Finite flowers and maximizing measures for generic Lipschitz functions on the Circle

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Abstract

Given the Circle endowed with the doubling map, we consider the problem of maximizing measures for Lipschitz functions. We provide a reduction result for the conjecture stating that in a dense open set in that space the maximizing measure is unique and supported by a closed periodic orbit. More precisely we show that it follows from a generic finite form in the Conze-Guivarc'h-Mane lemma. This unsolved point is discussed in the last section as well as other open questions.

1 Introduction

The question of *maximizing measures* is a relatively recent field that has raised from optimization questions in Ergodic Theory. A beginning general theory can be found in the works of Conze-Guivarc'h [8], Bousch-Mairesse [4] or Jenkinson [10], [11].

A natural context for this topic is topological dynamics, namely a compact metric space X with a continuous surjective transformation T. The dynamical system is then observed via the convex set $\mathcal{M}_T(X)$ of Borel T-invariant probability measures, with the natural weak-$*$ topology.

Fixing some continuous $f : X \to \mathbb{R}$, one introduces the variational problem :

$$\beta(f) = \sup\left\{\int f \, d\mu \mid \mu \in \mathcal{M}_T(X)\right\}$$

and more specifically aims at describing the measures realizing the maximum :

$$\mathcal{M}ax(f) = \left\{\mu \in \mathcal{M}_T(X) \mid \beta(f) = \int f \, d\mu\right\}.$$

As a first remark, the compactness of $\mathcal{M}_T(X)$ ensures that $\mathcal{M}ax(f)$ is non-empty. Then by desintegration it also contains ergodic measures. The problem of maximizing measures is, of course, interesting when $\mathcal{M}_T(X)$ is "large". This occurs for instance if the dynamical system satisfies an expansiveness property. In this setup numerical experiments suggest a rather striking degeneracy phenomenon, unprecisely formulated as : *most regular functions seem to have only periodic maximizing measures.* Heuristically and rather naturally, periodic measures seem to be the most economical ones.

The present paper is devoted to discussing that question. We decide to fix for the whole text the following setting, which in fact already contains the main difficulties :

Definition 1.1
Let the Circle $X = \mathbb{R}/\mathbb{Z}$ *be endowed with the map* $Tx = 2x \mod (1)$. *For any real function* f *on* X, *we use the notations* $Tf = f(T\ .\ )$ *and* $\tau f = f(\ .\ + 1/2)$.

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Introduce also our ambient functional space for the sequel.

Definition 1.2
Denote by $Lip(X)$ the space of real Lipschitz functions with the norm :

$$\|f\|_{Lip} = \|f\|_\infty + K(f), \text{ where } K(f) = \sup\left\{ \frac{|f(x)-f(y)|}{Dist(x,y)} \mid x \neq y \right\}.$$

As a first point, the validity of the "generic periodic character" of maximizing measures was supported by a consistent example due to Bousch.

Theorem 1.3 *(Bousch)* [1]
For $0 \leq t < 1$, let $f_t(x) = \cos 2\pi(x+t)$. Then for all t, $\mathcal{M}ax(f_t)$ is a singleton consisting in a Sturmian measure. Moreover this measure is periodic except for t in a zero-dimensional set.

Mention that the proof of this result is difficult in the sense that delicate numerical estimates are employed. Attempts in order to extend this result to functional spaces were first considered in $C^0(X)$, equipped with the usual supremum norm. Following from general duality arguments, the answer was in fact rather surprising.

Theorem 1.4 *(Bousch)* [2]
For a dense G_δ-set in $C^0(X)$, the maximizing measure is unique with full support.

This result recalls that a generic function in $C^0(X)$ is rather far from the usual intuition. The problem was then transfered to the family of Hölder spaces $(H^\alpha(X))_{0<\alpha\leq 1}$, endowed with their natural topology. In the Lipschitz setting, which we restrict to, it can be formulated as :

Conjecture 1.5
For at least a dense open set in $Lip(X)$, the maximizing measure is unique and supported by a closed periodic orbit.

This can be stated within a more general framework and for the $(H^\alpha(X))_{0<\alpha\leq 1}$. Let us detail known partial results. A weak form of the conjecture was proved by Contreras-Lopes-Thieullen [7] for Circle maps, namely for an infinite-codimensional subspace of each $H^\alpha(X)$. A similar result in the context of Anosov diffeomorphisms was shown by Lopes-Thieullen [12].

For hyperbolic systems and using fine orbital analysis, Hunt-Yuan [9] have shown that maximizing periodic measures are C^1-stably maximizing, whereas any function having a diffusive maximizing measure can be C^1-perturbed so that this measure is not maximizing. However the point would be to show that periodic maximizing measures are possible to obtain by perturbation, but this remains open.

Mention that the conjecture was established in a symbolic setting by Bousch [2] for a functional space "adapted" to the problem of Maximizing Measures, namely the space of Walters' functions.

Back to our setup and the space $Lip(X)$ and as mentioned by various authors, a fundamental difference between a regular functions space such as $Lip(X)$ and $C^0(X)$ is the existence of some reduction lemma implying for any $f \in Lip(X)$ that the measures of $\mathcal{M}ax(f)$ are characterized via their support. A proof can be found in [8], [4], [10] or [11]. This can be stated as follows :

Lemma 1.6 *(Conze-Guivarc'h-Mane)*
Let $f \in Lip(X)$. Then there exist φ and r in $Lip(X)$ such that f can be decomposed as the sum $f = \beta(f) + (T\varphi - \varphi) - r$, with $r \geq 0$, $r.\tau r = 0$.

The previous assertion is explained by the property that $r.\tau r$ is identically 0, which ensures that $\cap_{n\geq 0} T^{-n}(r^{-1}\{0\}) \neq \emptyset$. Thus $r^{-1}\{0\}$ carries elements of $\mathcal{M}_T(X)$ and since $r \geq 0$, one gets that $\mu \in \mathcal{M}ax(f) \Leftrightarrow \mathrm{Supp}(\mu) \subset r^{-1}\{0\}$. In view of theorem (1.4), this property cannot hold in $C^0(X)$. Mention also that the above decomposition of f is not necessarily unique (see [1]).

This lemma is somehow the starting point of all studies on maximizing measures for regular functions. The analysis is then naturally divided in two parts :

1. Explicit the way from f to r.

2. Determine the class of $T-$invariant measures with support in $r^{-1}\{0\}$.

Concerning the first point, the link between f and r provided by the usual proofs of the lemma is rather abstract, since are used either Schauder-Tychonov fixed point theorem or cluster values of fixed points of a family of contracting operators. This path was made "concrete" by Bousch [1] in the case of the family $(f_t)_{0\leq t<1}$, but this is not an easy task. It appears that for all t, f_t can be decomposed in Lipschitz functions as $f_t = \beta(f_t) + (T\varphi_t - \varphi_t) - r_t$, where $r_t \geq 0$ and $r_t^{-1}\{0\}$ is a semi-circle. The next step of the proof is that a T-invariant measure with support in a semi-circle is a Sturm measure. This makes a link with the second part.

Advances in the second direction can be considered as *downhill* results. Should naturally appear in the decomposition given by the lemma the case when $r^{-1}\{0\}$ is not a semi-circle but a more general anti-symmetric union of intervals of total length $1/2$. Such a domain was called a "2-flower" in [5] and shown to generically support only periodic $T-$invariant probability measures. This result was a consequence of a general study on the dynamics of locally contracting maps of the Interval. With respect to conjecture (1.5), it then treats the case when $r^{-1}\{0\}$ is a "good" finite 2-flower.

Content of the paper. Moving one step upwards in the above perspective, we deal with all finite 2-flowers. More precisely, we prove :

Theorem 1.7
Let $f \in Lip(X)$ have a decomposition in elements of $Lip(X)$ of the form $f = \beta(f) + (T\varphi - \varphi) - r$, with $r \geq 0$, $r.\tau r = 0$ and $r^{-1}\{0\}$ has finitely many connected components.

Then for all neighbourhood U of f, there exist a non-empty open ball $V \subset U$ and a T-invariant measure μ supported by a closed periodic orbit such that $\mathcal{M}ax(g) = \{\mu\}$ for $g \in V$.

