphi(c) = c$. In the independent case, the purely atomic situation reduces to P_X a Dirac mass, as follows from the relation (obtained when conditioning with respect to the first step) :

$$P_X(A) = \sum_{\varphi \in S} P(\varphi_\omega = \varphi) P_X(\varphi^{-1}(A)),\ A \in \mathcal{B}(\mathbb{R}).$$

Indeed, if there exists an atom, then the latter implies that the non-empty finite set E of points defining an atom of maximal mass is stable under any φ^{-1}. Finiteness of an orbit under iterations of an affine map forces any $c \in E$ to be a fixed point of any $\varphi \in S$.

This is far from true in the general ergodic context. Fixing r and any $\alpha \in L^1$ with countable support, when setting $b = \alpha - r\alpha \circ T$, we have $X = \alpha$. As a result, P_X can be discrete with even non-finite support. Moreover, as we shall see later, determining the conditions under which P_X is continuous can be a non-degenerate problem.

Mention here a recipe for building non-trivial examples of discrete laws when $r(\omega) = \lambda \in (0, 1)$ is algebraic. Let for instance $\lambda = 0,618...$ be the inverse of the Golden Mean, i.e. $\lambda^2 + \lambda - 1$=0. Taking $g \in L^1$ with countable support and $b = g + g \circ T - g \circ T^2$, then $b = (g + (1 + \lambda)g \circ T) - \lambda(g \circ T + (1 + \lambda)g \circ T^2)$. This means that $X(\omega) = g(\omega) + (1 + \lambda)g(T\omega)$.

More generally, if $\sum_{k=0}^{p} \alpha_k \lambda^{p-k} = 0$, $p \geq 1$, let $b(\omega) = \sum_{k=0}^{p} \alpha_k g(T^k\omega)$, where $g \in L^1$ has countable support. Then $X(\omega) = \sum_{n=0}^{p-1} g(T^n\omega)(\sum_{k=0}^{n} \alpha_k \lambda^{n-k})$, as $X(\omega) - r(\omega)X(T\omega) = b(\omega)$.

Recall also the link between the existence of atoms and the Fourier transform. We define :

$$\hat{P}_X(t) = \int_{\mathbb{R}} e^{2i\pi tx} dP_X(x),\ t \in \mathbb{R}.$$

If P_X is continuous, then, as a consequence of Wiener's theorem :

$$\frac{1}{R} \int_0^R |\hat{\mu}(t)| dt \to 0, \text{ as } R \to +\infty.$$

A more precise information of local regularity is when P_X is a Rajchman measure, meaning that $\hat{P}_X(t) \to 0$, as $t \to +\infty$. Equivalently, $tX \mod 1 \to_{\mathcal{L}} Leb_{\mathbb{T}}$, as $t \to +\infty$. A classical example of continuous non-Rajchman measures is the uniform measure on the triadic Cantor set. The present paper furnishes a natural class of such measures.

Content of the article. We study the special case when the dynamics is given by an irrational rotation on the 1-torus, with functions b and r locally constant on some finite partition in intervals. For obvious complexity reasons, P_X is singular, even of zero-dimensional support, so it remains to decide between continuous singularity and pure atomicity. We show that the latter is equivalent to the simultaneous satisfaction of a finite number of explicit algebraic equations. Generically, P_X appears to be continuous, but also not a Rajchman measure. In the last section, we discuss another approach of the continuity problem for general systems.

2 The case of the Circle

Let $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ be the 1-torus, with uniform measure $Leb_{\mathbb{T}}$ and an irrational rotation T of angle $\alpha \in (0,1)$. We recall classical material about continued fractions; see for example Khinchin's book [3]. The angle α can be expanded in infinite continued fraction :

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [0, a_1, a_2, \cdots],$$

where the partial quotients $(a_i)_{i\geq 1}$ are obtained by iterations of the Gauss map, starting from α. The successive truncations $[0, a_1, a_2, \cdots, a_n] = p_n/q_n$, $n \geq 1$, are the convergents of α. The (p_n) and (q_n) check the same recursive relation:

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 0,$$

with $p_0 = 0, p_{-1} = 1$ and $q_0 = 1, q_{-1} = 0$. Classical inequalities are (cf [3], chap. 1) :

$$\frac{1}{2q_{n+1}} \leq \frac{1}{q_n + q_{n+1}} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}},$$

where $\|x\|$ is the distance from x to $\mathbb{Z}$. Our purpose is to establish the following result.

Theorem 2.1.
Let T be a rotation of angle $\alpha = [0, a_1, a_2, \cdots] \notin \mathbb{Q}$ on $\mathbb{T}$.

