

The singularity spectrum of the Fish’s boundary

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Abstract. Let $\mathcal{M}(\mathbb{T}^1, T)$ be the convex set of Borel probability measures on the Circle \mathbb{T}^1 invariant under the action of the transformation $T : x \mapsto 2x \pmod{1}$. Its projection on the complex plane by the application $\mu \mapsto \int e^{2i\pi x} d\mu(x)$ is a compact convex of the unit disc, symmetric with respect to the x -axis, called the “Fish” by T. Bousch [3]. Seeing the boundary of the upper half-Fish as a function, we focus on its local regularity. We show that its multifractal spectrum is concentrated at ∞ , but that every pointwise regularity $\alpha \in [1, \infty]$ is realized in a non-denumerable and dense set of points. The results rely on fine properties of Sturm measures.

1. Introduction

Multifractal analysis describes the fine local structure of functions or measures. On typical examples, the pointwise regularity exponent varies erratically from one point to another, and the level sets corresponding to a given regularity are usually fractal sets. The purpose of Multifractal analysis is to determine the Hausdorff dimension of these sets.

The interest for multifractal analysis came from fluid mechanics and also dynamical systems, see among many references [12, 14, 21]. Since then, multifractal analysis has developed in many contexts, for instance in Probability Theory [15, 1] (see [11, 2, 16] for other examples). In this article, we consider the example of a graph naturally appearing in an optimization problem in Ergodic Theory.

The notion of regularity we discuss in the sequel is the following. Given a real function $f \in L_{loc}^\infty$ on an open interval I and $x_0 \in I$, recall that f belongs to $C^\alpha(x_0)$,

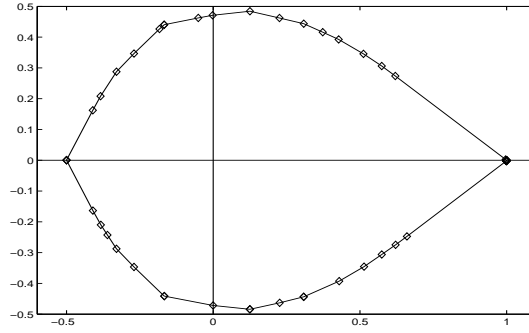


FIGURE 1. The Fish.

for some $\alpha \geq 0$, if there exist a polynomial P of degree at most $\lfloor \alpha \rfloor$ and a constant $C > 0$ such that locally :

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \quad (1)$$

The local regularity of f at x_0 is measured by the *pointwise Hölder exponent* :

$$h_f(x_0) = \sup\{\alpha \geq 0 \mid f \in \mathcal{C}^\alpha(x_0)\}.$$

The relevant information is then provided by the *spectrum of singularities* d_f of f , which is the application :

$$d_f : s \in [0, \infty] \longmapsto \text{Dim}_{\mathcal{H}}\{x_0 \in I \mid h_f(x_0) = s\},$$

where $\text{Dim}_{\mathcal{H}}$ stands for the Hausdorff dimension. We adopt the convention that $\text{Dim}_{\mathcal{H}}\emptyset = -\infty$.

We now detail the context of our example. Let \mathbb{T}^1 be the torus identified with \mathbb{R}/\mathbb{Z} and equipped with the transformation $T(x) = 2x \bmod (1)$. Introduce the convex set $\mathcal{M}(\mathbb{T}^1, T)$ of Borel probability measures on \mathbb{T}^1 invariant by T , endowed with the weak* topology. The *Fish* is the compact convex subset of \mathbb{C} , image of the following linear map :

$$\begin{aligned} \mathcal{M}(\mathbb{T}^1, T) &\longrightarrow \mathbb{C} \\ \mu &\longmapsto \int e^{2i\pi u} d\mu(u). \end{aligned}$$

The boundary of the Fish intersects the horizontal axis at the points $(-1/2, 0)$ and $(1, 0)$ and is symmetric with respect to this axis, since T commutes with the symmetry $x \mapsto -x$ on \mathbb{T}^1 . We shall then restrict our study to the upper half-Fish, whose boundary is denoted by \mathcal{F} . The function \mathcal{F} is a concave function :

$$\begin{aligned} [-1/2, 1] &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \mathcal{F}(x). \end{aligned}$$

The goal of this paper is to determine the pointwise Hölder exponent and the singularity spectrum of the function \mathcal{F} .

As a preliminary remark, let us mention that, with respect to Fourier coefficients, there is only one Fish. Indeed, let $k \geq 2$ and consider the k^{th} Fourier coefficient of the elements of $\mathcal{M}(\mathbb{T}^1, T)$, that is the linear map :

$$\begin{aligned} \mathcal{M}(\mathbb{T}^1, T) &\longrightarrow \mathbb{C} \\ \mu &\longmapsto \int e^{2i\pi ku} d\mu(u). \end{aligned}$$

Then the image of this map is also the Fish. This is a consequence of the fact that $\int e^{2i\pi ku} d\mu(u) = \int e^{2i\pi u} d(T_k\mu)(u)$, where $T_kx = kx \pmod{1}$ on \mathbb{T}^1 and $T_k\mu$ is T -invariant. Reciprocally, fixing $\nu \in \mathcal{M}(\mathbb{T}^1, T)$, there always exists some μ invariant under T such that $T_k\mu = \nu$, for instance we can consider

$$\mu = \frac{1}{k} \sum_{0 \leq j \leq k-1} \nu(\cdot/k + j/k).$$

The Fish was introduced by Bousch [3] and Jenkinson [17], who considered the question of finding maximizing measures for a degree one trigonometric polynomial $f_\omega : x \mapsto \cos 2\pi(x - \omega)$, $\omega \in \mathbb{R}/\mathbb{Z}$. More generally, fixing some continuous $f : \mathbb{T}^1 \rightarrow \mathbb{R}$, the initial problematic is given by the variational problem :

$$\beta(f) = \sup \left\{ \int f d\mu \mid \mu \in \mathcal{M}(\mathbb{T}^1, T) \right\},$$

where one aims at describing the measures realizing the maximum. Such measures, which always exist as $\mathcal{M}(\mathbb{T}^1, T)$ is compact, are called *maximizing measures* for f . The link with the Fish is simply that if z_ω and z_x are the vectors of \mathbb{R}^2 with respective affixes $e^{2i\pi\omega}$ and $e^{2i\pi x}$, then $\langle z_\omega, z_x \rangle = \cos 2\pi(x - \omega)$. Therefore a maximizing measure μ of the function $x \mapsto \cos 2\pi(x - \omega)$ is such that $\int e^{2i\pi u} d\mu(u)$ realizes the maximal orthogonal projection of the Fish on the line going through the origin and with angle $2\pi\omega$. We often adopt this point of view in the sequel.

The question of finding maximizing measures can be viewed as a statistical approach (via the Birkhoff Ergodic Theorem) to the difficult problem of studying the best pointwise growth of the ergodic sums $(f(x) + f(Tx) + \dots + f(T^{n-1}x))_{n \geq 0}$ of a function f . General presentations of the topic can be found in Conze-Guivarc'h [10], Bousch-Mairesse [4] or Jenkinson [17], [18]. See also [6].

Back to the regularity of \mathcal{F} , concavity implies that the pointwise Hölder exponent is always larger or equal to 1 and that \mathcal{F} is differentiable outside a at most denumerable subset. Here is our main result.

