On planar random walks in environments invariant by horizontal translations

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Abstract

We give a recurrence criterion for a model of planar random walk in environment invariant under horizontal translations. Some examples are next developed, for instance when the environment is produced by a dynamical system.

1 Introduction

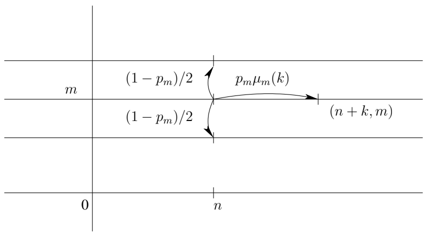
1.1 Presentation

We study the qualitative asymptotic behavior of a discrete time Markov chain $(X_n, Y_n)_{n\geq 0}$ on the lattice $\mathbb{Z}^2$. Vertical jumps are nearest neighbors, with equal probability, and the environment is invariant under horizontal translations. To precise the model, introduce parameters $(p_n)_{n\in\mathbb{Z}}$ and probability measures $(\mu_n)_{n\in\mathbb{Z}}$ with support on $\mathbb{Z}$, such that for some $\delta > 0$ and all $n \in \mathbb{Z}$:

$$\delta \leq p_n \leq 1-\delta,\ \mu_n(0) \leq 1-\delta \text{ and } \sum_{k\in\mathbb{Z}} |k|^3 \mu_n(k) \leq 1/\delta.$$

The last condition is introduced for having second order expansion of the characteristic functions uniform in $n \in \mathbb{Z}$. Letting $X_0 = Y_0 = 0$, the Markovian evolution is given by :

$$\mathbb{P}_{(n,m),(n+k,m)} = p_m \mu_m(k) \text{ and } \mathbb{P}_{(n,m),(n,m\pm 1)} = (1-p_m)/2.$$



When $p_m = 1/2$ and $\mu_m = (\delta_1 + \delta_{-1})/2$, we recover simple random walk in $\mathbb{Z}^2$. An important submodel, introduced by Campanino and Petritis in [1], is a model with oriented horizontal lines, corresponding to taking $p_n = p \in (0,1)$ and $\mu_n = \delta_{\varepsilon_n}$, where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a sequence of ± 1.

Let us detail known results, principally regarding this last model. For simplicity we say "i.i.d" for "a typical realization of an i.i.d. sequence". Campanino and Petritis [1] prove recurrence when $\varepsilon_n = (-1)^n$ and transience for $\varepsilon_n = 1_{n\geq 0} - 1_{n<0}$ and when the (ε_n) are i.i.d.. Guillotin-Plantard and Le Ny [5] show transience results when the (ε_n) form an independent family with marginals described by some dynamical system. Pene [8] proves transience when the (ε_n) are stationary, under a decorrelation condition. Castell, Guillotin-Plantard, Pene and Schapira [3] consider a

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similar model where the (ε_n) are i.i.d. and the vertical steps follow some measure μ in the domain of attraction of a centered stable distribution. A precise analysis of the annealed return times to 0 is performed, implying transience in this case. Devulder and Pene [4] consider an extension of the model introduced in [1] and establish transience when the (p_n) form an i.i.d. non-constant sequence and $\mu_n = \delta_{\varepsilon_n}$, with arbitrary (ε_n). Recently Campanino and Petritis [2], still considering their initial model, study a random perturbation of the periodic case for the (ε_n). The perturbation is chosen to decrease via some power law as the vertical coordinate increases. They show the existence of a critical parameter for recurrence/transience.

One may observe that most results on the model of Campanino-Petritis concern transience. As we shall detail later, the asymptotics of the random walk is related to the growth of the sums $(\varepsilon_1+\cdots+\varepsilon_n)$ and the frontier between recurrence and transience lies at logarithmic scales (precisely $\log n$). Hence recurrence is rare when the (ε_n) are stationary and a little of stochasticity is assumed.

1.2 Results

Let us present our results. Let $\eta_i = m_i p_i/(1-p_i)$, where m_i is the expectation of μ_i. This quantity can be interpreted when grouping in packets the successive horizontal steps of the random walk. Precisely, when arriving at the horizontal line i, the number Γ_i of steps on this line follows a geometrical law of parameter p_i. The displacement itself is $J_i = \sum_{1\le k\le \Gamma_i} \xi_k$, where the (ξ_k) are i.i.d. with law μ_i and are independent of Γ_i. Then $\eta_i = \mathbb{E}(J_i)$. For instance, for the model of Campanino-Petritis we have $\eta_i = (p/(1-p))\varepsilon_i$.

Introduce, for $k \le l$:

$$R_k^l = \sum_{k\le i\le l} \eta_i.$$

Set also $R_k^l = 0$ when $l < k$. The main quantities of interest for the sequel are the following.

Definition 1.1 *Introduce the strictly increasing functions :*

$$\varphi(n) = \sqrt{n^2 + \sum_{-n\le k\le l\le n} (R_k^l)^2} \quad \text{and} \quad \varphi_+(n) = \sqrt{n^2 + \sum_{-n\le k\le l\le n,\, kl>0} (R_k^l)^2}\,.$$

For large $x > 0$, let $\varphi^{-1}(x)$ be the unique integer n so that $\varphi(n) \le x < \varphi(n+1)$. Idem for $\varphi_+^{-1}(x)$.

Remark. — We have $n \le \varphi_+(n) \le \varphi(n) \le Cn^2$, so $c\sqrt{n} \le \varphi^{-1}(n) \le \varphi_+^{-1}(n) \le n$. An important property, detailed later, is that φ^{-1} and φ_+^{-1} are examples of regularly varying functions f in the sense that for all $C \ge 1$ there exists $K > 0$ so that $f(Cx) \le Kf(x)$.

Theorem 1.2 *The random walk is recurrent if and only if :*

$$\sum_{n\ge 1} \frac{1}{n^2} \frac{(\varphi^{-1}(n))^2}{\varphi_+^{-1}(n)} = +\infty.$$

Remark. — For simple random walk, one has $R_k^l = 0$, giving $\varphi_+^{-1}(n) = \varphi^{-1}(n) = n$. Recurrence hence follows from the divergence of the harmonic series. Notice that the generic term in the sum is always less than or equal to $n^{-2}\varphi_+^{-1}(n) \le n^{-1}$ and that simple random walk realizes equality. In this rough sense, all other random walks in the class considered are less recurrent.

Remark. — It is quite clear from the theorem that the asymptotics is not modified when changing a finite number of lines. Indeed, one gets some new $(\tilde{R}_k^l)$ which check $(\tilde{R}_k^l)^2 \le 2(R_k^l)^2 + C$. This gives $\tilde{\varphi}(n) \le C'\varphi(n)$ with also a similar symmetric inequality. As the same is true for φ_+, the regular variation properties of the inverse functions allow to conclude.

The difference of order between φ and φ_+ is related to symmetry reasons concerning the two half sequences $(\eta_n)_{n\ge 1}$ and $(\eta_n)_{n\le -1}$. Symmetry gives more transience, whereas antisymmetry implies more recurrence. This appears in the following statement.

Corollary 1.3
i) In the general case, a sufficient condition for transience is :

$$\sum_{n\geq 1} \frac{1}{\sqrt{n^2+\sum_{-n\leq k\leq l\leq n}(R_k^l)^2}} < +\infty.$$

ii) Antisymmetric model : $\eta_{-n}=-\eta_n$, $n\geq 1$. *The random walk is recurrent if and only if :*

$$\sum_{n\geq 1} \frac{1}{\sqrt{n^2+\sum_{1\leq k\leq l\leq n}(R_k^l)^2}} = +\infty.$$

iii) Symmetric model : $\eta_{-n}=\eta_n$, $n\geq 1$. *The random walk is transient if :*

$$\sum_{n\geq 1} \frac{1}{\sqrt{n^2+n\sum_{1\leq k\leq n}(R_1^k)^2}} < +\infty.$$

Remark. — Notice that the symmetric model can be seen as a model on $\mathbb{Z}\times\mathbb{N}$ with reflection on the horizontal axis $\mathbb{Z}\times\{0\}$. Let us point out that the sufficient conditions in *i*) and *iii*) are not necessary. Also the conditions appearing in *ii*) and *iii*) are not equivalent, as resulting from the following examples.

Proposition 1.4
i) In the general case, if $|R_1^n|+|R_{-n}^{-1}|=O(\log^{1/2} n)$, *then the random walk is recurrent.*

ii) In the antisymmetric case, if $|R_1^n|=O(\log n)$, *then the random walk is recurrent.*

iii) Let $1<\alpha\leq 2$ *and suppose that* $|R_1^n-\log^\alpha n|=O(\log^{\alpha-1} n)$.

- *If* $\eta_{-n}=\eta_n$ *for every* $n\geq 1$, *then the random walk is transient.*
- *If* $\eta_{-n}=-\eta_n$ *for every* $n\geq 1$, *then the random walk is recurrent.*

Remark. — The second case in item *iii*) is particularly interesting, because the random walk is recurrent but, roughly speaking, cannot come back when staying only North or only South. This implies an oscillating behavior (perhaps of sinusoidal or spiral type) for the random walk which would be interesting to quantify.

We next investigate the case when the (η_n) are produced by some dynamical system, first of all of quasi-periodic type and next with more randomness.

Proposition 1.5 *Take* $p_n=p\in(0,1)$ *and* $\mu_n=\delta_{\varepsilon_n}$ *with* $\varepsilon_n=1_{[0,1/2)}(n\alpha)-1_{[1/2,1)}(n\alpha)$. *Suppose that* $\alpha\notin\mathbb{Q}$ *has a continued fraction expansion* $[a_1,a_2,\cdots]$ *verifying :*

$$\sum_{n\geq 1}\frac{\log(1+a_n)}{a_1+\cdots+a_n}=+\infty.$$

Then the random walk is recurrent.

Remark. — The proof relies on the antisymmetry of $x\longmapsto 1_{[0,1/2)}(x)-1_{[1/2,1)}(x)$ and works without change for $\varepsilon_n=1_I(n\alpha)-1_{\mathbb{T}\backslash I}(n\alpha)$, where I is a finite union of intervals on the torus $\mathbb{T}$ with $\mathbb{T}\backslash I=-I$. The case of $\varepsilon_n=1_{[0,1/2)}(n\alpha+x)-1_{[1/2,1)}(n\alpha+x)$ for a general $x\in\mathbb{R}\backslash\mathbb{Z}$, for example $x=1/4$, is much more difficult. We are indeed in the critical zone and a specific study is a priori required, with finer estimates, especially lower bounds, on the ergodic sums associated to (ε_n). Notice that the condition on α is verified when the (a_n) are bounded. Also, defining the type of α as $\eta(\alpha)=\sup\{s\geq 1 \mid \liminf n^s \mathrm{dist}(n\alpha,\mathbb{Z})=0\}$, it is possible to have $\eta(\alpha)=a$ for any $a\in[1,+\infty]$ together with the condition of the proposition verified (simply choose a_n arbitrary large on a very sparse subsequence).

For the other application, we suppose that the (η_n) are a realization of strictly stationary process (for any $b \geq 0$ the law of $(\eta_n)_{a \leq n \leq a+b}$ is independent on $a \in \mathbb{Z}$). For $n \geq 1$, let us introduce the strong mixing coefficient :

$$\alpha(n) = \sup_{\substack{k \geq 1, A \in \sigma(\eta_1, \cdots, \eta_k) \\ B \in \sigma(\eta_l, l \geq k+n)}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The following proposition extends results in [1] and [4] relying on independence.

Proposition 1.6 *Let $\eta_n = \eta_n(\omega)$, where the $(\eta_n)_{n \in \mathbb{Z}}$ form a strictly stationary process. Suppose that $\mathbb{E}(\eta_0) = 0$ and that for some $\beta > 1$ and $\gamma > 1$:*

$$\alpha(n) = O(\log^{-\gamma} n) \text{ and } \liminf \frac{1}{N} \sum_{1 \leq n \leq N} \mathbb{P}(|R_1^n| \geq \log^\beta n) > 0.$$

Then the random walk is transient for almost-every ω.

Remark. — Mention that if replacing $\log^\beta n$ by n^ε in the statement of the proposition, then $\gamma > 0$ is sufficient in the proof. The second condition is satisfied if a central limit theorem is verified.

1.3 Notations

- The law of a random variable Γ is $\mathcal{G}(p)$ with $0 < p < 1$, if $\mathbb{P}(\Gamma = m) = (1-p)p^m$, $m \geq 0$.
- Let $\chi_n(t) = \sum_{k \in \mathbb{Z}} e^{itk} \mu_n(k)$, $m_n = \sum_{k \in \mathbb{Z}} k \mu_n(k)$, $m_{2,n} = \sum_{k \in \mathbb{Z}} k^2 \mu_n(k)$, $\mathrm{var}_n = m_{2,n} - m_n^2$.
- For $n \in \mathbb{Z}$, let φ_n be the characteristic function of $\sum_{1 \leq k \leq \Gamma} \xi_k$, where the $((\xi_k)_{k \geq 1}, \Gamma)$ are independent with $\xi_k \sim \mu_n$ and $\Gamma \sim \mathcal{G}(p_n)$. We have :

  $$\varphi_n(t) = \frac{1 - p_n}{1 - p_n \chi_n(t)} = r_n(t) e^{i a_n(t)},$$

  with argument (odd) and modulus (even), with O uniform in $n \in \mathbb{Z}$:

$$\left\{ \begin{array}{rcl} a_n(t) & = & \arctan\left(\frac{p_n \mathrm{Im}(\chi_n(t))}{1 - p_n \mathrm{Re}(\chi_n(t))}\right) = t \frac{p_n}{1-p_n} m_n + O(t^3), \\ r_n(t) & = & \frac{1-p_n}{(1 + p_n^2 |\chi_n|^2 - 2p_n \mathrm{Re}(\chi_n(t)))^{1/2}} = 1 - \frac{t^2}{2} \frac{p_n}{(1-p_n)^2}(m_{2,n} - p_n \mathrm{var}_n) + O(t^3). \end{array} \right. \tag{1}$$

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2 Reductions

2.1 A one-dimensional problem

As the second coordinate $(Y_n)_{n \geq 0}$ is simple random walk on $\mathbb{Z}$ (with waiting times at each site), the random walk is vertically recurrent. Introduce $0 = \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \cdots$, where:

$$\tau_k = \min\{n > \sigma_k \mid Y_n \neq 0\}, \; \sigma_{k+1} = \{n > \tau_k \mid Y_n = 0\}.$$

Set $D_n = X_{\sigma_n} - X_{\sigma_{n-1}}$. The $(D_n)_{n \geq 1}$ are independent and identically distributed, because the environment is invariant under horizontal translations.

Lemma 2.1 *The random walk $(X_n, Y_n)_{n \geq 0}$ is recurrent if and only $(\sum_{k=1}^n D_k)_{n \geq 1}$ is recurrent.*

Proof of the lemma :
We follow lemma 2.9 in [1]. If $(\sum_{k=1}^n D_k)_{n\geq 1}$ is recurrent, then so is (X_n, Y_n), because $(X_{\sigma_n}, Y_{\sigma_n}) = (\sum_{k=1}^n D_k, 0)$. In case of transience, using the invariance of the environment by horizontal translations, there is a constant C so that :

$$\sum_{n\geq 1} \mathbb{P}(X_{\sigma_n} = x) \leq C, \ x \in \mathbb{Z}.$$

Let $((\xi_k)_{k\geq 1}, \Gamma)$ be independent, and also from the (X_{σ_n}), such that $\xi_k \sim \mu_0$ and $\Gamma \sim \mathcal{G}(p_0)$. Then $(X_l)_{l\in[\sigma_k,\tau_k)}$ and $(X_{\sigma_k} + \sum_{1\leq m\leq l} \xi_m)_{0\leq l\leq \Gamma}$ have the same law. Set $H = \sum_{1\leq l\leq \Gamma} |\xi_l|$ and notice that it is integrable. Observe now (X_n, Y_n) can be 0 only for n in some $[\sigma_k, \tau_k)$ and we have :

$$\{\exists n \in [\sigma_k, \tau_k), \ X_n = 0\} \subset \{H \geq |X_{\sigma_k}|\}.$$

This provides :

$$\begin{aligned}\sum_{k\geq 1} \mathbb{P}(\exists n \in [\sigma_k, \tau_k), \ X_n = 0) \leq \sum_{k\geq 1} \mathbb{P}\left(H \geq |X_{\sigma_k}|\right) &\leq \sum_{x\in\mathbb{Z}} \sum_{k\geq 1} \mathbb{P}(X_{\sigma_k} = x)\mathbb{P}\left(H \geq |x|\right) \\ &\leq C \sum_{x\in\mathbb{Z}} \mathbb{P}(H \geq |x|) \leq C'\mathbb{E}(H) < +\infty.\end{aligned}$$

We conclude from the Borel-Cantelli lemma that (X_n, Y_n) is transient and this completes the proof of the lemma.