Given $N \geq 1$ points $d_0 < d_1 < \cdots < d_{N-1} < d_N = d_0$ on $\mathbb{T}$, consider on $\mathcal{D} = \{d_0, \cdots, d_{N-1}\}$ the partial order "$d_i \to d_j$ iff $d_j = T^p d_i$ for some $p \geq 0$". Partition $\mathcal{D} = \sqcup_{1\leq k\leq K}\mathcal{D}_k$ into maximal subsets $\mathcal{D}_k = \{d_{0,k} \to \cdots \to d_{m_k,k}\}$, with $m_k \geq 0$; define $p_k \geq 0$ by $d_{m_k,k} = T^{p_k} d_{0,k}$.

Let $b : \mathbb{T} \to \mathbb{R}$ and $r = \mathbb{T} \to (0,1)$ be constant on each interval $[d_i, d_{i+1})$, $0 \leq i < N$. Define $X(x) = \sum_{n\geq 0} b(T^n x) r_n(x)$, $x \in \mathbb{T}$, and denote by P_X the image of $Leb_{\mathbb{T}}$ by X. Then :

1. *Supp(P_X) has box-counting dimension zero, in particular $P_X \perp Leb$.*
2. *The measure P_X is continuous iff X is discontinuous at some $d_{0,k}$, $1 \leq k \leq K$. Otherwise X is constant on the intervals of the partition determined by $\{T^p d_{0,k},\ 0 \leq p \leq p_k,\ 0 \leq k \leq K\}$, hence Supp$(P_X)$ is finite, with at most $\sum_{1\leq k\leq K}(1 + p_k)$ elements.*
3. *If $a_n \geq 10N + 20N^2 \ln 13/(-\ln \|r\|_\infty)$ infinitely often, then P_X is not a Rajchman measure. If (a_n) is unbounded, then $t_n X \mod 1 \to_{\mathcal{L}} 0$, along a sequence of integers $(t_n) \to +\infty$.*

Proof of the theorem :
1) For any $n \geq 1$, $x \longmapsto \sum_{k=0}^{n-1} r_k(x) b(T^k x)$ is constant on each interval of the partition determined by $\cup_{0\leq k<n} T^{-k}\mathcal{D}$, so takes at most nN values. As $|\sum_{k\geq n} r_k(x) b(T^k x)| \leq \|r\|_\infty^n \|b\|_\infty/(1-$

$\|r\|_\infty)$, $Supp(P_X)$ can be covered for any $\varepsilon > 0$ by at most $-C \log \varepsilon$ balls of radius ε, for some constant $C > 0$. This gives the result.

2) In the present context of strict contractions, X is right-continuous and admits a left limit $X(x^-)$ at every $x \in \mathbb{T}$. Set $\Delta_k = X(d_{0,k}) - X(d_{0,k}^-)$ and $\mathcal{K} = \{1 \leq k \leq K,\ \Delta_k \neq 0\}$. Supposing that $\mathcal{K} \neq \emptyset$, we set $\Delta = \min_{k \in \mathcal{K}} |\Delta_k| > 0$. Choose also $\varepsilon > 0$ so that :

$$\min_{k \in \mathcal{K}} \inf_{\substack{x < d_{0,k} \leq y \\ |y-x| \leq \varepsilon}} |X(x) - X(y)| \geq \Delta/2. \tag{1}$$

Set $\rho_k^\pm = r_{p_k+1}(d_{0,k}^\pm)$, $1 \leq k \leq K$, and define $\rho_{\max/\min} = \max/\min\{\rho_k^\pm,\ 1 \leq k \leq K\}$. For the sequel, fix $M > \max\{p_1, \cdots, p_K\}$ such that :

$$\|X\|_\infty \|r\|_\infty^{M-1} < \frac{\Delta}{12N} \left(\frac{\rho_{\min}}{\rho_{\max}}\right)^{3N}. \tag{2}$$

For $1 \leq k \leq K$, call $(T^p d_{0,k})_{0 \leq p \leq p_k}$ the chain C_k. Choose $\gamma(M) > 0$ such that for any $x < y < x + \gamma(M)$, each interval $T^k(x, y]$, $k \geq 0$, meets at most one element of $\mathcal{D}$ and after covering the last element of a chain the (necessarily) first element of the next chain is not met until M steps.

Take $x \notin \cup_{l \geq 0} T^{-l}\mathcal{D}$ and $0 < \gamma_x < \min\{\gamma(M), \varepsilon\}$ such that if $x < y < x + \gamma_x$, then $T^k(x, y]$ meets no d_j, for $0 \leq k \leq M$. If $T^k(x, y]$ meets for the first time a chain, it thus has to be at the first element of the chain. For the moment, fix y like this. The choice of x, y is precised later.