THEOREM 1.1. *The singularity spectrum of \mathcal{F} is*

$$d_{\mathcal{F}}(s) = \begin{cases} -\infty & \text{if } s \in [0, 1), \\ 0 & \text{if } s \geq 1, \\ 1 & \text{if } s = +\infty. \end{cases}$$

More precisely :

1. For all x outside a zero Hausdorff-dimensional subset of $[-1/2, 1]$, there exist constants $C > 0$, $0 < \rho < 1$ and $0 < K \leq 1/2$ such that :

$$\forall h, |\mathcal{F}(x+h) - \mathcal{F}(x) - h\mathcal{F}'(x)| \leq C\rho^{|h|^{-K}}. \quad (2)$$

In particular, $h_{\mathcal{F}}(x) = \infty$.

2. For every $1 \leq s < +\infty$, the level set $\{x : h_{\mathcal{F}}(x) = s\}$ is an uncountable and dense subset of $[-1/2, 1]$ of Hausdorff dimension 0.

Hence, in general, \mathcal{F} is locally exponentially close to its tangent. Hence, although “the Fish has no edges” [3] (more precisely, it is strictly convex), its boundary is very flat. To some extent, this also confirms a remark made by Bousch that the Fish is well approximable by polygons with few edges. We also mention that our proof implies that the constants C, ρ, K depend on x and cannot be fixed locally in $[-1/2, 1]$.

In order to compute the pointwise Hölder exponents and the singularity spectrum of \mathcal{F} , we use a natural parametrization of the Fish’s boundary, given by Bousch [3].

Let $(\nu_t)_{t \in \mathbb{R} \setminus \mathbb{Z}}$ be the family of Sturm measures on \mathbb{T}^1 , where ν_t is the Sturm measure with rotation number t . Precise definitions are given in the next section. Theorem A and Corollary 2 of [3] imply that any map $x \mapsto \cos 2\pi(x - \omega)$ admits a unique maximizing measure which is a Sturm measure, and then that a bijective and bicontinuous parametrization of the upper half-Fish is given by :

$$\begin{aligned} [0, 1/2] &\longrightarrow \mathbb{C} = \mathbb{R}^2 \\ t &\longmapsto \int e^{2i\pi u} d\nu_t(u) =: I(t) = (x(t), y(t)). \end{aligned}$$

In the sequel the notations $t \mapsto I(t) = (x(t), y(t))$ are reserved to the above parametrization. Remark that when t increases from 0 to $1/2$, the graph of \mathcal{F} is described from the right side to the left side. Moreover, $\nu_0 = \delta_0$ and $\nu_{1/2} = 1/2(\delta_{1/3} + \delta_{2/3})$, explaining the extremal values.

It is shown by Bousch [3] that \mathcal{F} admits an angular point at $x(t)$ if and only if $t \in [0, 1/2] \cap \mathbb{Q}$. In particular the points $(-1/2, 0)$ and $(1, 0)$ are angular points of the Fish. The symmetry of the Fish with respect to the x -axis then implies :

$$-\infty < \mathcal{F}'_-(1) < \mathcal{F}'_+(-1/2) < +\infty. \quad (3)$$

We shall use these informations in the sequel.

The set of angular points of \mathcal{F} is denumerable and dense in $[-1/2, 1]$. Moreover, at such a point x_0 the Hölder exponent of \mathcal{F} is obviously equal to 1. We next deal with the non-angular points of \mathcal{F} , i.e. the real numbers $x(t)$ with $t \in [0, 1/2] \setminus \mathbb{Q}$. As a preliminary step, we precise the regularity of the maps $t \mapsto I(t)$ and $t \mapsto x(t)$.

THEOREM 1.2. *Let $c_0 = 2 \sum_{n \geq 1} n \sin(\pi 2^{-n-1}) > c_1 = \sum_{n \geq 1} n(1 - \cos(\pi 2^{-n}))$.*

1. *The map $t \mapsto I(t)$ is c_0 -Lipschitz, differentiable at $t \in [0, 1/2] \setminus \mathbb{Q}$ and left and right-differentiable but not differentiable at $t \in (0, 1/2) \cap \mathbb{Q}$. Also $I \mapsto I'(t)$ is continuous when restricted to $[0, 1/2] \setminus \mathbb{Q}$.*

2. The map $t \mapsto x(t)$ is a decreasing bi-Lipschitz homeomorphism from $[0, 1/2]$ onto $[-1/2, 1]$, verifying :

$$\text{for every } t \in [0, 1/2] \setminus \mathbb{Q}, \quad x'(t) \in \frac{1}{\sqrt{1 + (\mathcal{F}'(x(t)))^2}} [-c_0, -c_1].$$

If $t \in [0, 1/2] \setminus \mathbb{Q}$, denote by $(p_n/q_n)_{n \geq 0}$ its sequence of convergents. The regularity of \mathcal{F} at $x(t)$ is then read on the Diophantine properties of t .

THEOREM 1.3. *Let $t \in [0, 1/2] \setminus \mathbb{Q}$, with convergents $(p_n/q_n)_{n \geq 0}$. Introduce :*

$$M(t) = \liminf_{n \rightarrow +\infty} \frac{q_n}{\log_2 q_{n+1}}.$$

1. We have the following relation :

$$1 + M(t) = \sup\{\alpha \geq 0 \mid \exists C > 0, |\mathcal{F}(x(t) + h) - \mathcal{F}(x(t)) - h\mathcal{F}'(x(t))| \leq C h^\alpha\}.$$

This yields $h_{\mathcal{F}}(x(t)) \geq 1 + M(t)$.

2. Equality between $h_{\mathcal{F}}(x(t))$ and $1 + M(t)$ holds in the following situations :

(a) $M(t) = +\infty$.

(b) $M(t) \in \mathbb{R}^+ \setminus \{2m + 1 \mid m \geq 0\}$.

(c) $M(t) \in \{2m + 1 \mid m \geq 0\}$ and $\sup_n \{q_n^{M(t)+1} 2^{-q_n} q_{n+1}^{M(t)}\} = \infty$.

Fix $M_0 > 0$ (resp. $M_0 = 0$). If $t \in [0, 1/2] \setminus \mathbb{Q}$ is highly Liouville in the sense that

$$q_{n+1} \sim (2^{1/M_0})^{q_n} \quad (\text{resp. } q_{n+1} \sim 2^{q_n^2}), \quad (4)$$

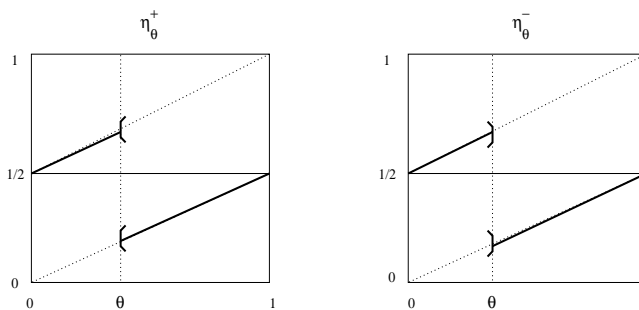
then $h_{\mathcal{F}}(x(t)) = 1 + M_0$. Since it is known that the set of Liouville numbers satisfying (4) for a given $M_0 > 0$ (resp. $M_0 = 0$) is an uncountable dense subset of $[-1/2, 1]$, item (2) of Theorem 1.1 is deduced from this remark.

2. On Sturm measures

We sum up the informations on the family of Sturm measures that are used in the sequel. Details can be found in Morse-Hedlund [20], Bullet-Sentenac [9] and Bousch [3]. The classical notion of rotation number for homeomorphisms of the Circle is introduced in Katok-Hasselblatt [19]. Proofs in the below discontinuous context are given in [7] and [5].