We conjecture that :

Conjecture 1.8
For a dense subset in $Lip(X)$, f has a decomposition in elements of $Lip(X)$ of the form $f = \beta(f) + (T\varphi - \varphi) - r$, with $r \geq 0$, $r.\tau r = 0$ and $r^{-1}\{0\}$ has finitely many connected components.

If r is as above, it is a simple remark (see the beginning of section (4)) that r can be perturbed so as to also satisfy $\#\{x \mid r(x) = r(x+1/2) = 0\} < \infty$. Observe that this condition, together with $r \geq 0$ and $r.\tau r = 0$, implies that $r^{-1}\{0\}$ is a finite and anti-symmetric union of intervals, thus adequate for the use of theorem (1.7). In the formulation of conjecture (1.8), "$r^{-1}\{0\}$ has finitely many connected components" can then be replaced by "$\#\{x \mid r(x) = r(x+1/2) = 0\} < \infty$".

Recall that the decomposition $f = \beta(f) + (T\varphi - \varphi) - r$ can be rewritten as :

$$\varphi(x) = -\beta(f) + \max_{Ty=x}(\varphi + f)(y).$$

Establishing this equality is the central point in the proof of lemma (1.6) and the finiteness assumption is equivalent to requiring that only finitely multiple nodes appear when considering inverse branches, namely only finitely many x satisfy $(\varphi + f)(x/2) = (\varphi + f)(x/2 + 1/2)$.

We thus reduce the proof of conjecture (1.5) to that of conjecture (1.8). Let us now emphasize a few aspects of the method we employ. Mention first that we intensely use the fact that the underlying space is $Lip(X)$ and not an arbitrary Hölder space. One key point in our proof consists in using some "finite time percolation property" on the graph of a Lipschitz function on the Interval. This allows to keep a *finite* 2-flower while perturbing a bad 2-flower. This lemma essentially follows from the fact that Lebesgue almost-all level sets of such a function are finite.

This geometrical approach on Lipschitz maps allows to use, via measure theoretical arguments with Lebesgue measure, the information given in [5] that the set of "good 2-flowers", namely supporting periodic orbits, and with $2m+1$ petals has total measure in the set of 2-flowers with $2m+1$ petals (as subset of $\mathbb{R}^{2m+1}$). A posteriori this approach appears to be rather flexible.

Plan of the article :

1. Introduction
2. Uniqueness part
3. Tools
   (a) Lemmas on Lipschitz functions
   (b) Finite anti-symmetric flowers on the Circle
4. Proof of the theorem
5. Concluding remarks and related open questions

2 Uniqueness part

As a first step in our study, we provide a short and constructive proof for the uniqueness part of the result. For other types of arguments and related discussions, see [9], [2] and [11].

Proposition 2.1
Let $f \in Lip(X)$ have a maximizing measure μ supported by a closed periodic orbit. Then every neighbourhood of f contains a non-empty open ball where μ is the unique maximizing measure.

Proof of the proposition :
Applying lemma (1.6) to f, we can replace f by r. We thus suppose that $f \leq 0$ and that the mentioned periodic orbit lays in $f^{-1}\{0\}$. For some periodic point x of period $q \geq 1$, the measure μ in $\mathcal{M}ax(f)$ can be written as :

$$\mu = \frac{1}{q} \sum_{l=0}^{q-1} \delta_{T^l x}.$$

Let arbitrary $0 < \varepsilon < \frac{1}{2} \min\{\text{Dist}(T^k x, T^l x) \mid 0 \leq k \neq l \leq q-1\}$ to be fixed later and introduce for all $y \in \mathbb{R}/\mathbb{Z}$ the map $\eta_y(z) = (\varepsilon^2 - \varepsilon \text{Dist}(z, y)) \vee 0$, making a small pick at y. Set next :

$$f_\varepsilon = f + \sum_{l=0}^{q-1} \eta_{T^l x}.$$

Obviously $\|f - f_\varepsilon\|_{Lip} = \varepsilon^2 + \varepsilon$. Fixing $0 < \delta < \varepsilon/2^q$, let then $\epsilon' > 0$ be such that any Lipschitz map g verifying $\|g - f_\varepsilon\|_{Lip} \leq \varepsilon'$ satisfies :

$$\begin{cases} \min_{0 \leq l \leq q-1} g(T^l x) > \max_{y \in \bigcap_{0 \leq k \leq q-1} [T^k x - \delta, T^k x + \delta]^c} g(y) + \delta\epsilon/2 \\ \\ g(T^k x) \geq g(y) + \frac{\varepsilon}{2} \text{Dist}(T^k x, y), \text{ for } y \in [T^k x - \varepsilon, T^k x + \varepsilon], \text{ for } 0 \leq k \leq q-1. \end{cases}$$

Taking a T-invariant Borel probability measure ν and g a map as above, we first use the general fact that $\nu(g) = \nu(S_q(g)/q)$, where $S_q(g) = \sum_{0 \leq l \leq q-1} T^l g$. We now focus on $S_q(g)$ instead of g. Considering any $y \in \mathbb{R} \backslash \mathbb{Z}$, suppose as a first case that no $0 \leq k, l \leq q-1$ satisfy $\text{Dist}(T^k x, T^l y) \leq \delta$. Then for all $0 \leq l \leq q-1$:

$$g(T^l y) < -\delta\varepsilon/2 + \min_{0 \leq k \leq q-1} g(T^k x) \leq -\delta\varepsilon/2 + \frac{1}{q} \sum_{0 \leq k \leq q-1} g(T^k x). \tag{1}$$

In a second case, let $0 \leq l \leq q-1$ be the smallest integer such that there exists $0 \leq k \leq q-1$ satisfying $\mathrm{Dist}(T^k x, T^l y) \leq \delta$. For $0 \leq r \leq q-1-l$ we have $\mathrm{Dist}(T^{l+r}y, T^{k+r}x) \leq 2^r \delta \leq 2^q \delta < \varepsilon$. Thus for $0 \leq r \leq q-1-l$:

$$g(T^{l+r}y) \leq g(T^{k+r}x) - \frac{\varepsilon}{2}\mathrm{Dist}(T^{l+r}y, T^{k+r}x). \tag{2}$$

Also if $-l \leq r \leq -1$ one has :

$$g(T^{l+r}y) < -\delta\varepsilon/2 + \min_{0 \leq s \leq q-1} g(T^s x) \leq g(T^{k+r}x) - \delta\varepsilon/2. \tag{3}$$

From (2) and (3) it results in this second case that :

$$\frac{1}{q} \sum_{0 \leq k \leq q-1} g(T^k y) \leq \frac{1}{q} \sum_{0 \leq k \leq q-1} g(T^k x),$$

where equality holds if and only if y lays in the orbit of x. Using (1), we conclude that $\nu(g) < \mu(g)$ if the support of ν is not contained in the orbit of x. This ends the proof of the proposition.

□

3 Tools

We intensely use in the sequel the following lemmas on Lipschitz functions, in the context of finite "2-flowers" on the Circle. The latter objects are introduced in a second part, as well as a result established in [5] on T-invariant measures with support a generic finite "2-flower".

3.1 Lemmas on Lipschitz functions

Recall that a Lipschitz map on the interval $[0,1]$ is differentiable Lebesgue-almost everywhere and is a primitive of its derivative, element of $L^\infty[0,1]$. Denote by λ Lebesgue measure on the Interval or on $\mathbb{R}$. Recall also that the direct image of a Borel set by a Lipschitz map is a Borel set.

Lemma 3.1
Let $f \in Lip[0,1]$ with $f(0)=0$, $f(1)=1$. Set $g(x) = \sup_{0 \leq t \leq x} f(t)$. Then the set of crests $\{0 \leq x \leq 1 \mid g(x) = f(x),\ f'(x) \text{ exists and } f'(x) > 0\}$ has positive Lebesgue measure.

