Dynamics of injective quasi-contractions

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Abstract

We study injective locally contracting maps of the Interval. After giving an upper-bound on the number of ergodic components, we show that generically finitely many periodic orbits attract the whole dynamics and that this picture is stable under perturbation. In relation with the problem of maximizing measures for regular maps, we next consider a class of probability measures on the Circle invariant by $\times p$ generalizing the family of Sturm measures and show its generic periodic character. In a second half we detail the structure of order-preserving locally contracting maps on the Circle. The rotation number is shown to be generically rational and the transformations having a given rational rotation number are explicited. We also count the periodic attractors. We then deduce for a model with three pieces on the Interval the existence of measurable conjugacies with three-intervals exchange transformations in non-periodic cases.

1 Introduction

This paper is motivated by the study of maximizing measures. This optimization problem in Ergodic Theory is naturally formulated in a topological setting. Fix a compact metric space X endowed with a continuous transformation T and the set $\mathcal{M}_T(X)$ of T-invariant Borel probability measures. Given some real continuous function f, one considers the variational problem :

$$\beta(f) = \sup \left\{ \int f \; d\mu \mid \mu \in \mathcal{M}_T(X) \right\}.$$

One then asks for the measures realizing the maximum and more specifically for their stochastic properties. This problem is related, via Birkhoff's Theorem, to finding points $x \in X$ ensuring the best growth of the ergodic sums $(\sum_{0 \le k \le n-1} f(T^k x))_{n \ge 0}$. For a detailed presentation of this topic and a beginning general theory, see Bousch-Mairesse [3], Conze-Guivarc'h [8] or Jenkinson [13].

A central question in this field concerns the role of periodic measures for regular maps when the dynamical system is of expansive type. A typical context is $X = \mathbb{R}/\mathbb{Z}$ with $Tx = 2x \mod (1)$ and a conjecture is that *for at least a dense open set in the set of Lipschitz functions, the maximizing measure is unique and supported by a periodic orbit.* See Contreras-Lopes-Thieullen [7] and Hunt-Yuan [12] for weak forms of this conjecture. This general picture is for instance suggested by the work of Bousch [2], considering the one-parameter family $(f_t)_{t \in [0,1)}$ with $f_t(x) = \cos 2\pi(x+t)$. The result is that for all t the maximizing measure of f_t is unique and Sturmian ; moreover except for t in a zero-dimensional set, this measure is periodic. The proof fundamentally uses the following reduction lemma (valid in a more general context) due to Conze-Guivarc'h-Mane (see [8]) :

Lemma 1.1 *Let $f \in Lip(X)$. There exists Lipschitz maps φ and r such that :*

$$f = \beta(f) + (\varphi \circ T - \varphi) - r, \text{ with } r \ge 0 \text{ and } r(.)r(. + 1/2) = 0.$$

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The conditions on r provide a characterization of the maximizing measures via their support. More precisely one checks that μ is maximizing for f if and only if $\mathrm{Supp}(\mu) \subset r^{-1}\{0\}$. In the study of the cosinus family, Bousch shows that for each f_t there exist only two points, diametrically opposed, verifying $r(x) = r(x+1/2) = 0$ and thus that $r^{-1}\{0\}$ is a semi-circle $[\alpha, \alpha+1/2]$. Finally the argument is that T-invariant Borel probability measures with support in such domains are known to be exactly the family of Sturm measures (see Bullet-Sentenac [6]).

In a dynamical approach to the above conjecture, a first step is to consider the case when r admits only finitely many x satisfying the equalities $r(x) = r(x+1/2) = 0$. Then $r^{-1}\{0\}$ is some *finite flower* F of total length 1/2 and not necessarily a semi-circle. Omitting boundary questions, the problem of describing the class of T-invariant measures with support in F can be reformulated using the right-inverse η of T defined by $T \circ \eta = Id$ and $Im(\eta) = F$. Since $\mu = T\mu$ and $\mathrm{Supp}(\mu) \subset \bar{F}$ is equivalent to $\mu = \eta\mu$, one is lead to studying the dynamics of η which appears to be locally affine and 1/2-contracting. This class of maps can be seen as a contracting variant of the family of Interval Exchange Transformations, namely the bijective piecewise translations of the Interval, introduced by Keane [15]. These considerations are the main reason for the present work.

Enlarging the perspective, the above maps belong to the more general class of injective and locally contracting maps of the interval $[0, 1)$, or *injective quasi-contractions.* Fixing an integer $m \geq 1$ and $0 = D_0 < D_1 < \cdots < D_m < D_{m+1} = 1$, a quasi-contraction T of the Interval is defined on each $[D_i, D_{i+1})$ by $T(x) = \Gamma_i(x)$, where each Γ_i is γ-Lipschitzian for some $0 < \gamma < 1$. These maps occur in various contexts, for instance in Differential Dynamics (see Alsed-Gambaudo-Mumbr [1] and Gambaudo-Tresser [11] when $m = 1$) and in Physics, since heuristically contraction properties ensure of some form of stability. Extending the work of Feely-Chua [9], from Signal Theory, on the map $x \longmapsto \{\gamma x + \alpha\}$, where $0 < \gamma < 1$, Bugeaud-Conze [5] have shown that its dynamics can be described via a concatenation algorithm similar to the Farey tree. Fixing γ, then for almost all α the transformation admits a unique periodic attractor. In the non-periodic cases, topological semi-conjugacies are then deduced with irrational rotations. Another point was to extend these results to general locally contracting and order-preserving maps of the Circle.

Content of the article. We study general injective quasi-contractions with $m + 1$ pieces, $m \geq 1$, on the Interval. The injectivity requirement is discussed in the next section.

1. We first provide a general analysis on the structure of an injective quasi-contraction which allows to bound from above the number of ergodic components by $2m$. We next prove a degeneracy result : for an open set of full Lebesgue measure in the set of parameters, the corresponding maps admit finitely many periodic orbits whose union of the attraction domains is the whole interval. Mention as a general fact that dilating maps are analyzed via inverse branches considered through transfer operators having then contraction properties. In our context we naturally look, in an elementary way, at forward images. Our approach is simple and geometric and seems better suited than the usual coding point of view (adopted when $m = 1$ for instance in [11] in terms of Sturm sequences) for situations far from rotations. This allows to slant the orbital complexity which is that of an Interval Exchange Transformation. In fact we show that generically the complexity stops growing in finite time. We complete this section with a perturbation result.

2. In a second part we solve the problem of invariant measure discussed above. More generally we consider the Borel probability measures on the Circle invariant by $Tx = px \mod (1)$ and with support in some *finite flower.* We prove that in general such measures are periodic and that the number of ergodic measures is less than the number of petals of the flower.

3. In a last part we consider a class of injective quasi-contractions that are order-preserving on the Circle $\mathbb{R}/\mathbb{Z}$. After observing the validity of the Poincaré theory on the rotation number, we first deduce from the first part of this paper that this number is generically rational. We then completely explicit the transformations having a given rational number and count the number of periodic attractors. As a corollary we study a model of injective quasi-contractions on the Interval with 3 pieces and show in the non-periodic cases the existence of mesurable conjugacies with interval exchange transformations.

2 Injective quasi-contractions and generic properties

Let us define injective quasi-contractions on the interval $[0,1)$.

Definition 2.1

1. *Let $m \geq 0$ and $\mathcal{D} = \{(D_i)_{1\leq i\leq m} \mid 0 < D_1 < \cdots < D_m < 1\} \subset \mathbb{R}^m$. To $D \in \mathcal{D}$ associate the partition $\mathcal{P}(D)$ in left-closed and right-open intervals. Set $D_0 = 0$ and $D_{m+1} = 1$.*
2. *Fix $0 \leq \gamma < 1$ and suppose to be given injective γ-Lipschitz maps $\Gamma_i : [0,1] \longrightarrow [0,1]$, for $0 \leq i \leq m$. For $D \in \mathcal{D}$ define a right-continuous map $T : [0,1) \to [0,1)$ by :*

$$Tx = \Gamma_i(x), \text{ for } x \in [D_i, D_{i+1}),$$

under the open condition on $D \in \mathbb{R}^m$: $\mathcal{C} = \mathcal{D} \cap \{T$ is injective and $TD_i^\pm \neq TD_j^\pm$, for $i \neq j,\ 1 \leq i \leq m,\ 0 \leq j \leq m+1\}$, where we set $Tx^- = T(x^-)$ for all $x \in [0,1)$.

3. *If $n \geq 1$ and $(\eta_j)_{1\leq j\leq n} \in \{0, \cdots, m\}^n$, write $\Gamma_{\eta_n,\cdots,\eta_1}$ for $\Gamma_{\eta_n} \circ \cdots \circ \Gamma_{\eta_1}$.*
4. *The sequence $(X_n = T^n[0,1))_{n\geq 0}$ is non-increasing. Set $X_\infty = \cap_{n\geq 0} X_n$.*
5. *Set $E = [0,1)\backslash T[0,1)$. This set is a disjoint union of intervals : $E = \cup_{1\leq i\leq m} F_i$.*

In the whole text, we say that $x^\pm$ is periodic if for some $q \geq 1$, one has $T^q x^\pm = x^\pm$. We show that in a generic sense, measurably and topologically, the dynamics of an injective quasi-contraction is degenerated. Denote by λ_m Lebesgue measure in $\mathbb{R}^m$ and let $\lambda = \lambda_1$.

Theorem 2.2

1. *For any $D \in \mathcal{C}$, the associated quasi-contraction T admits at most $2m$ ergodic components.*
2. *There is an open set $\Omega \subset \mathcal{C}$ with $\lambda_m(\mathcal{C}\backslash\Omega) = 0$ such that for $D \in \Omega$, the corresponding T admits a finite* **stable** *partition $\mathcal{P}(D')$ in intervals finer than $\mathcal{P}(D)$: every atom of $\mathcal{P}(D')$ is sent by T* **into** *another atom of $\mathcal{P}(D')$. As a result and more precisely :*
   - *The injective quasi-contraction T has $1 \leq n(T) \leq 2m$ attractive periodic orbits. If all Γ_i are increasing then $n(T) \leq m+1$.*
   - *The domain of attraction of each periodic orbit is a finite union of intervals with positive length and the union of the domains is $[0,1)$.*
   - *There exists a constant $C(T)$ such that for all $x \in [0,1)$, some periodic point y satisfies for all $n \geq 0$: $dist(T^n x, T^n y) \leq C(T)\ \gamma^n$.*

Remark. — The upper-bound $2m$ on $n(T)$ is not expected to be optimal, whereas the value $m+1$ is a natural candidate. It is easily reached for instance if $\Gamma_i[D_i, D_{i+1}) \subset [D_i, D_{i+1})$ for all $0 \leq i \leq m$.

2.1 Abstract structure of general injective quasi-contractions

We prove the first part of theorem (2.2) by giving a general description of $X_n = T^n[0,1)$, for $n \geq 1$. We proceed inductively :

- For $n = 1$, let $\mathcal{G}(1)$ be the set of disjoint intervals $(T[D_i, D_{i+1}))_{0\leq i\leq m}$. Denote by $\mathcal{R}_1$ the set of D_i in X_1. For $i \in \mathcal{R}_1$, let U_i^1 be the interval of $\mathcal{G}(1)$ containing D_i and $U_{i,\pm}^1$ its two halfs, putting D_i on the right side interval (the left side interval may be empty). Note that if for instance $U_{i,+}^1$ contains D_{i+1}, we replace this interval by $[D_i, D_{i+1})$, similar considerations holding on the left side of D_i. Then the elements of $\mathcal{G}(1)$, with the exception of the $(U_i^1)_{i\in\mathcal{R}_1}$, are sent into one another (without being cut) by T. For $i \in \mathcal{R}_1$, each of the intervals $U_{i,+}^1$ and $U_{i,-}$ has the same properties. Let $V_{i,\pm}^1$ be the two intervals of $\mathcal{G}(1)$ such that $TU_{i,\pm}^1 \subset V_{i,\pm}^1$.

As a result and when considering the intervals distincts of the $(U_i^1)_{i\in\mathcal{R}_1}$, together with the $(U_{i,-}^1, U_{i,+}^1)_{i\in\mathcal{R}_1}$, the set $\mathcal{G}(1)$ has a natural graph structure, where an edge corresponds to applying T. Starting from any interval, the sequence of visited intervals is either eventually periodic or gets to some interval (only one by injectivity) which is sent in some U_i^1 and that has a non-empty intersection with both $U_{i,-}^1$ and $U_{i,+}^1$. One then stops the graph. Call next *type I graph* a subgraph consisting in a periodic limit cycle together with its bassin of attraction and *type II graph* a subgraph where all intervals end in some interval which is sent in a U_i, with $i \in \mathcal{R}_1$, and has a non-empty intersection with $U_{i,-}^1$ and $U_{i,+}^1$. Denote by $\mathcal{T}_I(1)$ and $\mathcal{T}_{II}(1)$ the set of type I graphs and type II graphs at time 1 respectively.

- For $n \geq 1$, the set of disjoint intervals forming X_{n+1} is denoted by $\mathcal{G}(n+1)$ and is deduced from $\mathcal{G}(n)$ as follows. Replace first all intervals of $\mathcal{G}(n)$ by their images by T. Observe that the attracting limit cycle of a type I graph remains globally unchanged. Let next $\mathcal{R}_{n+1} \subset \mathcal{R}_n$ be the subset of discontinuities in X_{n+1}. For $i \in \mathcal{R}_{n+1}$, let $U_i^{n+1} = U_{i,-}^{n+1} \cup U_{i,+}^{n+1}$ be the interval containing D_i, with the same conventions as above. Let $V_{i,\pm}^{n+1}$ be the two intervals in $\mathcal{G}(n+1)$ such that $TU_{i,\pm}^{n+1} \subset V_{i,\pm}^{n+1}$. Then $\mathcal{G}(n+1)$ is defined as before with graphs of types I and II at time n respectively denoted by $\mathcal{T}_I(n+1)$ and $\mathcal{T}_{II}(n+1)$.