□

We thus concentrate on the recurrence properties of $(\sum_{k=1}^n D_k)_{n\geq 1}$. Set $D = D_1$ and :

$$\chi_D(t) = \mathbb{E}(e^{itD}), t \in \mathbb{R}.$$

In case of aperiodicity, i.e. when the support of the law of D generates $\mathbb{Z}$, an analytical recurrence theorem of Kesten and Spitzer, cf Spitzer [9], says that $(\sum_{k=1}^n D_k)_{n\geq 1}$ is transient if and only if :

$$\int_0^\eta \mathrm{Re}\left(\frac{1}{1-\chi_D(t)}\right) \, dt < +\infty, \ \text{for some } \eta > 0. \tag{2}$$

Our hypotheses imply that the subgroup of $(\mathbb{Z}, +)$ generated by the support of the law of D is not reduced to $\{0\}$. Up to considering some sublattice $m\mathbb{Z} \times \mathbb{Z}$, $m > 0$, we assume aperiodicity.

We therefore focus on the estimation of χ_D near the origin, which is now the only singularity of $1/(1-\chi_D)$. We fix $0 < t < \eta$ and omit the dependence in t to lighten the notations.

2.2 Local time and contour of a Galton-Watson tree

For simplicity, we first change notations. Denote now by $(Y_n)_{n\geq 0}$ simple symmetric random walk on $\mathbb{Z}$, starting at 0. Let $\sigma = \min\{k \geq 1 \mid Y_k = 0\}$ be the return time to 0. As in [1], we later make use of the standard properties $\mathbb{P}(\sigma > x) = O(1/\sqrt{x})$ and :

$$\mathbb{E}(s^\sigma) = 1 - \sqrt{1-s^2}, \ 0 \leq s \leq 1. \tag{3}$$

Grouping the successive horizontal steps of the random walk, observe that D can be written as :

$$D = \sum_{k=0}^{\sigma-1} \left(\sum_{u=1}^{\Gamma_k} \xi_u^{(k)}\right),$$

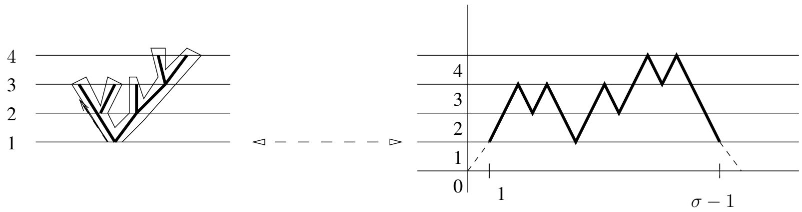
where, conditionally on the (Y_l), the $((\xi_u^{(k)})_{u\geq 1, k\geq 0}, (\Gamma_k)_{k\geq 0})$ are independent with $\xi_u^{(k)} \sim \mu_{Y_k}$ and $\Gamma_k \sim (\mathcal{G}(p_{Y_k}))$. This furnishes :

$$\begin{aligned}\chi_D(t) &= \mathbb{E}(\mathbb{E}(e^{itD} \mid (Y_l))) = \mathbb{E}\left(\prod_{k=0}^{\sigma-1} \mathbb{E}\left(e^{it\sum_{u=1}^{\Gamma_k} \xi_u^{(k)}} \mid (Y_l)\right)\right) = \mathbb{E}\left(\prod_{k=0}^{\sigma-1} \varphi_{Y_k}(t)\right) \quad (4)\\ &= \frac{1}{2}\mathbb{E}\left(\varphi_0(t)\prod_{n\geq 1}(\varphi_n(t))^{N_n}\right) + \frac{1}{2}\mathbb{E}\left(\varphi_0(t)\prod_{n\geq 1}(\varphi_{-n}(t))^{N_n}\right),\end{aligned}$$

where we have distinguished between positive and negative excursions and set :

$$N_n = \#\{0 < k < \sigma,\ |Y_k| = n\}.$$

To detail N_n, recall first that positive excursions of simple random walk can be described via the contour process of a critical Galton-Watson tree $(Z_n)_{n\geq 1}$, with root $Z_1 = 1$ and offspring distribution $\mathcal{G}(1/2)$. See for instance Le Gall [6]. This is illustrated by the following picture :



As shown in the left-hand side, start from the root of the tree and turn clockwise. We recover a positive excursion of simple random walk on the right-hand side in the time interval $[1, \sigma - 1]$, by associating to each ascending/descending movement a ± 1 step. One easily counts that the total number of visits at level $n \geq 1$ is $N_n = Z_n + Z_{n+1}$. This gives :

$$\varphi_0(t)\prod_{n\geq 1}(\varphi_n(t))^{N_n} = \varphi_0(t)\prod_{n\geq 1}(\varphi_n(t))^{Z_n+Z_{n+1}} = \prod_{n\geq 1}[\varphi_n(t)\varphi_{n-1}(t)]^{Z_n}.$$

We therefore obtain $\chi_D(t) = (\chi_+(t) + \chi_-(t))/2$, setting :

$$\chi_+(t) = \mathbb{E}\left[\prod_{n\geq 0}[\varphi_n(t)\varphi_{n+1}(t)]^{Z_{n+1}}\right] \text{ and } \chi_-(t) = \mathbb{E}\left[\prod_{n\geq 0}[\varphi_{-n}(t)\varphi_{-n-1}(t)]^{Z_{n+1}}\right]. \quad (5)$$

3 Continued fraction expansions

The Markovian structure of a Galton-Watson tree allows to develop χ_+ and χ_- in continued fractions. We first recall some classical facts about general continued fractions.

3.1 SP-continued fractions and their convergents

For sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$, we employ the following notation for a generalized finite continued fraction :

$$[b_0; (a_1, b_1); (a_2, b_2); \cdots; (a_n, b_n)] = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{\cdots}{\cdots + \cfrac{a_n}{b_n}}}}.$$

In the present article, we use this notation for Sleszynski-Pringsheim (SP) continued fractions. Finite SP-continued fractions are obtained by successively applying to an initial $z_0 \in \mathbb{C}$ in the unit disc maps of the form $z \longmapsto a/(b+z)$, with complex numbers such that $|a|+1 \leq |b|$, therefore preserving the unit disc. We write $[b_0; (a_1, b_1); (a_2, b_2); \cdots]$ for infinite SP-continued fractions. The latter are converging, by the Sleszynski-Pringsheim theorem (see [7]). We won't use the result directly, but largely reproduce the ideas from the proof.

For $n \geq 0$, the finite SP-continued fraction $[b_0; (a_1, b_1); (a_2, b_2); \cdots ; (a_n, b_n)]$ can be written as A_n/B_n, where the (A_n) and (B_n) satisfy the following recursive relations :

$$\begin{cases} A_n = b_n A_{n-1} + a_n A_{n-2},\ A_{-1} = 1,\ A_0 = b_0, \\ \\ B_n = b_n B_{n-1} + a_n B_{n-2},\ B_{-1} = 0,\ B_0 = 1. \end{cases}$$

In order to study convergence, we require the classical determinant, for $n \geq 1$:

$$\begin{aligned} A_n B_{n-1} - A_{n-1} B_n &= (-a_n)(A_{n-1}B_{n-2} - A_{n-2}B_{n-1}) = \cdots \\ &= (-1)^n a_1 \cdots a_n (A_0 B_{-1} - A_{-1} B_0) = (-1)^{n+1} a_1 \cdots a_n. \end{aligned}$$

3.2 Development of χ_+ and χ_-

Focusing on χ_+, cf (5), set :

$$\chi_+^{(N)}(t) = \mathbb{E}\left(\prod_{n=0}^{N} [\varphi_n(t)\varphi_{n+1}(t)]^{Z_{n+1}} \right).$$

By dominated convergence (by 1, using that $Z_n = 0$ for large n, a.s.) we have $\chi_+^{(N)}(t) \to \chi_+(t)$. Recall now a classical way of describing the Galton-Watson tree $(Z_n)_{n\geq 1}$ with root $Z_1 = 1$ and offspring distribution $\mathcal{G}(1/2)$. Fixing a bi-indexed collection $(R_{k,l})_{k\geq 1, l\geq 1}$ of independent random variables with distribution $\mathcal{G}(1/2)$, one can recursively write :

$$Z_{N+1} = \sum_{k=1}^{Z_N} R_{k,N+1},\ \ N \geq 1.$$

The generating function of $\mathcal{G}(1/2)$ being $s \longmapsto (2-s)^{-1}$, omitting the t, we obtain :

$$\chi_+^{(N)} = \mathbb{E}\left[\prod_{n=0}^{N-1} [\varphi_n\varphi_{n+1}]^{Z_{n+1}} \left(\frac{1}{2 - \varphi_N\varphi_{N+1}} \right)^{Z_N} \right] = \mathbb{E}\left[\prod_{n=0}^{N-2} [\varphi_n\varphi_{n+1}]^{Z_{n+1}} \left(\frac{\varphi_{N-1}\varphi_N}{2 - \varphi_N\varphi_{N+1}} \right)^{Z_N} \right].$$

Iterating the procedure, this leads to :

$$\begin{aligned} \chi_+^{(N)} &= \mathbb{E}\left[\varphi_0\varphi_1 \left(\frac{\varphi_1\varphi_2}{2 - \frac{\varphi_2\varphi_3}{2 - \frac{\cdots}{2 - \varphi_N\varphi_{N+1}}}} \right)^{Z_2} \right] \\ &= [0; (\varphi_0\varphi_1, 2); (-\varphi_1\varphi_2, 2); \cdots ; (-\varphi_{N-1}\varphi_N, 2 - \varphi_N\varphi_{N+1})] = \varphi_0 f_+^{(N)}, \end{aligned}$$

with $f_+^{(N)}(t) = [0; (1, 2/\varphi_1(t)); (-1, 2/\varphi_2(t)); \cdots ; (-1, 2/\varphi_N(t) - \varphi_{N+1}(t))]$.

In the framework of the previous section, take $a_1 = 1$, $a_n = -1$ for $n \geq 2$ and $b_0 = 0$, $b_n = 2/\varphi_n$, for $n \geq 1$. Call h_n the continued fraction of length n. As $A_n B_{n-1} - A_{n-1} B_n = 1$, $n \geq 1$, and $A_0 = 0$, we obtain the equality :

$$h_N(t) = \frac{A_0(t)}{B_0(t)} + \sum_{k=1}^{N} \left(\frac{A_k(t)}{B_k(t)} - \frac{A_{k-1}(t)}{B_{k-1}(t)} \right) = \sum_{k=1}^{N} \frac{1}{B_k(t)B_{k-1}(t)}.$$

Idem, with the modified term $\tilde{B}_N = \tilde{b}_N B_{N-1} - B_{N-2}$, where $\tilde{b}_N = 2/\varphi_N - \varphi_{N+1}$, we get :

$$f_+^{(N)}(t) = \sum_{k=1}^{N-1} \frac{1}{B_k(t)B_{k-1}(t)} + \frac{1}{\tilde{B}_N(t)B_{N-1}(t)}. \tag{6}$$

We now show that both $h_N(t)$ and $f_+^{(N)}(t)$ converge, as $N \to +\infty$, to the same $f_+(t)$, where :

$$f_+(t) = \sum_{k\geq 1} \frac{1}{B_k(t)B_{k-1}(t)}. \tag{7}$$

This way, from $B_n = (2/\varphi_n)B_{n-1} - B_{n-2}$, $n \geq 2$, and $B_1 = (2/\varphi_1)B_0 + B_{-1}$ we have :

$$|B_n| \geq 2|B_{n-1}| - |B_{n-2}|, \ n \geq 1.$$

This gives $|B_n|-|B_{n-1}| \geq |B_{n-1}|-|B_{n-2}| \geq \cdots \geq |B_0|-|B_{-1}| = 1$. Hence $|B_n| \geq n+1$. Similarly $|\tilde{B}_N| \geq (2-1)|B_{N-1}| - |B_{N-2}| \geq 1$. This implies the desired convergence.

Defining symmetrically some f_- associated to χ_-, we obtain :

$$\chi_D = \frac{\varphi_0}{2}(f_+ + f_-). \tag{8}$$

4 First estimates

4.1 General lower bound on Re$(1 - \chi_D)$

Proposition 4.1 *We have Re$(1 - \chi_D(t)) \geq \delta^2 t$.*

Proof of the proposition :
We shall suppose $\delta > 0$ small enough. We have $\chi_D = (\varphi_0/2)(f_+ + f_-)$. Hence :

$$\mathrm{Re}(1 - \chi_D(t)) \geq 1 - |\chi_D(t)| \geq 1 - (|f_+(t)| + |f_-(t)|)/2.$$

We prove that $|f_\pm(t)| \leq 1 - \delta^2 t$. By (1), using $m_{2,n} \geq \delta$, for small t uniformly in n :

$$r_n(t) \leq 1 - \frac{t^2}{4}\frac{\delta}{(1-\delta)^2} m_{2,n}(1-(1-\delta)) \leq 1 - \delta^3 t^2/(4(1-\delta)^2) \leq 1 - \delta^4 t^2.$$

Therefore for $n \geq 1$:

$$|B_n| \geq \frac{2}{r_n}|B_{n-1}| - |B_{n-2}| \geq \frac{2}{1-\delta^4 t^2}|B_{n-1}| - |B_{n-2}|.$$

Introduce the real sequence (u_n) such that $u_{-1} = 0$, $u_0 = 1$ and $u_n = (2/(1-\delta^4 t^2))u_{n-1} - u_{n-2}$. Define next $v_n = |B_n| - u_n$, which verifies $v_{-1} = v_0 = 0$ and :

$$v_n \geq \frac{2}{1-\delta^4 t^2} v_{n-1} - v_{n-2}.$$

As $v_n - v_{n-1} \geq (2/(1-\delta^4 t^2) - 2)v_{n-1} + v_{n-1} - v_{n-2}$, the condition "$v_{n-1} \geq v_{n-2} \geq 0$" is transmitted recursively. Hence v_n is non-decreasing and checks $v_n \geq 0$, i.e. $|B_n| \geq u_n$. Also :

$$u_n = \frac{\rho_+^{n+1} - \rho_-^{n+1}}{\rho_+ - \rho_-} = \rho_+^n \sum_{k=0}^{n} \rho_+^{-2k}, \ n \geq -1,$$

with $\rho_\pm = (1-\delta^4 t^2)^{-1}(1 \pm \sqrt{1-(1-\delta^4 t^2)^2}) = 1 \pm \delta^2 \sqrt{2}\, t + O(t^2)$, satisfying $\rho_+\rho_- = 1$. Then, remarking that $u_n = \rho_+^n \sum_{0\leq k\leq n} \rho_+^{-2k}$, we obtain :

$$|f_+| \leq \sum_{k\geq 1} \frac{1}{|B_k B_{k-1}|} \leq \sum_{k\geq 1} \frac{1}{u_k u_{k-1}} \quad \leq \quad \sum_{k\geq 1} \frac{1}{\rho_+^{2k-1}} \left(\frac{1}{\sum_{j=0}^{k-1} \rho_+^{-2j}} - \frac{1}{\sum_{j=0}^{k} \rho_+^{-2j}} \right) \frac{1}{\rho_+^{-2k}}.$$

We conclude that $|f_+| \leq \rho_+(1 - 1/\sum_{k\geq 0} \rho_+^{-2k}) = \rho_+(1-(1-\rho_+^{-2})) = \rho_- \leq 1 - \delta^2 t$, for t small enough. Idem for f_-. This concludes the proof of the proposition.