DEFINITION 2.1. *For $0 \leq \theta < 2$, the closed semi-circle $[\theta/2, \theta/2 + 1/2] \subset \mathbb{T}^1$ supports one and only one Borel T -invariant probability measure. Such a measure is ergodic and is called a Sturm measure.*

Distinct semi-circles may support the same Sturm measure, so a parametrization of these measures by the family of semi-circles is not intrinsic. In order to get a proper parametrization, we need the notion of rotation number of a Sturm measure.

FIGURE 2. Graphs of $\eta_{\theta,+}$ and $\eta_{\theta,-}$.

First, a natural way of constructing the Sturm measure with support in $[\theta/2, \theta/2 + 1/2]$ is to introduce the transformations $\eta_{\theta,+}$ and $\eta_{\theta,-}$ of \mathbb{T}^1 verifying $T \circ \eta_{\theta,\pm} = Id$ and defined by :

$$\eta_{\theta,\pm}(x) = \frac{1}{2}(x + \epsilon_{\theta}(x)), \quad \forall x \neq \theta \pmod{1},$$

where $\epsilon_{\theta}(x) \in \{0, 1\}$ is chosen so that $\eta_{\theta,\pm}(x) \in (\theta/2, \theta/2 + 1/2)$. Complete the definition by setting $\eta_{\theta,+}(\theta) = \theta/2$ and $\eta_{\theta,-}(\theta) = \theta/2 + 1/2$. The graphs of $\eta_{\theta,\pm}$ are plotted in Figure 2.

Notice that $\eta_{\theta,+}$ is right-continuous, whereas $\eta_{\theta,-}$ is left-continuous. Concretely, $\eta_{\theta,+}$ acts on \mathbb{T}^1 as follows : the Circle \mathbb{T}^1 is cut into an interval at θ , is linearly contracted by a $1/2$ and then rotated to the semi-circle $[\theta/2, \theta/2 + 1/2]$. For $\eta_{\theta,-}$, the image interval is this time $(\theta/2, \theta/2 + 1/2]$. The transformations $\eta_{\theta,\pm}$ are examples of quasi-contracting maps. More on this topic can be found in [5, 8, 13].

It is an observation (see [5], Lemma 3.2) that a Borel measure μ on \mathbb{T}^1 is T -invariant and with support in $[\theta/2, \theta/2 + 1/2]$ if and only if it is invariant under either $\eta_{\theta,+}$ or $\eta_{\theta,-}$. The maps $\eta_{\theta,+}$ and $\eta_{\theta,-}$ are order-preserving transformations of the Circle \mathbb{T}^1 . They admit a rotation number, the same one for both, written as $t \in \mathbb{R}/\mathbb{Z}$. Recall that any order-preserving transformation $\chi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ has a rotation number τ_{χ} . This quantity measures the average speed of rotation under iterations and is defined as :

$$\tau_{\chi} = \lim_{n \rightarrow +\infty} \frac{1}{n} (\tilde{\chi})^n(x) \pmod{1},$$

where $\tilde{\chi} : \mathbb{R} \rightarrow \mathbb{R}$ is any lift of χ . This limit is independent of x .

Reciprocally, it can be shown that for any $t \in \mathbb{R}/\mathbb{Z}$, there is a closed interval of parameters θ for which the applications $\eta_{\theta,+}$ and $\eta_{\theta,-}$ have rotation number t . Moreover, all these applications correspond to a unique Sturm measure. This measure will be written as ν_t in the sequel. Let us detail the relations between the rational character of t , the corresponding parameters θ and the support of ν_t in \mathbb{T}^1 :

1. When $t \in [0, 1) \setminus \mathbb{Q}$, there is a unique $\theta_t \in [0, 1)$ such that $\text{supp}(\nu_t) \subset [\theta_t/2, \theta_t/2 + 1/2]$. In this case ν_t is diffusive and its support is a minimal and uniquely ergodic Cantor set.

2. Suppose now that $t = p/q \in [0, 1)$, $p \wedge q = 1$.
- (a) There is a closed interval $[\theta_{p/q}^-, \theta_{p/q}^+]$ of length $1/(2^q - 1)$ such that :

$$\text{supp}(\nu_{p/q}) \subset [\theta/2, \theta/2 + 1/2], \text{ for } \theta \in [\theta_{p/q}^-, \theta_{p/q}^+].$$

The points $\theta_{p/q}^+$ and $\theta_{p/q}^-$ are q -periodic and lay in the same T -orbit. In this case, $\nu_{p/q}$ is the T -invariant periodic measure supported by this orbit. Mention that $\theta_{p/q}^+/2$ is periodic under T , whereas $\theta_{p/q}^+/2 + 1/2$ is not. Symmetrically, $\theta_{p/q}^-/2 + 1/2$ is periodic under T , whereas $\theta_{p/q}^-/2$ is not. In order to unify the proofs, we set $\theta_t^+ = \theta_t^- = \theta_t$, when $t \in [0, 1/2] \setminus \mathbb{Q}$.

- (b) If $0 \leq p/q < p'/q' < 1$ are adjacent rationals, in the sense that $p'q - pq' = 1$, we will use the information that :

$$\theta_{p'/q'}^- - \theta_{p/q}^+ = (2^q - 1)^{-1}(2^{q'} - 1)^{-1}, \quad (5)$$

given in the proof of Lemma 2 of Bullett-Sentenac [9]. It is a consequence of the following relations : $T^q(\theta_{p'/q'}^-) = \theta_{p'/q'}^+$ and $T^{q'}(\theta_{p/q}^+) = \theta_{p/q}^-$.

3. Another property is that $\cup_{p/q \in [0, 1), p \wedge q = 1} [\theta_{p/q}^-, \theta_{p/q}^+]$ has full measure in $[0, 1)$. As a corollary, the mapping $\theta \mapsto t$ is a continuous non-decreasing Devil staircase from $[0, 1)$ to $[0, 1)$

We finally develop the connexions between the maps $\eta_{\theta, \pm}$ and the rational character of the rotation number t of the Sturm measure ν_t . For any $\gamma \in \mathbb{T}^1$, introduce first the open semi-circle $U_\gamma = (\gamma/2 + 1/2, \gamma/2) \subset \mathbb{T}^1$ complementary to $[\gamma/2, \gamma/2 + 1/2]$. We sum up some results contained in Lemma 3.2 and Proposition 4.8 of [5] :

PROPOSITION 2.1.

1. The sets $(\eta_{\theta, +}^n(U_\theta))_{n \geq 0}$ are all disjoint and their union has full Lebesgue measure in \mathbb{T}^1 .
2. If $t \in [0, 1] \setminus \mathbb{Q}$, then the sets $(\eta_{\theta_t, +}^n(U_{\theta_t}))_{n \geq 0}$ are intervals. Moreover, each one can be written as $\eta_{\theta_t, +}^n(U_{\theta_t}) = (\eta_{\theta_t, -}^{n+1}(\theta_t), \eta_{\theta_t, +}^{n+1}(\theta_t))$, and has length 2^{-n-1} .
3. Let $q \geq 1$. A real number θ is in the closure of $\eta_{\theta, +}^{q-1}(U_\theta)$ if and only if there exists $0 \leq p < q$ with $p \wedge q = 1$ such that the rotation number of $\eta_{\theta, +}$ is p/q . This property is equivalent to saying that $\theta \in \cup_{0 \leq p < q: p \wedge q = 1} [\theta_{p/q}^-, \theta_{p/q}^+]$. In this case, the sets $\eta_{\theta, +}^{q-1}(U_\theta)$ are not always intervals. More precisely,

$$\begin{cases} 0 \leq n \leq q-1 : (\eta_{\theta, +})^n(U_\theta) = ((\eta_{\theta, -})^{n+1}(\theta), (\eta_{\theta, +})^{n+1}(\theta)) \\ n \geq q : (\eta_{\theta, +})^n(U_\theta) = ((\eta_{\theta, -})^{n+1}(\theta), (\eta_{\theta, -})^{n+1-q}(\theta)) \\ \cup ((\eta_{\theta, +})^{n+1-q}(\theta), (\eta_{\theta, +})^{n+1}(\theta)). \end{cases} \quad (6)$$

In the extremal cases :

$$\eta_{\theta_{p/q}^+, +}^q(\theta_{p/q}^+) = \theta_{p/q}^+ \text{ and } \eta_{\theta_{p/q}^-, -}^q(\theta_{p/q}^-) = \theta_{p/q}^-.$$

Finally, for every $\theta \in (\theta_{p/q}^-, \theta_{p/q}^+)$ and $n \geq 1$, we have $\eta_{\theta,-}^n(\theta) = 2^{-n}\theta + Z_n$, for some fixed Z_n , which depends only on p/q .