Proof of the lemma :
Observe first that g is non-decreasing and also lays in $Lip[0,1]$. Indeed for any x, $h > 0$ and if one has $g(x+h) > g(x)$, then :

$$g(x+h) \leq f(x) + K(f)h \leq g(x) + K(f)h.$$

Thus g is differentiable almost everywhere, $g' \in L^\infty[0,1]$ and $g(x) = \int_0^x g'(t)\, dt$ for $0 \leq x \leq 1$. As $g(0) = 0$ and $g(1) \geq 1$, the set $A = \{g' > 0\}$ has positive Lebesgue measure. Next at almost every x in A both maps f and g are differentiable and $g(x+h) - g(x) \sim hg'(x)$, as $h \to 0$.

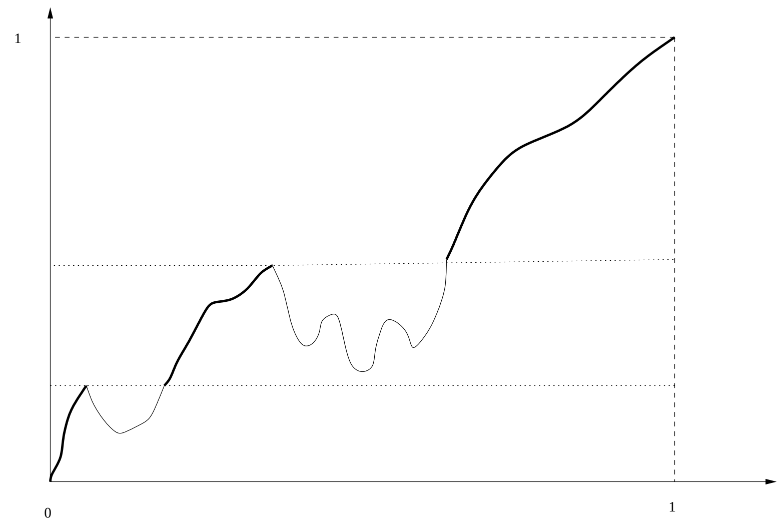
For such a x, since $g'(x) > 0$, we have for small $h > 0$ that $\sup_{[x,x+h]} f(t) > g(x)$. Thus $f(x) \geq g(x)$ and therefore $f(x) = g(x)$. Next :

$$\begin{aligned} \frac{1}{h}\left(g(x+h) - g(x)\right) &= \frac{1}{h}\left(\sup_{[x,x+h]} f - f(x)\right) = \frac{1}{h}\left(f(x+t_h) - f(x)\right) \\ &\leq \frac{1}{t_h}\left(f(x+t_h) - f(x)\right), \end{aligned}$$

for some $0 < t_h \leq h$, using that $f(x+t_h) - f(x) \geq 0$. This gives $f'(x) \geq g'(x) > 0$ and ends the proof of the lemma. □

Remark 1. — The lemma is false in the context of continuous functions. One may consider for instance a devil staircase.

Remark 2. — An illustration of the set of crests of a function as in lemma (3.1) is :



The following lemma and the idea for the given proof are due to T. Bousch.

Lemma 3.2
Let $f \in Lip[0,1]$. Then :

1. *For $\lambda-$almost all $y \in \mathbb{R}$, the set $f^{-1}\{y\}$ is finite.*
2. *If $f'(0)$ exists, then for $\lambda-$almost all $\theta \in \mathbb{R}$ the set $\{x \in [0,1] \mid f(x) = f(0) + \theta x\}$ is finite.*

Proof of the lemma :
As a first step, observe that the second case reduces to the first one. Indeed let $\varepsilon > 0$ and $\eta > 0$ be such that $(f(x) - f(0))/x \in [f'(0) - \varepsilon, f'(0) + \varepsilon]$ for $x \in [0, \eta]$. For $\theta \in \mathbb{R} \backslash [f'(0) - \varepsilon, f'(0) + \varepsilon]$, we get that $f(x) \neq f(0) + \theta x$ for $x \in (0, \eta]$.

As $(f(x) - f(0))/x$ lays in $Lip[\eta, 1]$, the first point then applies on $[\eta, 1]$. The result then holds for almost all $\theta \in \mathbb{R} \backslash [f'(0) - \varepsilon, f'(0) + \varepsilon]$. Since this is true for any $\varepsilon > 0$, this completes the proof.

We now focus on the first point of the lemma. For $\varepsilon > 0$ consider the enlarged graph :

$$G_\varepsilon = \{(x, y) \mid f(x) - \varepsilon \leq y \leq f(x) + \varepsilon\}.$$

The area of this graph is obviously $\lambda_2(G_\varepsilon) = 2\varepsilon$, using Fubini's Theorem and vertical integration, denoting by λ_2 planar Lebesgue measure. We now proceed to a horizontal integration :

$$\lambda_2(G_\varepsilon) = \int_{-\infty}^{+\infty} N_\varepsilon(y)\ dy,$$

where $N_\varepsilon(y) = \lambda(f^{-1}[y - \varepsilon, y + \varepsilon])$ is the length of the y-level set of G_ε. This quantity is measurable as well as $M_\varepsilon(y)$ that we define to be the cardinal of the maximal $2\varepsilon/(K(f) + 1)$-separated family of points in $f^{-1}\{y\}$ and also at distance $2\varepsilon/(K(f) + 1)$ from 0 and 1. Then the definition of $K(f)$ implies the inequality :

$$N_\varepsilon(y) \geq \frac{2\varepsilon}{K(f) + 1}\ M_\varepsilon(y).$$

We therefore get :

$$\int_{-\infty}^{+\infty} M_\varepsilon(y)\ dy \le K(f) + 1.$$

Since $M_\varepsilon(y)$ non-decreases to $\#(f^{-1}\{y\})$, this last quantity is integrable and then $\lambda-$almost surely finite. This concludes the proof of the lemma.

□

As an application of lemma (3.2), the next result describes some finite time percolation property on the graph of a Lipschitz function.

Lemma 3.3
Let $f \in Lip[0,1]$ be such that $f'(0)$ exists. Fixing $\eta > 0$, there are $N \ge 0$ and a family of paths $(\psi_\theta)_{\theta\in\Theta}$ in $Lip[0,1]$ satisfying :

- *For all $\theta \in \Theta$, $\psi_\theta(0) = f(0)$ and $\|\psi_\theta - f(0)\|_{Lip} \le \eta$.*
- *For all $\theta \in \Theta$, $\#\{0 < x \le 1 \mid \psi_\theta(x) = f(x)\} = N$.*
- *If $N \ge 1$, denote by $0 < x_1(\theta) < \cdots < x_N(\theta) \le 1$ the intersection points of the graphs of ψ_θ and f, for $\theta \in \Theta$. Then $\{(x_1(\theta), \cdots, x_N(\theta)) \mid \theta \in \Theta\}$ has positive Lebesgue measure in $\mathbb{R}^N$.*

Proof of the lemma :
By lemma (3.2), for $\lambda-$almost all θ the line $x \longmapsto f(0) + \theta x$, $0 < x \le 1$, intersects the graph of f at finitely many points. Let $N \ge 0$ and $A \subset [-\eta, \eta]$ with $\lambda(A) > 0$ be such that this line cuts the graph of f at exactly N points $0 < x_1(\theta) < \cdots < x_N(\theta) \le 1$, when $\theta \in A$. In the sequel we "break" the previous trajectories to create a set of intersection points of positive Lebesgue measure in $\mathbb{R}^N$ when $\theta \in \Theta$ varies.