□

Remark. — This general result is independent from the distribution of the (p_n) and (μ_n). It gives the upper bound $\mathrm{Re}(1/(1-\chi_D(t))) \leq 1/\mathrm{Re}(1-\chi_D(t)) \leq 2/(\delta t)$, not sufficient to decide the question of integrability. This proposition will allow to remove terms of order at most t in the final part of the proof. One may also notice that the expression of u_n furnishes :

$$|B_n| \geq \frac{n+1}{(1-\delta^4 t^2)^n} \sum_{k=0}^{n/2} \frac{1}{2k+1} \binom{n}{2k} (1-(1-\delta^4 t^2)^2)^k.$$

For small t, this gives a lower bound for $|B_n|$ of the form $(n+1)(1+\alpha n^2 t^2)$, $\alpha > 0$.

4.2 Isolating the main term

Let us start from the relation $\chi_D(t) = \mathbb{E}(\prod_{k=0}^{\sigma-1} \varphi_{Y_k}(t))$, extracted from (4). Our aim now is to replace the (φ_n) by the (ψ_n), where we set :

$$\psi_n(t) = e^{ia_n(t)} w_n(t),\ w_n(t) = \cos a_n(t) \left(\frac{1 - ip_n \mathrm{Im}(\chi_n(t))/(1-p_n\mathrm{Re}(\chi_n(t))}{1 - itp_n m_n/(1-p_n)} \right). \tag{9}$$

First of all, recalling the exact value of $a_n(t)$ given in (1) and that $\cos(\arctan x) = 1/\sqrt{1+x^2}$, we observe that :

$$|\psi_n(t)| = \frac{1}{\sqrt{1+\left(\frac{p_n \mathrm{Im}(\chi_n(t))}{1-p_n \mathrm{Re}(\chi_n(t))}\right)^2}} \frac{\sqrt{1+\left(\frac{p_n \mathrm{Im}(\chi_n(t))}{1-p_n \mathrm{Re}(\chi_n(t))}\right)^2}}{\sqrt{1+\frac{p_n^2 m_n^2}{(1-p_n)^2} t^2}} = \left(1 + \frac{p_n^2 m_n^2}{(1-p_n)^2} t^2\right)^{-1/2} \leq 1.$$

We now set $g(t) = \mathbb{E}(\prod_{k=0}^{\sigma-1} \psi_{Y_k}(t))$ and estimate the difference $g(t) - \chi_D(t)$:

$$\begin{aligned} |g - \chi_D| &= \left| \mathbb{E}\left[\left(\prod_{k=0}^{\sigma-1} w_{Y_k} - \prod_{k=0}^{\sigma-1} r_{Y_k} \right) \prod_{k=0}^{\sigma-1} e^{ia_{Y_k}} \right] \right| \leq \mathbb{E}\left(\left| \prod_{k=0}^{\sigma-1} w_{Y_k} - \prod_{k=0}^{\sigma-1} r_{Y_k} \right| \right) \\ &\leq \mathbb{E}\left(\left| \prod_{k=0}^{\sigma-1} w_{Y_k} - 1 \right| \right) + \mathbb{E}\left(\left| \prod_{k=0}^{\sigma-1} r_{Y_k} - 1 \right| \right) = (A) + (B). \end{aligned}$$

From (1), for some constant $c > 0$ and small t, uniformly in n, we have $1 - ct^2 \leq r_n(t)$. Using also $r_n(t) \leq 1$ and (3), we get :

$$(B) = \mathbb{E}\left(1 - \prod_{k=0}^{\sigma-1} r_{Y_k}(t)\right) \leq \mathbb{E}\left(1-(1-ct^2)^\sigma\right) = \sqrt{1-(1-ct^2)^2} = \sqrt{2c}\,t + O(t^2).$$

For (A), denoting by "arg" the principal determination of the argument, write :

$$\begin{aligned} (A) &\leq \mathbb{E}\left(\left| \left(\prod_{k=0}^{\sigma-1} |w_{Y_k}| - 1 \right) \prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} \right| \right) + \mathbb{E}\left(\left| \prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} - 1 \right| \right) \\ &\leq \mathbb{E}\left(1 - \prod_{0\leq k<\sigma} |w_{Y_k}| \right) + \mathbb{E}\left(\left| \prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} - 1 \right| \right). \end{aligned}$$

For the first term, for another constant $c > 0$ we have for small t, uniformly in n :

$$|w_n(t)| = \left(1 + \frac{p_n^2 m_n^2}{(1-p_n)^2} t^2\right)^{-1/2} \geq \left(1 + ct^2\right)^{-1/2} \geq 1 - ct^2,$$

giving that is this term is bounded from above by $\mathbb{E}\left(1 - (1-ct^2)^\sigma\right) = \sqrt{1-(1-ct^2)^2} \leq \sqrt{2c}\, t + O(t^2)$. For the second term, observe first that, uniformly in n :

$$|\arg(w_n(t))| = \left|\arctan\left(\frac{p_n \mathrm{Im}(\chi_n(t))}{1 - p_n \mathrm{Re}(\chi_n(t)}\right) - \arctan\left(\frac{p_n m_n t}{1-p_n}\right)\right| = O(t^3). \tag{10}$$

From the fact that $\mathbb{P}(\sigma > 1/t) = O(\sqrt{t})$, as $t \to 0$, we now get :

$$\mathbb{E}\left(\left|\prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} - 1\right|\right) \leq \mathbb{E}\left(\left|\prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} - 1\right| 1_{\sigma \leq \frac{1}{t^2}}\right) + \mathbb{E}\left(\left|\prod_{k=0}^{\sigma-1} e^{i\arg(w_{Y_k})} - 1\right| 1_{\sigma > \frac{1}{t^2}}\right).$$

The second term is thus less than or equal than $c't + 2\mathbb{P}(\sigma > 1/t^2) \leq c''t$. The conclusion is that :

$$\chi_D(t) = g(t) + O(t), \text{ with } g(t) = \mathbb{E}\left(\prod_{k=0}^{\sigma-1} \psi_{Y_k}(t)\right). \tag{11}$$

The purpose of the previous modification was to obtain, using (1) :

$$\begin{aligned}
\frac{1}{\psi_n(t)} &= \frac{1}{\cos a_n(t)} e^{-ia_n(t)} \frac{1 - itp_n m_n/(1-p_n)}{1 - ip_n \mathrm{Im}(\chi_n(t))/(1 - p_n \mathrm{Re}(\chi_n(t))} \\
&= (1 - i\tan a_n(t)) \frac{1 - itp_n m_n/(1-p_n)}{1 - ip_n \mathrm{Im}(\chi_n(t))/(1 - p_n \mathrm{Re}(\chi_n(t))} \\
&= (1 - ip_n \mathrm{Im}(\chi_n(t))/(1 - p_n \mathrm{Re}(\chi_n(t))) \frac{1 - itp_n m_n/(1-p_n)}{1 - ip_n \mathrm{Im}(\chi_n(t))/(1 - p_n \mathrm{Re}(\chi_n(t))} \\
&= 1 - itp_n m_n/(1-p_n).
\end{aligned}$$

Now, since $|\psi_n| \leq 1$, in the same way as for χ_D (cf (8)), we can write :

$$g(t) = \frac{\psi_0(t)}{2}(g_+(t) + g_-(t)), \tag{12}$$

where both g_+ and g_- can be developed as infinite SP-continued fractions. As a result, from (11) and (12) and the fact that $\psi_0(t) = 1 + O(t)$ we finally get :

$$\chi_D(t) = \frac{1}{2}(g_+(t) + g_-(t)) + O(t). \tag{13}$$

Focusing on g_+ (this is symmetric for g_-) and setting $\rho_n = 2p_n m_n/(1-p_n)$, we have :

$$g_+(t) = \sum_{n \geq 1} \frac{1}{B_n(t) B_{n-1}(t)},$$

where (using that $2/\psi_n(t) = 2 - it\rho_n$) the (B_n) satisfy $B_{-1} = 0$, $B_0 = 1$, $B_1 = (2 - it\rho_1)B_0 + B_{-1}$ and $B_n = (2 - it\rho_n)B_{n-1} - B_{n-2}$, $n \geq 2$. We now have the following estimates.

Proposition 4.2
1) $|B_n| \geq 2|B_{n-1}| - |B_{n-2}|$, $n \geq 1$. *Also* $n \longmapsto |B_n| - |B_{n-1}| \geq 1$ *is non-decreasing and* $|B_n| \geq n+1$.

2) We have the upper bound, for $n \geq 1$ *:*

$$\sum_{k>n} \frac{1}{|B_k B_{k-1}|} \leq \frac{n+1}{|B_n|^2}.$$

Proof of the proposition :
The first point is treated as before. For the second one, we first prove that $(n+1)|B_{n+1}| \geq (n+2)|B_n|,\ n \geq 1$. We will use it in the equivalent form $|B_n|/(|B_{n+1}| - |B_n|) \leq n+1$. Write :

$$\begin{aligned}(n+1)|B_{n+1}| - (n+2)|B_n| &\geq (n+1)(2|B_n| - |B_{n-1}|) - (n+2)|B_n| \\ &= n|B_n| - (n+1)|B_{n-1}|.\end{aligned}$$

Recursively, $(n+1)|B_{n+1}| - (n+2)|B_n| \geq |B_1| - 2|B_0| \geq 0$. Now :

$$\begin{aligned}\left|\sum_{k>n} \frac{1}{B_k B_{k-1}}\right| \leq \sum_{k>n} \frac{1}{|B_k B_{k-1}|} &\leq \sum_{k>n} \left(\frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}\right) \frac{1}{|B_k| - |B_{k-1}|} \\ &\leq \sum_{k>n} \left(\frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}\right) \frac{1}{|B_{n+1}| - |B_n|}.\end{aligned}$$

Finally $|\sum_{k>n} 1/(B_k B_{k-1})| \leq 1/(|B_n|(|B_{n+1}| - |B_n|)) \leq (n+1)/|B_n|^2$.

□

5 Precise study of the convergents of a continued fraction

We now perform exact computations. Recall that $g_+(t) = \lim_{n\to+\infty} A_n(t)/B_n(t)$, where :

$$\begin{cases} A_n = (2 - it\rho_n)A_{n-1} + a_n A_{n-2},\ A_{-1} = 1,\ A_0 = 0, \\ B_n = (2 - it\rho_n)B_{n-1} + a_n B_{n-2},\ B_{-1} = 0,\ B_0 = 1, \end{cases}$$

with $a_1 = 1$ and $a_n = -1$ for $n \geq 2$. Introduce the matrices :

$$M_t = \begin{pmatrix} 2-it & -1 \\ 1 & 0 \end{pmatrix},\ A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that $A^n = \begin{pmatrix} n+1 & -n \\ n & -n+1 \end{pmatrix},\ n \in \mathbb{Z}$, and $B^n = B,\ n \geq 0$. Rewrite the recursions as :

$$\begin{cases} \begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 2-it\rho_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ n \geq 1, \\ \begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 2-it\rho_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = M_{t\rho_n} \cdots M_{t\rho_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ n \geq 1. \end{cases}$$

Using obvious notations and writing $B_n = B_n(t, \rho_1, \cdots, \rho_n)$, remark that :

$$A_n = B_{n-1}(t, \rho_2, \cdots, \rho_n). \tag{14}$$

Hence, we only have to focus on (B_n). Developing according to the positions of the B's :

$$\begin{aligned}\begin{pmatrix} B_n - (n+1) \\ B_{n-1} - n \end{pmatrix} &= \sum_{r=1}^{n} (-it)^r \sum_{1\leq k_1 < \cdots < k_r \leq n} \prod_{j=1}^{r} \rho_{k_j} A^{n-k_r} B A^{k_r - k_{r-1} - 1} \cdots B A^{k_1 - 1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sum_{r=1}^{n} (-it)^r \sum_{1\leq k_1 < \cdots < k_r \leq n} \prod_{j=1}^{r} \rho_{k_j} \begin{pmatrix} n-k_r+1 & 0 \\ n-k_r & 0 \end{pmatrix} \cdots \begin{pmatrix} k_2 - k_1 & 0 \\ k_2 - k_1 - 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_1 - 1 \end{pmatrix} \\ &= \sum_{r=1}^{n} (-it)^r \sum_{1\leq k_1 < \cdots < k_r \leq n} \prod_{j=1}^{r} \rho_{k_j} \begin{pmatrix} n-k_r+1 \\ n-k_r \end{pmatrix} (k_r - k_{r-1}) \cdots (k_2 - k_1) k_1.\end{aligned}$$

Consequently :

$$B_n - B_{n-1} = 1 + \sum_{r=1}^{n} (-it)^r \sum_{1\le k_1<\cdots<k_r\le n} k_1(k_2-k_1)\cdots(k_r-k_{r-1})\rho_{k_1}\cdots\rho_{k_r}. \tag{15}$$

We now perform r successive Abel transforms on the right-hand side. This way, for $k \le l$, set $R_k^l = \rho_k + \cdots + \rho_l$, with $R_k^l = 0$ if $l < k$. Fixing k_{r-1}, n, and as $\rho_{k_r} = R_{k_r}^n - R_{k_r+1}^n$, we write :

$$\begin{aligned}
\sum_{k_{r-1}<k_r\le n} \rho_{k_r}(k_r - k_{r-1}) &= \sum_{k_{r-1}<k_r\le n} R_{k_r}^n (k_r - k_{r-1}) - \sum_{k_{r-1}+1<k_r\le n+1} R_{k_r}^n (k_r - 1 - k_{r-1}) \\
&= \sum_{k_{r-1}<k_r\le n} R_{k_r}^n (k_r - k_{r-1}) - \sum_{k_{r-1}<k_r\le n} R_{k_r}^n (k_r - 1 - k_{r-1}) \\
&= \sum_{k_{r-1}<k_r\le n} R_{k_r}^n.
\end{aligned}$$

We make this transformation in the right-hand side of (15). Repeat next this manipulation with :

$$\sum_{k_{r-2}<k_{r-1}\le k_r} \rho_{k_{r-1}}(k_{r-1} - k_{r-2}),$$

fixing this time all variables except k_{r-1} and taking $R_{k_{r-1}}^{k_r-1}$ in place of $R_{k_r}^n$. Doing this successively for all variables $k_r, \cdots, k_1$, we arrive at :

$$B_n - B_{n-1} = 1 + \sum_{r=1}^{n} (-it)^r \sum_{1\le k_1<\cdots<k_r\le n} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \cdots R_{k_r}^n. \tag{16}$$

As a result, summing in n, this furnishes :

$$B_n = n + 1 + \sum_{r=1}^{n} (-it)^r \sum_{1\le k_1<\cdots<k_{r+1}\le n+1} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \cdots R_{k_r}^{k_{r+1}-1}. \tag{17}$$

Similarly, via (14) :

$$A_n = n + \sum_{r=1}^{n-1} (-it)^r \sum_{2\le k_1<\cdots<k_{r+1}\le n+1} R_{k_1}^{k_2-1} R_{k_2}^{k_3-1} \cdots R_{k_r}^{k_{r+1}-1}.$$

The previous two equalities next furnish :

$$B_n - A_n \;=\; 1 + \sum_{r=1}^{n} (-it)^r \sum_{2\le k_2<\cdots<k_{r+1}\le n+1} R_1^{k_2-1} \cdots R_{k_r}^{k_{r+1}-1} = \sum_{0\le r\le n} (-it)^r \Delta_r^n, \tag{18}$$

where we set :

$$\Delta_r^n = \sum_{1\le k_1<\cdots<k_r\le n} R_1^{k_1} R_{k_1+1}^{k_2} \cdots R_{k_{r-1}+1}^{k_r}.$$

We convene that $\Delta_0^n = 1$ and $\Delta_r^n = 0$ as soon as $r > n$ or $r < 0$.