The following result will be used several times :

PROPOSITION 2.2.

Let $t \in [0, 1)$ and fix $n \geq 1$. Then $\eta_{\theta,-}^n(\theta)$ tends to $\eta_{\theta_t^+,-}^n(\theta_t^+)$, as $\theta \rightarrow \theta_t^+$ with $\theta > \theta_t^+$. If $t \in (0, 1]$, then $\eta_{\theta,+}^n(\theta)$ tends to $\eta_{\theta_t^-,+}^n(\theta_t^-)$, as $\theta \rightarrow \theta_t^-$ with $\theta < \theta_t^-$.

Proof. Consider the first situation. As θ_t^+ is the only discontinuity of $\eta_{\theta_t^+,-}$, the result is clear in the case when $\theta_t^+ \neq \eta_{\theta_t^+,-}^k(\theta_t^+)$, for $1 \leq k \leq n-1$. Suppose then that this is not true and let k be such that $\theta_t^+ = \eta_{\theta_t^+,-}^k(\theta_t^+)$, whereas $\theta_t^+ \neq \eta_{\theta_t^+,-}^l(\theta_t^+)$, for $1 \leq l < k$. By item (3) of Proposition 2.1, t would be of the form r/k . Then, the condition $\theta_t^+ = \eta_{\theta_t^+,-}^k(\theta_t^+)$ implies that $\theta_t^+ = \theta_{r/k}^+ = \theta_{r/k}^-$, which is impossible. \square

3. Preliminary results

As a first remark, \mathcal{F} is obtained by integrating twice a sum of Dirac masses.

LEMMA 3.1. In $(-1/2, 1)$, the second derivative \mathcal{F}'' of \mathcal{F} in the sense of Distributions is a sum of Dirac masses at the angular points of \mathcal{F} :

$$\mathcal{F}'' = \sum_{t \in (0, 1/2) \cap \mathbb{Q}} [\mathcal{F}'_+(x(t)) - \mathcal{F}'_-(x(t))] \delta_{x(t)}.$$

Proof. From Corollary 1 of [3], the maximizing measure of $x \mapsto \cos 2\pi(x - \omega)$ is periodic for λ -almost all $\omega \in [0, 1/2]$. A reformulation is that for λ -almost all $\omega \in [0, 1/2]$, the maximal orthogonal projection of the Fish on the straight line going through 0 and with angle $2\pi\omega$ is realized by an angular point. Thus for $-1/2 < a < b < 1$:

$$\sum_{x(p/q) \in (a, b)} \text{atan}(\mathcal{F}'_-(x(p/q))) - \text{atan}(\mathcal{F}'_+(x(p/q))) = \text{atan}(\mathcal{F}'_+(a)) - \text{atan}(\mathcal{F}'_-(b)),$$

which can be rewritten as :

$$\sum_{x(p/q) \in (a, b)} \int_{\mathcal{F}'_+(x(p/q))}^{\mathcal{F}'_-(x(p/q))} \frac{1}{1+u^2} du = \int_{\mathcal{F}'_-(b)}^{\mathcal{F}'_+(a)} \frac{1}{1+u^2} du$$

and equivalently :

$$\cup_{x(p/q) \in (a, b)} [\mathcal{F}'_+(x(p/q)), \mathcal{F}'_-(x(p/q))] = [\mathcal{F}'_-(b), \mathcal{F}'_+(a)], \lambda - \text{a.s.},$$

where the union is disjoint. Consequently :

$$\sum_{x(p/q) \in (a, b)} \mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q)) = \mathcal{F}'_+(a) - \mathcal{F}'_-(b),$$

which implies the lemma. \square

We now show that the angular defect of \mathcal{F} at each angular point $x(p/q)$, with $p \wedge q = 1$, has exact order $q2^{-q}$.

PROPOSITION 3.1. *There is a constant $C > 0$ such that for any $p/q \in (0, 1/2)$, with $p \wedge q = 1$:*

$$\frac{1}{C} q2^{-q} \leq \mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q)) \leq C q2^{-q}. \quad (7)$$

The proof of Proposition 3.1 is based on the next lemma. As in Bousch [3], we introduce the exit time $E_\gamma(x) \in \mathbb{N} \cup \{\infty\}$ of the semi-circle $[\gamma/2, \gamma/2 + 1/2]$ for a point $x \in \mathbb{T}^1$, under iterations of T . This map belongs to $L^1(\mathbb{T}^1)$. We denote, for every $\gamma \in [0, 1)$ the quantity

$$J(\gamma) = \int e^{2i\pi u} E_\gamma(u) du. \quad (8)$$

LEMMA 3.2.

1. For $t \in [0, 1/2)$:

$$\lim_{p/q \rightarrow t, t < p/q, p \wedge q = 1} \frac{2^q}{q} \left(J(\theta_{p/q}^+) - J(\theta_{p/q}^-) \right) = \xi_t^+, \quad (9)$$

$$\text{where } \xi_t^+ = \sum_{n \geq 1} 2^{-n} e^{2i\pi \eta_{\theta_t^+, -}^n} \left(1 - e^{2i\pi 2^{-n}} \right).$$

2. For $t \in (0, 1/2]$:

$$\lim_{p/q \rightarrow t, t > p/q, p \wedge q = 1} \frac{2^q}{q} \left(J(\theta_{p/q}^+) - J(\theta_{p/q}^-) \right) = \xi_t^-, \quad (10)$$

$$\text{where } \xi_t^- = \sum_{n \geq 1} 2^{-n} e^{2i\pi \eta_{\theta_t^-, +}^n} \left(e^{-2i\pi 2^{-n}} - 1 \right).$$

Proof. Fixing $0 \leq \gamma < 1$, we first rewrite $J(\gamma)$. It is readily checked that for $u \in \mathbb{T}^1$, $E_\gamma(u)$ is the integer $n \geq 0$ such that $u \in \eta_{\gamma, +}^n(U_\gamma)$, quantity defined λ -a.s. Consequently :

$$J(\gamma) = \sum_{n \geq 1} n \int_{\eta_{\gamma, +}^n(U_\gamma)} e^{2i\pi u} du. \quad (11)$$