We suppose that $N \ge 1$, otherwise there is nothing to prove. Observe that $\lambda-$almost surely on A, the map f is differentiable at each $x_i(\theta)$ with $f'(x_i(\theta)) \neq \theta$ for all $1 \le i \le N$. Indeed and as a first point, f is differentiable at λ-almost every point of $[0,1]$. Let $B \subset (0,1]$ with $\lambda(B) = 0$ be the set of x where $f'(x)$ does not exist. Remark that $\{\exists\, 1 \le i \le N \mid x_i(\theta) \in B\} \subset \{\theta \in \varphi(B)\}$, where $\varphi(x) = (f(x) - f(0))/x$. Since $\varphi \in Lip[\delta, 1)$ for any $\delta > 0$, we get :

$$\lambda(\varphi(B)) = \lim_{\delta\to 0} \lambda(\varphi(B \cap [\delta, 1])) \le \lim_{\delta\to 0} \int_{B\cap[\delta,1]} |\varphi'(x)|\ dx \le \lim_{\delta\to 0} K(\varphi_{|[\delta,1]}) \int_B dx = 0.$$

Thus for almost every $\theta \in [-\eta, \eta]$, all $f'(x_i(\theta))$ exist, $1 \le i \le N$. As a second point, we have $\lambda\{\varphi(C)\} = 0$, where $C = \{x \mid \varphi'(x) = 0\}$. Since $f'(x) = \varphi(x) + x\varphi'(x)$, if for some $1 \le i \le N$ one has both $\varphi(x_i(\theta)) = \theta$ and $f'(x_i(\theta)) = \theta$, we get $\varphi'(x_i(\theta)) = 0$. This would give $x_i(\theta) \in C$ and $\theta \in \varphi(C)$. The two assertions mentioned above are then proved. Deleting a null-set, we assume that these conditions are verified for all points in A.

Since $\lambda(A) > 0$, take next a density point $\theta_0 \in A$. Precisely :

$$\lim_{\delta\to 0} \frac{\lambda(A \cap [\theta_0 - \delta, \theta_0 + \delta])}{\lambda([\theta_0 - \delta, \theta_0 + \delta])} = 1.$$

From the previous discussion, the intersections of the line $x \longmapsto f(0) + \theta_0 x$ with the graph of f are all transverse. Therefore when $\theta \to \theta_0$ and θ stays in A, we observe that $x_i(\theta) \to x_i(\theta_0)$, for all $1 \le i \le N$. Precising the picture, suppose for instance that $f'(0) < \theta_0$. Then necessarily $f'(x_i(\theta_0)) > \theta_0$ for i odd and $f'(x_i(\theta_0)) < \theta_0$ for i even, $1 \le i \le N$.

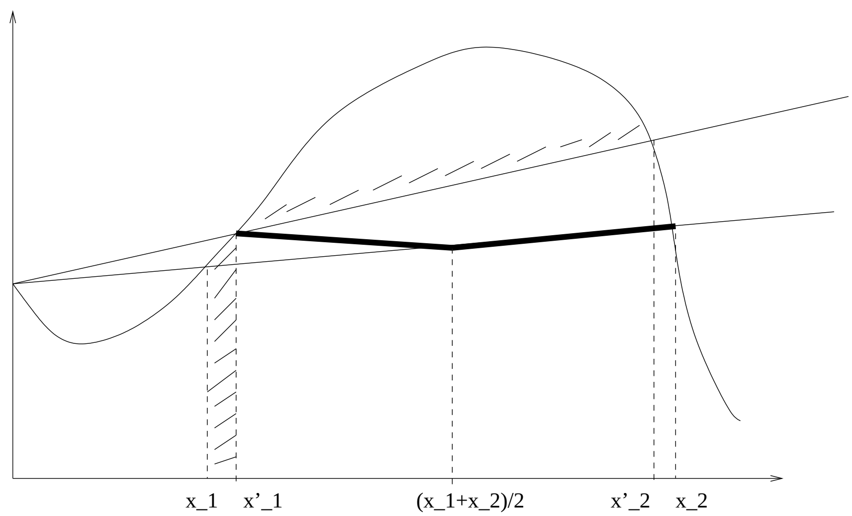
Observe next that for $\theta < \theta_0 < \theta'$ both in A and close to θ_0 and since $f'(x_i(\theta_0)) \neq \theta_0$ for $1 \le i \le N$, the intersection of the graph of f with the lines $t \longmapsto f(0) + t\theta$ and $t \longmapsto f(0) + t\theta'$ hold in distinct half-planes among $\{x < x_i(\theta_0)\}$ and $\{x > x_i(\theta_0)\}$, for $1 \le i \le N$. Choose $0 < \eta' \le \eta$ such that this property holds for any $\theta < \theta < \theta'$ both in A and with $\theta' - \theta \le \eta'$ and all $1 \le i \le N$.

We deduce that for such a couple (θ,θ') we have $x_i(\theta) < x_i(\theta')$ for i odd (since $f'(x_i(\theta_0)) > \theta_0$) and $x_i(\theta) < x_i(\theta')$ for i even (since $f'(x_i(\theta_0)) < \theta_0$). Focus now on the interval $[x_1(\theta_0), x_2(\theta_0)]$. We build a path $\psi(\theta',\theta)$ from $(x_1(\theta'), f(x_1(\theta')))$ to $(x_2(\theta), f(x_2(\theta)))$ with a low Lipschitz constant and not touching the graph of f except at its extremities.

- Define $\psi(\theta',\theta)$ on $[x_1(\theta'), (x_1(\theta)+x_2(\theta))/2]$ by :

$$t \longmapsto \frac{\frac{1}{2}\left[f(x_2(\theta)) + f(x_1(\theta))\right] - f(x_1(\theta'))}{\frac{1}{2}(x_2(\theta)+x_1(\theta)) - x_1(\theta')}(t - x_1(\theta')) + f(x_1(\theta')).$$

  On $[(x_1(\theta)+x_2(\theta))/2, x_2(\theta)]$ set $t \longmapsto \theta(t - x_1(\theta)) + f(x_1(\theta))$. A picture is the following one, where $x_1(\theta)$, $x_2(\theta)$, $x_1(\theta')$ and $x_2(\theta')$ are respectively replaced by x_1, x_2, x_1' and x_2' :



  Since the graph of f is above the line $t \longmapsto f(0) + t\theta'$ on $[x_1(\theta'), x_2(\theta')]$, the path $\psi(\theta',\theta)$ is "protected" by this line on this interval. The intersections of this path with the graph of f are then reduced to its extremities.

- In a similar way, define a path $\psi(\theta,\theta')$ from $(x_1(\theta), f(x_1(\theta)))$ to $(x_2(\theta'), f(x_2(\theta')))$, first on $[x_1(\theta), (x_1(\theta)+x_2(\theta))/2]$ by $t \longmapsto \theta(t - x_1(\theta)) + f(x_1(\theta))$ and second on the interval $[(x_1(\theta)+x_2(\theta))/2, x_2(\theta')]$ by :

$$t \longmapsto \frac{f(x_2(\theta')) - \frac{1}{2}(f(x_2(\theta)) + f(x_1(\theta)))}{x_2(\theta') - \frac{1}{2}(x_2(\theta)+x_1(\theta))}(t - (x_1(\theta)+x_2(\theta))/2) + \frac{f(x_1(\theta)) + f(x_2(\theta))}{2}.$$

  As above, the intersections of this path with the graph of f are reduced to its extremities.

The same treatment is then made between points $x_i(\theta_0)$ and $x_{i+1}(\theta_0)$ for all $1 \le i \le N-1$. Setting $A_+ = [\theta_0, \theta_0 + \eta'] \cap A$ and $A_- = [\theta_0 - \eta', \theta_0] \cap A$, this provides a family of paths $(\psi_0(\theta^0) \to \psi_1(\theta^1,\theta^2) \to \cdots \to \psi_{N-1}(\theta^{N-1},\theta^N) \to \psi_N(\theta^N))$ on the interval $[0,1]$, defining $\psi_0(\theta^0)$ as the linear path from $(0, f(0))$ to $(x_1(\theta^0), f(x_1(\theta^0)))$ and $\psi_N(\theta^N)$ as the linear path from $(x_N(\theta^N), f(x_N(\theta^N)))$ to $(1, f(0)+\theta^N)$, and where the sequence of angles $(\theta^i)_{0\le i\le N}$ belongs either to $A_+ \times A_- \times A_+ \cdots$ or to $A_- \times A_+ \times A_- \cdots$.

We next choose $\eta' > 0$ small enough so that the Lipschitz distance between such a path and $f(0)$ is less than 2η and also $\min\{\lambda(A_+), \lambda(A_-)\} > 0$.