Proposition 5.1 *i)* $|B_n - A_n|^2 = \sum_{r=0}^n t^{2r} K_r(n)$, *with* $K_0(n) = 1$ *and :*

$$K_r(n) = \sum_{1=k_1\le l_1<\cdots<k_r\le l_r\le n} (R_1^{l_1} R_{k_2}^{l_2} \cdots R_{k_r}^{l_r})^2 (1 + 2(n - l_r)) 2^{N((k_i,l_i)_{1\le i\le r})},$$

where $N((k_i, l_i)_{1\le i\le r}) = \#\{2 \le i \le r \mid k_i \ge l_{i-1} + 2\}$.

ii) $|B_n|^2 = \sum_{r=0}^n t^{2r} L_r(n)$, *with* $L_0(n) = (n+1)^2$ *and :*

$$L_r(n) = \sum_{1\le k_1\le l_1<\cdots<k_r\le l_r\le n} (R_{k_1}^{l_1} R_{k_2}^{l_2} \cdots R_{k_r}^{l_r})^2 (2k_1-1)(1+2(n-l_r)) 2^{N((k_i,l_i)_{1\le i\le r})}.$$

iii) $Re((B_n - A_n)\overline{B}_n) = \sum_{r=0}^{n} t^{2r} M_r(n)$*, with* $M_0(n) = n+1$ *and :*

$$M_r(n) = \sum_{1\le k_1\le l_1<\cdots<k_r\le l_r\le n} (R_{k_1}^{l_1} R_{k_2}^{l_2} \cdots R_{k_r}^{l_r})^2 (1+2(n-l_r)) 2^{N((k_i,l_i)_{1\le i\le r})}.$$

iv) $Im((B_n - A_n)\overline{B}_n) = -\sum_{r=0}^{n-1} t^{2r+1} N_r(n)$*, with :*

$$N_r(n) = \sum_{1=k_1\le l_1<\cdots<k_{r+1}\le l_{r+1}\le n} R_1^{l_1} (R_{k_2}^{l_2} \cdots R_{k_{r+1}}^{l_{r+1}})^2 (1+2(n-l_{r+1})) 2^{N((k_i,l_i)_{1\le i\le r+1})}.$$

Remark. — Point *i)* induces a similar formula for $|B_n - B_{n-1}|^2$, because of the identity $(B_n - B_{n-1})(t,\rho_1,\cdots,\rho_n) = (B_n - A_n)(t,\rho_n,\cdots,\rho_1)$, resulting from (16) and (18).

Remark. — It results from *ii)* that if $(\Omega,\mathcal{F},T,\mu)$ is an ergodic dynamical system and $f \in L^\infty(\mu)$, then the main Lyapunov exponent of the matrix :

$$M(\omega) = \begin{pmatrix} 2+if(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

defined as $\gamma_{\max}(M) = \lim n^{-1}\int_\Omega \|M(T^{n-1}\omega)\cdots M(\omega)\|\, d\mu(\omega)$ equals :

$$\gamma_{\max}(M) = \frac{1}{2}\lim \frac{1}{n} \sup_{1\le r\le n} \log \sum_{1\le k_1\le l_1<\cdots<k_r\le l_r\le n} (R_{k_1}^{l_1})^2 \cdots (R_{k_r}^{l_r})^2,$$

with $R_k^l = \sum_{k\le j\le l} f(T^j\omega)$. One may obtain some variational formula for this exponent.

5.1 Proof of *i)*

Starting from $B_n - A_n = \sum_{0\le r\le n} (-it)^r \Delta_r^n$, we compute :

$$\begin{aligned} |B_n - A_n|^2 = \sum_{r=0}^{n} t^{2r} (\Delta_r^n)^2 \quad &+ \quad 2 \sum_{0\le r<r'\le n/2} (-1)^{r+r'} t^{2r+2r'} \Delta_{2r}^n \Delta_{2r'}^n \\ &+ \quad 2 \sum_{0\le r<r'\le (n-1)/2} (-1)^{r+r'} t^{2r+2r'+2} \Delta_{2r+1}^n \Delta_{2r'+1}^n. \end{aligned}$$

Hence $|B_n - A_n|^2 = \sum_{r=0}^{n} t^{2r} (\Delta_r^n)^2 + 2\sum_{0\le r<r'\le n,\ r'-r \text{ even}} (-1)^{(r-r')/2} t^{r+r'} \Delta_r^n \Delta_{r'}^n$. Thus :

$$\begin{aligned} |B_n - A_n|^2 &= \sum_{r=0}^{n} t^{2r} \left[(\Delta_r^n)^2 + 2\sum_{1\le p\le r} (-1)^{(r+p)/2-(r-p)/2} \Delta_{r+p}^n \Delta_{r-p}^n \right] \\ &= \sum_{r=0}^{n} t^{2r} \left[\sum_{-r\le p\le r} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n \right]. \qquad (19) \end{aligned}$$

Call $K_r(n)$ the term between brackets. We check that it equals the formula given in *i)*. The latter is a consequence of the recursion :

$$\begin{cases} K_1(n) = \sum_{1\le k\le n} (R_1^k)^2 (1+2(n-k)), \\ K_r(n) = \sum_{1\le k\le n} (R_1^k)^2 \left[\theta^k K_{r-1}(n-k) + 2\sum_{k<l\le n} \theta^l K_{r-1}(n-l) \right],\ r\ge 2. \end{cases} \qquad (20)$$

In the second formula, θ denotes the shift $\theta((\rho_n)_{n\geq 0}) = (\rho_{n+1})_{n\geq 0}$ and for a function $\psi((\rho_n)_{n\geq 0})$ we write $\theta\psi$ for $\psi \circ \theta$. For the first relation, simply compute :

$$\begin{aligned} K_1(n) = (\Delta_1^n)^2 - 2\Delta_0^n \Delta_2^n &= \left(\sum_{1\leq k\leq n} R_1^k \right)^2 - 2 \sum_{1\leq k<l\leq n} R_1^k R_{k+1}^l \\ &= \sum_{1\leq k\leq n} (R_1^k)^2 + 2 \sum_{1\leq k<l\leq n} R_1^k (R_1^l - R_{k+1}^l). \end{aligned}$$

As $R_1^l - R_{k+1}^l = R_1^k$ and the number of l in $(k, n]$ is $n-k$, we get the expression for $K_1(n)$.

We turn to the proof of the recursive relation. This needs some preparation. Taking general $p \geq 1$, $q \geq 1$, we have :

$$\Delta_p^n \Delta_q^n = \sum_{\substack{1\leq k_1<\cdots<k_p\leq n \\ 1\leq k_1'<\cdots<k_q'\leq n}} (R_1^{k_1} \cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1'} \cdots R_{k_{q-1}'+1}^{k_q'}).$$

Discuss now according to the relative positions of k_1 and k_1' :

$$\begin{aligned} \Delta_p^n \Delta_q^n = & \sum_{1\leq k\leq n} (R_1^k)^2 \theta^k \Delta_{p-1}^{n-k} \theta^k \Delta_{q-1}^{n-k} \\ & + \sum_{\substack{1\leq k_1<\cdots<k_p\leq n \\ k_1<k_1'<\cdots<k_q'\leq n}} R_1^{k_1} (R_{k_1+1}^{k_2} \cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1} + R_{k_1+1}^{k_1'})(R_{k_1'+1}^{k_2'} \cdots R_{k_{q-1}'+1}^{k_q'}) \\ & + \sum_{\substack{1\leq k_1'<\cdots<k_q'\leq n \\ k_1'<k_1<\cdots<k_p\leq n}} (R_1^{k_1'} + R_{k_1'+1}^{k_1})(R_{k_1+1}^{k_2} \cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1'} \cdots R_{k_{q-1}'+1}^{k_q'}). \end{aligned}$$

Regrouping terms, this can be rewritten more concisely as :

$$\begin{aligned} \Delta_p^n \Delta_q^n &= \sum_{1\leq k\leq n} (R_1^k)^2 \left[\theta^k \Delta_{p-1}^{n-k} \sum_{k\leq l\leq n} \theta^l \Delta_{q-1}^{n-l} + \theta^k \Delta_{q-1}^{n-k} \sum_{k<l\leq n} \theta^l \Delta_{p-1}^{n-l} \right] \\ &+ \sum_{1\leq k\leq n} R_1^k \left[\theta^k \Delta_{p-1}^{n-k} \theta^k \Delta_q^{n-k} + \theta^k \Delta_p^{n-k} \theta^k \Delta_{q-1}^{n-k} \right]. \end{aligned}$$

Let $r \geq 2$. We now plug this expression in :

$$K_r(n) = \sum_{-r\leq p\leq r} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n. \tag{21}$$

This gives :

$$\begin{aligned} K_r(n) &= \sum_{-r+1\leq p\leq r-1} (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n + 2(-1)^r \Delta_{2r}^n \\ &= \sum_{1\leq k\leq n} (R_1^k)^2 \sum_{-r+1\leq p\leq r-1} (-1)^p \left[\theta^k \Delta_{r+p-1}^{n-k} \sum_{k\leq l\leq n} \theta^l \Delta_{r-p-1}^{n-l} + \theta^k \Delta_{r-p-1}^{n-k} \sum_{k<l\leq n} \theta^l \Delta_{r+p-1}^{n-l} \right] \\ &+ 2(-1)^r \Delta_{2r}^n + 2 \sum_{1\leq k\leq n} R_1^k \left[\sum_{-r+1\leq p\leq r-1} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \theta^k \Delta_{r-p}^{n-k} \right]. \end{aligned}$$

The last line is $2\sum_{1\leq k\leq n} R_1^k \left[\sum_{-r+1\leq p\leq r} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \theta^k \Delta_{r-p}^{n-k} \right]$. The inner brackets are 0, so :

$$K_r(n) = \sum_{1\le k\le n} (R_1^k)^2 \left[\theta^k K_{r-1}(n-k) + 2 \sum_{-r+1\le p\le r-1} (-1)^p \theta^k \Delta_{r+p-1}^{n-k} \sum_{k<l\le n} \theta^l \Delta_{r-p-1}^{n-l}\right].$$

Set $m = n-k$ and $Z_r(m) = \sum_{-r\le p\le r} (-1)^p \Delta_{r+p}^m \sum_{1\le l\le m} \theta^l \Delta_{r-p}^{m-l}$. We therefore have :

$$K_r(n) = \sum_{1\le k\le n} (R_1^k)^2 \left[\theta^k K_{r-1}(n-k) + 2\theta^k Z_{r-1}(n-k)\right].$$

We shall show that :

$$Z_r(m) = \sum_{1\le k\le m} \theta^k K_r(m-k),\ r\ge 1. \tag{22}$$

To complete the proof of (20), we apply this equality to $Z_{r-1}(n-k)$. First of all, with $0\le p\le r-1$:

$$\begin{aligned}
\Delta_{r+p}^m \sum_{1\le l\le m} \theta^l \Delta_{r-p}^{m-l} &= \sum_{\substack{1\le k_1<\cdots<k_{r+p}\le m\\ 1\le l_1<l_2<\cdots<l_{r-p+1}\le m}} R_1^{k_1}\cdots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}} \\
&= \sum_{\substack{1\le k_1<\cdots<k_{r+p}\le m\\ k_1<l_2<\cdots<l_{r-p+1}\le m}} R_1^{k_1} R_{k_1+1}^{k_2}\cdots R_{k_{r+p-1}+1}^{k_{r+p}} R_{k_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}} \\
&+ \sum_{\substack{1\le k_1<\cdots<k_{r+p}\le m\\ k_1<l_1<\cdots<l_{r-p+1}\le m}} R_1^{k_1} R_{k_1+1}^{k_2}\cdots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}} \\
&+ \sum_{\substack{1\le l_1<\cdots<l_{r-p+1}\le m\\ l_1<k_1<\cdots<k_{r+p}\le m}} (R_1^{l_1} + R_{l_1+1}^{k_1}) R_{k_1+1}^{k_2}\cdots R_{k_{r+p-1}+1}^{k_{r+p}} R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}}.
\end{aligned}$$

In short :

$$\begin{aligned}
\Delta_{r+p}^m \sum_{1\le l\le m} \theta^l \Delta_{r-p}^{m-l} &= \sum_{1\le k\le m} R_1^k \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k\le l\le m} \theta^l \Delta_{r-p}^{m-l} + \theta^k \Delta_{r-p}^{m-k} \sum_{k<l\le m} \theta^l \Delta_{r+p-1}^{m-l}\right] \\
&+ \sum_{1\le k\le m} \theta^k \Delta_{r+p}^{m-k} \theta^k \Delta_{r-p}^{m-k}.
\end{aligned}$$

Consequently :

$$\begin{aligned}
Z_r(m) &= (-1)^r \left[m\Delta_{2r}^m + \sum_{1\le l\le m} \theta^l \Delta_{2r}^{m-l}\right] + \sum_{1\le k\le m} \sum_{-r+1\le p\le r-1} (-1)^p \theta^k \Delta_{r+p}^{m-k} \theta^k \Delta_{r-p}^{m-k}. \\
&+ \sum_{1\le k\le m} R_1^k \sum_{-r+1\le p\le r-1} (-1)^p \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k\le l\le m} \theta^l \Delta_{r-p}^{m-l} + \theta^k \Delta_{r-p}^{m-k} \sum_{k<l\le m} \theta^l \Delta_{r+p-1}^{m-l}\right].
\end{aligned}$$

Recognizing some $\theta^k K_r(m-k)$, i.e. using (21), we get :

$$
\begin{aligned}
Z_r(m) &= (-1)^r \left[m\Delta_{2r}^m - \sum_{1\le l\le m} \theta^l \Delta_{2r}^{m-l} \right] + \sum_{1\le k\le m} \theta^k K_r(m-k) \\
&+ \sum_{1\le k\le m} R_1^k \sum_{-r+1\le p\le r-1} (-1)^p \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k\le l\le m} \theta^l \Delta_{r-p}^{m-l} \right] \\
&+ \sum_{1\le k\le m} R_1^k \sum_{-r+2\le p\le r} (-1)^{p+1} \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k< l\le m} \theta^l \Delta_{r-p}^{m-l} \right].
\end{aligned}
$$

Therefore :

$$
\begin{aligned}
Z_r(m) &= (-1)^r \left[m\Delta_{2r}^m - \sum_{1\le l\le m} \theta^l \Delta_{2r}^{m-l} - \sum_{1\le k\le m} R_1^k \left(\theta^k \Delta_{2r-1}^{m-k}(m-k+1) + \sum_{k<l\le m} \theta^l \Delta_{2r-1}^{m-l} \right) \right] \\
&+ \sum_{1\le k\le m} \theta^k K_r(m-k) + \sum_{1\le k\le m} R_1^k \sum_{-r+1\le p\le r} (-1)^p \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k\le l\le m} \theta^l \Delta_{r-p}^{m-l} \right] \\
&+ \sum_{1\le k\le m} R_1^k \sum_{-r+1\le p\le r} (-1)^{p+1} \left[\theta^k \Delta_{r+p-1}^{m-k} \sum_{k<l\le m} \theta^l \Delta_{r-p}^{m-l} \right].
\end{aligned}
$$

The last two terms are $\sum_{1\le k\le m} R_1^k [\sum_{-r+1\le p\le r} (-1)^p \theta^k \Delta_{r+p-1}^{m-k} \theta^k \Delta_{r-p}^{m-k}]$ and the brackets are equal to 0. It remains to show that the first bracket above is also 0. It equals :

$$
-\sum_{1\le l\le m} \theta^l \Delta_{2r}^{m-l} + \sum_{1\le k\le m} (k-1) R_1^k \theta^k \Delta_{2r-1}^{m-k} - \sum_{1\le k<l\le m} R_1^k \theta^l \Delta_{2r-1}^{m-l}.
$$

In the last term, write $R_1^k = R_1^l - R_{k+1}^l$. We conclude from the observation that :

$$
\sum_{1\le k<l\le m} R_1^l \theta^l \Delta_{2r-1}^{m-l} = \sum_{1\le k\le m} (k-1) R_1^k \theta^k \Delta_{2r-1}^{m-k} \text{ and } \sum_{1\le k<l\le m} R_{k+1}^l \theta^l \Delta_{2r-1}^{m-l} = \sum_{1\le l\le m} \theta^l \Delta_{2r}^{m-l}.
$$

This completes the proof of i).