Let $p/q \in (0, 1/2)$ with $p \wedge q = 1$ and fix γ and γ' such that $\theta_{p/q}^- < \gamma < \gamma' < \theta_{p/q}^+$. Using (6) and (11), we write $J(\gamma') - J(\gamma) = A + B + C$, where :

$$\begin{aligned} A &= \sum_{n=1}^{q-1} n \left(\int_{\eta_{\gamma', -}^{n+1}(\gamma')} e^{2i\pi u} du - \int_{\eta_{\gamma, -}^{n+1}(\gamma)} e^{2i\pi u} du \right) \\ B &= \sum_{n \geq 0} (n+q) \left(\int_{\eta_{\gamma', -}^{n+q+1}(\gamma')} e^{2i\pi u} du - \int_{\eta_{\gamma, -}^{n+q+1}(\gamma)} e^{2i\pi u} du \right) \\ C &= \sum_{n \geq 0} (n+q) \left(\int_{\eta_{\gamma', +}^{n+q+1}(\gamma')} e^{2i\pi u} du - \int_{\eta_{\gamma, +}^{n+q+1}(\gamma)} e^{2i\pi u} du \right). \end{aligned}$$

Consider first A . We use two informations:

- For $\gamma \in (\theta_{p/q}^-, \theta_{p/q}^+)$ and $n \geq 1$, $\eta_{\gamma,-}^n(\gamma) = 2^{-n}\gamma + Z_n$, by proposition 2.1.
- For $\gamma \in [\theta_{p/q}^-, \theta_{p/q}^+]$ and $1 \leq n \leq q-1$, $\eta_{\gamma,+}^{n+1}(\gamma) - \eta_{\gamma,-}^{n+1}(\gamma) = 2^{-n-1}$.

Hence

$$\begin{aligned} A &= \sum_{n=1}^{q-1} n \left(\int_{2^{-n-1}\gamma' + Z_{n+1} - 2^{-n-1}}^{2^{-n-1}\gamma' + Z_{n+1}} e^{2i\pi u} du - \int_{2^{-n-1}\gamma + Z_{n+1} - 2^{-n-1}}^{2^{-n-1}\gamma + Z_{n+1}} e^{2i\pi u} du \right) \\ &= \sum_{n=1}^{q-1} n e^{2i\pi \eta_{\gamma,+}^{n+1}(\gamma)} \left(e^{2i\pi 2^{-n-1}(\gamma' - \gamma)} - 1 \right) \int_{-2^{-n-1}}^0 e^{2i\pi u} du. \end{aligned}$$

Since $|e^{iu} - 1| \leq |u|$ and $\gamma' - \gamma < \theta_{p/q}^+ - \theta_{p/q}^- = 1/(2^q - 1)$, we deduce the upper-bound :

$$|A| \leq \frac{\pi}{2(2^q - 1)} \sum_{n \geq 1} n 4^{-n}. \quad (12)$$

We now show that B and C have a higher order. Using the same informations as above, we get :

$$\begin{aligned} B &= \frac{1}{2i\pi} \sum_{n \geq 0} (n+q) \left[e^{2i\pi \eta_{\gamma,-}^{n+1}(\gamma)} \left(e^{2i\pi 2^{-n-1}(\gamma' - \gamma)} - 1 \right) \right. \\ &\quad \left. - e^{2i\pi \eta_{\gamma,-}^{n+q+1}(\gamma)} \left(e^{2i\pi 2^{-n-q}(\gamma' - \gamma)} - 1 \right) \right]. \end{aligned}$$

Let then $t \in [0, 1/2)$ and suppose that $p/q \rightarrow t$, with $p/q > t$, $p \wedge q = 1$. We fix $n \geq 0$. By proposition 2.2, $\eta_{\gamma,-}^{n+1}(\gamma) \rightarrow \eta_{\theta_t^+,-}^{n+1}(\theta_t^+)$, when $q \rightarrow +\infty$. Moreover, still for $q \rightarrow +\infty$, the quantity $e^{2i\pi 2^{-n-1}(\gamma' - \gamma)} - 1$ is equivalent to $2i\pi 2^{-n-1}(\gamma' - \gamma)$ and the last term in the right-hand side above has a strictly lower order.

Therefore, uniformly in $\theta_{p/q}^- < \gamma < \gamma' < \theta_{p/q}^+$:

$$\frac{B}{q(\gamma' - \gamma)} \longrightarrow \sum_{n \geq 0} 2^{-n-1} e^{2i\pi \eta_{\theta_t^+,-}^{n+1}(\theta_t^+)}, \text{ as } p/q \rightarrow t, t < p/q, p \wedge q = 1. \quad (13)$$

In a similar way, uniformly in $\theta_{p/q}^- < \gamma < \gamma' < \theta_{p/q}^+$:

$$\frac{C}{q(\gamma' - \gamma)} \longrightarrow - \sum_{n \geq 0} 2^{-n-1} e^{2i\pi \left(\eta_{\theta_t^+,-}^{n+1}(\theta_t^+) + 2^{-n-1} \right)}, \text{ as } p/q \rightarrow t, t < p/q, p \wedge q = 1. \quad (14)$$

Using finally the fact that $\theta_{p/q}^+ - \theta_{p/q}^- = 1/(2^q - 1)$ with (12), (13) and (14), the first item is proved.

The second item is shown in the same way. \square

We now move to Proposition 3.1.

Proof. For every irreducible rational p/q , let $[\omega_{p/q,-}, \omega_{p/q,+}]$ be such that for every $\omega \in [\omega_{p/q,-}, \omega_{p/q,+}]$, the maximizing measure of $x \mapsto \cos 2\pi(x - \omega)$ is $\nu_{p/q}$. Then :

$$\omega_{p/q,+} - \omega_{p/q,-} = \text{atan}(\mathcal{F}'_-(x(p/q))) - \text{atan}(\mathcal{F}'_+(x(p/q))).$$

Remembering (3), we have $-\infty < \mathcal{F}'_-(1) \leq \mathcal{F}'_+(x) < \mathcal{F}'_-(x) \leq \mathcal{F}'_+(-1/2) < +\infty$. Hence, $\omega_{p/q,+} - \omega_{p/q,-}$ is proportional to $\mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q))$ with constants not depending on p/q . We thus focus on $\omega_{p/q,+} - \omega_{p/q,-}$.

From Bousch [3], for any ω , the maximizing measure of $x \mapsto \cos 2\pi(x - \omega)$ has support in $[\gamma, \gamma + 1/2]$ and the parameter ω is uniquely determined in terms of γ by the two conditions (Proposition p. 503 and Remark p. 506 in [3]) :

$$e^{2i\pi(\omega+1/4)} \perp J(\gamma) \quad \text{and} \quad |\gamma + 1/4 - \omega| \leq 0,111, \quad (15)$$

where the first expression is a shorthand notation for orthogonality of the corresponding vectors of the plane. It is also known that $J(\gamma) \neq 0$. Theorem B in Bousch [3] indicates that the correspondance $\gamma \mapsto \omega$ is a homeomorphism with a modulus of continuity of the form $Kx \log(1/x)$. If $p/q \in (0, 1/2)$, $p \wedge q = 1$, since $\theta_{p/q}^+ - \theta_{p/q}^- = 1/(2^q - 1)$, we get $\omega_{p/q,+} - \omega_{p/q,-} \leq C q2^{-q}$ for some universal constant $C > 0$. This gives the right-hand side inequality in (7).