We finally observe that the intersection points $(x_1(\theta^1), \cdots, x_N(\theta^N))$ describe a set of positive Lebesgue measure in $\mathbb{R}^N$, when the corresponding path varies in the above family. Indeed remark first that the Lebesgue measure in $\mathbb{R}^N$ of the set of $(\theta^1, \cdots, \theta^N)$ is at least equal to $2(\min\{\lambda(A_+), \lambda(A_-)\})^N > 0$. Next for small $\varepsilon > 0$, one can decrease $\eta' > 0$ in such a way that for all $1 \leq i \leq N$, $A_+ \subset \varphi([x_i(\theta_0) - \varepsilon, x_i(\theta_0) + \varepsilon])$. Thus :

$$\begin{aligned} \lambda\{x_i(\theta) \mid \theta \in A_+\} &= \lambda\{\varphi^{-1}(A_+) \cap [x_i(\theta_0) - \varepsilon, x_i(\theta_0) + \varepsilon]\}. \\ &\geq \frac{1}{1 + K_i} \lambda(A_+) > 0, \text{ where } K_i = K(\varphi_{| [x_i(\theta_0)-\varepsilon, x_i(\theta_0)+\varepsilon]}). \end{aligned}$$

The same holds for A_-, so the assertion is proved. This completes the proof of the lemma.

□

Remark. — As for lemma (3.1), the previous lemma should not hold true for any continuous function, for instance a typical realization of the Standard Brownian motion.

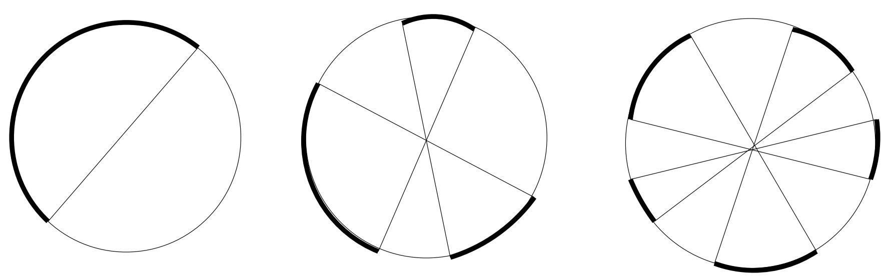
3.2 Finite anti-symmetric flowers on the Circle

Fixing an element $f \in Lip(X)$ and a reduction $f = \beta(f) + (T\varphi - \varphi) - r$ given by lemma (1.6), we now focus on the level set $r^{-1}\{0\}$. The case under consideration is when $r^{-1}\{0\}$ is a finite and anti-symmetric union of intervals. Such a domain was called a 2-flower in [5], where were defined $p-$flowers in the context of the transformation $x \longmapsto px \mod (1)$, with $p \geq 2$. Since "2-flowers" have an odd number of petals, we slightly simplify the definition.

Definition 3.4
Fix $m \geq 0$ and let $S_{2m+1} = \{(s_i)_{1 \leq i \leq 2m+1} \mid 0 \leq s_1 < s_2 < \cdots < s_{2m+1} < s_1 + 1/2 \leq 1\} \subset \mathbb{R}^{2m+1}$. For $s \in S_{2m+1}$ and $k \in \{0, 1\}$, define a "2-flower" $Fl(s,k)$ with $2m+1$ petals on $\mathbb{R}\backslash\mathbb{Z}$ as follows :

$$Fl(s,k) = \left[\cup_{1 \leq i \leq m} \left([s_{2i-1}, s_{2i}] \cup \left([s_{2i}, s_{2i+1}] + \frac{1}{2}\right)\right) \cup \left([s_{2m+1}, s_1] + \frac{1}{2}\right)\right] + \frac{k}{2}. \qquad (4)$$

Remark. — Examples of 2-flowers with respectively one, three and five petals and respectively $k = 0$, $k = 0$ and $k = 1$:



Considering the T-invariant measures with support in a 2-flower with $2m + 1$ petals, we have :

Theorem 3.5 [*5*]
Let $m \geq 0$, $k \in \{0, 1\}$. There is an open set $\Omega_{2m+1} \subset S_{2m+1}$ of full Lebesgue measure in S_{2m+1} such that for $s \in \Omega_{2m+1}$, the set of ergodic $T-$invariant Borel probability measures with support in $Fl(s,k)$ consists in at most $2m + 1$ periodic measures. These measures are locally constant functions of s on Ω_{2m+1}.

4 Proof of the theorem

We begin the proof of theorem (1.7). Let $f \in Lip(X)$ be decomposed in $Lip(X)$ as a sum :

$$f = \beta(f) + (T\varphi - \varphi) - r, \tag{5}$$

where $r \geq 0$, $r.\tau r = 0$ and $r^{-1}\{0\}$ has finitely many connected components, and thus is a finite union of intervals. The property $r.\tau r = 0$ ensures that $r^{-1}\{0\}$ contains a finite anti-symmetric flower F. Up to adding to f the map $x \longmapsto -\varepsilon \operatorname{Dist}(x, F)$ for arbitrary small $\varepsilon > 0$, we suppose that $r^{-1}\{0\}$ is *exactly* an anti-symmetric 2-flower. Write it as F. If this domain supports a T-invariant periodic probability measure, then the result follows from proposition (2.1). We thus assume that F carries no such measure.

For some $m \geq 0$, F is a 2-flower with $2m+1$ petals in the sense of definition (3.4). Without loss of generality, suppose that there exist $(s_i)_{1\leq i\leq 2m+1} \in S_{2m+1}$ and F can be written as :

$$F = \cup_{1\leq i\leq m}\left([s_{2i-1}, s_{2i}] \cup \left([s_{2i}, s_{2i+1}] + \frac{1}{2}\right)\right) \cup \left([s_{2m+1}, s_1] + \frac{1}{2}\right).$$

Starting with $[s_1, s_2]$, denote by $(I_i)_{1\leq i\leq 2m+1}$ the ordered sequence on the Circle of the above intervals. Remark that all are at a positive distance of each other. We write $I_i = [t_i, u_i]$. Observe for the sequel that for all $1 \leq i \leq 2m+1$, one has $t_i + 1/2 = u_{i+m}$ and thus $u_i + 1/2 = t_{i+m+1}$, the indices being taken modulo $2m+1$. We now start a closer analysis.

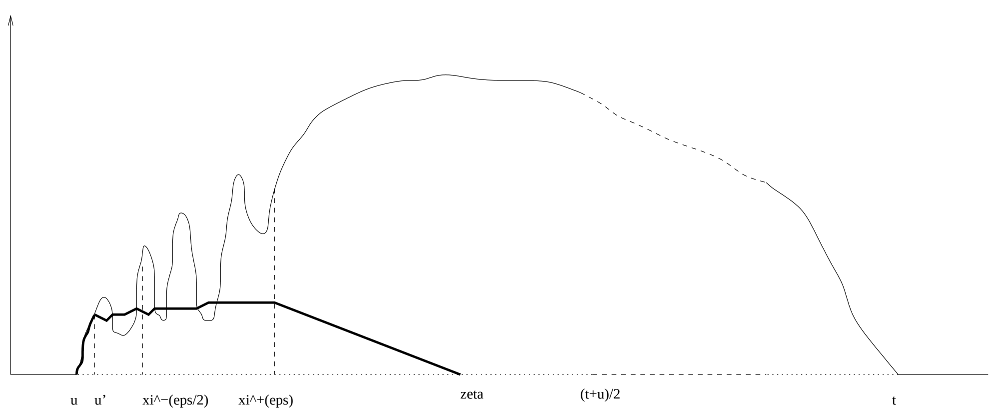
Step 1. Fix the interval $[u_1, t_2]$, simplified as $[u, t]$. Recall that $r(x) > 0$ for $x \in (u, t)$. Let $u < u'$ be close to u and be a crest for r on $[u, u']$, that is $r(u') = \sup_{u\leq y\leq u'} r(y)$ and $r'(u') > 0$. By lemma (3.1) and since $r > 0$ on (u, t), the set of crests has positive Lebesgue measure on every interval $[u, u+\varepsilon]$, $\varepsilon > 0$.