5.2 Proof of ii)

Set $\tilde\Delta_r^n = \sum_{0\le k_0<k_1<\cdots k_r\le n} R_{k_0+1}^{k_1} R_{k_1+1}^{k_2} \cdots R_{k_{r-1}+1}^{k_r}$, with $\tilde\Delta_0^n = n+1$ and $\tilde\Delta_r^n = 0$ whenever $r > n$ or $r < 0$. From (17), we get :

$$
B_n = \sum_{0\le r\le n} (-it)^r \tilde\Delta_r^n. \tag{23}
$$

Exactly in the same way as for deriving relation (19), we have :

$$
|B_n|^2 = \sum_{r=0}^n t^{2r} \left[\sum_{-r\le p\le r} (-1)^p \tilde\Delta_{r+p}^n \tilde\Delta_{r-p}^n \right] =: \sum_{r=0}^n t^{2r} L_r(n).
$$

To compute $L_r(n)$ we first observe that :

$$
\tilde\Delta_{r+p}^n \tilde\Delta_{r-p}^n = \sum_{0\le k\le n} \left[\theta^k \Delta_{r+p}^{n-k} \sum_{k\le l\le n} \theta^l \Delta_{r-p}^{n-l} + \theta^k \Delta_{r-p}^{n-k} \sum_{k<l\le n} \theta^l \Delta_{r+p}^{n-l} \right].
$$

Replacing in $L_r(n)$, this allows to write, recognizing $K_r(n)$ and $Z_r(n)$ given in (22) :

$$\begin{aligned} L_r(n) &= \sum_{0\le k\le n}\sum_{-r\le p\le r}(-1)^p\left[\theta^k\Delta^{n-k}_{r+p}\sum_{k\le l\le n}\theta^l\Delta^{n-l}_{r-p}+\theta^k\Delta^{n-k}_{r-p}\sum_{k<l\le n}\theta^l\Delta^{n-l}_{r+p}\right] \\ &= \sum_{0\le k\le n}\theta^k K_r(n-k)+2\sum_{0\le k\le n}\theta^k Z_r(n-k) \\ &= \sum_{0\le k\le n}\theta^k K_r(n-k)+2\sum_{0\le k<l\le n}\theta^l K_r(n-l)=\sum_{0\le k\le n}(1+2k)\theta^k K_r(n-k). \quad (24)\end{aligned}$$

This completes the proof of *ii*).

5.3 Proof of *iii*)

From (18) and (23), we deduce :

$$(B_n-A_n)\overline{B}_n=\left[\sum_{0\le r\le n}(-it)^r\Delta^n_r\right]\left[\sum_{0\le r\le n}(it)^r\tilde{\Delta}^n_r\right]. \quad (25)$$

Therefore :

$$\begin{aligned}\mathrm{Re}((B_n-A_n)\overline{B}_n) &= \sum_{0\le r\le n/2}(-t^2)^r\Delta^n_{2r}\sum_{0\le r'\le n/2}(-t^2)^{r'}\tilde{\Delta}^n_{2r'} \\ &+ \sum_{0\le r\le (n-1)/2}(-1)^r t^{2r+1}\Delta^n_{2r+1}\sum_{0\le r'\le (n-1)/2}(-1)^{r'}t^{2r'+1}\tilde{\Delta}^n_{2r'+1}.\end{aligned}$$

Consequently :

$$\begin{aligned}\mathrm{Re}((B_n-A_n)\overline{B}_n) &= \sum_{0\le r\le n}t^{2r}\Delta^n_r\tilde{\Delta}^n_r+\sum_{0\le r<r'\le n/2}(-1)^{r+r'}t^{2r+2r'}(\Delta^n_{2r}\tilde{\Delta}^n_{2r'}+\tilde{\Delta}^n_{2r}\Delta^n_{2r'}) \\ &+ \sum_{0\le r<r'\le (n-1)/2}(-1)^{r+r'}t^{2r+2r'+2}(\Delta^n_{2r+1}\tilde{\Delta}^n_{2r'+1}+\tilde{\Delta}^n_{2r+1}\Delta^n_{2r'+1}) \\ &= \sum_{0\le r\le n}t^{2r}\Delta^n_r\tilde{\Delta}^n_r+\sum_{\substack{0\le r<r'\le n\\ r'-r \text{ even}}}(-1)^{(r'-r)/2}t^{r+r'}(\Delta^n_r\tilde{\Delta}^n_{r'}+\tilde{\Delta}^n_r\Delta^n_{r'}) \\ &= \sum_{0\le r\le n}t^{2r}\left[\sum_{-r\le p\le r}(-1)^p\Delta^n_{r+p}\tilde{\Delta}^n_{r-p}\right]=:\sum_{0\le r\le n}t^{2r}M_r(n).\end{aligned}$$

As $\tilde{\Delta}^n_r=\Delta^n_r+\sum_{1\le k\le n}\theta^k\Delta^{n-k}_r$, using $K_r(n)$ and the value of $Z_r(n)$ in (22), we have :

$$M_r(n) = K_r(n)+Z_r(n)=K_r(n)+\sum_{1\le k\le n}\theta^k K_r(n-k)=\sum_{0\le k\le n}\theta^k K_r(n-k).$$

This ends the proof of *iii*).

5.4 Proof of *iv*)

Starting from (25) and (18), we have :

$$
\begin{aligned}
\mathrm{Im}((B_n - A_n)\overline{B}_n) &= \sum_{0\le r\le n/2} (-1)^r t^{2r}\Delta_{2r}^n \sum_{0\le r'\le (n-1)/2} (-1)^{r'} t^{2r'+1}\tilde{\Delta}_{2r'+1}^n \\
&- \sum_{0\le r\le (n-1)/2} (-1)^r t^{2r+1}\Delta_{2r+1}^n \sum_{0\le r'\le n/2} (-1)^{r'} t^{2r'}\tilde{\Delta}_{2r'}^n \\
&= \sum_{\substack{0\le r\le n/2\\ 0\le r'\le (n-1)/2}} t^{2r+2r'+1}(-1)^{r+r'}(\Delta_{2r}^n\tilde{\Delta}_{2r'+1}^n - \tilde{\Delta}_{2r}^n\Delta_{2r'+1}^n). \\
&= \sum_{\substack{0\le r\le n\\ 0\le r'\le n, r-r' \text{ odd}}} (-1)^r(-1)^{(r+r'-1)/2} t^{r+r'}\Delta_r^n\tilde{\Delta}_{r'}^n \\
&= -\sum_{0\le r\le n-1} t^{2r+1}\left[\sum_{-r-1\le p\le r} (-1)^p \Delta_{r+p+1}^n \tilde{\Delta}_{r-p}^n\right] =: -\sum_{0\le r\le n-1} t^{2r+1} N_r(n).
\end{aligned}
$$

Use now that $\tilde{\Delta}_r^n = \Delta_r^n + \sum_{1\le k\le n}\theta^k\Delta_r^{n-k}$ and $\sum_{-r-1\le p\le r}(-1)^p\Delta_{r+p+1}^n\Delta_{r-p}^n = 0$ to get :

$$
\begin{aligned}
N_r(n) &= (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta_{2r+1}^{n-k} + \sum_{-r\le p\le r}(-1)^p\Delta_{r+p+1}^n\sum_{1\le k\le n}\theta^k\Delta_{r-p}^{n-k} \\
&= (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta_{2r+1}^{n-k} + \sum_{1\le l\le n} R_1^l \sum_{-r\le p\le r}(-1)^p\theta^l\Delta_{r+p}^{n-l}\sum_{l<k\le n}\theta^k\Delta_{r-p}^{n-k} \\
&+ \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta_{r-p}^{n-k}\sum_{k\le l\le n} R_1^l\theta^l\Delta_{r+p}^{n-l}.
\end{aligned}
$$

As a result, separating in the last term the sums corresponding to $k = l$ and to $k < l$, together with expressions of $K_r(n)$ and $Z_r(n)$ given by (21) and (22) respectively, we obtain :

$$
\begin{aligned}
N_r(n) &= \sum_{1\le k\le n} R_1^k\left[\theta^k K_r(n-k) + \theta^k Z_r(n-k)\right] + (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta_{2r+1}^{n-k} \\
&+ \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta_{r-p}^{n-k}\sum_{k<l\le n} R_1^l\theta^l\Delta_{r+p}^{n-l}. \qquad (26)
\end{aligned}
$$

We next define and focus on :

$$
\begin{aligned}
O_r(n) &= (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta_{2r+1}^{n-k} + \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta_{r-p}^{n-k}\sum_{k<l\le n}(R_1^k + R_{k+1}^l)\theta^l\Delta_{r+p}^{n-l} \\
&= \sum_{1\le k\le n} R_1^k \sum_{-r\le p\le r}(-1)^p\theta^k\Delta_{r-p}^{n-k}\sum_{k<l\le n}\theta^l\Delta_{r+p}^{n-l} + (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta_{2r+1}^{n-k} \\
&+ \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta_{r-p}^{n-k}\theta^k\Delta_{r+p+1}^{n-k} \\
&= \sum_{1\le k\le n} R_1^k\theta^k Z_r(n-k) + \sum_{1\le k\le n}\left[\sum_{-r\le p\le r-1}(-1)^p\theta^k\Delta_{r-p}^{n-k}\theta^k\Delta_{r+p+1}^{n-k}\right],
\end{aligned}
$$

where for the last equality, we have used the expression for $Z_r(n)$ given by (22) and simplified the term corresponding to $p = r$ in the last sum. The terms between brackets are 0. Together with (26), we arrive at :

$$
N_r(n) = \sum_{1\le k\le n} R_1^k\left[\theta^k K_r(n-k) + 2\sum_{k<l\le n}\theta^l K_r(n-l)\right]. \qquad (27)
$$

This completes the proof of *iv*).

6 End of the proof of the theorem

6.1 Preliminaries

Let us begin with a lemma on the regular variation of φ^{-1} and φ_+^{-1}.

Lemma 6.1 *i) For all $C \geq 1$, there exists a constant $K > 0$ so that for large x :*

$$K\varphi^{-1}(Cx) \leq \varphi^{-1}(x) \leq \varphi^{-1}(Cx) \text{ and } K\varphi_+^{-1}(Cx) \leq \varphi_+^{-1}(x) \leq \varphi_+^{-1}(Cx).$$

ii) There is a constant $c > 0$ so that for all n : $\varphi(n+1) \leq c\varphi(n)$ and $\varphi_+(n+1) \leq c\varphi_+(n)$.

Proof of the lemma :
$i)$ Functions are increasing, giving the second inequalities. For the first ones, consider the case of φ, that of φ_+ being similar. It is enough to prove the result for $C = \sqrt{2}$. Indeed, if $\varphi^{-1}(x) \geq K\varphi^{-1}(\sqrt{2}x)$ and now $C \geq 1$ is arbitrary, choose m so that $(\sqrt{2})^m \geq C$. Then $\varphi^{-1}(x) \geq K^m\varphi^{-1}((\sqrt{2})^m x) \geq K^m\varphi^{-1}(Cx)$. Now :

$$\begin{aligned}
\sum_{1\leq k\leq l\leq 2n} (R_k^l)^2 &\geq \sum_{1\leq k\leq l\leq n} (R_k^l)^2 + \sum_{1\leq k\leq n<l\leq 2n} (R_k^n + R_{n+1}^l)^2 \\
&\geq \sum_{1\leq k\leq l\leq n} (R_k^l)^2 + n\sum_{1\leq k\leq n} (R_k^n)^2 + n\sum_{n<l\leq 2n} (R_{n+1}^l)^2 + 2\sum_{1\leq k\leq n} R_k^n \sum_{n<l\leq 2n} R_{n+1}^l.
\end{aligned}$$

As a result :

$$\begin{aligned}
\sum_{1\leq k\leq l\leq 2n} (R_k^l)^2 \geq\ & \sum_{1\leq k\leq l\leq n} (R_k^l)^2 + n\sum_{1\leq k\leq n} (R_k^n)^2 - \left(\sum_{1\leq k\leq n} R_k^n\right)^2 \\
&+ n\sum_{n<l\leq 2n} (R_{n+1}^l)^2 - \left(\sum_{n+1\leq l\leq 2n} R_{n+1}^l\right)^2 + \left(\sum_{1\leq k\leq n} R_k^n + \sum_{n<l\leq 2n} R_{n+1}^l\right)^2.
\end{aligned}$$

Let us now compute $n\sum_{1\leq k\leq n}(R_k^n)^2 - \left(\sum_{1\leq k\leq n} R_k^n\right)^2$. It equals :

$$\begin{aligned}
(n-1)\sum_{1\leq k\leq n} (R_k^n)^2 - 2\sum_{1\leq k<k'\leq n} R_k^n R_{k'}^n &= (n-1)\sum_{1\leq k\leq n} (R_k^n)^2 - 2\sum_{1\leq k\leq k'\leq n} R_k^n R_{k'+1}^n \\
&= (n-1)\sum_{1\leq k\leq n} (R_k^n)^2 + \sum_{1\leq k\leq k'\leq n} (R_k^{k'})^2 \\
&- \sum_{1\leq k\leq n} (R_k^n)^2(n-k+1) - \sum_{1\leq k\leq n} (R_{k+1}^n)^2 k.
\end{aligned}$$

Remark that $\sum_{1\leq k\leq n}(R_k^n)^2(n-k+1) + \sum_{1\leq k\leq n}(R_{k+1}^n)^2 k = n\sum_{1\leq k\leq n}(R_k^n)^2$. Hence :

$$n\sum_{1\leq k\leq n} (R_k^n)^2 - \left(\sum_{1\leq k\leq n} R_k^n\right)^2 = \sum_{1\leq k\leq l\leq n-1} (R_k^l)^2.$$

Similarly :

$$n\sum_{n<l\leq 2n} (R_{n+1}^l)^2 - \left(\sum_{n+1\leq l\leq 2n} R_{n+1}^l\right)^2 = \sum_{n+2\leq k\leq l\leq 2n} (R_k^l)^2 \geq 0.$$

Finally, this furnishes :

$$\sum_{1\le k\le l\le 2n}(R_k^l)^2 \ge \sum_{1\le k\le l\le n}(R_k^l)^2+\sum_{1\le k\le l\le n-1}(R_k^l)^2.$$

As a result :

$$\varphi(4n)^2=16n^2+\sum_{1\le k\le l\le 4n}(R_k^l)^2\ge 2\left(n^2+\sum_{1\le k\le l\le n}(R_k^l)^2\right)=2\varphi(n)^2.$$

This concludes the proof of this item.

ii) Simply remark that :

$$\begin{aligned}\sum_{1\le k\le l\le n+1}(R_k^l)^2 &\le \sum_{1\le k\le l\le n}(R_k^l)^2+2\sum_{1\le k\le n}(R_k^n)^2+2\sum_{1\le k\le n+1}\rho_{n+1}^2\\ &\le 3\sum_{1\le k\le l\le n}(R_k^l)^2+2(n+1)\|\rho\|_\infty^2\le C\varphi_+^2(n).\end{aligned}$$

The case of $\sum_{1\le k\le l\le n+1}(R_{-l}^{-k})^2$ is treated similarly, furnishing the result for φ_+. The case of φ is similar, remarking that $(R_{-n-1}^{n+1})^2\le cn^2$.