We now prove the other direction. Suppose that an infinite sequence of distinct rationals $(p_n/q_n)_{n \geq 0}$ in $(0, 1/2)$, with $p_n \wedge q_n = 1$, is such that $\frac{2^{q_n}}{q_n} (\mathcal{F}'_-(x(p_n/q_n)) - \mathcal{F}'_+(x(p_n/q_n))) \rightarrow 0$ as $n \rightarrow +\infty$. By the remarks above, this is equivalent to :

$$\frac{2^{q_n}}{q_n} (\omega_{p_n/q_n,+} - \omega_{p_n/q_n,-}) \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (16)$$

Up to extraction, we suppose that $p_n/q_n \rightarrow t \in [0, 1/2)$, $t < p_n/q_n$, $p_n \wedge q_n = 1$. In this case we shall use (9). The other case $p_n/q_n \rightarrow t \in (0, 1/2]$, $t > p_n/q_n$, $p_n \wedge q_n = 1$ is treated similarly, using (10). By (15), we have

$$\begin{cases} e^{2i\pi(\omega_{p_n/q_n,-}+1/4)} \perp J(\theta_{p_n/q_n}^-) \\ e^{2i\pi(\omega_{p_n/q_n,+}+1/4)} \perp J(\theta_{p_n/q_n}^+), \end{cases}$$

hence the angular variation between $J(\theta_{p_n/q_n}^+)$ and $J(\theta_{p_n/q_n}^-)$ equals $\omega_{p_n/q_n,+} - \omega_{p_n/q_n,-}$. But by (9), and provided that $\xi_t^+ \neq 0$, $J(\theta_{p_n/q_n}^+) - J(\theta_{p_n/q_n}^-)$ is of order $q_n 2^{-q_n}$. As both $J(\theta_{p_n/q_n}^+)$ and $J(\theta_{p_n/q_n}^-)$ tend to $J(\theta_t^+) \neq 0$ (since $\gamma \mapsto E_\gamma$ is continuous in $L^1(\mathbb{T})$, see Lemma p. 505 of [3]), it is enough to prove that $\xi_t^+ \neq 0$ and that ξ_t^+ and $J(\theta_t^+)$ do not have the same directions, so that (9) and (16) will be in contradiction.

We develop a numerical argument, already used in [3]. Set :

$$u_n = e^{2i\pi\eta_{\theta_t^+}^n, -(\theta_t^+)} (1 - e^{2i\pi 2^{-n}}), \quad n \geq 1.$$

With these notations, $\xi_t^+ = \sum_{n \geq 1} 2^{-n} u_n$. Remark that $|u_1| = 2$ and $|u_2| = \sqrt{2}$. We introduce the straight line \mathcal{D}_t supported by $e^{2i\pi(\theta_t^+/2)}$ and denote by *proj* the

orthogonal projection on \mathcal{D}_t and by $proj_{\perp}$ the orthogonal projection on $(\mathcal{D}_t)^{\perp}$. One checks that :

$$\begin{aligned} 1 - proj(u_2)/4 - \sum_{n \geq 3} 2^{-n} 2 \sin(2\pi 2^{-n-1}) &\leq |proj(\xi_t^+)| \\ proj_{\perp}(u_2)/4 + \sum_{n \geq 3} 2^{-n} 2 \sin(2\pi 2^{-n-1}) &\geq |proj_{\perp}(\xi_t^+)|. \end{aligned}$$

We next show that $|proj(\xi_t^+)| > |proj_{\perp}(\xi_t^+)|$ (which implies that $\xi_t^+ \neq 0$). Since $\sum_{n \geq 3} 2^{-n} 2 \sin(2\pi 2^{-n-1}) \leq \pi/24$, this amounts to proving that $1 - proj(u_2)/4 > proj_{\perp}(u_2)/4 + \pi/12$. Setting $\alpha = proj(u_2)$, we have to check that $4 > \alpha + \sqrt{2 - \alpha^2} + \pi/3$, $\alpha \in [0, \sqrt{2}]$. The right-hand side is maximal for $\alpha = 1$ and the proof reduces to $2 > \pi/3$, which is true.

Consequently, as claimed above, $|proj(\xi_t^+)| > |proj_{\perp}(\xi_t^+)|$. This yields :

$$Arg(\xi_t^+) \in [\theta_t^+/2 - 0, 125, \theta_t^+/2 + 0, 125] \pmod{(1/2)}.$$

Let $\omega_{t,+}$ be such that the maximizing measure of $x \mapsto \cos 2\pi(x - \omega_{t,+})$ has its support contained in $[\theta_t^+/2, \theta_t^+/2 + 1/2]$. If ξ_t^+ and $J(\theta_t^+)$ have the same direction, then $\omega_{t,+} = Arg(\xi_t^+) \pmod{(1/2)}$. Thus $\omega_{t,+} \in [\theta_t^+/2 - 0, 125, \theta_t^+/2 + 0, 125] \pmod{(1/2)}$. But by (15), we get $\omega_{t,+} \in [\theta_t^+/2 + 0, 139, \theta_t^+/2 + 0, 361]$. The two conditions on $\omega_{t,+}$ are not compatible, hence the contradiction. \square

For p/q with $p \wedge q = 1$, still denote by $[\omega_{p/q,-}, \omega_{p/q,+}]$ the interval of ω such that the maximizing measure of $x \mapsto \cos 2\pi(x - \omega)$ is $\nu_{p/q}$. As a by-product of the proof, we obtain :

COROLLARY 3.1. *There is a constant $C > 0$ such that for $p/q \in (0, 1/2)$ with $p \wedge q = 1$:*

$$\frac{1}{C} q 2^{-q} \leq \omega_{p/q,+} - \omega_{p/q,-} \leq C q 2^{-q}.$$

Moreover, if $p/q \rightarrow t$ with $p \wedge q = 1$ and $p/q < t \in (0, 1/2)$ (or $p/q > t \in [0, 1/2)$), then $(2^q/q)(\omega_{p/q,+} - \omega_{p/q,-})$ converges to a real number in $(0, +\infty)$.

Corollary 3.1 answers a question posed by Jenkinson in [17]. We now turn to another preliminary study, concerning the analysis of the regularity of $t \mapsto I(t)$ and $t \mapsto x(t)$. The next proposition implies Theorem 1.2.

PROPOSITION 3.2.

1. *Let $t \in [0, 1/2)$. Then the map $u \mapsto I(u)$ is right-differentiable at t and :*

$$I'_+(t) = \sum_{n \geq 1} n e^{2i\pi\eta_{\theta_t^+,-}^{n+1}(\theta_t^+)} \left(e^{2i\pi 2^{-n-1}} - 1 \right).$$

2. *Let $t \in (0, 1/2]$. Then $u \mapsto I(u)$ is left-differentiable at t and :*

$$I'_-(t) = \sum_{n \geq 1} n e^{2i\pi\eta_{\theta_t^+,-}^{n+1}(\theta_t^-)} \left(1 - e^{-2i\pi 2^{-n-1}} \right).$$

3. Let $t \in [0, 1/2] \setminus \mathbb{Q}$. Then $u \mapsto I(u)$ is differentiable at t and :

$$I'(t) = \sum_{n \geq 1} n \left(e^{2i\pi\eta_{\theta_t^+, +}^{n+1}(\theta_t)} - e^{2i\pi\eta_{\theta_t^-, -}^{n+1}(\theta_t)} \right). \quad (17)$$

4. The application $u \mapsto I(u)$ is not differentiable at $t \in (0, 1/2) \cap \mathbb{Q}$ and $t \mapsto I'(t)$ is continuous in restriction to $[0, 1/2] \setminus \mathbb{Q}$.