We now make the following observations :

- The sequence of sets $(\mathcal{R}_n)_{n\geq 1}$ is non-increasing and is therefore eventually constant to some limit set written as $\mathcal{R}$. Denote also by $\mathcal{R}' \subset \mathcal{R}$ the elements D_i such that $U_{i,-}^n \neq \emptyset$ for all n. Recall that $U_{i,+}^n \neq \emptyset$ for $i \in \mathcal{R}$ and for all n. The set $\mathcal{R}'$ is also eventually constant. Let N be such that these properties hold for $n \geq N$, as well as $U_{i,-}^n = \emptyset$ for $i \in \mathcal{R}\backslash\mathcal{R}'$.

- Next if some non-empty $U_{i,\epsilon}^n$ with $\epsilon \in \{\pm\}$ and $i \in \mathcal{R}$ branches on a type I graph, then observe that this is true for all $U_{i,\epsilon}^p$ with $p \geq n$, since the branching is made on images of the same type I graph. Thus the number of such (i,ϵ) is non-decreasing and consequently eventually constant. Denote then by J the set of (i,ϵ) with $\epsilon \in \{\pm\}$ and $i \in \mathcal{R}$ such that $U_{i,\epsilon}^n$ do not branch on a type I graph. Let N be such that these properties hold for $n \geq N$.

- Each type I graph contracts to a periodic orbit laying in its closure. Remark that if $m \geq 1$ then at least one the boundaries of any interval of X_n has the form $T^k D_s^\pm$ with $s \notin \{0, m+1\}$. Thus the periodic orbit corresponding to each type I graph is exactly the ω-limit set of some $D_s^\pm$ with $s \notin \{0, m+1\}$. Distinct such periodic orbits require distinct discontinuities of this form. Also if distinct type I graphs admit the same limit periodic orbit and since T is injective and each D_i, $1 \leq i \leq m$ is a true discontinuity for T, then one type I graphs must contain some $U_{i,-}^n$ and the other one the interval $U_{i,+}^n$. Thus the number of distinct type I graphs is bounded from above by $4m$. We only use for the moment that it is bounded. Suppose next that no other "new" type I graph appears after time N.

- For n large enough we assert that no periodic point is contained in the closure of a type II graph. If not, a periodic orbit (say of length q) would be entirely contained in the closure of the family of type II graphs. Denote by $D_{i_1}, \cdots, D_{i_p}$ the discontinuities such that each corresponding type II graph contains a periodic point. Since the lengths of the intervals in $\mathcal{G}(n)$ containing the D_{i_j} tend to 0, as $n \to +\infty$, for some $(\epsilon_j)_{1\leq j\leq p} \in \{\pm\}^p$ the $D_{i_j}^{\epsilon_j}$ are periodic points in the periodic orbit. More precisely, either no D_{i_j} is a periodic point and then all $D_{i_j}^-$ are periodic, or all D_{i_j} are periodic, since T is right-continuous. As T is injective, all D_{i_j} belong to the closure of the boundary of $T^{N+q}[0,1)$. Take then $n \geq N+q$. In the first case, all $U_{i_1,-}^n$ are non-empty and obviously define a type I graph under iterations. In the second case, either all $U_{i_j,-}^n = \emptyset$, in which case the same holds with the $U_{i_j,+}^n$, or there exists $U_{i_{j_0},-}^n \neq \emptyset$ and then $U_{i_{j_0},+}^n = \{D_{i_{j_0}}\}$ since $D_{i_{j_0}}$ is a the boundary point. However the sequence of intervals starting at $U_{i_{j_0},+}^n$ defines a type I graph. In any case we get a contradiction. Suppose next that N is large enough so that type II graphs for time $n \geq N$ contain no periodic point.

- For $(i,\epsilon) \in J$, the ω-limit set of D_i^ϵ contains no periodic orbit. Indeed, $U_{i,\epsilon}^N$ branches on a type II graph. If the forward orbit of D_i^ϵ never falls in some $U_{i',\epsilon'}^N$ that branches in a type

I graph, then the result holds since the orbit of D_i^ϵ is contained in type II graphs whose closure do not contain any periodic orbit. If on the contrary the forward orbit of D_i^ϵ falls in an interval $U^N_{i',\epsilon'}$ branching in a type I graph, consider the first time p when this holds. Up to taking forward images of D_i^ϵ (with then indices in J), assume also that between times 1 and p, the images of D_i^ϵ are at a positive distance of all D_j with $j \in \mathcal{R}$. Thus for n large enough and since the lengths of the intervals decrease, the interval $U^n_{i,\epsilon}$ will not be cut by forward iterations and will branch on a type I graph. This contradicts the definition of J.

As a result, the number of periodic orbits is less or equal to $2m - \mathrm{card}(J)$. Next, Borel diffusive probability measures must be supported by type II graphs. We show that the number of ergodic ones is less or equal to $\mathrm{card}(J)$. First, only branches starting from $V^N_{i,\epsilon}$ with $(i,\epsilon) \in J$ can be weighted by a $T-$invariant Borel probability measure. Following an idea of Keane [15], let $\mu_1, \cdots, \mu_q$ be q T-invariant ergodic diffusive Borel probability measures. Let $(A_i)_{1\le i\le q}$ be disjoint $T-$invariant Borel sets such that $\mu_i(A_j) = \delta_{i,j}$. Define $\mu = \frac{1}{q}(\mu_1 + \cdots + \mu_q)$ and fix $0 < \epsilon < \min\{1/q, 1/2\}$. Introduce then :

$$B_i^n = \{P \in \mathcal{T}_{II}(n) \text{ with } \mu(A_i \cap P) \geq (1-\varepsilon)\mu(P),\ \mu(P) > 0\}.$$

By regularity of the diffusive measure μ, if n is large enough, then B_i^n is non-empty and even, setting $Y^n = \left(\cup_{P\in\mathcal{T}_{II}(n)} P\right) \cup \setminus \cup_{i=1}^q B_i^n$, we have :

$$\mu(A_i \Delta B_i^n) < \varepsilon, \text{ for } 1 \le i \le q, \text{ and } \mu(Y^n) < \varepsilon.$$

Let now $P \in B_i^n$. Using the T-invariance of A_i, we get $\mu(T^{-1}P \cap A_i) = \mu(T^{-1}(P \cap A_i)) = \mu(P \cap A_i) \geq (1-\varepsilon)\mu(P) = (1-\varepsilon)\mu(T^{-1}P)$. As a result, at least one element of the type II graph where P lays and that branches to P also belongs to B_i^n. Repeating this procedure backward, at least one element of the form $V^n_{i',\epsilon'}$, with $(i',\epsilon') \in J$, belongs to B_i^n. This element does not belong to any other B_k^n, since otherwise the fact that $A_i \cap A_k = \emptyset$ implies $\mu(V^n_{i',\epsilon'}) \geq 2(1-\varepsilon)\mu(V^n_{i',\epsilon'})$, whereas $\mu(V^n_{i',\epsilon'}) > 0$ and $\varepsilon < 1/2$. Thus $q \le \mathrm{card}(J)$.

This concludes the proof of the first part of the theorem.

□

2.2 Proof of the second part of theorem (2.2)

In the context of theorem (2.2), we begin with an elementary analysis suggesting our strategy. Denote by A^o the interior of a set A.

Lemma 2.3

1. *We have* $\lim_{n\to+\infty} \lambda(X_n) = \lambda(X_\infty) = 0$.
2. *For* $n \geq 0$, $[0,1) = T^{n+1}[0,1) \cup_{0\le i\le n} T^i E$, *disjointly. Also* $(T^n F_i)_{n\ge 0, 1\le i\le m+2}$ *are disjoint.*
3. *Set* $\Omega = \{D \in \mathcal{C} \mid \forall\ 1 \le i \le m,\ \exists\ (k_i, l_i) \in \mathbb{N} \times \{1, \cdots, m+2\}$ *with* $D_i \in (T^{k_i} F_{l_i})^o\}$. *Then* Ω *is open. If* $D \in \Omega$, *the conclusion of theorem* (2.2) *holds.*

Proof of the lemma :
1) We have $\lambda(T^n[0,1)) \to \lambda(X_\infty)$ and $X_\infty = \cap_{n\ge1} T^n[0,1) = \cap_{n\ge0} T(T^n[0,1)) \subset TX_\infty$. Therefore :

$$\lambda(X_\infty) \le \lambda(TX_\infty) \le \sum_{0\le i\le m} \int_{X_\infty\cap[D_i,D_{i+1})} |\Gamma_i'(t)|\ dt \le \gamma\ \lambda(X_\infty).$$

Consequently $\lambda(X_\infty) = 0$.

2) As $[0,1) = T[0,1) \cup E$ disjointly, we get recursively $[0,1) = T^{n+1}[0,1) \cup_{0\le i\le n} T^i E$ disjointly, since T is injective. Next if $T^n F_i \cap T^{n'} F_{i'} \neq \emptyset$ with $n \geq n'$, then $n = n'$ otherwise $T^{n-n'} Fi \cap F_{i'} \neq \emptyset$ (as T is injective) and $F_{i'} \subset [0,1)\backslash T[0,1)$. Then $F_i \cap F_{i'} \neq \emptyset$ and thus $i = i'$.

3) If $D \in \Omega$, observe that for $1 \le i \le m$, $D_i \notin \{T^n D_j^{\pm} \mid 0 \le j \le m+1,\ 1 \le n \le K\}$ with $K = \max_{1 \le i \le m} k_i$. Thus the boundaries of all $T^{k_i} F_{l_i}$ are locally continuous functions of D and thus Ω is open. Fix now $D \in \Omega$ with a collection of (k_i, l_i) and set $D' = \{T^{-n} D_i \mid 1 \le i \le m,\ 0 \le n < k_i\}$. The elements of D' are all distinct. Denote by $\mathcal{P}(D') = (U_k)$ the corresponding finite partition of $[0,1)$. For $1 \le i \le m$ let U_{n_i} and U_{m_i} be the components respectively on the left side and on the right side of D_i.

Taking any U_k, we now show that $(TU_k)^o$ is contained in another atom of $\mathcal{P}(D')$. First note that $(TU_k)^o$ is an interval, since U_k contains no D_i in its interior. Next if some $T^{-l} D_j \in (TU_k)^o$ with $0 \le l < k_j$, then $l \le k_j - 2$ as $T^{-k_j} D_j$ does not exist. However this would imply $T^{-l-1} D_j \in (U_k)^o$, contradicting the definition of D'.

Writing $U_k = [x, y)$, there exist $0 \le i, j \le m+1$, $l_i \ge 0$, $l_j \ge 0$ with $x = T^{-l_i} D_i$ and $y = T^{-l_j} D_j$, setting $l_0 = 0$ and $l_{m+1} = 0$. Except if $m = 0$ in which case T contracts to a unique fixed point, either i or j is in $\{1, \cdots, m\}$. Since T is continuous at any $T^{-l} D_k$ with $l \ge 1$ and since $TD_i \notin D'$ for $0 \le i \le m+1$, U_k is never cut under iterations of T and is a positive distance from the boundary of the atom of $\mathcal{P}(D')$ where it lies for times larger than $\max\{k_i, k_j\} + 1$. Writing U_{i_l} for the atom containing $T^l U_k$, the sequence (i_l) contains a term among the $\{n_i, m_i,\ \text{for } 1 \le i \le m\}$ and is necessarily preperiodic. Observe that the periodic part of $(i_l)_{l \ge 0}$ begins at a time $\le K$ and the period is $\le 2mK$, where $K = \max_{1 \le i \le m} k_i$. An embedded compacts argument gives that U_k is attracted exponentially fast to a periodic orbit. Also the ω-limit set of any x is the one of some $D_i^{\pm}$, $1 \le i \le m$. Hence there are at most $2m$ periodic orbits. If all Γ_i are increasing then the ω-limit set of any point $x \in [0,1)$ is the one of some D_i, $0 \le i \le m$, and the upper-bound on $n(T)$ is $m+1$ in this case.

□

Remark. — When $m = 1$, the inequality $n(T) \le 2$ is contained in [11] or [10]. The same upper-bounds as the ones we give in the general case would also follow easily.

We begin the proof of theorem (2.2). We first build a denumerable family $\mathcal{F}$ of zero-Lebesgue measure subsets in $\mathcal{C}$. We next show that $\lambda_m(\mathcal{C} \backslash (\Omega \cup \mathcal{F})) = 0$, giving then $\lambda_m(\mathcal{C} \backslash \Omega) = 0$.

Step 1 — Fixing $1 \le i \le m$, $0 \le j \le m+1$, $n \ge 1$ and $(\eta_k)_{1 \le k \le n} \in \{0, \cdots, m\}^n$, observe that :

$$\{D \in \mathcal{C} \mid D_i = \Gamma_{\eta_n, \cdots, \eta_1}(D_j)\}$$

has zero Lebesgue measure in $\mathcal{C}$. This is obvious if $i \ne j$, as this is the graph of a Lipschitz map. If $i = j$, then D_i is the fixed point of $\Gamma_{\eta_n, \cdots \eta_1}$ in $[0,1]$, since $n \ge 1$. Define $\mathcal{F}$ as the family of subsets of $\mathcal{C}$ obtained when i, j, n and $(\eta_k)_{1 \le k \le n}$ vary.

Step 2 — Let $D \in (\Omega \cup \mathcal{F})^c$. We make preliminary remarks on T.