□

Recall from (13) that $\chi_D(t)=(g_+(t)+g_-(t))/2+O(t)$. We have $g_+(t)=\lim_{n\to+\infty}A_n^+(t)/B_n^+(t)$ and $g_-(t)=\lim_{n\to+\infty}A_n^-(t)/B_n^-(t)$, where we restore the dependency with respect to g_+ and g_-. We will apply the Kesten-Spitzer theorem, recalled in (2), using the equality :

$$\mathrm{Re}\left(\frac{1}{1-\chi_D}\right)=\frac{\mathrm{Re}(1-\chi_D)}{|1-\chi_D|^2}.$$

It will also be important to remember Proposition 4.1, furnishing $\mathrm{Re}(1-\chi_D)(t)\ge\delta^2 t$.

Proposition 6.2 *For some constant $K\ge 1$, which can be made explicit in terms of $\|\rho\|_\infty$:*

$$\frac{1}{K\varphi_+^{-1}(1/t)}\le Re(1-\chi_D(t))\le\frac{K}{\varphi_+^{-1}(1/t)}\ \text{ and }\ \frac{1}{K\varphi^{-1}(1/t)}\le|1-\chi_D(t)|\le\frac{K}{\varphi^{-1}(1/t)}.$$

Proposition 6.2 is proved in the next three sections. The theorem now follows from the proposition. Indeed, using the Kesten-Spitzer theorem, the random walk is recurrent if and only if :

$$\int_0^\eta\frac{(\varphi^{-1}(1/t))^2}{\varphi_+^{-1}(1/t)}\,dt=+\infty.$$

Cutting the above integral with respect to the intervals $[1/(n+1),1/n]$ of length $1/(n(n+1))\in(n^{-2}/2,n^{-2})$ and using that $\varphi^{-1}(1/t)$ (resp. $\varphi_+^{-1}(1/t)$) has order exactly $\varphi^{-1}(n)$ (resp. $\varphi_+^{-1}(n)$) on such an interval, by lemma 6.1, we obtain the formulation given in the statement of the theorem.

6.2 Control of the numerator

Fix $t>0$ and n. We have $g_+(t)=A_n^+(t)/B_n^+(t)+\sum_{k>n}(1/(B_k^+B_{k-1}^+))$ and similarly for g_-. This furnishes the equalities :

$$\begin{aligned}2\mathrm{Re}(1-\chi_D) &= \mathrm{Re}\left(1-\frac{A_n^+}{B_n^+}+1-\frac{A_n^-}{B_n^-}-\sum_{k>n}\left(\frac{1}{B_k^+B_{k-1}^+}+\frac{1}{B_k^-B_{k-1}^-}\right)\right)+O(t)\\ &= \frac{\mathrm{Re}[(B_n^+-A_n^+)\overline{B_n^+}]}{|B_n^+|^2}+\frac{\mathrm{Re}[(B_n^--A_n^-)\overline{B_n^-}]}{|B_n^-|^2}\\ &- \mathrm{Re}\sum_{k>n}\left(\frac{1}{B_k^+B_{k-1}^+}+\frac{1}{B_k^-B_{k-1}^-}\right)+O(t).\end{aligned}$$

1) Towards proving a lower-bound, we use Proposition 4.2 and 5.1. We get :

$$\begin{aligned}
2\mathrm{Re}(1-\chi_D) &\geq \frac{\mathrm{Re}[(B_n^+ - A_n^+)\overline{B_n^+}] - (n+1)}{|B_n^+|^2} + \frac{\mathrm{Re}[(B_n^- - A_n^-)\overline{B_n^-}] - (n+1)}{|B_n^-|^2} + O(t) \\
&\geq \frac{1}{n+1}\left(\frac{\sum_{1\leq r\leq n} t^{2r}\frac{M_r^+(n)}{n+1}}{1+\sum_{1\leq r\leq n} t^{2r}\frac{L_r^+(n)}{(n+1)^2}} + \frac{\sum_{1\leq r\leq n} t^{2r}\frac{M_r^-(n)}{n+1}}{1+\sum_{1\leq r\leq n} t^{2r}\frac{L_r^-(n)}{(n+1)^2}}\right) + O(t).
\end{aligned}$$

Remark that $L_r^{\pm}(n) \leq 2(n+1)M_r^{\pm}(n)$ and next use that $x \longmapsto x/(1+2x)$ is increasing to obtain, for some constant $C>0$:

$$\begin{aligned}
2\mathrm{Re}(1-\chi_D) &\geq \frac{1}{n+1}\left(\frac{\sum_{1\leq r\leq n} t^{2r}\frac{M_r^+(n)}{n+1}}{1+2\sum_{1\leq r\leq n} t^{2r}\frac{M_r^+(n)}{(n+1)^2}} + \frac{\sum_{1\leq r\leq n} t^{2r}\frac{M_r^-(n)}{n+1}}{1+2\sum_{1\leq r\leq n} t^{2r}\frac{M_r^-(n)}{(n+1)^2}}\right) - Ct. \\
&\geq \frac{1}{n+1}\left(\frac{t^2\frac{M_1^+(n)}{n+1}}{1+2t^2\frac{M_1^+(n)}{(n+1)^2}} + \frac{t^2\frac{M_1^-(n)}{n+1}}{1+2t^2\frac{M_1^-(n)}{(n+1)^2}}\right) - Ct.
\end{aligned}$$

Choose now n so that $n/2 = \varphi_+^{-1}(1/t)$. For some constant $c_0>0$:

$$t^2 \geq \frac{c_0}{\varphi_+^2(n/2)} = \frac{c_0}{n^2/4 + \sum_{1\leq k\leq l\leq n/2}(R_k^l)^2 + \sum_{1\leq k\leq l\leq n/2}(R_{-l}^{-k})^2}.$$

We may assume that $\sum_{1\leq k\leq l\leq n/2}(R_k^l)^2 \geq \sum_{1\leq k\leq l\leq n/2}(R_{-l}^{-k})^2$. Next :

$$\frac{M_1^+(n)}{n+1} = 2\sum_{1\leq k\leq l\leq n}(R_k^l)^2\left(1-\frac{l+1/2}{n+1}\right) \geq \sum_{1\leq k\leq l\leq n/2}(R_k^l)^2.$$

As a result :

$$t^2\frac{M_1^+(n)}{n+1} \geq c_0\frac{\sum_{1\leq k\leq l\leq n/2}(R_k^l)^2}{n^2/4 + 2\sum_{1\leq k\leq l\leq n/2}(R_k^l)^2}.$$

– Case 1 : $\sum_{1\leq k\leq l\leq n/2}(R_k^l)^2 \geq n^2$. Then $t^2M_1^+(n)/(n+1) \geq c_0/4$ and this implies that :

$$2\mathrm{Re}(1-\chi_D(t)) \geq \frac{c_0/4}{(n+1)(1+c_0/2)} - Ct = \frac{c_1}{n+1} - Ct.$$

If $t < c_1/(2C(n+1))$, then $2\mathrm{Re}(1-\chi_D(t)) \geq c_1/(2(n+1))$. On the contrary, if $t \geq c_1/(2C(n+1))$, then recall that $\mathrm{Re}(1-\chi_D(t)) \geq \delta^2 t$. This gives $\mathrm{Re}(1-\chi_D(t)) \geq \delta^2 c_1/(2C(n+1))$.

– Case 2 : $\sum_{1\leq k\leq l\leq n/2}(R_k^l)^2 < n^2$. We obtain $t^2 \geq c_0/(4n^2)$ and conclude that $\mathrm{Re}(1-\chi_D(t)) \geq \delta^2\sqrt{c_0}/(2n)$. This completes the proof of the lower bound.

2) We now prove the other direction. For some constant $C>0$, using Proposition (4.2) and that $|B_n^{\pm}| \geq n+1$ we have :

$$\begin{aligned}
2\mathrm{Re}(1-\chi_D(t)) &\leq \frac{\mathrm{Re}[(B_n^+ - A_n^+)\overline{B_n^+}]}{|B_n^+|^2} + \frac{\mathrm{Re}[(B_n^- - A_n^-)\overline{B_n^-}]}{|B_n^-|^2} + \frac{1}{n+1} + \frac{1}{n+1} + Ct \\
&\leq \frac{1}{n+1}\left(1 + \sum_{1\leq r\leq n} t^{2r}\frac{M_r^+(n)}{n+1} + 1 + \sum_{1\leq r\leq n} t^{2r}\frac{M_r^-(n)}{n+1} + 2\right) + Ct.
\end{aligned}$$

Using Lemma 6.1, choose now n so that $n = \varphi_+^{-1}(1/(\kappa t))$, for some constant $\kappa \geq 1$ so that :

$$t^2 \leq \left(\frac{1}{4}\right) \frac{1}{n^2 + \sum_{1\leq k\leq l\leq n}(R_k^l)^2 + \sum_{1\leq k\leq l\leq n}(R_{-l}^{-k})^2}.$$

From $\frac{M_r^+(n)}{n+1} \leq 2\left(2\sum_{1\leq k\leq l\leq n}(R_k^l)^2\right)^r$ and the symmetric inequality for $\frac{M_r^-(n)}{n+1}$ we obtain :

$$2\mathrm{Re}(1-\chi_D(t)) \leq \frac{1}{n+1}\left(1 + \sum_{1\leq r\leq n} 2^{-r} + 1 + \sum_{1\leq r\leq n} 2^{-r} + 2\right) + Ct \leq \frac{6}{n+1} + Ct.$$

As $t \leq 1/(\kappa n)$, for some constant $K > 0$, we have $\mathrm{Re}(1-\chi_D(t)) \leq K/n$. We conclude by Lemma 6.1, saying that n has the same order as $\varphi_+^{-1}(t)$.

6.3 Denominator

In a first step, let us compute :

$$\begin{aligned}\left|1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-}\right|^2 = \frac{|B_n^+ - A_n^+|^2}{|B_n^+|^2} + \frac{|B_n^- - A_n^-|^2}{|B_n^-|^2} &+ 2\frac{\mathrm{Re}[(B_n^+ - A_n^+)\overline{B_n^+}]\mathrm{Re}[(B_n^- - A_n^-)\overline{B_n^-}]}{|B_n^+ B_n^-|^2} \\ &+ 2\frac{\mathrm{Im}[(B_n^+ - A_n^+)\overline{B_n^+}]\mathrm{Im}[(B_n^- - A_n^-)\overline{B_n^-}]}{|B_n^+ B_n^-|^2}.\end{aligned}$$

Putting all terms with $|B_n^+|^2|B_n^-|^2$ as denominator, call W_n the numerator :

$$\begin{aligned}W_n = |B_n^+ - A_n^+|^2|B_n^-|^2 + |B_n^- - A_n^-|^2|B_n^+|^2 &+ 2\mathrm{Re}[(B_n^+ - A_n^+)\overline{B_n^+}]\mathrm{Re}[(B_n^- - A_n^-)\overline{B_n^-}] \\ &+ 2\mathrm{Im}[(B_n^+ - A_n^+)\overline{B_n^+}]\mathrm{Im}[(B_n^- - A_n^-)\overline{B_n^-}].\end{aligned}$$

By proposition 5.1 we get :

$$\begin{aligned}W_n &= \sum_{\substack{0\leq r\leq n\\0\leq r'\leq n}} t^{2r+2r'}\left(K_r^+(n)L_{r'}^-(n) + L_r^+(n)K_{r'}^-(n) + 2M_r^+(n)M_{r'}^-(n)\right) \\ &+ 2\sum_{\substack{0\leq r\leq n-1\\0\leq r'\leq n-1}} t^{2(r+r'+1)}N_r^+(n)N_{r'}^-(n).\end{aligned}$$

Consequently :

$$\begin{aligned}W_n &= \sum_{0\leq m\leq 2n} t^{2m}\sum_{0\leq r\leq m}\left(K_r^+(n)L_{m-r}^-(n) + L_r^+(n)K_{m-r}^-(n) + 2M_r^+(n)M_{m-r}^-(n)\right) \\ &+ 2\sum_{1\leq m\leq 2n-1} t^{2m}\sum_{0\leq r\leq m-1} N_r^+(n)N_{m-1-r}^-(n).\end{aligned}$$

We now focus on $2N_r^+(n)N_{m-1-r}^-(n)$. Using (27) and setting $(k,k)=0$ and $(k,l)=1$ for $k<l$:

$$2N_r^+(n)N_{m-1-r}^-(n) = \sum_{\substack{1\leq k\leq l\leq n\\1\leq k'\leq l'\leq n}} 2R_1^k R_{-k'}^{-1} 2^{(k,l)+(k',l')}\theta^l K_r^+(n-l)\theta^{-l'}K_{m-1-r}^-(n-l').$$

Write $2R_1^kR_{-k'}^{-1} = (R_{-k'}^{-1} + R_1^k)^2 - (R_1^k)^2 - (R_{-k'}^{-1})^2$. Then $2N_r^+(n)N_{m-1-r}^-(n)$ equals :

$$\sum_{\substack{1\le k\le l\le n\\ 1\le k'\le l'\le n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')}\theta^l K_r^+(n-l)\theta^{-l'} K_{m-1-r}^-(n-l')$$
$$- \left(\sum_{1\le k\le l\le n} (R_1^k)^2 2^{(k,l)}\theta^l K_r^+(n-l)\right)\left(\sum_{1\le k'\le l'\le n} 2^{(k',l')}\theta^{-l'} K_{m-1-r}^-(n-l')\right)$$
$$- \left(\sum_{1\le k\le l\le n} 2^{(k,l)}\theta^l K_r^+(n-l)\right)\left(\sum_{1\le k'\le l'\le n} (R_{-k'}^{-1})^2 2^{(k',l')}\theta^{-l'} K_{m-1-r}^-(n-l')\right).$$

Due to (20), the latter can be rewritten as :

$$\sum_{\substack{1\le k\le l\le n\\ 1\le k'\le l'\le n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')}\theta^l K_r^+(n-l)\theta^{-l'} K_{m-1-r}^-(n-l')$$
$$- K_{r+1}^+(n)\left(\sum_{1\le l\le n} (2l-1)\theta^{-l} K_{m-1-r}^-(n-l)\right) - K_{m-r}^-(n)\left(\sum_{1\le k\le n} (2k-1)\theta^k K_r^+(n-k)\right).$$

Using (24) and the definitions of K_r^+ and M_r^+ given in Proposition 5.1, we have :

$$L_r^+(n) = \sum_{0\le k\le n} (2k+1)\theta^k K_r^+(n-k) = 2M_r^+(n) - K_r^+(n) + \sum_{1\le k\le n} (2k-1)\theta^k K_r^+(n-k).$$

Using also a similar expression for L_{m-r-1}^-, we get :

$$\begin{aligned} 2N_r^+(n)N_{m-1-r}^-(n) &= \sum_{\substack{1\le k\le l\le n\\ 1\le k'\le l'\le n}} (R_1^k + R_{-k'}^{-1})^2 2^{(k,l)+(k',l')}\theta^l K_r^+(n-l)\theta^{-l'} K_{m-1-r}^-(n-l') \\ &- K_{r+1}^+(n)\left(L_{m-r-1}^-(n) - 2M_{m-r-1}^-(n) + K_{m-r-1}^-(n)\right) \\ &- K_{m-r}^-(n)\left(L_r^+(n) - 2M_r^+(n) + K_r^+(n)\right). \end{aligned}$$