We consider $X(x) - X(y)$. This way, let $0 = t_0 < s_1 < t_1 < s_2 < t_2 < \cdots$, where, for $i \geq 0$, the $[t_i, s_{i+1})$ are the maximal time intervals of k where $T^k(x, y]$ meets no chain. For $i \geq 1$, the $(T^k(x, y])_{k \in [s_i, t_i)}$ cover some chain, say C_{l_i}, with $d_{0,l_i} \in T^{s_i}(x, y]$ and $d_{m_{l_i}, l_i} \in T^{t_i - 1}(x, y]$.

Introduce $r_n(x) = r_{s_n - t_{n-1}}(T^{t_{n-1}} x)$, $n \geq 1$. We define $n_0 \geq 1$ as the first integer n such that $l_n \in \mathcal{K}$. First of all, we can write :

$$X(x) - X(y) = r_1(x)(X(T^{s_1} x) - X(T^{s_1} y)).$$

In a recursion, suppose now that for some $1 \leq n < n_0$:

$$X(x) - X(y) = r_1(x) \cdots r_n(x) \sum_{0 \leq u < n} \rho_1^* \cdots \rho_{n-1}^* (X(x_u^n) - X(x_{u+1}^n)), \tag{3}$$

with points $T^{s_n} x = x_0^n \leq x_1^n \leq \cdots \leq x_n^n = T^{s_n} y$ and $\rho_i^* = \rho_{l_i}^\pm$. Since $T^{s_n} x < d_{0,l_n} \leq T^{s_n} y$, let v be the index such that $x_v^n < d_{0,l_n} \leq x_{v+1}^n$. Adding d_{0,l_n} to the $(x_i^n)_{0 \leq i \leq n}$ gives $n+2$ points, written in their natural order as $(y_u^n)_{0 \leq u \leq n+1}$. Since $n < n_0$, we split in the following way the term for $u = v$ in (3) :

$$\begin{aligned} X(x_v^n) - X(x_{v+1}^n) &= X(x_v^n) - X(d_{0,l_n}^-) + X(d_{0,l_n}) - X(x_{v+1}^n) \\ &= X(y_v^n) - X(y_{v+1}^{n,-}) + X(y_{v+1}^n) - X(y_{v+2}^n). \end{aligned}$$

Set $\rho_n^* = \rho_{l_n}^-$ if $u \leq v$ and $\rho_n^* = \rho_{l_n}^+$ if $u \geq v+1$. For $u \neq v$:

$$\begin{aligned} X(y_u^n) - X(y_{u+1}^n) &= \rho_{l_n}^* (X(T^{t_n - s_n} y_u^n) - X(T^{t_n - s_n} y_{u+1}^n)) \\ &= \rho_{l_n}^* r_{n+1}(x)(X(T^{s_{n+1} - s_n} y_u^n) - X(T^{s_{n+1} - s_n} y_{u+1}^n)). \end{aligned}$$

Now, in the same way :

$$X(y_v^n) - X(y_{v+1}^{n,-}) = \rho_{l_n}^- r_{n+1}(x)(X(T^{s_{n+1}-s_n} y_v^n) - X(T^{s_{n+1}-s_n} y_{v+1}^{n,-})).$$

As $T^{s_{n+1}-s_n} y_{v+1}^n = T^{s_{n+1}-t_n+1} d_{m_{l_n},l_n}$ and $s_{n+1} - t_n + 1 \geq 1$, from the continuity of X at any $T^k d_{m_{l_n},l_n}$, $k \geq 1$, we get $X(T^{s_{n+1}-s_n} y_{v+1}^{n,-}) = X(T^{s_{n+1}-s_n} y_{v+1}^n)$. We can now finally set $x_u^{n+1} = T^{s_{n+1}-s_n} y_u^n$, $0 \leq u \leq n+1$, and we obtain when replacing in (3) that the latter is satisfied with n replaced by $n+1$. As a result, the formula is true for $n = n_0$:

$$X(x) - X(y) = r_1(x) \cdots r_{n_0}(x) \left[\sum_{0 \leq u < n_0} \rho_1^* \cdots \rho_{n_0-1}^* (X(x_u) - X(x_{u+1})) \right], \tag{4}$$

with, simplifying notations, points $T^{s_{n_0}} x = x_0 \leq x_1 \leq \cdots \leq x_{n_0} = T^{s_{n_0}} y$ and $\rho_i^* = \rho_{l_i}^{\pm}$. Again $T^{s_{n_0}} x < d_{0,l_{n_0}} \leq T^{s_{n_0}} y$ and let v be the index such that $x_v < d_{0,l_{n_0}} \leq x_{v+1}$.