5. Let $c_0 = 2 \sum_{n \geq 1} n \sin(\pi 2^{-n-1}) > c_1 = \sum_{n \geq 1} n(1 - \cos(\pi 2^{-n}))$. Then $t \mapsto I(t)$ is c_0 -Lipschitz. Moreover, $t \mapsto x(t)$ is a decreasing bi-Lipschitz homeomorphism from $[0, 1/2]$ on $[-1/2, 1]$, with :

$$x'(t) \in -\frac{1}{\sqrt{1 + (\mathcal{F}'(x(t)))^2}} [c_0, c_1], \quad t \in [0, 1/2] \setminus \mathbb{Q}.$$

Proof. We prove (1). Consider adjacent rationals $0 \leq p/q < p'/q' < 1/2$, with $p'q - pq' = 1$ (therefore $p/q - p'/q' = 1/(qq')$). For a function f defined on \mathbb{T}^1 and $n \geq 0$, introduce the ergodic sum $S_n f(x) = \sum_{k=0}^{n-1} f(T^k x)$. Recall that $\nu_{p/q}$ is the T -invariant Sturm measure supported by the orbit of $\theta_{p/q}^+$ and $\theta_{p/q}^-$. We can write :

$$\begin{aligned} I(p/q) &= \int e^{2i\pi u} d\nu_{p/q}(u) = \frac{1}{q'} \int (S_{q'} e^{2i\pi \cdot})(u) d\nu_{p/q}(u) \\ &= \frac{1}{qq'} \sum_{0 \leq n < q, 0 \leq m < q'} e^{2i\pi 2^{n+m} \theta_{p/q}^+}. \end{aligned}$$

Similarly :

$$I(p'/q') = \frac{1}{qq'} \sum_{0 \leq n < q, 0 \leq m < q'} e^{2i\pi 2^{n+m} \theta_{p'/q'}^-}.$$

Combining the previous two equalities together with (5), we deduce that :

$$\begin{aligned} qq'(I(p/q) - I(p'/q')) &= \sum_{0 \leq n < q, 0 \leq m < q'} e^{2i\pi 2^{n+m} \theta_{p/q}^+} \left(1 - e^{2i\pi \frac{2^{n+m} - q - q'}{(1-2^{-q})(1-2^{-q'})}} \right) \\ &= \sum_{1 \leq n \leq q, 1 \leq m \leq q'} e^{2i\pi 2^{q+q'-n-m} \theta_{p/q}^+} \left(1 - e^{2i\pi \frac{2^{-n-m}}{(1-2^{-q})(1-2^{-q'})}} \right). \end{aligned}$$

Let now $t \in [0, 1/2)$. We shall show that if adjacent rationals $p/q < p'/q'$ verify $t < p/q < p'/q'$ and tend to t , then $qq'(I(p/q) - I(p'/q'))$ converges.

First, the generic expression in (18) is bounded by $8\pi 2^{-n-m}$ and the term between brackets tends to $(1 - e^{2i\pi 2^{-n-m}})$, as $\min\{q, q'\} \rightarrow +\infty$.

Second, as recalled in the section of Sturm measures, $2^{q+q'} \theta_{p/q}^+ = \theta_{p'/q'}^- \pmod{1}$, which yields $2^{q+q'-n-m} \theta_{p/q}^+ = \eta_{\theta_{p'/q'}^-, -}^{n+m}(\theta_{p'/q'}^-) \pmod{1}$. Then, by proposition 2.2, for fixed n and m , $\eta_{\theta_{p'/q'}^-, -}^{n+m}(\theta_{p'/q'}^-) \rightarrow \eta_{\theta_t^+, -}^{n+m}(\theta_t^+)$.

Finally, when $t < p/q < p'/q'$, $p'q - pq' = 1$, $p'/q' - t \rightarrow 0$:

$$\begin{aligned} qq'(I(p/q) - I(p'/q')) &\longrightarrow -Z_t^+ = - \sum_{n \geq 1, m \geq 1} e^{2i\pi \eta_{\theta_t^+, -}^{n+1}(\theta_t^+)} \left(e^{2i\pi 2^{-n-m}} - 1 \right) \\ &= - \sum_{n \geq 1} n e^{2i\pi \eta_{\theta_t^+, -}^{n+1}(\theta_t^+)} \left(e^{2i\pi 2^{-n-1}} - 1 \right). \quad (18) \end{aligned}$$

Fixing $h > 0$, the Farey construction of the rationals gives the existence of an increasing bi-infinite sequence $(p_s/q_s)_{s \in \mathbb{Z}}$ of rationals checking :

$$\left\{ \begin{array}{l} t < p_s/q_s < t + h, \text{ for } s \in \mathbb{Z}, \\ p_s/q_s \rightarrow t, \text{ as } s \rightarrow -\infty, \\ p_s/q_s \rightarrow t + h, \text{ as } s \rightarrow +\infty, \\ p_{s+1}q_s - p_sq_{s+1} = 1, \text{ for } s \in \mathbb{Z}. \end{array} \right.$$

We now decompose the increment $I(t) - I(t+h)$ as :

$$I(t) - I(t+h) = \sum_{s \in \mathbb{Z}} I(p_s/q_s) - I(p_{s+1}/q_{s+1})$$

Observe that $\min_{s \in \mathbb{Z}} q_s \rightarrow \infty$ as $h \rightarrow 0$. Consequently, combining (18) with the fact that $\sum_{s \in \mathbb{Z}} \frac{1}{q_s q_{s+1}} = h$ by construction, the previous uniform calculus easily implies that $(I(t) - I(t+h))/h - Z_t^+ \rightarrow 0$ when $h \rightarrow 0^+$. Hence $u \mapsto I(u)$ is right-differentiable at t with right-derivative Z_t^+ . The proof of point (2) is similar.

Consider items (3) and (4). Let $t \in [0, 1/2] \setminus \mathbb{Q}$. Then $\theta_t^- = \theta_t^+ = \theta_t$ and $\eta_{\theta_t, +}^n(\theta_t) - \eta_{\theta_t, -}^n(\theta_t) = 2^{-n}$, $n \geq 2$. Thus $Z_t^- = Z_t^+$ and $u \mapsto I(u)$ is differentiable at t . The continuity of $u \mapsto I'(u)$ when restricted to $[0, 1/2] \setminus \mathbb{Q}$ is a consequence of the remark that for fixed $n \geq 2$, the maps $u \mapsto \eta_{\theta_u, +}^n(\theta_u)$ and $u \mapsto \eta_{\theta_u, -}^n(\theta_u)$ are continuous at t .

Assume now that $t = r/s \in [0, 1/2] \cap \mathbb{Q}$, $r \wedge s = 1$. Since $(x(r/s), \mathcal{F}(x(r/s)))$ is an angular point for \mathcal{F} , the only possibility for I to be differentiable at r/s is to have $Z_{r/s}^+ = Z_{r/s}^- = 0$. Hence, in order to show that $u \mapsto I(u)$ is not differentiable at r/s , it is enough to show that $Z_{r/s}^+ \neq 0$.

Section 2 on Sturm measures gives that for $n \geq 2$, the intervals $[\eta_{\theta_{r/s}^+, -}^n(\theta_{r/s}^+), \eta_{\theta_{r/s}^+, -}^n(\theta_{r/s}^+) + 2^{-n}]$ are contained in $[\theta_{r/s}^+/2, \theta_{r/s}^+/2 + 1/2]$. Thus, using (18), the orthogonal projection of the vector with affix $Z_{r/s}^+$ on the line spanned by $e^{2i\pi(\theta_{r/s}^+/2)}$ is not zero. This concludes (4).