For small $\varepsilon > 0$, define :

$$\begin{cases} \xi^+(\varepsilon) = \sup\{u < y < (t+u)/2 \mid r(y) = \varepsilon\} \\ \\ \xi^-(\varepsilon) = \inf\{u < y < (t+u)/2 \mid r(y) = \varepsilon\}. \end{cases}$$

Both quantities tend to u, as $\varepsilon \to 0$. Fixing $\varepsilon > 0$, take a crest $u < u' < \xi^-(\varepsilon/2)$ and apply lemma (3.3) on the interval $[u', \xi^+(\varepsilon)]$, with a precision parameter $\eta > 0$ chosen so that all typical paths ψ are strictly below r at the beginning of the path and stay at a Lipschitz distance less than $r(u')/2$ from the constant $r(u')$ on $[u', \xi^+(\varepsilon)]$. Let $N \geq 0$ be the integer given by the lemma. Since the set of u' has positive measure, we take N valid for a set of u' of positive measure.

Fixing $\varepsilon' > 0$, complete finally each ψ on $[u, u']$ by r and on some $[\xi^+(\varepsilon), \zeta(\varepsilon', \psi)]$ by the line $t \longmapsto \psi(\xi^+(\varepsilon)) + \varepsilon'(t - \xi^+(\varepsilon))$, where $\zeta(\varepsilon', \psi) = \xi^+(\varepsilon) - \psi(\xi^+(\varepsilon))/\varepsilon'$. Denote next by $[u, v] = [u, \zeta(\varepsilon', \psi)]$ the total interval of definition and still by ψ the global path. A summary picture on the graph of r is the following one :



Adjusting parameters $\varepsilon > 0$, $\varepsilon' > 0$ and $\eta > 0$, the Lipschitz norm of $(\psi - r(u'))$ on $[u', v]$ can be taken as small as desired, as well as $\mathrm{Dist}(u, v)$, uniformly on ψ. The counterpart is that there is no control on N. We just require it to be finite.

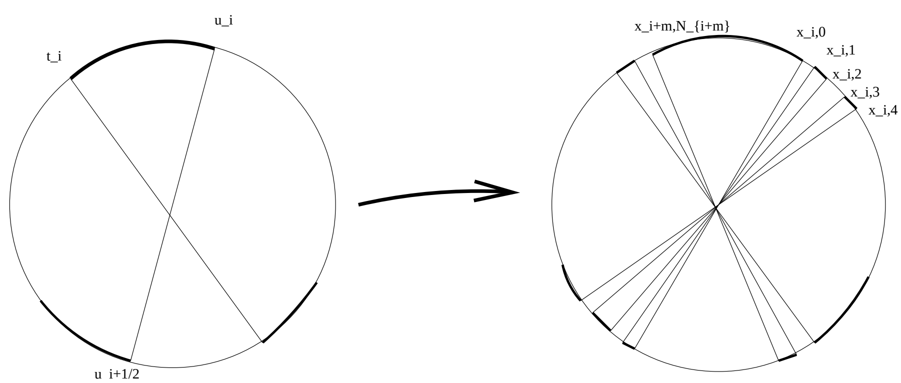
Step 2. The same procedure is repeated on each interval $[u_i, t_{i+1}]$, $1 \leq i \leq 2m+1$. As a summary, fixing $\delta > 0$, for all $1 \leq i \leq 2m+1$ there exist integers $N_i \geq 0$, quantities u'_i, v_i and typical paths ψ_i on $[u_i, v_i]$ such that for $1 \leq i \leq 2m+1$:

- For all ψ_i, one has $\mathrm{Dist}(u_i, v_i) \leq \delta$ and $\|\psi - r(u'_i)\|_{Lip,[u'_i,v_i]} \leq \delta$.
- Each ψ_i intersects the graph of r on $[u'_i, v_i]$ at exactly $N_i + 1$ points (including u'_i), written as $x_{i,0} < x_{i,1} < \cdots < x_{i,N_i}$. The set of such points has positive Lebesgue measure in $\mathbb{R}^{N_i+1}$ when ψ_i varies within its set of typical paths.

Since each interval is treated independently, we deduce that the set $\{x_{i,j} \mid 1 \leq i \leq 2m+1,\ 0 \leq j \leq N_i\}$ has positive Lebesgue measure in $\mathbb{R}^N$, when all ψ_i vary within their respective sets of typical paths and where :

$$N = \sum_{1 \leq i \leq 2m+1} (N_i + 1).$$

At this step, let us precise that each (N_i+1) is odd, since $r(t) > \psi_i(t)$ on $(x_{i,0}, x_{i,1})$. Thus N is odd. For all $1 \leq i \leq 2m+1$, replace $[t_i, u_i]$ by the intervals $[x_{i+m,0}, x_{i+m,1}]+1/2$, $[x_{i+m,2}, x_{i+m,3}]+1/2$, $\cdots$, $[x_{i+m,N_{i+m}-2}, x_{i+m,N_{i+m}-1}]+1/2$, close to t_i, the central and large interval $[x_{i+m,N_{i+m}}+1/2, x_{i,0}]$ and the intervals $[x_{i,1},\ x_{i,2}]$, $[x_{i,3}, x_{i,4}]$, $\cdots$, $[x_{i,N_i-1}, x_{i,N_i}]$, close to u_i. An illustrating picture is as follows :



The collection of such intervals, when $1 \leq i \leq 2m+1$ varies, determines a 2-flower with N petals, since $N = \sum_{1 \leq i \leq 2m+1}(N_{i+m}/2 + N_i/2 + 1)$. Denote by $\mathcal{F}$ the set of obtained 2-flowers. By the previous study, the set of the parameters of these 2-flowers has positive Lebesgue measure in the set S_N. In view of theorem (3.5), we deduce that for a set of parameters of positive measure (thus non-empty), such 2-flowers contain a periodic orbit.

Step 3. Given any 2-flower F_0 in $\mathcal{F}$, we show that $-r$ (and thus f) can be perturbed in the Lipschitz norm and added a coboundary so that its maximizing measures are supported by this 2-flower.

In this direction, fix i and consider at the same time u_i and its symmetric point $u_i + 1/2 = t_{i+m+1}$. Extending ψ_i by 0 outside $[u_i, v_i]$, define :

$$-r_i = -r + \pi_i + \tau\pi_i, \text{ where } \pi_i = (\psi_i \wedge r)1_{[u_i,v_i]}.$$

We shall describe precisely the shape of $-r_i$. First of all $-r_i$ is only changed on the intervals $[u_i, v_i]$ and $[u_i, v_i] + 1/2$. Observe that :

- On $[u_i, v_i]$: $-r_i$ is equal to 0 on $[u_i, x_{i,0}]$, $[x_{i,1}, x_{i,2}]$, $[x_{i,3}, x_{i,4}]$, $\cdots$, $[x_{i,N_i-1}, x_{i,N_i}]$, is < 0 on $(x_{i,N_i}, (u_i+t_{i+1})/2]$ and equal to $(\psi_i - r) < 0$ on $(x_{i,0}, x_{i,1})$, $(x_{i,2}, x_{i,3})$, $\cdots$, $(x_{i,N_i-2}, x_{i,N_i-1})$.
- On $[u_i + 1/2, v_i + 1/2]$: $-r_i$ is equal to $\tau\psi_i$ on $[x_{i,0}, x_{i,1}] + 1/2$, $[x_{i,2}, x_{i,3}] + 1/2$, $\cdots$, $[x_{i,N_i-1}, x_{i,N_i}]+1/2$ and $[x_{i,N_i}, v_i]+1/2$, equal to τr ($\leq \tau\psi_i$) on $[u_i, x_{i,0}]+1/2$, $[x_{i,1}, x_{i,2}]+1/2$, $\cdots$, $[x_{i,N_i-2}, x_{i,N_i-1}] + 1/2$.

Therefore the map $-r_i$ is what we wish on $[u_i, v_i]$, but not a priori on $[u_i, v_i] + 1/2$. Focusing on $[u_i, v_i] + 1/2$, we show that with a small perturbation on a slightly larger interval, the map $-r_i$ is ≤ 0 everywhere on this larger interval and identically 0 on the $[x_{i,0}, x_{i,1}] + 1/2$, $[x_{i,2}, x_{i,3}] + 1/2$, $\cdots$, $[x_{i,N_i-1}, x_{i,N_i}] + 1/2$ and $[x_{i,N_i}, v_i] + 1/2$.