Now, using (1), by definition, $|X(x_v) - X(x_{v+1})| \geq \Delta/2$, whereas, as before, for $u \neq v$:

$$X(x_u) - X(x_{u+1}) = \rho_{l_{n_0}}^* r_{n_0+1}(x)(X(T^{s_{n_0+1}-s_{n_0}} x_u) - X(T^{s_{n_0+1}-s_{n_0}} x_{u+1})).$$

Since M verifies $r_{n_0+1}(x) = r_{s_{n_0+1}-t_{n_0}}(T^{t_{n_0}} x) \leq \|r\|_\infty^{M-1}$, when calling A the term between brackets in (4), we deduce from the previous considerations that :

$$\begin{aligned} |A| &\geq \frac{\Delta}{2}(\rho_{\min})^{n_0-1} - 2\|X\|_\infty (n_0 - 1)(\rho_{\max})^{n_0} r_{n_0+1}(x) \\ &\geq \frac{(\rho_{\min})^{n_0}}{2} \left[\Delta - 4n_0 \|X\|_\infty \left(\frac{\rho_{\max}}{\rho_{\min}} \right)^{n_0} \|r\|_\infty^{M-1} \right]. \end{aligned} \tag{5}$$

Suppose P_X purely atomic. Let x be a Lebesgue density point in some atom ($Leb_{\mathbb{T}}$ a.-e. point is such a point), not in the countable set $\cup_{l \geq 0} T^{-l}\mathcal{D}$. Choose n large enough so that $3\|q_n\alpha\| < \gamma_x$ and take $y \in x + (2\|q_n\alpha\|, 3\|q_n\alpha\|)$ verifying $X(x) = X(y)$. This is possible, as the proportion of points in $x + (0, 3\|q_n\alpha\|)$ lying in the same atom as x tends to one, as $n \to +\infty$.

Recall that the $(0, \|q_n\alpha\|) + k\alpha$, $0 \leq k < q_{n+1}$, are disjoint and, as a classical consequence of the identity $q_n\|q_{n+1}\alpha\| + q_{n+1}\|q_n\alpha\| = 1$, that the $x + (0, 2\|q_n\alpha\|) + k\alpha$, $0 \leq k < q_{n+1}$, cover $\mathbb{T}$, each point belonging to at most two intervals.

As a result, the Circle $\mathbb{T}$ is covered by the $T^k(x, y]$, $0 \leq k < q_{n+1}$, and each point of $\mathbb{T}$ is covered at most 3 times. We deduce that the $T^k(x, y]$ will pass at most three times in chains C_z, $z \notin \mathcal{K}$, before finally meeting a chain whose index is in $\mathcal{K}$. Therefore $n_0 \leq 3N$. From (5) :

$$|A| \geq \frac{(\rho_{\min})^{n_0}}{2} \left[\Delta - 12N\|X\|_\infty \left(\frac{\rho_{\max}}{\rho_{\min}} \right)^{3N} \|r\|_\infty^{M-1} \right] > 0,$$

using property (2) of M. Since $A \neq 0$ and $r_1(x) \cdots r_{n_0}(x) \neq 0$, we get a contradiction in (4) with the fact that $X(x) - X(y) = 0$.

In the other direction, suppose that $\Delta_k = 0$, $1 \leq k \leq K$. The set $\{T^p d_{0,k},\ 0 \leq p \leq p_k,\ 1 \leq k \leq K\}$, the union of the chains, gives a partition of $\mathbb{T}$ into $\sum_{k=1}^K (1+p_k)$ intervals. We show that X is constant on each piece. This way, let $M > 2 + \max\{p_1, \cdots, p_k\}$ and take the corresponding

$\gamma(M) > 0$. Take $x < y$ interior to the same interval of the partition, with $x < y < x + \gamma(M)$. Considering the orbit $T^k(x, y]$, $k \geq 0$, if a chain is met for the first time, then it is at the first element of the chain. As $\Delta_k = 0$ for all $1 \leq k \leq K$, formula (3) is true for all $n \geq 1$:

$$X(x) - X(y) = r_1(x) \cdots r_n(x) \sum_{0 \leq u < n} \rho_1^* \cdots \rho_{n-1}^* (X(x_u^n) - X(x_{u+1}^n)),$$

with, using the same notations for time intervals, points $T^{s_n} x = x_0^n \leq x_1^n \leq \cdots \leq x_n^n = T^{s_n} y$ and $\rho_i^* = \rho_{l_i}^{\pm}$. As $r_k(x) \leq \|r\|_\infty^{M-1} \leq \|r\|_\infty$, we get :

$$|X(x) - X(y)| \leq \|r\|_\infty^n \times n\rho_{\max}^{n-1} \times 2\|X\|_\infty.$$

As this goes to 0, as $n \to +\infty$, we get $X(x) = X(y)$. Hence X is locally constant, hence constant, on each interval of the partition. This concludes the proof of point 2).