- Any equality $D_i = T^l D_j^{\pm}$ with $1 \le i \le m$, $0 \le j \le m+1$ and $l \ge 1$ is impossible.
- Setting $I = \{1, \cdots, m\}$ there is $\emptyset \ne J \subset I$ such that for $i \in J$, then $D_i \in \cap_{k \ge 0} (T^k [0,1))^o$ and for $i \in I \backslash J$, there is $(k_i, l_i) \in \mathbb{N} \times \{1, \cdots, m+2\}$ such that $D_i \in (T^{k_i} F_{l_i})^o$.
- Set $K = \max_{i \in I \backslash J} k_i$ and denote by $G_1, \cdots, G_p$ the intervals forming $T^K E$. Each G_i is not cut under iterations of T. Also $\lambda(\cup_{n \ge K} T^n E) = 1 - \lambda(\cup_{0 \le n < K} T^n E)$.
- Since the $(T^n G_j)_{1 \le j \le p, n \ge 0}$ form the remaining mass to be added after time K and since $\text{Dist}(D_i, G_k) > 0$ for $i \in J$ and $1 \le k \le p$, any D_i with $i \in J$ admits arbitrary close intervals among the $(T^n G_j)_{1 \le j \le p, n \ge 0}$ on its left side and on its right side.

Lemma 2.4
For all $i \in J$ there exist arbitrary large n and $1 \le j \le p$ such that the interval $T^n G_j$ is arbitrary close to D_i, lies on its right side and verifies :

$$\frac{\lambda(T^n G_j)}{Dist(D_i, T^n G_j)} \ge \kappa = \frac{1-\gamma}{p} > 0. \tag{1}$$

Proof of the lemma :
Introduce the set $Z \subset \{1, \cdots, p\}$ of indices such that for $j \in Z$, G_j admits iterates converging to D_i on its right side. Set $p' = \text{Card}(Z)$ and note that $1 \leq p' \leq p$. Fix $\varepsilon > 0$ such that all approximations of D_i in $[D_i, D_i + \varepsilon]$ are made by iterates of G_j with $j \in Z$. Suppose now that for some N inequality (12) is false for all $n \geq N$ and $j \in Z$.

Let $T^{v_0}G_{u_0}$ be a best record iterate on the right side of D_i with $u_0 \in Z$ and a record time $v_0 \geq N$. Let $(T^{v_k}G_k)_{k \in Z \backslash \{u_0\}}$ be the first iterates of each G_k, with $k \in Z \backslash \{w_0\}$, standing between D_i and $T^{v_0}G_{u_0}$. For such a k, notice that $v_k \geq N$. Finally observe that the possible mass that can be added between D_i and $T^{v_0}G_{u_0}$ by iterates of $(G_j)_{j \in Z}$ is at most :

$$\begin{aligned}
\sum_{k \in Z \backslash \{u_0\}} \frac{1}{1-\gamma} \lambda(T^{v_k}G_k) + \frac{\gamma}{1-\gamma} \lambda(T^{v_0}G_{u_0}) &\leq \sum_{k \in Z} \frac{1}{1-\gamma} \lambda(T^{v_k}G_k) \\
&< \frac{\kappa}{1-\gamma} \sum_{k \in Z} \text{Dist}(D_i, T^{v_k}G_k) \\
&< \frac{\kappa p'}{1-\gamma} \text{Dist}(D_i, T^{v_0}G_{u_0}).
\end{aligned}$$

Since $\kappa p'/(1-\gamma) \leq \kappa p/(1-\gamma) \leq 1$, this brings a contradiction.

□

Step 3 — We prove that $\lambda_m(\mathcal{C} \backslash (\Omega \cup \mathcal{F})) = 0$. Supposing the contrary, there exists $J \subset I$ with $J \neq \emptyset$ such that the following set A verifies $\lambda_m(A) > 0$:

$$A = \left\{ D \in \mathcal{C} \backslash (\Omega \cup \mathcal{F}) \mid \{D_j\}_{j \in I \backslash J} \subset \cup_{n \geq 0} (T^n E)^o, \ \{D_j\}_{j \in J} \cap (\cup_{n \geq 0} T^n E) = \emptyset \right\}.$$

Then A admits a density point D^0. Denoting by $B_m(a, r)$ the Euclidian ball in $\mathbb{R}^m$ of center a and radius $r > 0$, one has :

$$\lim_{r \to 0} \frac{\lambda_m(B(D^0, r) \cap A)}{\lambda_m(B(D^0, r))} = 1. \tag{2}$$

We use for D^0 the quantities defined in *Step 2*. We add an exponent $(\ )^0$ to mark the dependence in D^0, except for T for simplicity. We now make arbitrary small perturbations of D^0. Note that the corresponding map will remain injective and the images of the wandering intervals cannot overlap. Also each D_i with $i \in I \backslash J$ remains strictly in the iterior of $T^{k_i} F_{l_i}$.

Fix $i \in J$. There exists $1 \leq j \leq p$ and arbitrary large n such that $T^n G_j$ is on the right side of D_i^0 with $\lambda(T^n G_j^0) \geq \kappa \ \text{Dist}(D_i^0, T^n G_j^0)$. Set $G = G_j$, $d_n = \text{Dist}(D_i^0, T^n G^0)$ and $l_n = \lambda(T^n G^0)$. Without loss of generality, assume that :

$$d_n = \min\{\text{Dist}(D_k^0, T^l G^0) \mid 1 \leq k \leq m, \ 0 \leq l \leq n\},$$

since the lengths of the $(T^l G^0)_{l \geq 0}$ decrease. Denote by $(e_k)_{1 \leq k \leq m}$ the canonical basis of $\mathbb{R}^m$ and consider the path $D^\eta = D^0 + \eta e_i$, $\eta \geq 0$. All the corresponding quantities for the transformation get some $(\ )^\eta$. Introduce an integer q such that $2\gamma^q(1 + \kappa/4) \leq 1/2$. Since the boundaries of G^0 are of the form $T^s(D_u^0)^\pm$, $s \geq 1$, we make the following observations for $0 \leq l \leq n$ and small η :

$$\begin{aligned}
\text{Dist}(D_i^\eta, T^n G^\eta) &\leq d_n - (1 - \gamma^n)\eta & (3) \\
\lambda(T^n G^\eta) &\geq l_n - 2\gamma^n \eta & (4) \\
\text{Dist}(D_k^\eta, T^l G^\eta) &\geq \text{Dist}(D_k^0, T^l G^0) - \gamma^l \eta \geq d_n - \gamma^l \eta, \text{ if } k \in \{1, \cdots, m\} \backslash \{i\}. & (5)
\end{aligned}$$

Also for any $T^l G^\eta$ with $0 \leq l < n$ and close to D_i we have $\text{Dist}(D_i^\eta, T^l G^\eta) \geq d_n + \eta(1-\gamma)$ if it is on its left side and $\text{Dist}(D_i^\eta, T^l G^\eta) \geq \text{Dist}(D_i^\eta, T^n G^\eta)$ if it stands on its right side, as $T^l G$ is protected by $T^n G$. Then it results from (3), (4), (5) and $l_n \geq \kappa d_n$ that :

$$\eta_1 = \inf\{\eta \geq 0 \mid D_i^\eta \in T^n G^\eta,\ \mathrm{Dist}(D_i^\eta, \partial(T^n G^\eta)) = \kappa d_n/4\}$$

exists and verifies $\eta_1 \leq (1-\gamma^n)^{-1} d_n(1+\kappa/4) \leq 2d_n(1+\kappa/4)$ for n large enough. Also for $q \leq l \leq n$ and $k \in I\backslash\{i\}$, we have $\mathrm{Dist}(D_k^{\eta_1}, T^l G^{\eta_1}) \geq d_n - \gamma^l \eta_1 \geq d_n/2$. Similarly for $q \leq l \leq n$ we have $\mathrm{Dist}(D_i^{\eta_1}, T^l G^{\eta_1}) \geq \min\{d_n, \mathrm{Dist}(D_i^{\eta_1}, \partial(T^n G^{\eta_1})\} \geq d_n(1 \wedge \kappa/4)$. These inequalities evidently remain true for $1 \leq l < q$ if n is large enough since q is fixed and $d_n \to 0$.

Finally all boundaries of the $(T^l G)_{0 \leq l \leq n}$ are at most 1−Lipschitz maps of D and thus the ball $B(D^{\eta_1}, \eta_2)$ with $\eta_2 = (d_n/4) \wedge (\kappa d_n/8))$ is entirely contained in A^c as $D_i \in T^n G$ in that case. As a result we get the inequality :

$$\frac{\lambda_m(B(D^0, \eta_1+\eta_2) \cap A^c)}{\lambda_m(B(D^0, \eta_1+\eta_2))} \quad \geq \quad \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^m \geq \left(\frac{\kappa \wedge 2}{\kappa \wedge 2 + 16(1+\kappa/4)}\right)^m > 0. \tag{6}$$

As this lower bound is independent on n and $\eta_1 + \eta_2 \to 0$, as $n \to +\infty$, this contradicts (2) and concludes the proof of the theorem.

□

Remark 1 — In the proof of theorem (2.2) the injectivity assumption essentially simplifies the description of $[0,1)\backslash T^n[0,1))$, $n \geq 1$, which consists in considering the forward orbits of the wandering intervals defined by E. When suppressing injectivity and among other difficulties the number of pieces of $E_n = T^n[0,1)\backslash T^{n+1}[0,1)$ can increase with n and this corresponds to p becoming large in inequality (1). It is however natural to conjecture that without supposing injectivity the dynamics still consists in attracting periodic orbits. For instance lemma (2.3) reconducts with only slight changes in the formulation.

Remark 2 — Theorem (2.2) provides an algorithm for concretely computing the dynamics of a generic injective quasi-contraction.

1. Compute the $(T^n F_i)_{1 \leq i \leq m+2}$, $n \geq 0$, until each D_j is covered, $1 \leq j \leq m$. Theorem (2.2) implies that this process stops almost surely. Write then $D_j \in T^{k_j} F_{l_j}$ with $1 \leq j \leq m$.
2. Reversing the calculations, we get the $T^{-k} D_j$, with $0 \leq k \leq k_j$ and $1 \leq j \leq m$. Then $\mathcal{D} = \{T^{-k} D_j,\ 0 \leq k \leq k_j,\ 1 \leq j \leq m\}$ defines a stable partition of $[0,1)$. Considering the orbit of any point in each atom allows to locate and count the periodic attractors.

For a non-generic injective quasi-contraction having diffusive ergodic components, the first step in the above algorithm does not stop. It may be possible to extend the first part of the proof of theorem (2.2) to describe the structure of the transformation via an algorithm generalizing the usual one associated to continued fractions and the sequence of best approximations. This would probably provide natural isomorphisms with maps in the class of interval exchange transformations.

2.3 Structural stability

We show that perturbations of a generic injective quasi-contraction still have the same behaviour as the one described by theorem (2.2). For any map $\delta : [0,1) \to \mathbb{R}$, set $T_\delta(x) = T(x) + \delta(x)$.

Theorem 2.5
Let Ω be the open set of lemma (2.3) and theorem (2.2). For $D \in \Omega$ denote by T the corresponding transformation and suppose that $T[0,1) \subset (0,1)$. Then for $\varepsilon > 0$, there is $\eta > 0$ such that any sequence of maps $\delta = (\delta_n)_{n \geq 1}$ with $\|\delta_n\|_\infty \leq \eta$ for all $n \geq 0$ and $\|\delta_n\|_\infty \to 0$ verifies :

1. *For every $x \in [0,1)$, the sequence $(x_n)_{n \geq 0}$ defined by $x_0 = x$ and $x_{n+1} = T_{\delta_n(x_n)} x_n$ converges to a periodic orbit of T.*
2. *Every interval I in the attraction domain of a periodic orbit $\mathcal{O}$ for T contains a sub-interval J verifying $|J| \geq (1-\varepsilon)|I|$ such that for $x \in J$ one has Dist$(x_n, T^n x) \to 0$, as $n \to +\infty$.*

Proof of the theorem :
As in lemma (2.3) assume that there exist $k_i \geq 0$ and $l_i \in \{1, \cdots, m+2\}$ for $1 \leq i \leq m$ such that $D_i \in \text{int}(T^{k_i} E_{l_i})$. Then the partition $\mathcal{P}(D')$ is stable, where :

$$D' = \{T^{-k} D_i \mid 0 \leq k \leq k_i,\ 1 \leq i \leq m\}.$$

Take η small enough so that the image segments by T_{δ_n} remain disjoint and contained in $(0,1)$. Suppose first that each δ_n is a constant map still written δ_n. Build D'' by replacing $T^{-k} D_i$ by $T_{\delta_1}^{-1} \cdots T_{\delta_k}^{-1} D_i$ in the definition of D'. These quantities exist if η is small enough and are perturbations. Also if η is small enough, the condition on the length in the second part of the theorem is satisfied. We show that an atom of $\mathcal{P}(D'')$ converges under iterations of the perturbed transformations to the same periodic orbit for T as the corresponding atom of $\mathcal{P}(D')$ under iterations of T.

Recall that the definition of Ω implies that any D_u cannot be written as $T^k D_v^{\pm}$ with $k \geq 1$. Therefore if $a = [T^{-l_i} D_i, T^{-l_j} D_j)$ is an atom of $\mathcal{P}(D')$, then for $l \geq \min\{l_i, l_j\} + 1$, at least one boundary point of $T^l a$ is at a positive distance of the boundary of the atom of $\mathcal{P}(D')$ where $T^l a$ lies. For $l \geq \tau_a := \max\{l_i, l_j\} + 1$, both boundaries share this property. Since the elements of the orbit of $T^l a$ that lie in a given atom of $\mathcal{P}(D')$ form a decreasing sequence of intervals, there finally exists $\eta_a > 0$ such that the distance between a boundary of $T^l a$ and the boundary of the atom where $T^l a$ lies is either 0 or $\geq \eta_a$. The last case is verified for both boundaries when $l \geq \tau_a$. This is then also true for the distance between $T^l a$ and any D_i for $l \geq \tau_a$.