This way, fix some $\varepsilon'' > 0$ and define a map κ_i on $[x_{i_0}, v_i] + 1/2$ as $\tau\psi_i$, on $[u_i + 1/2, x_{i,0} + 1/2]$ as the constant $r(x_{i,0})$ and by a line $t \longmapsto \varepsilon''(t - u_i - 1/2) + r(x_{i,0})$ of slope ε'' on the interval $[w_i, u_i + 1/2]$, where $w_i = (u_i + 1/2) - r(x_{i,0})/\varepsilon''$. Extend it by 0 elsewhere.

Then $-r_i - \kappa_i$ is ≤ 0 everywhere on $[w_i, v_i + 1/2]$, since $r \leq r(x_{i,0})$ on the interval $[u_i, x_{i,0}]$, as $x_{i,0}$ was chosen to be a *crest*. This is where this information is fundamentally used. Also the map is identically 0 on the intervals $[x_{i,0}, x_{i,1}] + 1/2$, $[x_{i,2}, x_{i,3}] + 1/2$, $\cdots$, $[x_{i,N_i-1}, x_{i,N_i}] + 1/2$ and $[x_{i,N_i}, v_i] + 1/2$. Observe finally that the Lipschitz norm of κ_i can be made arbitrary small, according to ψ_i, $r(x_{i,0})$ and ε''.

As a result, the map $-r_i - \kappa_i = -r + \pi + \tau\pi - \kappa_i$ is ≤ 0 and such that its zero level set in the neighbourhood of the right side of $[t_i, u_i]$ and on the left side of $[t_{i+m+1}, u_{i+m+1}]$ is that of F_0. All changes with respect to r have been made on two intervals respectively containing u_i and $u_i + 1/2$ and each of length bounded by $\mathrm{Dist}(w_i, v_i + 1/2)$.

Step 4. Fix a precision $\delta_0 > 0$ and an integer M such that $2^{-M}\|r\|_{Lip} \leq \delta_0$. In this section we make the assumption that the orbits and the symmetric orbits (that is orbits $+1/2$) for time ≥ 1 of all t_i and all u_i, $1 \leq i \leq 2m+1$, never intersect the boundary of F, until getting outside F. Recall that none of these boundary points can both be periodic and have its orbit contained in F, since F contains no periodic point. Bounding time by $\min\{M, \text{exit of } F\}$, the minimal distance between points in such pieces of orbits and the boundary of F is positive.

In the sequel all perturbations of r occur only on arbitrary small intervals around the $(T^j u_i)$ and the $(T^j u_i) + 1/2$, for $1 \leq i \leq 2m+1$ and $0 \leq j \leq M$.

Beginning the story, let $\delta > 0$ be as in *Step 2*, but also larger than $\max_{1\leq i\leq 2m+1} \mathrm{Dist}(w_i, u_i + 1/2)$ and $\max_{1\leq i\leq 2m+1} \|\kappa_i\|_{Lip}$, and to be fixed later. For small $\delta > 0$, the following map is a Lipschitz perturbation of $-r$:

$$-\tilde{r} = -r - \sum_{1\leq i\leq 2m+1} \kappa_i.$$

Define $-R = -r + \sum_{1\leq i\leq 2m+1}(-\kappa_i + \pi_i + \tau\pi_i)$. For small $\delta > 0$, this map is ≤ 0 and such that its zero level set is exactly $F_0 \in \mathcal{F}$. Write then :

$$-\tilde{r} = -R - \sum_{1\leq i\leq 2m+1} (\pi_i + \tau\pi_i).$$

Remark that for all $1 \leq i \leq 2m+1$, one has $\pi_i + \tau\pi_i = T\pi_i^1$ for some π_i^1 verifying :

$$\|\pi_i^1\|_\infty = \|\pi_i\|_\infty \text{ and } K\left(\pi_i^1\right) = \frac{1}{2}K(\pi_i).$$

Write next :

$$-\tilde{r} = -R - \left(\sum_{1\leq i\leq 2m+1} T\pi_i^1 - \pi_i^1\right) - \left(\sum_{1\leq i\leq 2m+1} \pi_i^1\right).$$

We now treat independently the case of each π_i^1, for $1 \leq i \leq 2m+1$. First, the support of π_i^1 is included in an interval of length at most 2δ and containing the point $2u_i$. This leads to the following discussion :

- If $2u_i \notin F$ and since $-R(2u_i) < 0$ we get, if δ is small enough, that the map $-R - \pi_i^1$ is still < 0 on the support of π_i^1 and this support lays at a positive distance of F.
- If $2u_i \in F$, then it is not a boundary point of F by assumption. We write :

$$-\pi_i^1 = -(\pi_i^1 + \tau\pi_i^1) + \tau\pi_i^1.$$

The support of $\tau\pi_i^1$ is around $2u_i + 1/2 \notin F$ and the reasoning of the previous case is reconducted with $\tau\pi_i^1$. Next for some π_i^2, one has $\pi_i^1 + \tau\pi_i^1 = T\pi_i^2$. This provides :

$$-\pi_i^1 = -(T\pi_i^2 - \pi_i^2) + \tau\pi_i^1 - \pi_i^2.$$

Considering next π_i^2 in the last case, one can iterate this procedure. Finally from this algorithm we obtain two situations :

- Either there exists $1 \leq p \leq M$ such that $2u_i \in F, \cdots, 2^{p-1}u_i \in F$, distinct from all boundary points of F, and $2^p u_i \notin F$. In this case the map $-\pi_i^1$ can be decomposed as the sum :

$$-\pi_i^1 = -\sum_{k=2}^{p} \left(T\pi_i^k - \pi_i^k\right) - \sum_{k=1}^{p-1} \tau\pi_i^k - \pi_i^p,$$

where, for small enough $\delta > 0$, the support of all $\tau\pi_i^k$ for $1 \leq k \leq p-1$ fall outside F, as well as that of π_i^p.

- Or $2^j u_i \in F$ for all $1 \leq j \leq M$. In this case :

$$-\pi_i^1 = -\sum_{k=2}^{M} \left(T\pi_i^k - \pi_i^k\right) - \sum_{k=1}^{M-1} \tau\pi_i^k - \pi_i^M.$$

Observe that $\|\pi_i^M\|_{Lip} \leq \|\pi_i\|_\infty + 2^{-M}K(\pi_i) \leq \|\pi_i\|_\infty + 2^{-M}(K(r) + K(\psi_i)) \leq 2\delta_0$ for small $\delta > 0$. Thus π_i^M has a small Lipschitz norm.

As a result, for each $1 \leq i \leq 2m+1$ the map $-\pi_i^1$ can be decomposed as a sum of Lipschitz maps of the form :

$$-\pi_i^1 = (Tg_i - g_i) + h_i + \epsilon_i,$$

in such a way that $-R + h_i < 0$ on the support of h_i and ϵ_i has a small Lipschitz norm. We therefore deduce the following equality :

$$-\tilde{r} = \left(-R + \sum_{1 \leq i \leq 2m+1} h_i\right) - \sum_{1 \leq i \leq 2m+1} \left[T(\pi_i^1 - g_i) - (\pi_i^1 - g_i)\right] + \sum_{1 \leq i \leq 2m+1} \epsilon_i.$$

For small $\delta > 0$, the support of $\sum_{1 \leq i \leq 2m+1} h_i$ is at a positive distance of F and the zero level set of the map $-R + \sum_{1 \leq i \leq 2m+1} h_i$ is exactly F_0. Since $\sum_{1 \leq i \leq 2m+1} \epsilon_i$ has arbitrary small Lipschitz norm, according to $\delta > 0$, we conclude that all the maximizing measures of :

$$f - \sum_{1 \leq i \leq 2m+1} (\kappa_i + \epsilon_i)$$

have their support contained in the 2-flower F_0. This ends the proof of this step, using proposition (2.1) to conclude the demonstration of theorem (1.7).