3) We examine the Rajchman character of P_X. Set $S_k(x) = -\sum_{l=0}^{k-1} \log r(T^l x)$, with $S_0 = 0$. Then $X(x) = \sum_{k \geq 0} e^{-S_k(x)} b(T^k x)$. Fixing n and $0 \leq m_n \leq a_{n+1}$, arbitrary for the moment :

$$\begin{aligned} X(x) &= \sum_{k=0}^{q_n - 1} e^{-S_k(x)} \sum_{m \geq 0} e^{-S_{mq_n}(T^k x)} b(T^{mq_n + k} x) \\ &= \sum_{k=0}^{q_n - 1} e^{-S_k(x)} \sum_{0 \leq m \leq m_n} e^{-S_{mq_n}(T^k x)} b(T^{mq_n + k} x) \qquad (6) \\ &+ \sum_{k=0}^{q_n - 1} e^{-S_k(x)} \sum_{m > m_n} e^{-S_{mq_n}(T^k x)} b(T^{mq_n + k} x). \qquad (7) \end{aligned}$$

Suppose n even (the other case is similar), so $q_n \alpha \mod 1$ is on the right side of 0 on the Circle. Consider (6) and $0 \leq k < q_n$, as well as $m \geq 1$. If $[T^{k+l} x, T^{k+l+(m-1)q_n} x]$ contains no d_i, for any $0 \leq l < q_n$, then $S_{mq_n}(T^k x) = m S_{q_n}(T^k x)$. Similarly, $b(T^{mq_n + k} x) = b(T^k x)$, whenever $[T^k x, T^{k+mq_n} x]$ contains no d_i. Introduce :

$$\Omega_n = \bigcup_{0 \leq k < 2q_n, 0 \leq i < N} - k\alpha - d_i + [-m_n q_n \alpha, 0],$$

of measure $\leq 2q_n N m_n \|q_n \alpha\| \leq 2N m_n / a_{n+1}$. For $x \notin \Omega_n$, one has $X(x) = Z_n(x) + R_n(x)$, with :

$$Z_n(x) = \sum_{k=0}^{q_n - 1} e^{-S_k(x)} b(T^k x) \frac{1 - e^{-(m_n + 1) S_{q_n}(T^k x)}}{1 - e^{-S_{q_n}(T^k x)}}, \ \|R_n\|_\infty \leq \frac{\|b\|_\infty \|r\|_\infty^{(m_n + 1) q_n}}{1 - \|r\|_\infty}.$$

For any $t_n > 0$, decomposing $e^{2i\pi t_n (Z_n + R_n)} - 1 = e^{2i\pi t_n Z_n}(e^{2i\pi t_n R_n} - 1) + e^{2i\pi t_n Z_n} - 1$ and using that $x \longmapsto e^{ix}$ is 1-Lipschitz on $\mathbb{R}$, we have :

$$\begin{aligned} |\hat{P}_X(t_n) - 1| &\leq \int_{\Omega_n^c} |e^{2i\pi t_n X} - 1| dx + 2|\Omega_n| \qquad (8) \\ &\leq \int_{\Omega_n^c} |e^{2i\pi t_n Z_n(x)} - 1| dx + t_n \|R_n\|_\infty |\Omega_n^c| + 4N m_n / a_{n+1}. \end{aligned}$$

Now, Z_n is constant on each interval of the partition determined by $\cup_{0\leq l<2q_n} T^{-l}\mathcal{D}$ and therefore takes at most $2Nq_n$ values. Fixing an integer $r_n \geq 4$, cut the torus $\mathbb{T}^{2Nq_n}$ in cubes of sides of length $1/r_n$. This gives $r_n^{2Nq_n}$ cubes. Considering the integers $\{nk,\ 0 \leq k \leq r_n^{2Nq_n}\}$, by the pigeonhole principle, there exists an integer nt_n, with $1 \leq t_n \leq r_n^{2Nq_n}$, such that $\|nt_n Z_n(x)\| \leq 1/r_n$, for all $x \in \mathbb{T}$. Replacing t_n by nt_n (arbitrary large) :

$$\begin{aligned} |\hat{P}_X(nt_n) - 1| &\leq |\Omega_n^c| 2\pi/r_n + nt_n\|R_n\|_\infty + 4Nm_n/a_{n+1} \\ &\leq 2\pi/r_n + nr_n^{2Nq_n} \frac{\|b\|_\infty \|r\|_\infty^{(m_n+1)q_n}}{1-\|r\|_\infty} + 4Nm_n/a_{n+1}. \end{aligned}$$