Let a' be the atom of $\mathcal{P}(D'')$ corresponding to a. First choose η small enough such that for all atom a and all $0 \leq l \leq \tau_a$ the distance between a boundary of $T_{\delta_l} \cdots T_{\delta_1} a'$ and the boundary of the atom of $\mathcal{P}(D'')$ where it lies is either 0 or $\geq \eta_a/4$. Assume also that the boundaries of $T_{\delta_{\tau_a}} \cdots T_{\delta_1} a'$ and of $T^{\tau_a} a$ differ from less than $\eta_a/4$. Finally impose :

$$\eta \leq \left(\frac{1-\gamma}{4}\right) \min\{\eta_a \mid a \in \mathcal{P}(D')\}.$$

Let now $y \in T_{\delta_{\tau_a}} \cdots T_{\delta_1} a'$ and $z \in T^{\tau_a} a$ be such that $\text{Dist}(y,z) \leq \eta_a/4$. We assert that the images of y by $(T_{\delta_{\tau_a+n}} \cdots T_{\delta_{\tau_a+1}})_{n \geq 1}$ and the iterates of z under $(T^n)_{n\geq 1}$ lie in the same intervals of $\mathcal{P}(D)$ and the distance between the corresponding images tends to zero. Indeed, for $n \geq 1$:

$$\text{dist}(T_{\delta_{\tau_a+n}} \cdots T_{\delta_{\tau_a+1}}, T^n z) \leq \gamma^n \, |y - z| + \sum_{s=0}^{n-1} \gamma^{n-s-1} \, \delta_{s+\tau_a} \quad (7)$$

$$\leq \frac{\eta_a}{4} + \frac{\eta_a}{4} \leq \frac{\eta_a}{2}. \quad (8)$$

Then (8) gives the first assertion and (7) the second one. To conclude the proof of the theorem, fix x and apply the result for constant maps with the sequence $(\delta_n)_{n\geq 0} := (\delta_n(x_n))_{n\geq 0}$.

□

3 A class of p-invariant measures on the Circle

We use theorem (2.2) to describe a class of Borel Probability measures on the Circle invariant by $\times p$ and with zero entropy. These measures generalize the family of Sturm measures (see Bullet-Sentenac [6]) and appear in the study of *maximizing measures*, as mentioned in the introduction.

Recall that Sturm measures are characterized by the fact that their support is contained in a semi-Circle (see [6]). We introduce a more general family of domains with symmetry properties.

3.1 Definition of a p-flower with m petals

Consider the Circle $S^1 = \mathbb{R}/\mathbb{Z}$ endowed with Lebesgue measure and the transformation $T(x) = px \mod (1)$, where $p \geq 2$ is a fixed integer.

Definition 3.1

1. *Fix $m \geq 1$. Let then :*

   - $S_{p,m} = \{(s_i)_{1\leq i\leq m} \mid s_1 < s_2 < \cdots < s_m < s_1 + 1/p,\ 0 \leq s_1 < 1/p\} \subset \mathbb{R}^m$,
   - $K_{p,m} = \{(k_i)_{1\leq i\leq m} \mid 0 \leq k_i \leq p-1,\ k_i \neq k_{i+1},\ \text{for } 1 \leq i \leq m-1,\ k_m \neq p - k_1\}$.

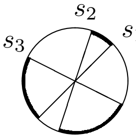
   Set also $s_{m+1} = s_1 + 1/p$ and $k_{m+1} = k_1 - 1$. For any $(s,k) \in S_{p,m} \times K_{p,m}$, define a "p-flower with m petals" as follows :

$$Fl(s,k) = \cup_{1\leq i\leq m} \left([s_i, s_{i+1}] + \frac{k_i}{p} \right). \tag{9}$$

2. *For $(s,k) \in S_{p,m} \times K_{p,m}$ and $\varepsilon = (\varepsilon_i)_{1\leq i\leq m} \in \{0,1\}^m$, define $Fl(s,k,\varepsilon)$ as in (9), but deleting $s_{i+1} + k_i/p$ if $\varepsilon_i = 0$ and $s_{i+1} + k_{i+1}/p$ if $\varepsilon_i = 1$, for $1 \leq i \leq m$. To $Fl(s,k,\varepsilon)$ associate the right-inverse $\eta_{s,k,\varepsilon}$ of T verifying for all $x \in \mathbb{R}/\mathbb{Z}$:*

$$\eta_{s,k,\varepsilon}(x) \in Fl(s,k,\varepsilon) \text{ and } p\eta_{s,k,\varepsilon}(x) = x \mod (1).$$

Remark. — For $p = 2$, $m = 1$ and $(s,k) \in S_{2,1} \times K_{2,1}$, then $Fl(s,k)$ is a closed semi-circle. An example of a 2-flower with 3 petals and $k = (k_1,k_2,k_3) = (0,1,0)$ is :



We consider below the set of T-invariant measures with support in a p-flower with m petals. These tend to be periodic. Observe that any finite union of periodic orbits is contained in the interior of some $Fl(s,k)$ for suitable m, s and k. Start with an elementary lemma.

Lemma 3.2
Let $(s,k) \in S_{p,m} \times K_{p,m}$ and μ be a Borel probability measure on S^1.

1. *Let $\mu = T\mu$. Then for any x there is at most one $0 \leq k \leq p-1$ with $\mu\{x + k/p\} > 0$.*
2. *Both $\mu = T\mu$ and Supp$(\mu) \subset Fl(s,k)$ hold if and only if there is $\varepsilon \in \{0,1\}^m$ with $\mu = \eta_{s,k,\varepsilon}\mu$.*

Proof of the lemma :
If $\mu\{x\} > 0$, then $T^{-n-n'}\{x\} \cap T^{-n}\{x\} \neq \emptyset$ for some $n \geq 0$ and $n' \geq 1$, since μ is a finite measure. Thus $T^{n'}x = x$. If for some $1 \leq k \leq p-1$ one has $\mu\{x + k/p\} > 0$, then $x + k/p$ is also periodic. As $x + k/p$ must be in the same orbit as x, this is a contradiction.

Consider the second part. If $\mu = \eta_{s,k,\varepsilon}\mu$, then $T\mu = (T \circ \eta_{s,k,\varepsilon})\mu = \mu$. Also $\text{Supp}(\mu) = \eta_{s,k,\varepsilon}^{-1}\text{Supp}(\mu) \subset Fl(s,k)$. Reciprocally and using the first part, either $s_{i+1} + k_i/p$ or $s_{i+1} + k_{i+1}/p$ has $\mu-$measure zero for $1 \leq i \leq m$. Choose next $Fl(s,k,\varepsilon)$ corresponding to the exclusion from $Fl(s,k)$ of any point with zero mass in each couple mentioned above. Then $\eta_{s,k,\varepsilon}\mu = \mu$.

□

Remark — If a T-invariant measure μ verifies $\text{Supp}(\mu) \subset Fl(s,k)$, then it need not be invariant with respect to all $\eta_{s,k,\varepsilon}$. For instance taking $m = 1$ and the right-inverse η of T defined with $(0, 1/p]$, observe that δ_0 verifies $T\delta_0 = \delta_0$ but $\eta\delta_0 = \delta_{1/p}$.

3.2 Invariant measures with support in a p-flower with m petals

Theorem 3.3
Let $p \geq 2$, $m \geq 1$ and $k \in K_{p,m}$. There is an open set $\Omega \subset S_{p,m}$ of full Lebesgue measure in $S_{p,m}$ such that for $s \in \Omega$, the ergodic $T-$invariant Borel probability measures with support in $Fl(s,k)$ are periodic and their number is $\leq m$. These measures are locally constant functions of s on Ω.

Proof of the theorem :
Fixing $s \in S_{p,m}$ lemma (3.2) implies that a T-invariant Borel probability measure with support in $Fl(s,k)$ is invariant with respect to some $\eta_{s,k,\varepsilon}$. Fix then $\varepsilon \in \{0,1\}^m$ and write η for $\eta_{s,k,\varepsilon}$. On $[0,1)$ observe that η is an injective quasi-contraction with $\{\Gamma_i\} \subset \{x \longmapsto x/p + i/p \mid 0 \leq i \leq p-1\}$ and $D = \{ps_j \mod (1) \mid 1 \leq j \leq m\} \backslash \{0\}$. Remark that if $ps_j \neq 0 \mod (1)$ for $1 \leq j \leq m$, then 0 is not a discontinuity of η when seen on the Circle. This condition can be assumed to be realized.

Applying now theorem (2.2) and using the notations of this theorem, there is an open set $\Omega \subset S_{p,m}$ of full Lebesgue measure in $S_{p,m}$, such that for $s \in \Omega$, then for $1 \leq i \leq m$, $ps_i \in (\eta^{k_i} F_{l_i})^o$, for some $k_i \geq 0$ and $1 \leq l_i \leq m+2$, where $(F_j)_{1 \leq j \leq m+2}$ are the wandering intervals of $[0,1) \backslash \eta([0,1))$.

When η is seen on the Circle, the partition defined by $\mathcal{D} = \{\eta^{-k}(ps_i) \mid 1 \leq i \leq m,\ 0 \leq k \leq k_i\}$ is stable under η. Then each point x under iterates of η converges to a periodic orbit and its ω-limit set verifies $\omega(x) \subset \cup_{1 \leq i \leq m} \omega(ps_i)$, as η is order-preserving on each interval of continuity. There are then at most m periodic orbits. Also, since $ps_i \in (\eta^{k_i} F_{l_i})^o$ for $1 \leq i \leq m$, each periodic point does not belong to $\mathcal{D}$. Thus all $\eta_{s,k,\varepsilon}$, when ε vary, have the same periodic orbits. Remark also that on Ω the periodic orbits vary continuously with s and the lengths of these orbits are locally constant. Thus those orbits are locally constant.

□

Remark — If $p = 2$ and $m = 1$, one obtains that for an open set of full Lebesgue measure in $s \in [0,1)$, the semi-circle $[s, s+1/2]$ is uniquely ergodic with respect to $x \longmapsto 2x \mod (1)$. This is a well-known result on Sturm measures (see Bousch-Mairesse [3] or Bullet-Sentenac [6]), in fact valid for all s. This will however follow from the study of the next section, since any right-inverse of $x \longmapsto 2x \mod (1)$ defined with a semi-circle is an "order-preserving quasi-contraction on the Circle" with one piece, in the sense of the further definition (4.4). If τ is its rotation number, the conclusion comes from the remark before proposition (4.16) if τ is rational and from proposition (4.3) if τ is irrational.

4 Order-preserving quasi-contractions on the Circle

We precise theorem (2.2) for a class of quasi-contraction on the Circle that preserve cyclic order. We fully detail the maps with a periodic structure.

As for diffeomorphisms of the Circle (see Katok-Hasselblatt [14], chapter 11), the main properties of the dynamics are contained in the rotation number (the average speed of rotation of the transformation). We provide a short theory adapted from the usual one of Poincaré. Proofs result from simple modifications of the ones in [14] and are omitted (see [4]).

4.1 Rotation number

We begin with a few definitions. The cyclic order of three distinct points a, b and c on the Circle $S^1 = \mathbb{R}/\mathbb{Z}$ is (a,b,c) if turning in the trigonometric sense and starting from a, one meets b before c. A map $f : S^1 \to S^1$ is order-preserving if the order of any triple is preserved by f.

In this section we fix $f : S^1 \to S^1$, an order-preserving injective and right-continuous map of degree one. By definition, there is an increasing and right-continuous map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for $x \in \mathbb{R}$ and $f = \tilde{f} \mod 1$. Such a $\tilde{f}$ is called a lift of f.

Proposition 4.1
The map f admits a rotation number $\tau(f)$ defined by :

$$\tau(f) = \lim_{n\to+\infty} \frac{1}{n}\left[\left(\tilde{f}\right)^n(x) - x\right] \mod (1), \tag{10}$$

where $x \in \mathbb{R}$ and $\tilde{f}$ is any lift of f. This quantity is independent on x and on the lift $\tilde{f}$. Also the limit is uniform in $x \in \mathbb{R}$.

We next distinguish the situations $\tau(f) \in \mathbb{Q}$ and $\tau(f) \notin \mathbb{Q}$. As for diffeomorphisms of the Circle, the behaviour is significantly different in each case.

Proposition 4.2

1. *We have $\tau(f) \in \mathbb{Q}$ if and only if f admits a periodic point.*
2. *If $\tau(f) = p/q$ with $p \wedge q = 1$, then all periodic points have period q.*
3. *Let $\tau(f) = p/q$ with $p \wedge q = 1$. If x is q-periodic, the order on the Circle of $\left\{x, f(x), \cdots, f^{q-1}(x)\right\}$ is that of $\{0, p/q, \cdots, (q-1)p/q\}$. In particular, the orbit of x induces intervals on the Circle that are sent one into another by f.*

Denote by R_α the rotation of angle α. When $\tau(f) \notin \mathbb{Q}$, then f is closely related to $R_{\tau(f)}$. Recall that for sufficiently regular diffeomorphisms of the Circle with an irrational rotation number, this latter one is a complete invariant for topological conjugacy.

Proposition 4.3
Set $\tau = \tau(f)$ and assume that $\tau \notin \mathbb{Q}$.