Denote by $Q_{r,m-r-1}(n)$ the first quantity on the right-hand side. Finally :

$$\begin{aligned} W_n &= \sum_{0\le m\le 2n} t^{2m} \sum_{0\le r\le m} \left(K_r^+(n)L_{m-r}^-(n) + L_r^+(n)K_{m-r}^-(n) + 2M_r^+(n)M_{m-r}^-(n)\right) \\ &+ \sum_{1\le m\le 2n-1} t^{2m} \sum_{0\le r\le m-1} Q_{r,m-r-1}(n) \\ &- \sum_{1\le m\le 2n-1} t^{2m} \sum_{0\le r\le m-1} K_{r+1}^+(n)\left(L_{m-r-1}^-(n) - 2M_{m-r-1}^-(n) + K_{m-r-1}^-(n)\right) \\ &- \sum_{1\le m\le 2n-1} t^{2m} \sum_{0\le r\le m-1} K_{m-r}^-(n)\left(L_r^+(n) - 2M_r^+(n) + K_r^+(n)\right). \end{aligned}$$

As a result, separating quantities corresponding to $m = 0$ and $m = 2n$ in the first term in the right-hand side and simplifying the $K_\cdot^+ L_\cdot^-$ and $K_\cdot^- L_\cdot^+$ with the corresponding terms of lines 3 and 4, we obtain the equality :

$$
\begin{aligned}
W_n &= 4(n+1)^2 + t^{4n}(K_n^+(n)L_n^-(n) + K_n^-(n)L_n^+(n) + 2M_n^+(n)M_n^-(n)) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\left(L_m^-(n) + L_m^+(n)\right) + 2\sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m} M_r^+(n)M_{m-r}^-(n) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m-1} Q_{r,m-r-1}(n) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m-1} K_{r+1}^+(n)\left(2M_{m-r-1}^-(n) - K_{m-r-1}^-(n)\right) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m-1} K_{m-r}^-(n)\left(2M_r^+(n) - K_r^+(n)\right).
\end{aligned}
$$

One notices that $(n+1)^2 + \sum_{1\le m\le 2n-1} t^{2m}L_m^+(n) = |B_n^+|^2$ and similarly for $|B_n^-|^2$. Also, using directly the expressions given in Proposition 5.1 :

$$
2M_r^+(n) - K_r^+(n) = \sum_{0\le k\le n} 2^{(0,k)}\theta^k K_r^+(n-k)
$$

and similarly $2M_{m-r-1}^-(n) - K_{m-r-1}^-(n) = \sum_{0\le k\le n} 2^{(0,k)}\theta^{-k}K_{m-r-1}^-(n-k)$. As $K_n^+(n)L_n^-(n) = K_n^-(n)L_n^+(n) = M_n^+(n)M_n^-(n) = \prod_{1\le k\le n}(\rho_k\rho_{-k})^2$ and recognizing expressions for $|B_n^+|^2$ and $|B_n^-|^2$ given by proposition 5.1, we deduce :

$$
\begin{aligned}
W_n &= 2(n+1)^2 + 4t^{4n}\prod_{1\le k\le n}(\rho_k\rho_{-k})^2 + |B_n^+|^2 + |B_n^-|^2 \\
&+ 2\sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m} M_r^+(n)M_{m-r}^-(n) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{0\le r\le m-1}\left(Q_{r,m-r-1}(n) + K_{m-r}^-(n)\sum_{0\le k\le n} 2^{(0,k)}\theta^k K_r^+(n-k)\right) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{1\le r\le m} K_r^+(n)\sum_{0\le k\le n} 2^{(0,k)}\theta^{-k}K_{m-r}^-(n-k).
\end{aligned}
$$

In order to regroup term of line two and the first term of line three, introduce :

$$
S_m(n) = \sum^{*}_{\substack{-n\le k_1\le l_1<\cdots<k_m\le l_m\le n \\ k_i\neq 0, l_i\neq 0}} (R_{k_1}^{l_1}\cdots R_{k_m}^{l_m})^2 2^{N((k_i,l_i)_{1\le i\le m})}\alpha_n(k_1,l_m),
$$

where $\sum^*$ means that if for some i we have $k_il_i < 0$, then we remove the term ρ_0 in $R_{k_i}^{l_i}$ and $\alpha_n(k_1,l_m) = (1+2(n-l_m))(1+2(n+k_1))$ if $k_1<0$, $l_m>0$, $\alpha_n(k_1,l_m) = (1+2(n-l_m))(n+1)$ if $k_1>0$, $l_m>0$, and $\alpha_n(k_1,l_m) = (1+2(n+k_1))(n+1)$ if $k_1<0$, $l_m<0$.

Observing that $S_{2n}(n) = 2\prod_{1\le k\le n}(\rho_k\rho_{-k})^2$, we conclude this preliminary computation with :

$$
\left|1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-}\right|^2 = \frac{W_n}{|B_n^+|^2|B_n^-|^2}, \tag{28}
$$

where :

$$
\begin{aligned}
W_n &= 2(n+1)^2 + |B_n^+|^2 + |B_n^-|^2 + 2t^{4n}\prod_{1\le k\le n}(\rho_k\rho_{-k})^2 + \sum_{1\le m\le 2n} t^{2m}S_m(n) \\
&+ \sum_{1\le m\le 2n-1} t^{2m}\sum_{1\le r\le m}\sum_{0\le k\le n} 2^{(0,k)}\left(K_r^+(n)\theta^{-k}K_{m-r}^-(n-k) + K_r^-(n)\theta^k K_{m-r}^+(n-k)\right).
\end{aligned}
$$

6.4 Control of the denominator

1) For the moment, let $t > 0$ and n be arbitrary. Towards proving a lower bound, for some constant $C > 0$, we have, using Proposition 4.2 :

$$\begin{aligned} 2|1-\chi_D| &\geq \left|1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-}\right| - \sum_{k>n}\left(\frac{1}{|B_k^+ B_{k-1}^+|} + \frac{1}{|B_k^- B_{k-1}^-|}\right) - Ct \\ &\geq \left|1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-}\right| - \frac{(n+1)(|B_n^+|/|B_n^-| + |B_n^-|/|B_n^+|)}{|B_n^+||B_n^-|} - Ct. \end{aligned}$$

Hence, by (28) :

$$2|1-\chi_D| \quad \geq \quad \frac{\sqrt{W_n} - (n+1)\left(\frac{|B_n^+|}{|B_n^-|} + \frac{|B_n^-|}{|B_n^+|}\right)}{|B_n^+||B_n^-|} - Ct.$$

Remark now that, still using Proposition 4.2 :

$$\begin{aligned} (n+1)^2\left(\frac{|B_n^+|}{|B_n^-|} + \frac{|B_n^-|}{|B_n^+|}\right)^2 &= (n+1)^2\frac{|B_n^-|^2}{|B_n^+|^2} + (n+1)^2\frac{|B_n^+|^2}{|B_n^-|^2} + 2(n+1)^2 \\ &\leq |B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2 \leq W_n. \end{aligned}$$

This gives, via (28) :

$$\begin{aligned} 2|1-\chi_D| &\geq \frac{\sqrt{W_n} - \sqrt{|B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2}}{|B_n^+||B_n^-|} - Ct \\ &\geq \frac{W_n - (|B_n^+|^2 + |B_n^-|^2 + 2(n+1)^2)}{2\sqrt{W_n}|B_n^+||B_n^-|} - Ct \geq \frac{\sum_{1\leq m\leq 2n} t^{2m} S_m(n)}{2\sqrt{W_n}|B_n^+||B_n^-|} - Ct. \end{aligned}$$

We obtain, observing that $W_n \leq 16(n+1)^2 + 16\sum_{1\leq m\leq 2n} t^{2m} S_m(n)$:

$$\begin{aligned} 2|1-\chi_D| &\geq \left(\frac{1}{n+1}\right)\frac{\sum_{1\leq m\leq 2n} t^{2m}\frac{S_m(n)}{(n+1)^2}}{8\sqrt{1+\sum_{1\leq m\leq 2n} t^{2m}\frac{S_m(n)}{(n+1)^2}}} \\ &\times \frac{1}{\sqrt{1+\sum_{1\leq r\leq n} t^{2r}\frac{L_r^+(n)}{(n+1)^2}}\sqrt{1+\sum_{1\leq r\leq n} t^{2r}\frac{L_r^-(n)}{(n+1)^2}}} - Ct. \end{aligned}$$

Remark next that :

$$\begin{aligned} &\left(1 + \sum_{1\leq r\leq n} t^{2r}\frac{L_r^+(n)}{(n+1)^2}\right)\left(1 + \sum_{1\leq r\leq n} t^{2r}\frac{L_r^-(n)}{(n+1)^2}\right) \\ = \;& 1 + \sum_{1\leq m\leq 2n} t^{2m}\left(\sum_{1\leq r\leq m-r}\frac{L_r^+(n)L_{m-r}^-(n)}{(n+1)^4}\right) \leq 4\left(1 + \sum_{1\leq m\leq 2n} t^{2m}\frac{S_m(n)}{(n+1)^2}\right). \end{aligned}$$

As a result we obtain :

$$2|1-\chi_D| \quad \geq \quad \left(\frac{1}{16(n+1)}\right)\frac{\sum_{1\leq m\leq 2n} t^{2m} S_m(n)/(n+1)^2}{1+\sum_{1\leq m\leq 2n} t^{2m} S_m(n)/(n+1)^2} - Ct.$$

As the map $x \longmapsto x/(1+x)$ is increasing, this gives :

$$2|1-\chi_D| \;\; \geq \;\; \left(\frac{1}{16(n+1)}\right) \frac{t^2 S_1(n)/(n+1)^2}{1+t^2 S_1(n)/(n+1)^2} - Ct.$$

Choose now n so that $n/2 = \varphi^{-1}(1/t)$. As a result, for some constant $c_0 > 0$:

$$t^2 \geq \frac{c_0}{\varphi^2(n/2)} = \frac{c_0}{n^2/4 + \sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2}. \tag{29}$$

Remark also that :

$$\frac{S_1(n)}{(n+1)^2} \geq \sum_{-n/2 \leq k \leq l \leq n/2, kl \neq 0}^{*} (R_k^l)^2.$$

– Case 1 : $\sum^*_{-n/2 \leq k \leq l \leq n/2, kl \neq 0} (R_k^l)^2 \geq n^2$. Observing that $(R_0^l)^2 \leq 2(R_1^l)^2 + C$, one easily gets that $\sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2 \leq c_1 \sum^*_{-n/2 \leq k \leq l \leq n/2, kl \neq 0} (R_k^l)^2 + c_2 n^2$. For generic constants $c_i > 0$:

$$t^2 \frac{S_1(n)}{(n+1)^2} \;\; \geq \;\; c_0 \frac{\sum^*_{-n/2 \leq k \leq l \leq n/2, kl \neq 0} (R_k^l)^2}{n^2/4 + \sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2} \geq c_0 \frac{\sum^*_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2}{c_3 n^2 + c_4 \sum^*_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2} \geq c_5 > 0.$$

Hence :

$$2|1-\chi_D| \;\; \geq \;\; \left(\frac{1}{16(n+1)}\right) \frac{c_5}{1+c_5} - Ct \geq c_6/n - Ct.$$

As a result, if $t \leq c_6/(2Cn)$ we get $2|1-\chi_D| \geq c_6/(2n)$. If $t > c_6/(2Cn)$, then as before, using Proposition 4.1, $|1-\chi_D| \geq \delta^2 t \geq \delta^2 c_6/(2Cn)$.

– Case 2 : $\sum^*_{-n/2 \leq k \leq l \leq n/2, kl \neq 0} (R_k^l)^2 < n^2$. This furnishes $\sum_{-n/2 \leq k \leq l \leq n/2} (R_k^l)^2 < c_7 n^2$ and next from (29), $t^2 \geq c_8/n^2$. Hence $|1-\chi_D| \geq \delta^2 t \geq \delta^2 \sqrt{c_8}/n$. This ends the proof of the lower bound.

2) We consider the other direction. Let $t > 0$ and n be arbitrary. For some constant $C > 0$, using Proposition 4.2 :

$$\begin{aligned} 2|1-\chi_D| \;\; &\leq \;\; \left| 1 - \frac{A_n^+}{B_n^+} + 1 - \frac{A_n^-}{B_n^-} \right| + \frac{1}{n+1} + \frac{1}{n+1} + Ct \\ &\leq \;\; \frac{\sqrt{W_n}}{|B_n^+||B_n^-|} + \frac{2}{n+1} + Ct \leq \frac{\sqrt{W_n}}{(n+1)^2} + \frac{2}{n+1} + Ct. \end{aligned}$$

Remark also that :

$$\begin{aligned} W_n \;\; &\leq \;\; 16(n+1)^2 \left(1 + \sum_{1 \leq m \leq 2n} t^{2m} S_m(n)/(n+1)^2 \right) \\ &\leq \;\; 32(n+1)^2 \left(1 + \sum_{1 \leq m \leq 2n} t^{2m} 4^m \left(\sum_{-n \leq k \leq l \leq n} (R_k^l)^2 \right)^m \right). \end{aligned}$$

Using Lemma 6.1, choose now n so that $n = \varphi^{-1}(1/(\kappa t))$, for some constant $\kappa \geq 1$ so that :

$$t^2 \leq \left(\frac{1}{8}\right) \frac{1}{n^2 + \sum_{-n \leq k \leq l \leq n} (R_k^l)^2}.$$

This gives $W_n \leq 32(n+1)^2 \left(1 + \sum_{1 \leq m \leq 2n} 2^{-m}\right)$ and therefore :

$$2|1-\chi_D| \quad \leq \quad \frac{8}{n+1} + \frac{2}{n+1} + Ct.$$

As $t \leq 1/(\kappa n)$, using lemma 6.1, $|1-\chi_D| \leq c_9/\varphi^{-1}(1/t)$. This ends the proof of proposition (6.2).