We shall impose $m_n \geq \ln(r_n^{2N})/(-\ln\|r\|_\infty)$, giving :

$$|\hat{P}_X(nt_n) - 1| \leq 2\pi/r_n + 4Nm_n/a_{n+1} + n\|r\|_\infty^{q_n} \frac{\|b\|_\infty}{1-\|r\|_\infty}. \tag{9}$$

If $r_n \geq 4\pi$ and $m_n \leq a_{n+1}/(10N)$, then $|\hat{P}_X(nt_n) - 1| \leq 1/2 + 2/5 + o(1) = 9/10 + o(1)$. Fixing $r_n = 13 > 4\pi$, then P_X is not a Rajchman measure whenever for infinitely many n, one can find an integer m_n satisfying the inequalities :

$$2N \ln r_n/(-\ln\|r\|_\infty) \leq m_n \leq a_{n+1}/(10N). \tag{10}$$

Since $r_n = 13$, this is thus true $a_{n+1}/(10N) \geq 1 + 2N\ln 13/(-\ln\|r\|_\infty)$, along a subsequence.

If the partial quotients are unbounded, take :

$$r_n = a_{n+1} \text{ and } m_n = [\sqrt{a_{n+1}}],$$

along a subsequence where $a_{n+1} \to +\infty$. Then (10) is true for large n. By (9), $\hat{P}_X(nt_n) \to 1$ along a subsequence $nt_n \to +\infty$. Next, for any integer $m \geq 1$, $|e^{2i\pi t_n mX} - 1| \leq m|e^{2i\pi t_n X} - 1|$. Keeping the same sequence (nt_n), relation (8) at time nt_n for mX gives :

$$|\hat{P}_X(mnt_n) - 1| \leq m \int_{\Omega_n^c} |e^{2i\pi nt_n X} - 1| dx + 2|\Omega_n|.$$

As before, the integral and $|\Omega_n|$ go to zero, as $n \to +\infty$, along the above mentioned subsequence. This completes the proof of point 3).

□

Remark. — Explicitly, P_X is purely atomic if and only if for all $1 \leq k \leq K$:

$$\sum_{i=0}^{p_k} \left[r_i(d_{0,k})b(T^i d_{0,k}) - r_i(d_{0,k}^-)b(T^i d_{0,k}^-)\right] + \left[r_{p_k+1}(d_{0,k}) - r_{p_k+1}(d_{0,k}^-)\right] X(Td_{m_k,k}) = 0.$$

Because of $X(Td_{m_k,k})$, this value may involve the whole orbit of $d_{0,k}$. On the contrary, when $r(x) = \lambda \in (0,1)$ and writing any maximal set as $\mathcal{D}_k = \{d_{0,k} \to_{p_{0,k}} \cdots \to_{p_{m_k-1,k}} d_{m_k,k}\}$, with integers $p_{i,k} \geq 1$ such that $d_{i+1,k} = T^{p_{i,k}} d_{i,k}$, this simplifies into :

$$\sum_{i=0}^{m_k} \lambda^{p_{0,k}+\cdots+p_{i-1,k}} \left[b(d_{i,k}) - b(d_{i,k}^-)\right] = 0,\ 1 \leq k \leq K.$$

Remark. — If for example all d_i are in distinct orbits, the condition of pure atomicity reduces to $b(d_i)-b(d_i^-)+[r(d_i)-r(d_i^-)]X(Td_i)=0$ and, when $r(x)$ is constant, to $b(d_i)-b(d_i^-)=0$, $0\le i<N$, i.e. b constant, thus giving $P_X=\delta_{b/(1-\lambda)}$. Proceeding as indicated in the Introduction, it is easy to build examples with any finitely supported law.

Remark. — Concerning point 3), we conjecture that P_X is never a Rajchman measure. Here is a classical situation where the result is true for any angle. Recall that a Pisot number $\rho>1$ is an algebraic integer, with Galois conjugates of modulus <1.