1. *If $\tilde{f}$ is a lift of f, then for all $(n_1, n_2) \geq 0$ and $(m_1, m_2) \in \mathbb{Z}^2$:*

$$n_1\tau + m_1 < n_2\tau + m_2 \text{ if and only if } \forall x \in \mathbb{R},\ \tilde{f}^{n_1}(x) + m_1 < \tilde{f}^{n_2}(x) + m_2.$$

2. *The map f is topologically semi-conjugated to R_τ : there is a non-decreasing, continuous and surjective map $h : [0,1) \to [0,1)$ such that $h \circ f = R_\tau \circ h$.*
3. *The map f is uniquely ergodic.*

4.2 Definition and first properties

We define order-preserving quasi-contraction on the Circle with $m + 1$ pieces. We modify definition (2.1) to take into account the topology of the Circle : the image of an interval is allowed to cover the point 0. Also and up to a change of origin, 0 will always be a discontinuity.

Definition 4.4
Let $m \geq 0$, $0 \leq \gamma < 1$ and strictly increasing γ-Lipschitzian maps $(\Gamma_i)_{0\leq i\leq m} : [0,1] \to \mathbb{R}$. Set $\mathcal{D} = \{D = (D_i)_{1\leq i\leq m} \mid 0 = D_0 < D_1 < \cdots < D_m < D_{m+1} = 1\}$. For $D \in \mathcal{D}$ and $\beta = (\beta_i)_{0\leq i\leq m}$, define $\tilde{T} : \mathbb{R} \to \mathbb{R}$ by $\tilde{T}(x+1) = \tilde{T}(x) + 1$ and on each $[D_i, D_{i+1})$ for $0 \leq i \leq m$ by :

$$\tilde{T}(x) = \Gamma_i(x) + \beta_i.$$

under the conditions $\mathcal{C} = \mathcal{D} \cap \{0 \leq \Gamma_0(0) + \beta_0,\ \Gamma_m(1) + \beta_m < \Gamma_0(0) + \beta_0 + 1,\ \Gamma_i(D_{i+1}) + \beta_i < \Gamma_{i+1}(D_{i+1}) + \beta_{i+1}$, for $0 \leq i \leq m\}$ on $(D, \beta) \in \mathbb{R}^{2m+1}$. Write $\mathcal{C}(\beta)$ the section of $\mathcal{C}$ corresponding to fixing β. Finally set :

$$T = \tilde{T} \mod (1).$$

From the previous subsection, T admits a rotation number $\tau(T)$. As T is defined by (D, β), we write $\tau(D, \beta)$ for $\tau(T)$. First, the same strategy as for proving theorem (2.2) gives that fixing β then generically for $(D, \beta) \in \mathcal{C}$, the transformation T admits a periodic point. Combined with proposition (4.2) we obtain the following result.

Theorem 4.5
For all $\beta \in \mathbb{R}^{m+1}$, there is an open set $\Omega(\beta) \subset \mathcal{C}(\beta)$ with $\lambda_m(\mathcal{C}(\beta)\backslash\Omega(\beta)) = 0$ such that $\tau(D,\beta) \in \mathbb{Q}$ for $D \in \Omega(\beta)$.

We precise below the transformations with a given rational rotation number. We consider first the continuity properties of the rotation number $\tau(D,\beta)$ in $\mathcal{C}$. Set $u = (1,\cdots,1) \in \mathbb{R}^{m+1}$.

Lemma 4.6

1. *The map $(D,\beta) \longmapsto \tau(D,\beta)$ is continuous on $\overline{\mathcal{C}}$.*
2. *For $(D,\beta) \in \overline{\mathcal{C}}$, the map $t \longmapsto \tau(D,\beta+tu)$ is non-decreasing and continuous from $[-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0]$ onto $[0,1]$.*

Proof the lemma :
1) The following classic idea can be found in [14]. Let $(D,\beta) \in \overline{\mathcal{C}}$ and denote by T the associated map. Let p/q and p'/q' be irreducible rational numbers such that $p'/q' < \tau(D,\beta) < p/q$ and U be the lift of T such that $-1 < U^q - Id - p \leq 0$. Remark that $U^q - Id - p < 0$, otherwise $\tau(D,\beta) = p/q$.

Let x be such that $T^l x$ is not a discontinuity for T, $0 \leq l \leq q-1$ (there is a finite set to avoid). If T' is associated to (D',β') and (D',β') is close to (D,β), the orbits $(T^l x)_{0\leq l\leq q-1}$ and $((T')^l x)_{0\leq l\leq q-1}$ are close. Thus the lift U' of T' compatible with U verifies $U^q(x) < p + x$. This gives $\tau(D',\beta') \leq p/q$. Symmetrically $\tau(D',\beta') \geq p'/q'$ for (D',β') close enough to (D,β). This concludes the proof of *1)*.

2) From *1)* the map $t \longmapsto \tau(D,\beta+tu)$ is continuous on $[-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0]$. For $t \in [-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0]$ let U_t be the lift as in definition (4.4) corresponding to the map of parameters $(D,\beta+tu)$. All U_t are compatible and verify $0 \leq U_t(0) \leq 1$. Remark that for $-\Gamma_0(0)-\beta_0 \leq t < t' \leq 1-\Gamma_0(0)-\beta_0$:

$$0 \leq \lim_{n\to+\infty} \frac{U_t^n(0)}{n} \leq \lim_{n\to+\infty} \frac{U_{t'}^n(0)}{n} \leq 1.$$

This implies that $t \longmapsto \tau(D,\beta+tu)$ is non-decreasing. Next if $t \in \{-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0\}$, then $\tau(D,\beta+tu) = 0 \mod (1)$. Therefore the map $t \longmapsto \tau(D,\beta+tu)$ is either equal to 0 on $[-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0]$ or its image is exactly $[0,1]$. The first case is impossible as for some values of $t \in (-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0)$ the graph of U_t is at a positive distance of $x \longmapsto x$ and $x \longmapsto x+1$. For such t we get $\tau(D,\beta+tu) \neq 0 \mod (1)$, using proposition (4.2).

□

Definition 4.7

1. *For $0 \leq r < 1$ define the closed and non-empty interval $[b^-_{D,\beta}(r), b^+_{D,\beta}(r)]$ as the set of parameters t in $[-\Gamma_0(0)-\beta_0, 1-\Gamma_0(0)-\beta_0]$ for which $\tau(D,\beta+tu) = r$.*
2. *Let $\mathcal{C}_0 = \overline{\mathcal{C}} \cap \{\Gamma_i(D_i)+\beta_i = \Gamma_{i-1}(D_i)+\beta_{i-1}\ , 1 \leq i \leq m\}$ be the set of parameters in $\overline{\mathcal{C}}$ for which the transformation makes no jump at D_i, $1 \leq i \leq m$.*
3. *Set $\mathcal{C}_0(p/q) = \{(D,\beta,t) \mid (D,\beta) \in \mathcal{C}_0,\ b^-_{D,\beta}(p/q) \leq t \leq b^+_{D,\beta}(p/q)\}$.*

We then get :

Proposition 4.8
Let $0 \leq p/q < 1$ with $p \wedge q = 1$. Then :

1. *We have $\tau(D,\beta) = p/q$ if and only if $\tilde{T}^q D_i^- \leq D_i + p \leq \tilde{T}^q D_i$ for some $0 \leq i \leq m$.*
2. *The maps $b^{\pm}_{D,\beta}(p/q)$ are continuous functions of $(D,\beta) \in \mathcal{C}_0$.*

3. For $(D,\beta)\in\mathcal{C}_0$ we have $b^-_{D,\beta}(p/q) < b^+_{D,\beta}(p/q)$. Let $T_{D,\beta,t}$ be associated with (D,β,t). Then the $T^l_{D,\beta,t}0^\pm$ for $1\le l\le q$, are continuous maps of $(D,\beta,t)\in\mathcal{C}_0(p/q)$. When $1\le l\le q-1$, these are distinct from each other and from 0. Also :

$$\tilde{T}^q_{D,\beta,t}(0^-)\le p\le \tilde{T}^q_{D,\beta,t}(0),$$

with $\tilde{T}^q_{D,\beta,t}(0^-)=p$ if and only if $t=b^+_{D,\beta}(p/q)$ and $\tilde{T}^q_{D,\beta,t}(0)=p$ if and only if $t=b^-_{D,\beta}(p/q)$.

Proof of the proposition :
1) By proposition (4.2) we have $\tau(D,\beta)=p/q$ if and only if $\tilde{T}^q-p$ admits a fixed point. As the graph of $\tilde{T}^q-p$ is increasing and piecewise contracting, this condition is equivalent to the existence of a discontinuity y for $\tilde{T}^q-p$ such that $\tilde{T}^q(y^-)\le p+y\le \tilde{T}^q(y)$.

Next y can be written as $y=\tilde{T}^{-l}D_k$ with $0\le l\le q-1$ and $0\le k\le m$, up to translating by an integer. Suppose that l is minimal. If $l=0$, then the result is proved. If $l\ge 1$ then y is not a discontinuity of $\tilde{T}$. Applying $\tilde{T}$ we obtain $\tilde{T}^q((\tilde{T}(y))^-)\le p+\tilde{T}(y)\le \tilde{T}^q(\tilde{T}(y))$ and $\tilde{T}(y)=\tilde{T}^{-l+1}D_k$ is a discontinuity of $\tilde{T}^q$. Repeating this operation gives the result.

2) Observe first that the continuity of the rotation number implies that $b^-_{D,\beta}(p/q)$ and $b^+_{D,\beta}(p/q)$ are respectively lower semi-continuous and upper semi-continuous maps of $(D,\beta)\in\overline{\mathcal{C}}$. Suppose now that 0 is the only discontinuity of $\tilde{T}$ on $[0,1)$, that is $(D,\beta)\in\mathcal{C}_0$.

Let $\delta>0$. For $(D',\beta')\in\mathcal{C}_0$ close to (D,β), the graph of $\tilde{T}_{D',\beta'+t'u}$ with $t'=b^+_{D,\beta}(p/q)-\delta$ is below the graph of $\tilde{T}_{D,\beta+tu}$ with $t=b^+_{D,\beta}(p/q)$. Thus $\tau(D',\beta'+t'u)\le p/q$ and then $b^+_{D',\beta'}(p/q)\ge b^+_{D,\beta}(p/q)-\delta$. This gives the continuity of $b^+_{D,\beta}(p/q)$. The same holds for $b^-_{D,\beta}(p/q)$.

3) For $(D,\beta,t)\in\mathcal{C}_0(p/q)$ we have $\tau(D,\beta+tu)=p/q$. Observe first that $\mathrm{dist}(\tilde{T}^l_{D,\beta,t}(0^\pm),\mathbb{Z})>0$, for $1\le l\le q-1$, otherwise $\tau(D,\beta+tu)$ would be of the form r/l, whereas p/q is irreducible. This gives the continuity of $\tilde{T}^l_{D,\beta,t}(0^\pm)$ on $\mathcal{C}_0(p/q)$ for $1\le l\le q$. Similarly all $T^l_{D,\beta,t}0^\pm$ for $1\le l\le q-1$ are distinct from each other.

Using point *1)* and the fact that 0 is the only discontinuity of $T_{D,\beta,t}$ we have $\tilde{T}^q_{D,\beta,t}(0^-)\le p\le\tilde{T}^q_{D,\beta,t}(0)$. As $\tilde{T}^q_{D,\beta,t}(0^-)<\tilde{T}^q_{D,\beta,t}(0)$, suppose for example that $\tilde{T}^q_{D,\beta,t}(0^-)<p\le\tilde{T}^q_{D,\beta,t}(0)$. By continuity and monotony, for some small $\delta>0$ we still have $\tilde{T}^q_{D,\beta,t+\delta}(0^-)<p\le\tilde{T}^q_{D,\beta,t+\delta}(0)$. Consequently $\tau(D,\beta+(t+\delta)u)=p/q$ and $b^-_{D,\beta}(p/q)\le t<t+\delta\le b^+_{D,\beta}(p/q)$.

This also proves that $t<b^+_{D,\beta}(p/q)$. Thus if $t=b^+_{D,\beta}(p/q)$ then $\tilde{T}^q_{D,\beta,t}(0^-)=p$. Reciprocally if this equality is verified then for any small $\delta>0$, we have $p<\tilde{T}^q_{D,\beta,t+\delta}(0^-)<\tilde{T}^q_{D,\beta,t+\delta}(0)$. Thus $\tau(D,\beta+(t+\delta)u)>p/q$ and this proves the other sense. The same holds for $b^-_{D,\beta}(p/q)$.

□

4.3 Transformations with a rational rotation number

In the sequel all D_i, $0\le i\le m$, play the same role. We introduce the expression of the transformation when D_i is taken as origin.

Definition 4.9
Let T be as in definition (4.4). For $0\le i\le m$, let T_i be the map deduced from T when taking D_i as origin. Denote by $\Gamma^i=(\Gamma^i_j)_{0\le j\le m}$ the reordered family of γ-Lipschitzian maps and by (D^i,β^i) the corresponding parameters. Set $\Gamma=\Gamma^0$.

Consider first the case of a zero rotation number.