Step 5. We now drop the assumption that the forward orbit of no t_i (and thus of no u_i) falls on the boundary of F for times ≥ 1, until exiting F. Mention that in *Step 4*, the quantity M depends on $\|r\|_{Lip}$. Since we next make Lipschitz perturbations on r as well as adding coboundaries, there is no control on the Lipschitz norm of r. We shall in fact show that one can suppose that the full forward orbit of each t_i never meets the boundary of F.

Following the previous procedure we construct a *single* perturbation of f, having a decomposition as in (5), such that $r^{-1}\{0\}$ is a finite 2-flower whose boundary points check this property. Then steps $1 \to 4$ can be reconducted for that map and provide the result.

Starting from f and r, first rename all $(t_i)_{1\leq i\leq 2m+1}$ and $(u_i)_{1\leq i\leq 2m+1}$ in a single sequence $(y_i)_{1\leq i\leq 4m+2}$. Suppose then that there exists a couple (i_1, i_2) such that y_{i_2} lays in the orbit of y_{i_1} and the corresponding piece of orbit is contained in F. Write this property as $i_1 \to i_2$. If some i_3 checks $i_2 \to i_3$, then $i_1 \to i_2 \to i_3$. Observe that all i_j are distinct, otherwise F would carry a periodic orbit and this was supposed not to be true (in the remarks preceding *Step 1*). Iterating the procedure, there exists $p \geq 2$ such that $i_1 \to i_2 \to \cdots \to i_p$ and the orbit of y_{i_p} and $y_{i_p} + 1/2$ contains no point in the boundary of F, until maybe getting out of F.

Rule then out the perturbation method of steps $1 \to 4$ but only for the couple of symmetric points $(y_{i_p}, y_{i_p} + 1/2)$. Thus, up to adding a small Lipschitz map and a coboundary, the point y_{i_p} is replaced by a family $(y_{i_p,0}, \cdots, y_{i_p,N_{i_p}})$ and the set of such possible points has positive Lebesgue measure in $\mathbb{R}^{N_{i_p}}$. The symmetric point $y_{i_p} + 1/2$ is replaced by $(y_{i_p,0} + 1/2, \cdots, y_{i_p,N_{i_p}} + 1/2)$.

We make the choice that no such point $y_{i_p,j}$ or $y_{i_p,j} + 1/2$ either contains a point y_{i_l} or $y_{i_p,k}$ or else $y_{i_p,k} + 1/2$ in its forward orbit or lays in the forward orbit of such a point. Observe that there is only a denumerable set of hyperplanes to avoid in $\mathbb{R}^{N_{i_p}}$.

In a next step, start using again the relation "$\to$". Remark then that it can involve at most, due to the above choice, the points $(y_i)_{1\leq i\leq 2m+1, i\neq i_p, i\neq i_q}$, where $y_{i_p} + 1/2 = y_{i_q}$. If there exists at least one cycle, one repeats the same procedure. Remark that the whole process ends in at most $2m+1$ steps. Therefore the sum of the perturbations of f (not involving the addition of coboundaries) remains a perturbation.

Thus the assumption enounced at the beginning of *Step 4* is verified and this completes the proof of the theorem.

□

We complete this section by indicating a consequence of theorem (1.7) in the spirit of [7], namely that conjecture (1.5) is true when replacing the natural topology in $Lip(X)$ by the strictly weaker topology of an arbitrary Hölder space $H^\alpha(X)$, $0 < \alpha < 1$. Recall that this topology is induced by the classical norm :

$$N^\alpha(f) = \|f\|_\infty + \sup\left\{\frac{|f(x) - f(y)|}{\mathrm{Dist}(x,y)^\alpha} \mid x \neq y\right\}.$$

We have the following result :

Theorem 4.1
In $Lip(X)$ and for all $0 < \alpha < 1$, there is a N^α-dense N^α-open set of maps f admitting a unique maximizing measure, which is supported by a closed periodic orbit.

Proof of the theorem :
Taking any $f \in Lip(X)$, consider a Lipschitz decomposition $f = \beta(f) + (T\varphi - \varphi) - r$ given by lemma (1.6), with $r \geq 0$ and $r.\tau r = 0$. Denoting by $(I_j)_{j\geq 0}$ the connected components of the open set $\{r > 0\}$, observe that :

$$r = \sum_{j\geq 0} r_j, \text{ with } r_j = 1_{I_j}\ r. \tag{6}$$

The convergence in the above sum holds for the supremum norm, since all I_j are disjoint and the sequence of lengths $(\lambda(I_j))_{j\geq 0}$ tends to 0. As the partial sums $(R_n)_{n\geq 0} = (\sum_{0\leq j\leq n} r_j)_{n\geq 0}$ are obviously Lipschitz bounded, the convergence in (6) also holds for any norm N^α, with $0 < \alpha < 1$, for instance applying lemma 17 of [7].

As a result, for n large enough, R_n is arbitrary N^α-close to r and thus also f_n to f, where we set $f_n = \beta(f) + (T\varphi - \varphi) - R_n$. For such a n, we have $R_n \geq 0$, $R_n.\tau R_n = 0$ and $R_n^{-1}\{0\}$ has finitely many connected components. Applying theorem (1.7), we get that f_n admits a Lipschitz perturbation (and thus a N^α-perturbation) with a maximizing periodic orbit.

As a final step, the proof of proposition (2.1) reconducts in an arbitrary Hölder setting, changing only (with the notations of the proof) for any $y \in \mathbb{R}\backslash\mathbb{Z}$ the "small pick function" η_y at y by the "$\alpha-$Hölder small pick function" at y defined by $z \longmapsto (\varepsilon^{1+\alpha} - \varepsilon \mathrm{Dist}(z,y)^\alpha) \vee 0$, for small $\varepsilon > 0$.

This completes the proof of the theorem.

□

Remark. — Let us stress that the perturbation step in the previous proof uses only the last part, namely r, of the decomposition $f = \beta(f) + (T\varphi - \varphi) - r$. On the contrary, the demonstration of theorem (1.7) was fondamentally using r as well as the coboundary part.

5 Concluding remarks and related open questions

The present study heavily uses the assumption of finiteness on the number of connected components of $r^{-1}\{0\}$. This allowed a geometrical approach which may have to be combined with a more abstract strategy to treat the case of an arbitrary $r^{-1}\{0\}$. Indeed observe for instance that taking any closed set F in some semi-circle $[a, a+1/2]$, then $G = F \cup ([a, a+1/2]\backslash F^0 + 1/2)$ is *exactly* the level set $r^{-1}\{0\}$ of some $r \in Lip(X)$ verifying $r \geq 0$ and $r.\tau r = 0$. Among difficulties, ∂G can have arbitrary large Lebesgue measure in $\mathbb{R}\backslash\mathbb{Z}$.

In the downhill perspective mentionned in the introduction and hoping that some degeneracy holds for maximizing measures, at least for a dense subset in $Lip(X)$, it would be interesting to determine geometrical conditions on such a level set ensuring that its maximal T-invariant compact subset is uniquely ergodic or at least has zero topological entropy. In a next step however, the difficulty would be that comes into play the a priori huge class of T-invariant measures with zero entropy.

We conclude this set of remarks by considering $C^1(X)$ endowed with its natural topology. Mention without proofs first that theorem (1.7) is valid for $C^1(X)$ and second that the analogue of conjecture (1.5) follows from the assumption :

Conjecture 5.1
There is a dense subset in $C^1(X)$ of maps f that can be decomposed as $f = \beta(f) + (T\varphi - \varphi) - r$, where φ is bounded and r is a **piecewise**-*C^1 map such that $r \geq 0$, $r.\tau r = 0$.*

Indeed, denote by H the subset of $\partial\{r > 0\}$ consisting in points x limit of a sequence in $(r^{-1}\{0\})\backslash\{x\}$. Taking $x \in H$, if r' exists and is continuous at x, then r can be neglected in the C^1-topology in a neighbourhood of x, since $r(x) = r'(x) = 0$. Under conjecture (5.1), only finitely many points in H would not satisfy this property and in fact the proof of theorem (1.6) can be reconducted. A very interesting result in direction of conjecture (5.1) is lemma 1 in [3].

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