Lemma 2.2.
Let T be a rotation of angle α on $\mathbb{T}$, $r(x)=\lambda\in(0,1)$, with $1/\lambda$ a Pisot number, and $b(x)\in\mathbb{Z}$, locally constant on a partition $\mathbb{T}=\sqcup_{0\le i<N}[d_i,d_{i+1})$. Then P_X is not a Rajchman measure.

Proof of the lemma :
In this case, $X(x)=\sum_{k\ge 0}\lambda^k b(T^k x)$. If $B\subset\mathbb{Z}$ denotes the finite set of values of b, then :

$$\mathrm{Supp}(P_X)\subset\left\{\sum_{k\ge 0}\lambda^k b_k,\ b_k\in B\right\}.$$

Classically, the latter self-similar set is a set of uniqueness for trigonometric series, hence cannot support a Rajchman measure; cf for example the general result of Varjú-Yu [6], Theorem 1.4.

For a more elementary proof, introduce the conjugates $\mu_1,\cdots,\mu_d$ of $1/\lambda$ and recall that $\lambda^{-n}+\mu_1^n+\cdots+\mu_d^n\in\mathbb{Z}$, $n\ge 0$. If P_X were a Rajchman measure, we would have in particular $\lambda^{-n}X \mod 1 \to_{\mathcal{L}} Leb_{\mathbb{T}}$, hence $\lambda^{-n}X\circ T^{-n} \mod 1\to_{\mathcal{L}} Leb_{\mathbb{T}}$. However, modulo 1 :

$$\lambda^{-n}X(T^{-n}x)\equiv\sum_{k=1}^{n}\lambda^{-k}b(T^{-k}x)+X(x)\equiv X(x)-\sum_{k=1}^{n}(\mu_1^k+\cdots+\mu_d^k)b(T^{-k}x).$$

The term on the right-hand side converges pointwise to the real random variable :

$$Y(x)=X(x)-\sum_{k\ge 1}(\mu_1^k+\cdots+\mu_d^k)b(T^{-k}x),$$

We would get $P_{Y \mod 1}=Leb_{\mathbb{T}}$, on $\mathbb{T}$. However, $Y_n(x)\to Y(x)$, as $n\to+\infty$, where :

$$Y_n(x)=\sum_{k=0}^{n}\lambda^k b(T^k x)-\sum_{k=1}^{n}(\mu_1^k+\cdots+\mu_d^k)b(T^{-k}x).$$

We have $\|Y-Y_n\|_\infty\le C\rho^n$, where $\rho=\max\{\lambda,|\mu_1|,\cdots,|\mu_d|\}<1$. Since Y_n takes at most $(2n+1)N$ values, we get $Leb(\mathrm{Supp}(P_Y)=0$. Hence P_Y on $\mathbb{R}$ is singular. Therefore $P_{Y \mod 1}$ is singular on $\mathbb{T}$ and in particular $P_{Y \mod 1}\ne Leb_{\mathbb{T}}$. This concludes the proof of the lemma.

□

3 A remark for general dynamical systems

For the general setting of the Introduction, we discuss in this last section another approach, relating the continuity of the measure P_X to a question on fixed points. We suppose the dynamical system ergodic and invertible.

Changing notations, write $\varphi_\omega = \psi_{\epsilon(\omega)}$, $\epsilon(\omega) \in \mathcal{S}$, where $\mathcal{S}$ is a countable set. For simplicity, we suppose that all affine maps ψ_j, $j \in \mathcal{S}$, are strict contractions. We shall use multi-indices $i = (i_0, \cdots, i_{n-1}) \in \mathcal{S}^n$, for $n \geq 1$. We also write $\psi_i = \psi_{i_0} \cdots \psi_{i_{n-1}}$.

Definition 3.1. *A multi-index $i \in \mathcal{S}^n$, $n \geq 1$, is minimal if $P((\epsilon, \cdots, T^{n-1}\epsilon) = i) > 0$ and for any strict prefix j of i, $fix(\psi_j) \neq fix(\psi_i)$. Let $\mathcal{M} = \{i \in \cup_{n\geq 1}\mathcal{S}^n,\ minimal\}$.*

Remark. — It is easily verified that $fix(\psi_i) = fix(\psi_j)$ if and only if $\psi_i \circ \psi_j = \psi_j \circ \psi_i$.

Lemma 3.2.
Suppose the map : i minimal $\longmapsto fix(\psi_i)$, from $\mathcal{M}$ to $\mathbb{R}$, injective. Then, either P_X is continuous or there exists $N \geq 1$ and $(i_0, \cdots, i_{N-1}) \in \mathcal{S}^N$ such that for a.-e. ω, $(\epsilon(T^n\omega))_{n\geq 0}$ is a left shift of the periodic sequence $(\overline{i_0, \cdots, i_{N-1}}, \cdots) \in \mathcal{S}^{\mathbb{N}}$, in which case $X(\Omega) = \{\psi_{i_k} \cdots \psi_{i_{N-1}}(c),\ 0 \leq k < N\}$, up to a null set, where $c = fix(\psi_{i_0} \cdots \psi_{i_{N-1}})$.