Proposition 4.10
For $(D,\beta)\in\overline{\mathcal{C}}$*, we have* $\tau(D,\beta)=0$ *if and only if* $(D,\beta)\in E(0):=\underset{0\le i\le m}{U}F_i(0)$*, where :*

$$F_i(0)=\{(D,\beta)\in\overline{\mathcal{C}}\mid \Gamma_0^i(0)+\beta_0^i\ge 0,\ \Gamma_m^i(1)+\beta_m^i\le 1\}.$$

The number of fixed points of T *is the number of* $0\le i\le m$ *such that* $(D,\beta)\in F_i(0)$*.*

Proof of the proposition :
According to proposition (4.8), $\tau(D,\beta)=0$ if and only if $\tilde{T}(D_i^-)\le D_i\le\tilde{T}(D_i)$ for some $0\le i\le m$. Taking D_i as origin, this can be written as $\tilde{T}_i(0^-)\le 0\le\tilde{T}_i(0)$ and corresponds to parameters in $F_i(0)$. Finally remark that if D_i and D_j are consecutive discontinuities with $\tilde{T}(D_i^-)\le D_i\le\tilde{T}(D_i)$ and $\tilde{T}(D_j^-)\le D_j\le\tilde{T}(D_j)$, there is exactly one fixed point for $\tilde{T}$ in $[D_i,D_j)$. □

Next turn to the case when $\tau(T)=p/q$ with $0<p/q<1$ and $p\wedge q=1$. Observe in such a case that the graph of $\tilde{T}$ is strictly contained in the tube determined by the lines $x\longmapsto x$ and $x\longmapsto x+1$. Then a non-empty part of this graph is *above* the horizontal line $y=1$. According to the position with respect to this line, we have the following discussion.

Definition 4.11

1. *Assume that* $0<\tau(T)<1$*. Set* $h=2i$ *when* $\tilde{T}(D_i)\le 1<\tilde{T}(D_{i+1}^-)$ *for* $0\le i\le m$ *and* $h=2i-1$ *when* $\tilde{T}(D_i^-)\le 1<\tilde{T}(D_i)$ *for* $1\le i\le m$*. We have* $0\le h\le 2m$*.*
2. *If* h *is odd, then* T *is an injective quasi-contraction on* $[0,1)$ *as in definition* (2.1) *with* $m+1$ *pieces, parameters* D' *with* $D'=D$ *and family of* γ*-Lipschitzian maps* $\Gamma'=(\Gamma'_i)_{0\le i\le m+1}$ *with* $\Gamma'_i=\Gamma_i+\beta_i \mod (1)$.
3. *If* $h=2h'$*, then* T *is an injective quasi-contraction on* $[0,1)$ *with* $m+2$ *pieces and parameters* D' *with* $D'=\{D_1,\cdots,D_{h'},\tilde{T}^{-1}(1),D_{h'+1},\cdots,D_m\}$ *and family of* γ*-Lipschitzian maps* $\Gamma'=(\Gamma'_i)_{0\le i\le m+1}$ *such that* $\Gamma'_i=\Gamma_i+\beta_i$ *for* $0\le i\le h$ *and* $\Gamma'_i=\Gamma_{i-1}+\beta_{i-1}\mod(1)$ *for* $h+1\le i\le m+1$*.*

To precise the condition in point *1)* of proposition (4.8) we need to consider the possible codes for $(T^lD_i^{\pm})_{0\le l\le q-1}$. Up to taking T_i one only needs to focus on $(T^l0^{\pm})_{0\le l\le q-1}$.

Lemma 4.12
Let $0<p/q<1$ *with* $p\wedge q=1$*. For* $0\le l\le q-1$ *let* $0\le n_l\le q-1$ *verify* $pn_l=l\mod(q)$*. If* $\tau(T)=p/q$*, then the order on the Circle of* $(T^l0^{\pm})_{0\le l\le q-1}$ *is :*

$$(0,T^{n_1}0^-,T^{n_1}0,\cdots,T^{n_i}0^-,T^{n_i}0,\cdots,T^{n_{q-1}}0^-,T^{n_{q-1}}0).$$

Proof of the lemma :
As $\tau(T)=p/q$, proposition (4.2) implies the existence of a q-periodic point x. The order of its orbit on the Circle is $(x,T^{n_1}x,\cdots,T^{n_i}x,\cdots,T^{n_{q-1}}x)$. As the intervals induced by this orbit are stable, the conclusion follows by considering the interval where 0 lies.

□

The previous lemma leads to the following definition.

Definition 4.13
Let $0<p/q<1$ *with* $p\wedge q=1$ *and let* $\tau(T)=p/q$*.*

1. *Mark the* $2q-1$ *intervals in* $[0,1)$ *induced by* $(T^l0^{\pm})_{0\le l\le q-1}$*. Precisely :*
   - $(T^{n_i}0,T^{n_{i+1}}0^-)$ *has number* $2i+1$*, for* $0\le i\le q-2$*,*
   - $[T^{n_i}0^-,T^{n_i}0]$ *has number* $2i$*, for* $1\le i\le q-1$*,*
   - $(T^{n_{q-1}}0,1)$ *has number* $2q-1$*.*

2. *The interval where D_j lies is denoted by d_j, $1 \le j \le m$.*

3. *If $h = 2h' - 1$ set $S(p/q, h) = \{(d_j)_{1\le j\le m} \mid 1 \le d_j \le d_{j+1} \le 2q-1,\ d_{h'} = 2(q-p)\}$, where $d_{m+1} = 2q-1$ if $h' = m$. Set $r_j = n_{[(d_j+1)/2]}$ and $r'_j = n_{[d_j/2]}$, for $1 \le j \le m$. Denote by :*

   - *$(\varepsilon_k)_{0\le k\le q-1}$ the code of $(R^k_{p/q}0)_{0\le k\le q-1}$ in the intervals $[0, r_1/q),\ \cdots,\ [r_m/q, 1)$.*
   - *$(\varepsilon'_k)_{0\le k\le q-1}$ the code of $(R^k_{p/q}0)_{0\le k\le q-1}$ in the intervals $(0, r'_1/q],\ \cdots,\ (r'_m/q, 1]$.*

4. *If $h = 2h'$ set $S(p/q, h) = \{(d_j)_{1\le j\le m} \mid 1 \le d_j \le d_{j+1} \le 2q-1,\ d_{h'} \le 2(q-p) \le d_{h'+1}\}$, where $d_{m+1} = 2q-1$ if $h' = m$ and $d_0 = 1$ if $h' = 0$. Set $r_j = n_{[(d_j+1)/2]}$ for $1 \le j \le h'$, $r_{h'+1} = n_{q-p}$ and $r_j = n_{[(d_{j-1}+1)/2]}$ for $h'+2 \le j \le m+1$. Let also $r'_j = n_{[d_j/2]}$ for $1 \le j \le h'$, $r'_{h'+1} = n_{q-p}$ and $r'_j = n_{[d_{j-1}/2]}$ for $h'+2 \le j \le m+1$. Denote by :*

   - *$(\varepsilon_k)_{0\le k\le q-1}$ the code of $(R^k_{p/q}0)_{0\le k\le q-1}$ in the intervals $[0, r_1/q),\ \cdots,\ [r_{m+1}/q, 1)$.*
   - *$(\varepsilon'_k)_{0\le k\le q-1}$ the code of $(R^l_{p/q}0)_{0\le k\le q-1}$ in the intervals $(0, r'_1/q],\ \cdots,\ (r'_{m+1}/q, 1]$.*

Remark. — Points *3)* and *4)* describe the fact that $\tilde{T}^q(0^-) \le p \le \tilde{T}^q(0)$. Then $S(p/q, h)$ lists the possible positions of D with respect to the $(T^l 0^\pm)_{0\le l\le q-1}$, noticing that $\tilde{T}(T^{q-1}0^-) \le 1 \le \tilde{T}(T^{q-1}0)$ and $[T^{q-1}0^-, T^{q-1}0]$ has number $2(q-p)$ since $n_{q-p} = q-1$. Finally $(\varepsilon_k)_{0\le k\le q-1}$ and $(\varepsilon'_k)_{0\le k\le q-1}$ are the codes of $(T^k0)_{0\le k\le q-1}$ and $(T^k0^-)_{0\le k\le q-1}$ respectively.

We next come to the main result of this section. Recall definitions (4.9), (4.11) and (4.13).

Theorem 4.14
Let $0 < p/q < 1$ with $p \wedge q = 1$. Then $\tau(D, \beta) = p/q$ if and only if :

$$(D, \beta) \in E(p/q) := \bigcup_{0\le i\le m} F_i(p/q),$$

where :

$$F_i(p/q) := \bigcup_{\substack{0\le h\le 2m,\\ d\in S(p/q,h)}} G_{i,\Gamma}(p/q, h, d)$$

and $G_i(p/q, h, d) = \{(D, \beta) \in \overline{\mathcal{C}} \mid (D^i, \beta^i) \in G_{0,\Gamma^i}(p/q, h, d)\}$ and finally $G_{0,\Gamma}(p/q, h, d) \subset \overline{\mathcal{C}}$ is defined by the inequalities :

$$\left\{\begin{array}{c} \Gamma'_{\varepsilon_{q-1},\cdots,\varepsilon_0}(0) \ge 0 \text{ and } \Gamma'_{\varepsilon'_{q-1},\cdots,\varepsilon'_0}(1) \le 1, \\ \\ \Gamma_{h'-1}(D_{h'}) + \beta_{h'-h''} \le 1 < \Gamma_{h'}(D_{h'+1-h''}) + \beta_{h'}, \text{ where } h = 2h' - h'',\ h'' \in \{0,1\}, \\ \\ D_j \in \left[\zeta_{d_j}, \zeta_{d_j+1}\right] \text{ for even } d_j \text{ and } D_j \in \left(\zeta_{d_j}, \zeta_{d_j+1}\right) \text{ for odd } d_j,\ 1 \le j \le m, \end{array}\right. \tag{11}$$

where $\zeta_{2t} = \Gamma_{\varepsilon'_{n_t-1},\cdots,\varepsilon'_0}(1)$ and $\zeta_{2t+1} = \Gamma_{\varepsilon_{n_t-1},\cdots,\varepsilon_0}(0)$.

Moreover for $0 \le i \le m$, $0 \le h, h' \le m$ and $d, d' \in S(p/q, h)$, the set $G_{i,\Gamma}(p/q, h, d)$ is non-empty and even has non-empty interior. If $(h, d) \ne (h', d')$, then $G_{i,\Gamma}(p/q, h, d) \cap G_{i,\Gamma}(p/q, h', d') = \emptyset$.

Proof of the theorem :
Step 1 — From proposition (4.8) we have $\tau(D, \beta) = p/q$ if and only if some D_i with $0 \le i \le m$ verifies point *1)* of this proposition. This means that $\tilde{T}^q_i 0^- \le p \le \tilde{T}^q_i 0$. Assuming then that $i = 0$, we show that $(D, \beta) \in G_{0,\Gamma}(p/q, h, d)$.

Recall the definition of h given in definition (4.11). The position of the graph of $\tilde{T}$ with respect to $y=1$ is then given by the second condition in (11). Next the order on the Circle of $(T^l 0^{\pm})_{0\le l\le q-1}$ is given by lemma (4.12). Following definition (4.13), the positions of the elements of D with respect to this sequence are given by some $d \in S(p/q,h)$. This gives the last set of conditions in (11). Finally the first line of (11) is for $\tilde{T}^q(0^-) \le p \le \tilde{T}^q(0)$. Thus $(D,\beta) \in G_{0,\Gamma}(p/q,h,d)$.

Step 2 — Reciprocally fix $0 \le i \le m$, $0 \le h \le 2m$ and $d \in S(p/q,h)$. Suppose for instance that $h = 2h'-1$. From the symmetric roles played by the $(D_i)_{0\le i\le m}$, it is enough to prove that $G_{0,\Gamma}(p/q,h,d)$ has non-empty interior and that for(D,β) in this set we have $\tau(D,\beta) = p/q$.

Recall definition (4.7), proposition (4.8) and the notations $\mathcal{C}_0$ and $T_{D,\beta,t}$. When $(D,\beta)\in\mathcal{C}_0$ then if D and β_0 are fixed, the $(\beta_i)_{1\le i\le m}$ are also fixed. Write then $T_{D,\beta_0,t}$ for $T_{D,\beta,t}$. We use the same convention for $b^{\pm}_{D,\beta}(p/q)$. Next define $\varphi : C \to C$, where $C = \{(s_i)_{1\le i\le m} \mid 0 \le s_1 \le \cdots \le s_m \le 1\}$, in the following way. For $s \in C$ consider the transformation $T_{s,0,\eta_s}$ with :

$$\eta_s = \frac{1}{2}(b^-_{s,0}(p/q) + b^+_{s,0}(p/q)).$$

Proposition (4.8) implies that $s \longmapsto \eta_s$ is continuous on C. Similarly $s \longmapsto (T^k_{s,0,\eta_s}0^{\pm})_{1\le k\le q-1}$ is continuous on C without collapse and each component remains far away from 0. Write as follows the ordered sequence on $[0,1)$:

$$0 < B_1(s) < \cdots < B_{2q-2}(s) < 1 \text{ and set } B_0 = 0,\ B_{2q-1} = 1.$$

Set $\varphi(s) = (s'_1,\cdots,s'_m)$, where $s'_{h'} = \tilde{T}^{-1}_{s,0,\eta_s}(1)$. This clearly is a continuous function of $s\in C$. Take now $i \ne h'$, let $N_i = \text{card}\{1\le j\le m \mid d_j = d_i\}$ and suppose that i is exactly the k^{th} index z such that $d_z = d_i$. If $d_i \ne d_{h'}$ we set :

$$s'_i = B_{d_i-1}(s) + k\left(\frac{B_{d_i}(s) - B_{d_i-1}(s)}{N_i+1}\right). \tag{12}$$

If $d_i = d_{h'}$ then $B_{d_{h'}-1}(s) < s'_{h'} < B_{d_{h'}}(s)$ as $\tilde{T}_{s,0,\eta_s}(T^{q-1}_{s,0,\eta_s}0^-) < 1 < \tilde{T}_{s,0,\eta_s}(T^{q-1}_{s,0,\eta_s}0)$. If for example $i < h'$, set $N_i = \text{card}\{1\le j\le h' \mid d_j = d_i\}$ and define s'_i as in (12), replacing $B_{d_i}(s)$ by $s'_{h'}$. As a result we get a continuous $\varphi : C \to C$ with $\varphi(C)\subset C^o$.