7 Proof of the applications

7.1 Proof of Corollary 1.3

i) As $\varphi^{-1}(1/t) \leq \varphi_+^{-1}(1/t)$ we get $(\varphi^{-1}(1/t))^2/\varphi_+^{-1}(1/t) \leq \varphi^{-1}(1/t)$. A sufficient condition for transience is therefore :

$$\int_0^\eta \varphi^{-1}(1/t)\ dt < +\infty.$$

On the interval $[1/\varphi(n+1), 1/\varphi(n)]$, $\varphi^{-1}(1/t)$ has exactly order n, because $\varphi(n)$ and $\varphi(n+1)$ have the same order, by Lemma (6.1), *ii*). The last condition is thus equivalent to the finiteness of :

$$\sum_{n\geq 1} \int_{1/\varphi(n+1)}^{1/\varphi(n)} n\ dt = \sum_{n \geq 1} n \left(\frac{1}{\varphi(n)} - \frac{1}{\varphi(n+1)} \right) = \lim_N \sum_{1 \leq n \leq N} \frac{1}{\varphi(n)} - \frac{N}{\varphi(N+1)}.$$

As $N \leq \varphi(N+1)$, finiteness is equivalent to $\sum_{n\geq 1}(1/\varphi(n)) < +\infty$.

ii), *iii*) We now consider the symmetric case $\rho_{-n} = \rho_n$, $n \geq 1$, and the antisymmetric case $\rho_{-n} = -\rho_n$, $n \geq 1$. In both situations :

$$\varphi_+^2(n) = n^2 + 2 \sum_{1 \leq k \leq l \leq n} (R_k^l)^2.$$

Next, for some $C > 0$, we have $C^{-1}\psi(n) \leq \varphi(n) \leq C\psi(n)$, where :

$$\begin{aligned}
\psi^2(n) &= n^2 + \sum_{1\leq k \leq l \leq n} (R_k^l)^2 + \sum_{1 \leq k,l \leq n} (R_{-k}^{-1} + R_1^l)^2 \\
&= n^2 + \sum_{1\leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + 2 \left(\sum_{1 \leq k \leq n} R_{-k}^{-1} \right) \left(\sum_{1 \leq k \leq n} R_1^k \right).
\end{aligned}$$

Therefore :

$$\begin{aligned}
\psi^2(n) &= n^2 + \sum_{1\leq k \leq l \leq n} (R_k^l)^2 + n \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + \left(\sum_{1 \leq k \leq n} (R_{-k}^{-1} + R_1^k) \right)^2 \\
&- \left(\sum_{1 \leq k \leq n} R_{-k}^{-1} \right)^2 - \left(\sum_{1 \leq k \leq n} R_1^k \right)^2. \\
&= n^2 + \sum_{1\leq k \leq l \leq n} (R_k^l)^2 + (n-1) \sum_{1 \leq k \leq n} [(R_{-k}^{-1})^2 + (R_1^k)^2] + \left(\sum_{1 \leq k \leq n} (R_{-k}^{-1} + R_1^k) \right)^2 \\
&- 2 \sum_{1 \leq k < l \leq n} R_{-k}^{-1} R_{-l}^{-1} - 2 \sum_{1 \leq k < l \leq n} R_1^k R_1^l.
\end{aligned}$$

Observe that :

$$
\begin{aligned}
-\sum_{1\le k<l\le n} 2R_1^k R_1^l &= -\sum_{2\le k\le l\le n} 2R_1^{k-1}R_1^l = \sum_{2\le k\le l\le n} [(R_k^l)^2 - (R_1^{k-1})^2 - (R_1^l)^2] \\
&= \sum_{2\le k\le l\le n} (R_k^l)^2 - (n-1)\sum_{1\le k\le n} (R_1^k)^2.
\end{aligned}
$$

As a result :

$$
\psi^2(n) = n^2 + \sum_{1\le k\le l\le n} 2^{(1,k)}[(R_k^l)^2 + (R_{-l}^{-k})^2] + \left(\sum_{1\le k\le n} (R_{-k}^{-1} + R_1^k)\right)^2.
$$

– In the antisymmetric case, we have $\varphi(n) \le C\psi(n) \le C\sqrt{2}\,\varphi_+(n)$. Hence the random walk is transient if and only if :

$$
\int_0^\eta \varphi_+^{-1}(1/t)\ dt < +\infty.
$$

Proceeding as for i), this last property is equivalent to $\sum_{n\ge1}(1/\varphi_+(n)) < +\infty$.

– In the symmetric case, starting from the beginning of the computation of $\psi^2(n)$ we have :

$$
\psi^2(n) = n^2 + \sum_{1\le k\le l\le n} (R_k^l)^2 + 2n\sum_{1\le k\le n} (R_1^k)^2 + 2\left(\sum_{1\le k\le n} R_1^k\right)^2.
$$

As $(n+1)\sum_{1\le k\le n}(R_1^k)^2 = \sum_{1\le k\le l\le n}(R_k^l)^2 + \left(\sum_{1\le k\le n} R_1^k\right)^2$, we get $\psi^2(n)$ and thus $\varphi^2(n)$ has exact order :

$$
n^2 + n\sum_{1\le k\le n} (R_1^k)^2.
$$

From point i), the random walk is therefore transient if :

$$
\sum_{n\ge1} \frac{1}{\sqrt{n^2 + n\sum_{1\le k\le n}(R_1^k)^2}} < +\infty.
$$

7.2 Proof of Proposition 1.4

i) It follows from the hypothesis that both $\varphi(n)$ and $\varphi_+(n)$ are $O(n\sqrt{\log n})$. Hence, for some constant $c > 0$ we have the lower bound :

$$
\frac{1}{n^2}\frac{(\varphi^{-1}(n))^2}{\varphi_+^{-1}(n)} \ge c\frac{1}{n^2}\frac{(n/\sqrt{\log n})^2}{n} \ge \frac{c}{n\log n}.
$$

Using Theorem 1.2, we deduce recurrence.

ii) We have $\varphi_+(n) = O(n\log n)$. We next use Corollary 1.3, ii).

iii) Using Corollary 1.3, ii) and iii), it is enough to show that :

$$
\sum_{n\ge1} \frac{1}{\sqrt{n^2 + n\sum_{1\le k\le n}(R_1^k)^2}} < +\infty, \text{ but } \sum_{n\ge1} \frac{1}{\sqrt{n^2 + \sum_{1\le k\le l\le n}(R_k^l)^2}} = +\infty.
$$

First of all, making successive integration by parts :

$$
\int_1^t \log^\alpha s\ ds = t\log^\alpha t - \alpha t\log^{\alpha-1} t + \alpha(\alpha-1)t(\log^{\alpha-2} t)(1+o(1))).
$$

Write now, for some $C > 0$:

$$\sum_{1\le k\le n} (R_1^k)^2 \ge \frac{1}{2} \sum_{1\le k\le n} \log^{2\alpha} k - Cn \log^{2\alpha-2} n \ge \frac{1}{2} n(\log^{2\alpha} n)(1+o(1)).$$

The first sum thus converges. On the other hand, from the hypothesis we have $R_k^l = \log^\alpha l - \log^\alpha k + O(\log^{\alpha-1} n)$, so we deduce that :

$$\begin{aligned}
\frac{1}{2} \sum_{1\le k\le l\le n} (R_k^l)^2 &\le \sum_{1\le k<l\le n} (\log^\alpha l - \log^\alpha k)^2 + O(n^2 \log^{2\alpha-2} n) \\
&\le (n+1) \sum_{1\le k\le n} \log^{2\alpha} k - 2 \sum_{1\le k<l\le n} \log^\alpha k \log^\alpha l + O(n^2 \log^{2\alpha-2} n) \\
&\le (n+1) \sum_{1\le k\le n} \log^{2\alpha} k - \left(\sum_{1\le k\le n} \log^\alpha k \right)^2 + O(n^2 \log^{2\alpha-2} n).
\end{aligned}$$

As a result, putting $\int_2^{n+1} \log^{2\alpha} t \, dt$ in $O(n^2 \log^{2\alpha-2} n)$:

$$\frac{1}{2} \sum_{1\le k\le l\le n} (R_k^l)^2 \le n \int_2^{n+1} \log^{2\alpha} t \, dt - \left(\int_1^n \log^\alpha t \, dt \right)^2 + O(n^2 \log^{2\alpha-2} n).$$

This finally furnishes :

$$\begin{aligned}
\frac{1}{2} \sum_{1\le k\le l\le n} (R_k^l)^2 &\le n \int_1^n \log^{2\alpha} t \, dt - \left(\int_1^n \log^\alpha t \, dt \right)^2 + O(n^2 \log^{2\alpha-2} n). \\
&\le n^2 \log^{2\alpha} n \left(1 - \frac{2\alpha}{\log n} + \frac{2\alpha(2\alpha-1)}{\log^2 n}(1+o(1)) \right) \\
&- n^2 \log^{2\alpha} n \left(1 - \frac{\alpha}{\log n} + \frac{\alpha(\alpha-1)}{\log^2 n}(1+o(1)) \right)^2 + O(n^2 \log^{2\alpha-2} n) \\
&\le \alpha^2 n^2 (\log^{2\alpha-2} n)(1+o(1)) + O(n^2 \log^{2\alpha-2} n).
\end{aligned}$$

Consequently the second series diverges. This completes the proof of the proposition.

7.3 Proof of Proposition 1.5

We have $\varepsilon_{-n} = -\varepsilon_n$, $n \ge 1$. Let (q_p) be the denominators of the convergence of α. Every integer $n < q_{p+1}$ can be decomposed as $n = b_0(n) + b_1(n)q_1 + \cdots b_p(n)q_p$, with $0 \le b_k(n) \le a_{k+1} - 1$ (this is called sometimes Ostrowski's decomposition; the proof relies on the relations $q_{k+1} = a_{k+1}q_k + q_{k-1}$ and works as for the Euclidian division). As $x \longmapsto 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$ has bounded variation, the Denjoy-Koksma inequality implies that for any u and any k that $|R_u^{u+q_k}| \le C$. As a result, $|R_1^n| \le C(a_1 + \cdots + a_{p+1})$. Hence, for $n < q_{p+1}$:

$$\sum_{1\le k\le l\le n} (R_k^l)^2 \le 2n \sum_{1\le k\le n} (R_1^k)^2 \le 2n^2 (a_1 + \cdots + a_{p+1})^2.$$

In the perspective of using Corollary 1.3 *ii*), we have for some $C > 0$:

$$\sum_{n\ge1} \frac{1}{\sqrt{\sum_{1\le k\le l\le n} (R_k^l)^2}} \ge C \sum_{p\ge1} \sum_{q_p\le n<q_{p+1}} \frac{1}{n(a_1+\cdots+a_{p+1})} \ge C \sum_{p\ge1} \frac{\log(q_{p+1}/q_p)}{(a_1+\cdots+a_{p+1})}.$$

Write then :

$$\sum_{p\geq 1} \frac{\log(q_{p+1}/q_p)}{(a_1+\cdots+a_{p+1})} \geq \frac{1}{2}\sum_{p\geq 2}\left[\frac{\log(q_{p+1}/q_p)}{(a_1+\cdots+a_{p+1})}+\frac{\log(q_p/q_{p-1})}{(a_1+\cdots+a_p)}\right].$$

Recall that $q_{p+1} = a_{p+1}q_p + q_{p-1}$. We also have $q_{p+1} \geq (1+a_{p+1}a_p)q_{p-1} \geq 2q_{p-1}$. Now, if $a_{p+1} \geq 2$, then $\log(q_{p+1}/q_p) \geq \log a_{p+1} \geq \frac{\log 2}{\log 3}\log(1+a_{p+1})$. If $a_{p+1} = 1$, then :

$$\frac{\log(q_{p+1}/q_p)}{(a_1+\cdots+a_{p+1})} + \frac{\log(q_p/q_{p-1})}{(a_1+\cdots+a_p)} \geq \frac{\log(q_{p+1}/q_{p-1})}{(a_1+\cdots+a_{p+1})} \geq \frac{\log 2}{(a_1+\cdots+a_{p+1})}.$$

As $\log 2 = \log(1+a_{p+1})$, this concludes the proof of the proposition. □

7.4 Proof of Proposition 1.6

Step 1. One can always decrease $\gamma > 1$ to get $\beta > \gamma > 1$. For $k \geq 1$, let $D_k = [u(k), v(k)]$, where $u(k) = k\log^\beta k$ and $v(k) = k\log^\beta k + k$, of length k. Introduce :

$$A_n = \frac{1}{n\log^{-\beta} n}\sum_{k=1}^{n\log^{-\beta} n} 1_{\left|R_{u(k)}^{v(k)}\right| \geq \log^\beta k}.$$

There exists $\alpha > 0$ such that for n large enough, $\mathbb{E}(A_n) \geq \alpha$. Next :

$$\mathrm{var}(A_n) \leq \frac{2}{n^2\log^{-2\beta} n}\sum_{0\leq k\leq l\leq n\log^{-\beta} n}\left|\mathrm{cov}\left(1_{\left|R_{u(k)}^{v(k)}\right|\geq \log^\beta k}, 1_{\left|R_{u(l)}^{v(l)}\right|\geq\log^\beta l}\right)\right|.$$

– The previous sum restricted to indices $k \leq l \leq k + k\log^{-\gamma} k$ is less than or equal to :

$$\sum_{0\leq k\leq n\log^{-\beta} n} k\log^{-\gamma} k \leq cn^2\log^{-2\beta-\gamma} n.$$

– For indices such that $l \geq k + k\log^{-\gamma} k$, the distance between D_k and D_l is at least :

$$l\log^\beta l - k\log^\beta k - k \geq k\log^\beta k + k\log^{\beta-\gamma} k - k\log^\beta k - k \geq \frac{1}{2}k\log^{\beta-\gamma} k.$$

It results that the corresponding sum is less than or equal to :

$$\sum_{0\leq k\leq n\log^{-\beta} n} n\log^{-\beta} n\alpha((k/2)\log^{\beta-\gamma} k) = O\left(n^2\log^{-2\beta-\gamma} n\right).$$

This thus gives $\mathrm{var}(A_n) = O(\log^{-\gamma} n)$. Hence $\sum_{n\geq 1}\mathrm{var}(A_{2^n}) = O(\sum_{n\geq 1} n^{-\gamma}) < +\infty$. Therefore, almost surely, $A_{2^n} - \mathbb{E}(A_{2^n}) \to 0$ and $A_{2^n} \geq \alpha/2 > 0$ for n large enough.

Step 2. Denote by $k_1, \cdots, k_N$ the k's in $[1, n\log^{-\beta} n]$ so that $\left|R_{u(k)}^{v(k)}\right| \geq \log^\beta k$. We have :

$$N \geq \frac{\alpha}{2}n\log^{-\beta} n,$$

for large n along the sequence (2^m). For each $1 \leq r \leq N$, introduce two intervals I_r and J_r of length $\log^\beta k_r/(10(1+\|\rho\|_\infty))$ centered in $u(k_r)$ and $v(k_r)$ respectively. The length of $D_{k_r}\cup I_r\cup J_r$ is equivalent to $k_r \leq n\log^{-\beta} n$. Choose an interval $K_r \subset [1, n]$ of length $\geq n/4$ in the complementary of the latter union.

Observe now that J_r can intersect at most three $I_{r'}$ with $r' > r$. Indeed, as r' increases from the value r, the center of $I_{r'}$ makes steps of size $\geq \log^\beta k_r$. At most $k_r\log^{-\beta} k_r$ steps are necessary to cross $I_r \cup D_{k_r} \cup J_r$. At the end of the crossing, the length of the $I_{r'}$ are at most :

$$\log^\beta(k_r + \log^\beta k_r + k_r\log^{-\beta} k_r) = \log^\beta k_r + o(1).$$

In a similar way, I_r intersects at most three $J_{r'}$ with $r' < r$. Taking into account these overlaps and which k or l is in some I_r or J_r, we have :

$$\sum_{1\le r\le N}\left(\sum_{k\in I_r, l\in K_r}(R^l_k)^2+\sum_{k\in J_r, l\in K_r}(R^l_k)^2\right)\le 6\sum_{1\le k\le l\le n}(R^l_k)^2.$$

In this notation, K_r is supposed for simplicity to be on the right side of $I_r\cup D_{k_r}\cup J_r$. For each $l\in K_r$, either $|R^l_{u(r)}|\ge (1/2)\log^\beta k_r$ or $|R^l_{v(r)}|\ge (1/2)\log^\beta k_r$ and when one inequality is true, one has $|R^l_w|\ge (1/4)\log^\beta k_r$, $w\in I_r$, in the first case and $|R^l_w|\ge (1/4)\log^\beta k_r$, $w\in J_r$, in the second case. As a result, the left-hand side of the previous inequality checks, for constants $c_i>0$:

$$\ge c_1\sum_{r=1}^N n\log^\beta k_r \log^{2\beta} k_r \ge c_2 nN\log^{3\beta} N\ge c_3 n^2\log^{2\beta} n.$$

We now conclude. Cutting in dyadic blocks, for M large enough :

$$\sum_{k\ge M}\frac{1}{\varphi_+(k)}\le \sum_{n\ge n_0}\sum_{2^n\le k<2^{n+1}}\frac{1}{\varphi_+(2^n)}\le C\sum_{n\ge n_0}\frac{2^n}{2^n n^\beta}<+\infty.$$

Applying Corollary 1.3 i), we get the result.

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