Proof of the lemma :
If P_X is purely atomic, let c and $A = \{X = c\}$, with $P_X(A) > 0$. On A, let $\tau \geq 1$ be the return time, a.-e. defined. Then, restricting to sequences appearing with positive probability, $(\epsilon(\omega), \cdots, \epsilon(T^{\tau(\omega)-1}\omega))$ is minimal, as $c = \psi_{\epsilon(\omega)} \cdots \psi_{\epsilon(T^{\tau(\omega)-1}\omega)}(c)$ and if $c = \psi_{\epsilon(\omega)} \cdots \psi_{\epsilon(T^{m-1}\omega)}(c)$ for some $m < \tau(\omega)$, then $X(T^m\omega) = c$, by injectivity, contradicting the definition of $\tau(\omega)$.

Since for a.-e. $\omega \in A$, $(\epsilon(\omega), \cdots, \epsilon(T^{\tau(\omega)-1}\omega))$ is minimal and c is the corresponding fixed point, the hypothesis implies that there exists $N \geq 1$ and $(i_0, \cdots, i_{N-1}) \in \mathcal{S}^N$ such that $\tau(\omega) = N$ and $(\epsilon(\omega), \cdots, \epsilon(T^{N-1}\omega)) = (i_0, \cdots, i_{N-1})$, for a.-e. ω in A. Also, clearly, $X = c$, a.-e. on A.

By ergodicity and invertibility, we now have, up to a null set, $\Omega = \sqcup_{0\leq k<N} T^k A$. Then, for a.-e. ω, the sequence $(\epsilon(T^n\omega))_{n\geq 0}$ is periodic, being a left shift of $(\overline{i_0, \cdots, i_{N-1}}, \cdots)$, depending on the $0 \leq k < N$ for which $\omega \in T^k A$. It is now quite evident that the values taken by X with positive probability are the $\psi_{i_k} \cdots \psi_{i_{N-1}}(c)$, $0 \leq k < N$.

□

Remark. — The condition of the Lemma is verified if $X(\omega) = \sum_{n\geq 0} \lambda^n b(T^n\omega)$, when $b = \pm 1$ and $0 < \lambda < 1$ is not a root of a polynomial with $0, \pm 1$ as coefficients. Indeed, let $\epsilon = (\epsilon_0, \cdots, \epsilon_{n-1})$ and $\delta = (\delta_0, \cdots, \delta_{m-1})$ be minimal, with $n \leq m$. If $fix(\psi_\epsilon) = fix(\psi_\delta)$, then :

$$\frac{1}{1-\lambda^n} \sum_{k=0}^{n-1} \lambda^k \epsilon_k = \frac{1}{1-\lambda^m} \sum_{k=0}^{m-1} \lambda^k \delta_k,$$

or $(1-\lambda^m) \sum_{k=0}^{n-1} \lambda^k \epsilon_k = (1-\lambda^n) \sum_{k=0}^{m-1} \lambda^k \delta_k$. We rewrite this as :

$$\begin{aligned}
\sum_{k=0}^{n-1} \lambda^k (\epsilon_k - \delta_k) &= \lambda^m \sum_{k=0}^{n-1} \lambda^k \epsilon_k - \lambda^n \sum_{k=0}^{m-1} \lambda^k \delta_k + \sum_{k=n}^{m-1} \lambda^k \delta_k \\
&= \left(\sum_{k=n}^{m-1} \lambda^k \delta_k - \lambda^n \sum_{k=0}^{m-n-1} \lambda^k \delta_k \right) + \left(\lambda^m \sum_{k=0}^{n-1} \lambda^k \epsilon_k - \lambda^n \sum_{k=m-n}^{m-1} \lambda^k \delta_k \right).
\end{aligned}$$

On the right-hand side, there are only powers of λ that are $\geq n :$ between n and $m-1$ in the first parenthesis and between m and $n+m-1$ in the second one. As λ is not a root of a polynomial with $0, \pm 2$ coefficients, it is necessary on the left-hand side that $\epsilon_k = \delta_k$, $0 \leq k < n$. Therefore ϵ is a prefix of δ, which wouldn't be minimal, unless $n = m$. Thus $\epsilon = \delta$.

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