From the Brouwer fixed point theorem φ admits a fixed point $s\in C$ and thus $s\in C^o$. Then $T_{s,0,\eta_s}$ is such that $\tau(T_{s,0,\eta_s}) = p/q$. The $(T^k_{s,0,\eta_s}0^{\pm})_{0\le k\le q-1}$ are all distinct and $\tilde{T}^q_{s,0,\eta_s}(0^-) < p < \tilde{T}^q_{s,0,\eta_s}(0)$. Also the positions of $(D_i)_{1\le i\le m}$ with respect to $(T^k_{s,0,\eta_s}0^{\pm})_{0\le k\le q-1}$ are exactly specified by d. Those points are distinct from each other and from the boundaries of the interval where they lie. Perturbing $(s,0,\eta_s)$ in such a way that $\tilde{T}(D_{h'})^- < 1 < \tilde{T}(D_{h'})$ gives parameters $(D,\beta)\in\mathcal{C}$ contained in the interior of $G_{0,\Gamma}(p/q,h,d)$. The case when h is even is treated in the same way.

Step 3 — We prove by connexity that for (D,β) in the interior of $G_{0,\Gamma}(p/q,h,d)$, then $\tau(D,\beta) = p/q$, $\tilde{T}^q(0^-) < p < \tilde{T}^q(0)$, the positions of the $(D_i)_{1\le i\le m}$ with respect to the $(T^k 0^{\pm})_{0\le k\le q-1}$ are specified by d and the position of $y=1$ with respect to the graph of $\tilde{T}$ is given by h. Denote by H the parameters in the interior of $G_{0,\Gamma}(p/q,h,d)$ having this property.

By the previous study H is non-empty and the same argument of perturbation gives that H is open. We now show that H is closed in the interior of $G_{0,\Gamma}(p/q,h,d)$. If $((D,\beta)_n)_{n\ge0}\subset H$ converges to (D,β) in the interior of $G_{0,\Gamma}(p/q,h,d)$, then the expressions associated to $(D,\beta)_n$ in the inequalities defining H converge to the formal expressions associated to (D,β) which verify strict inequalities. Thus the expressions of D and $(T^k 0^{\pm})_{0\le k\le q-1}$ corresponding to $(D,\beta)_n$ are still valid for (D,β). Consequently H is closed in the interior of $G_{0,\Gamma}(p/q,h,d)$ and then H is exactly the interior of $G_{0,\Gamma}(p/q,h,d)$.

As a corollary we get that if $(h,d)\ne(h',d')$ then $G_{0,\Gamma}(p/q,h,d)\cap G_{0,\Gamma}(p/q,h',d') = \varnothing$. This concludes the proof of the theorem.

□

Remark. — The case $m = 0$ of theorem (4.14) and when the maps are linear is due to Bugeaud and Conze [5]. The case when $m = 1$ in a linear context is also treated in this article but with $\tilde{T}$ continuous at D_1 and $\tilde{T}(D_1) = 1$. Introducing contraction coefficients $0 \le \gamma_0, \gamma_1 < 1$, the map $\tilde{T}$ can be written as :

$$\tilde{T}x = \gamma_i x + \beta_i, \text{ on } [D_i, D_{i+1}).$$

The previous condition is $\gamma_0 D_1 + \beta_0 = \gamma_1 D_1 + \beta_1 = 1$ or equivalently $D_1 = (1 - \beta_0)/\gamma_0$ and $\beta_1 = 1 - (1 - \beta_0)\gamma_1/\gamma_0$. Choosing next β_0 as parameter as in [5], in (11) only the first line remains and one deduces the following result, correcting the formula of [5].

Theorem 4.15
Let $m = 1$ and $\tilde{T}$ be as above. Set $T = \tilde{T} \mod (1)$ and let for $0 \le p/q < 1$ with $p \wedge q = 1$, $(\varepsilon_k)_{0 \le k \le q-1}$ be the code of $(R^k_{p/q} 0)_{0 \le k \le q-1}$ in the intervals $\{[0, 1 - p/q), [1 - p/q, 1)\}$. Then $\tau(T) = p/q$ if and only if :

$$\beta_0 \in \left[\frac{\sum_{k=0}^{q-1} \varepsilon_k \left(\gamma_{\varepsilon_k} \cdots \gamma_{\varepsilon_{q-2}}\right)}{\sum_{k=0}^{q-1} \left(\gamma_{\varepsilon_k} \cdots \gamma_{\varepsilon_{q-2}}\right)}, \frac{1 + \gamma_{\varepsilon_{q-1}} \cdots \gamma_{\varepsilon_1} (1 - \gamma_{\varepsilon_0}) + \sum_{k=1}^{q-2} \varepsilon_k \left(\gamma_{\varepsilon_k} \cdots \gamma_{\varepsilon_{q-2}}\right)}{1 + \gamma_{\varepsilon_{q-1} \cdots \gamma_{\varepsilon_1}} + \sum_{k=1}^{q-2} \left(\gamma_{\varepsilon_k} \cdots \gamma_{\varepsilon_{q-2}}\right)} \right]$$

In this case T has a unique periodic orbit and it has period q.

Remark. — Consider the question of counting the periodic orbits when $\tau(T) \in \mathbb{Q}$. As in lemma (2.3) the set $\mathcal{D} = \{T^{-k} D_i \mid 0 \le k \le k_i,\ 1 \le i \le m\}$ induces a stable partition on the Circle. There are then less than $m+1$ periodic orbits. We now give an algorithm for counting the periodic orbits.

Proposition 4.16
Let $0 < p/q < 1$ with $p \wedge q = 1$. Let $(D, \beta) \in \mathcal{C}$ verify $\tau(D, \beta) = p/q$. Using theorem (4.14), let $I = \{0 \le i \le m \mid (D, \beta) \in F_i(p/q)\}$. For $i \in I$ let h^i and d^i be such that $(D, \beta) \in G_{i,\Gamma}(p/q, h^i, d^i)$.

1. *For $i \in I$ set $A_i = \left\{ j \in I \mid d^i_j \text{ is even} \right\}$. If $A_i \ne \emptyset$ there is exactly one $j \in A_i$ such that :*

   $$n_{d^i_j/2} = \min \left\{ n_{d^i_k/2} \mid k \in A_i \right\}.$$

   Write then $i \to j$. Reciprocally if $j \in I$ there is at most one $i \in I$ such that $i \to j$.

2. *The set I decomposes into disjoint closed cycles of length $\le q$ for the relation " $\to$ ". The number of distinct periodic orbits for T is the number of cycles in I.*

Proof of the proposition :
Let us reason modulo 1. Using proposition (4.2) let x be a fixed point for $\tilde{T}^q - p$. The number of distinct q-periodic orbits is the number of fixed points of $\tilde{T}^q - p$ in the interval $[x, \tilde{T}^{n_1} x)$. This is also the number of discontinuities $w \in [x, T^{n_1} x)$ of $\tilde{T}^q - p$ verifying :

$$\tilde{T}^q(w^-) \le p + w \le \tilde{T}^q(w). \tag{13}$$

Let next $x < y \le T^{n_1} x$ be the fixed point consecutive to x. Call $w_0 \in [x, y)$ the unique discontinuity of $\tilde{T}^q - p$ satisfying (13). We write $w_0 = \tilde{T}^{-l_0} D_{i_0}$ for some D_{i_0} and a minimal $0 \le l_0 \le q - 1$. Then D_{i_0} is uniquely determined as the $(D_j)_{0 \le j \le m}$ are distinct. If $l_0 > 0$ then by minimality of l_0, for $0 \le k < l_0$ the point $\tilde{T}^k(\tilde{T}^{-l_0}(D_{i_0}))$ does not belong to the $(D_j)_{0 \le j \le m}$. Successive iterations give that for $0 \le k \le l_0$:

$$\tilde{T}^q((\tilde{T}^k(w_0))^-) \le p + \tilde{T}^k(w_0) \le \tilde{T}^q(\tilde{T}^k(w_0)).$$

Consequently for $0 \leq k \leq l_0$ the unique discontinuity of $\tilde{T}^q - p$ in $[\tilde{T}^k(x), \tilde{T}^k(y))$ satisfying (13) is $\tilde{T}^k(w_0)$. For $k = l_0$ we have $\tilde{T}^q(D_{i_0}^-) \leq p + D_{i_0} \leq \tilde{T}^q(D_{i_0})$ giving that $i_0 \in I$. Another application of $\tilde{T}$ leads to :

$$\tilde{T}^q\left(\tilde{T}(D_{i_0}^-)\right) \leq p + \tilde{T}(D_{i_0}^-) < p + \tilde{T}(D_{i_0}) \leq \tilde{T}^q\left(\tilde{T}(D_{i_0})\right).$$

Consequently if w_1 is the discontinuity of $\tilde{T}^q - p$ satisfying (13) in $[\tilde{T}^{l_0+1}(x), \tilde{T}^{l_0+1}(y))$ then $\tilde{T}(D_{i_0}^-) \leq w_1 \leq \tilde{T}(D_{i_0})$. Write $w_1 = \tilde{T}^{-l_1}D_{i_1}$ with a minimal $0 \leq l_1 \leq q-1$ and some uniquely determined D_{i_1}. As above $i_1 \in I$ and for $0 \leq k \leq l_1$, $\tilde{T}^k(w_1)$ is the unique discontinuity of $\tilde{T}^q - p$ verifying (13) in $[\tilde{T}^{k+1+l_0}(x), \tilde{T}^{k+1+l_0}(y))$.

Remark now that $D_{i_1} \in [\tilde{T}^{l_1+1}(D_{i_0}^-), \tilde{T}^{l_1+1}(D_{i_0})]$, whose associated number is even, following definition (4.13). As $[\tilde{T}^u(D_{i_0}^-), \tilde{T}^u(D_{i_0})) \subset [\tilde{T}^{l_0+u}(x), \tilde{T}^{l_0+u}(y))$ for $1 \leq u \leq l_1+1$, we get that i_1 is the only $j \in J$ such that $D_j \in [T^u(D_{i_0}^-), T^u(D_{i_0}))$, for some $1 \leq u \leq l_1+1$. Indeed D_i must satisfy (13) and l_1 is minimal. We thus get $i_0 \to i_1$.

Repeating this procedure we obtain a sequence $(i_0 \to i_1 \to \cdots \to i_t \to i_0) \subset I$ which is a cycle since $[T^q(x), T^q(y)) = [x, y)$. This also gives $t+1 \leq q$. Proceding in this way for each pair of consecutive q-periodic points in $[x, T^{n_1}x]$ we obtain non-empty and disjoint cycles in I in bijection with the periodic orbits. Finally if $i \in I$ then D_i verifies (13) and belongs to some iterate of the interval determined by two consecutive periodic points in $[x, T^{n_1}x]$. Thus i belongs to a cycle. This concludes the proof of the proposition.

□

Remark. — The case $\tau(T) \notin \mathbb{Q}$ can be considered using the monotony of the rotation number for fixed D. As in theorem (4.14) the set of parameters (D, β) verifying $\tau(D, \beta) = r$ can be decomposed into a union of polyhedrons, each one of codimension at least one. However this union is not denumerable as soon as $m \geq 1$. More details can be found in [4].

4.4 Pictures

We provide a numerical illustration of the decomposition in theorem (4.14) and proposition (4.16). For the sake of simplicity we restrict to the linear case. Commented pictures for the case $m = 0$ can then be found in Bugeaud-Conze [5]. In the context of definition (4.4), we consider here the situation when $m = 1$.

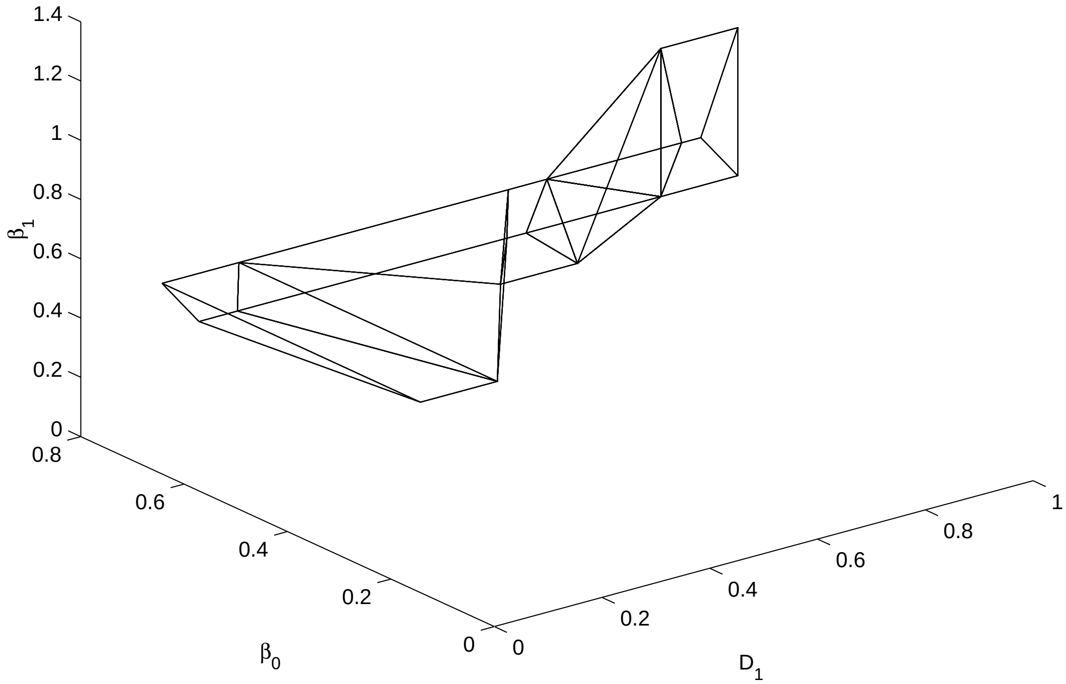
Consider a map $T = \tilde{T} \mod (1)$, where :

$$\tilde{T} = \begin{cases} \frac{x}{2} + \beta_0, \text{ on } [0, D_1), \\ \\ \frac{x}{2} + \beta_1, \text{ on } [D_1, 1). \end{cases}$$

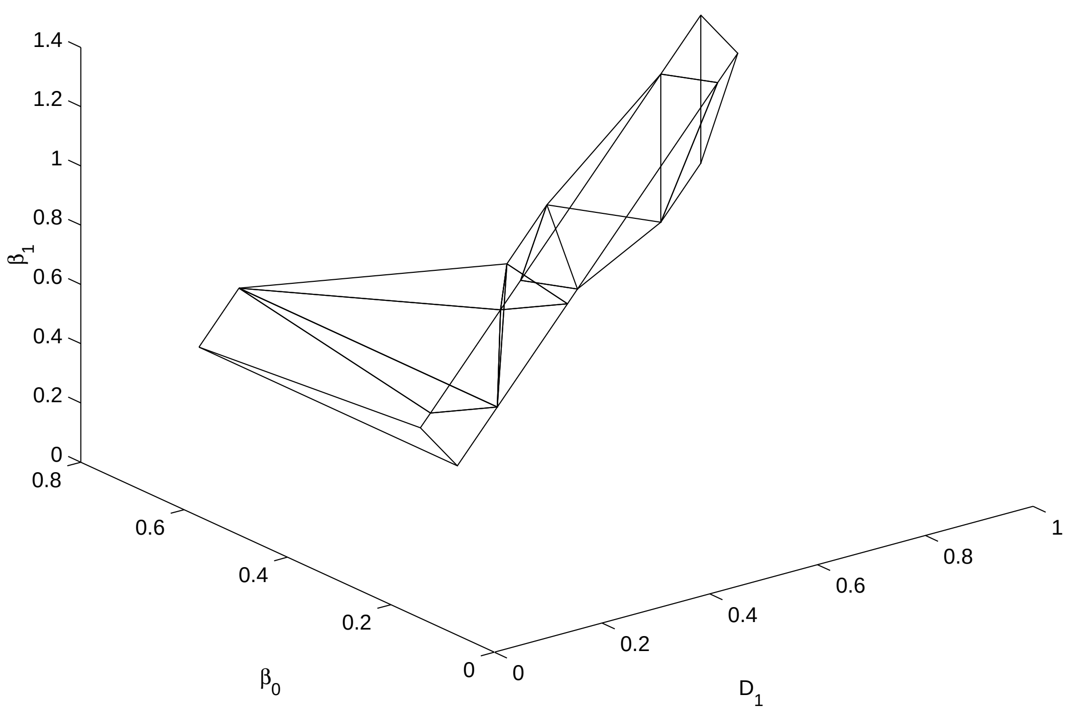
The parameters are $(D_1, \beta_0, \beta_1) \in \mathbb{R}^3$. Below, these are respectively represented on the X-axis, Y-axis and the Z-axis. We look at the case when $\tau(D_1, \beta_0, \beta_1) = 1/3$. Recall then that there is one or two 3-periodic orbits.

1. $F_0(1/3)$.
2. $F_1(1/3)$.
3. $E(1/3) = F_0(1/3) \cup F_1(1/3)$.
4. Parameters giving two distinct 3-periodic orbits.

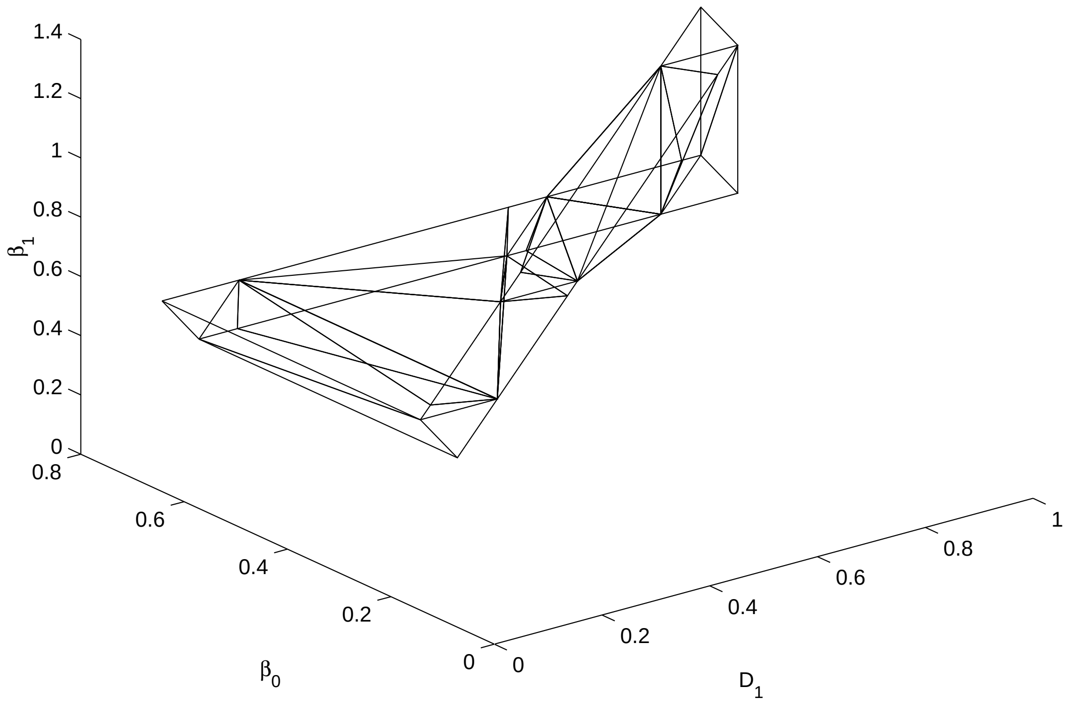
$F_0(1/3)$


1.4
1.2
1
0.8
0.6
0.4
0.2
0
β1
0.8
0.6
0.4
0.2
0
β0
0
0.2
0.4
0.6
0.8
1
D1

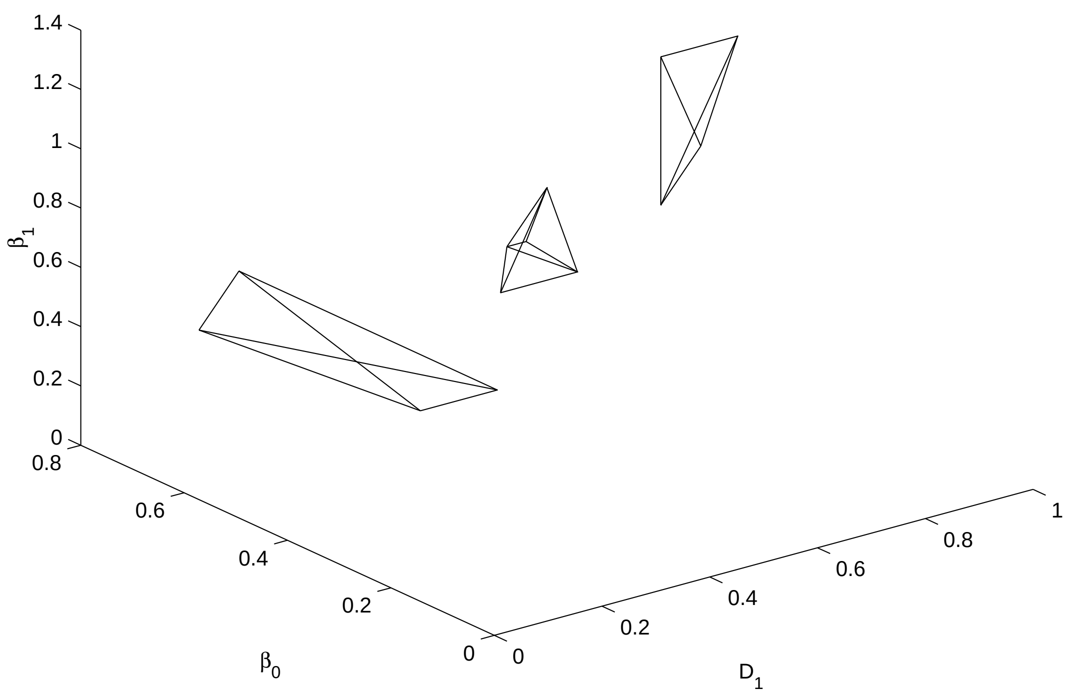
$F_1(1/3)$


1.4
1.2
1
0.8
0.6
0.4
0.2
0
β1
0.8
0.6
0.4
0.2
0
β0
0
0.2
0.4
0.6
0.8
1
D1

τ(D,β)=1/3



τ(D,β)=1/3 and 2 distinct 3−periodic attractors



4.5 Injective quasi-contractions with three pieces on the Interval

We apply the study on order-preserving quasi-contractions on the Circle to the analysis of a class of injective locally increasing quasi-contractions with three pieces on $[0, 1)$, as in definition (2.1). Writing as $(0, 1, 2)$ the three segments, we restrict to the case when the images segments are ordered as $(2, 1, 0)$. A classical construction shows that such an injective quasi-contraction can be seen as the first return map of an order-preserving quasi-contraction on the Circle with four pieces.

Definition 4.17

1. *Let T be as in definition (2.1) with $m = 2$, $D = (D_i)_{1\le i\le 2} \in \mathcal{D}$, a set of increasing $\gamma-$Lipschitzian maps $\Gamma_T = (\Gamma_i)_{0\le i\le 2}$, where $0 \le \gamma < 1$. Denote by $\mathcal{C}'$ the component of $\mathcal{C}$ corresponding to the fact that the image segments are ordered as $(2, 1, 0)$.*
2. *Set $u_0 = -\sqrt{\Gamma_1(D_2) - \Gamma_1(D_1)}$, $u_1 = 0$, $u_2 = D_1$ and $u_3 = D_2$. Let $R : [u_0, 1) \longmapsto [u_0, 1)$ be the order-preserving transformation with 4 pieces defined by :*
   - $\Gamma_R = (\Gamma', \Gamma_0, \Gamma'', \Gamma_2)$
   - $R_{|[u_1,u_2)\cup[u_3,1)} = T_{|[u_1,u_2)\cup[u_3,1)}$,
   - *The pair (Γ', Γ'') (not unique) is such that Γ'' contracts $[u_1, u_2)$ on $[u_0, u_1)$, Γ' contracts $[u_0, u_1)$ on $[\Gamma_1(D_1), \Gamma_1(D_2))$ and $\Gamma' \circ \Gamma'' = \Gamma_1$ on $[u_1, u_2)$.*

   Then T is the first return map of R in $[0, 1)$: if $x \in [0, 1)$ and $t(x) = \inf\{n \ge 1 \mid R^n(x) \in [0, 1)\}$ then $T(x) = R^{t(x)}(x)$. Remark that $t(x) \in \{1, 2\}$, $x \in [0, 1)$.

We then get :

Theorem 4.18
For T be as in definition (4.17), only two excluding cases can occur :

1. *There exists a periodic orbit. Then T admits at most* **two** *distinct periodic orbits and no diffusive T-invariant measure.*
2. *The map T has no periodic orbit. Then T is uniquely ergodic, measurably conjugated and topologically semi-conjugated to a possibly degenerated interval exchange transformation with 3 pieces and permutation $(0, 1, 2) \longrightarrow (2, 1, 0)$ on $[0, 1)$.*

Moreover each case is non-empty, but the first one holds for at least an open set of full Lebesgue measure in $\mathcal{C}'$. The subcases with exactly one or two periodic orbits have non-empty interior in $\mathcal{C}'$.

Proof of the theorem :
First, an example with two periodic orbits is when interval i is sent in interval $\sigma(i)$, where σ is the permutation $(0, 1, 2) \longrightarrow (2, 1, 0)$. This provides a fixed point and a 2-periodic orbit. If all intervals are sent in interval 2, then there is a unique fixed point. Both situations are clearly realized by open subsets in $\mathcal{C}'$.

Next observe that T admits a periodic orbit if and only if R admits one, that is if and only if $\tau(R) \in \mathbb{Q}$. We show that in this case there are less than two periodic orbits. Denote by (D', β) the parameters of R. As in proposition (4.16) and after renormalization let I be the set of integers i such that $(D', \beta) \in F_i(p/q)$.

Remark that $0 \in [R(D'_2)^-, R(D'_2)]$ and $D'_1 \in [R(D'_3)^-, R(D'_3)]$. Thus if both 0 and D'_2 lay in I they belong to the same cycle of length ≥ 2. The same holds for D'_1 and D'_3. From proposition (4.16) the number of periodic orbits of R is ≤ 2. This is also true for T.

Next the situation when R has no periodic orbit is non-empty. Indeed for some configurations, R admits a fixed point and for other ones there is no fixed point. This gives respectively $\tau(R) = 0$ and $0 < \tau(R) < 1$. The continuity of the rotation number implies that irrational values must be taken. This then provides parameters for which R (and thus T) has no periodic point.

In this case, proposition (4.3) implies that R and therefore T are uniquely ergodic. Let then h be a linearizing map as in proposition (4.3), point 2), that topologically semi-conjugates R with $R_{\tau(R)}$. Then T is topologically semi-conjugated and measurably conjugated to the first return map of $R_{\tau(R)}$ into $h([D'_1, 1))$. This map is a possibly degenerated interval exchange transformation with three pieces associated to the permutation $(0, 1, 2) \longrightarrow (2, 1, 0)$.

□

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