Consider (5). We first prove that $t \mapsto I(t)$ is a Lipschitz map. Via (18), for any adjacent rationals $p/q < p'/q'$, one has $|I(p/q) - I(p'/q')| \leq 8\pi|p/q - p'/q'|$. The Farey construction of the rationals implies that this relation is valid between any neighbour points on an arbitrary thin net of $[0, 1/2]$. This shows that $t \mapsto I(t)$ is 8π -Lipschitz. Consequently, $t \mapsto I(t)$ is absolutely continuous and then the integral of its derivative. The result follows from the remark that for every $t \in [0, 1/2] \setminus \mathbb{Q}$, we easily verify using (17) that :

$$|I'(t)| \leq \sum_{n \geq 1} 2n \sin(2\pi 2^{-n-2}).$$

Finally, for $t \in [0, 1/2] \setminus \mathbb{Q}$, we have $x'(t) = Re(I'(t))$. A first remark is that $|I'(t)|$ is uniformly strictly positive. Indeed for every $n \geq 2$, the interval $[\eta_{\theta_t, -}^n(\theta_t), \eta_{\theta_t, +}^n(\theta_t)]$ is contained in the semi-circle $[\theta_t/2, \theta_t/2 + 1/2]$. Projecting orthogonally the vector

with affix $I'(t)$ on the line spanned by $e^{2i\pi(\theta_t/2)}$ gives :

$$|I'(t)| \geq \sum_{n \geq 1} n(1 - \cos(\pi 2^{-n})) > 0.$$

Since $|x'(t)| = \cos(\text{atan}(\mathcal{F}'(x(t))))|I'(t)|$ and $x'(t) \leq 0$, the proof is complete. \square

4. Proof of the main Theorems

4.1. *Proof of Theorem 1.1* Fix $t \in [0, 1/2] \setminus \mathbb{Q}$ and recall that \mathcal{F} is differentiable at $x(t)$. Let $h \in \mathbb{R}$ and introduce the difference for $|h|$ small enough

$$\Delta_t(h) = \mathcal{F}(x(t)) - \mathcal{F}(x(t+h)) + \mathcal{F}'(x(t)) \cdot (x(t) - x(t+h)).$$

Remark that $\Delta_t(h) \geq 0$, as \mathcal{F} is concave. By Lemma 3.1, $\Delta_t(h)$, which is a second-order difference, depends only on the angular variation of \mathcal{F} at the irreducible rational points lying in $[t, t+h]$. This implies :

$$\Delta_t(h) \leq \sum_{p/q \in (t, t+h), p \wedge q = 1} |x(p/q) - x(t+h)| (\mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q))).$$

Then, using Proposition 3.1 ($C > 0$ is the constant appearing in (7)), we get :

$$\Delta_t(h) \leq C |x(t) - x(t+h)| \sum_{p/q \in (t, t+h), p \wedge q = 1} q 2^{-q}. \quad (19)$$

Introduce the convergents $(p_k/q_k)_{k \geq 0}$ of t . From Jarnik's Theorem (see Falconer [11]), for every $t \notin A$, where $A \subset [0, 1/2]$ is a set of Hausdorff dimension 0, there exist an integer $N_t \geq 1$ and a constant $C_t > 0$ such that $q_{k+1} \leq C_t (q_k)^{N_t}$, for all k . We focus on such a $t \notin A$.

Fixing $h > 0$, let then k be such that $t < p_{2k+1}/q_{2k+1} \leq t+h < p_{2k-1}/q_{2k-1}$. By definition of the convergents and k , for some universal constant $C_0 > 0$:

$$\sum_{p/q \in (t, t+h), p \wedge q = 1} q 2^{-q} \leq C_0 q_{2k+1} 2^{-q_{2k+1}}. \quad (20)$$

By Proposition 3.2, we can choose $C' > 0$ such that $t \mapsto x^{-1}(t)$ is C' -Lipschitz. Then, remembering that $t \rightarrow x(t)$ is a decreasing map, we obtain that $x(t) - x(t+h) \geq x(t) - x(p_{2k+1}/q_{2k+1})$, implying :

$$x(t) - x(t+h) \geq C' |t - p_{2k+1}/q_{2k+1}| \geq \frac{C'}{2q_{2k+1}q_{2k+2}} \quad (21)$$

$$\geq \left(\frac{C'}{2C_t} \right) \frac{1}{(q_{2k+1})^{1+N_t}}. \quad (22)$$

Set next $C'' = C'/(2C_t)$. We inject relations (20) and (22) in (19), using the remark that the map $u \mapsto u 2^{-u}$ is decreasing for $u \geq 1/\log 2$:

$$\begin{aligned} \Delta_t(h) &\leq CC_0 |x(t) - x(t+h)| \left(\frac{C''}{x(t) - x(t+h)} \right)^{1/(1+N_t)} 2^{-\left(\frac{C''}{x(t) - x(t+h)} \right)^{1/(1+N_t)}} \\ &\leq CC_0 (C'')^{1/(1+N_t)} |x(t) - x(t+h)|^{N_t/(1+N_t)} 2^{-\left(\frac{C''}{x(t) - x(t+h)} \right)^{1/(1+N_t)}} \\ &\leq C''' 2^{-\left(\frac{C''}{x(t) - x(t+h)} \right)^{1/(1+N_t)}}, \end{aligned}$$

which exactly implies (2) (C''' depends on t). The case $h < 0$ is treated similarly.

As $t \mapsto x(t)$ is Lipschitz (by Theorem (1.2)), the image of A by $t \mapsto x(t)$ also has zero Hausdorff-dimension. Hence, outside a set of Hausdorff-dimension zero, (2) holds true. This completes the proof of item (1) of Theorem 1.1.

Recall that item (2) of Theorem 1.1. follows from Theorem 1.3 and the remarks made in the introduction.

4.2. *Proof of Theorem 1.3* Fix $t \in [0, 1/2] \setminus \mathbb{Q}$ and consider the convergents $(p_n/q_n)_{n \geq 1}$ associated with t . Define :

$$E(t) = \{M \geq 0 \mid \sup_n \{q_n^{M+1} 2^{-q_n} q_{n+1}^M\} < +\infty\} \quad (23)$$

and set $M(t) = \sup E(t)$. Remark that this definition of $M(t)$ coincides with the one given in the introduction : $M(t) = \liminf_{n \rightarrow +\infty} q_n / \log_2 q_{n+1}$.

Let $h > 0$ and k be such that $t < p_{2k+1}/q_{2k+1} \leq t+h < p_{2k-1}/q_{2k-1}$. Again, the case $h < 0$ is similar. Let now $M \in E(t)$. From (21), we obtain that

$$|x(t) - x(t+h)|^M \geq \left(\frac{C'}{2q_{2k+1}q_{2k+2}} \right)^M.$$

It follows then from (20) that :

$$\sum_{p/q \in (t, t+h), p \wedge q = 1} q 2^{-q} \leq C_0 \frac{2^M}{(C')^M} [(q_{2k+1})^{1+M} (q_{2k+2})^M 2^{-q_{2k+1}}] (x(t) - x(t+h))^M.$$

Since $M \in E(t)$, the quantity between brackets is bounded in k , thus (19) clearly implies that for some constant C_M (depending on t and M),

$$\Delta_t(h) \leq C_M |x(t) - x(t+h)|^{1+M}.$$

Hence, $h_{\mathcal{F}}(x(t)) \geq 1 + M$, for every $M \in E(t)$ and finally $h_{\mathcal{F}}(x(t)) \geq 1 + M(t)$. This obviously implies that $h_{\mathcal{F}}(x(t)) = +\infty$ when $M(t) = +\infty$ (this yields item (2a) of Theorem 1.3).

In order to prove items (2b) and (2c) of Theorem 1.3, we now study the converse inequality. Suppose that $M(t) < +\infty$. Notice that it is very classical to see that the set of points satisfying $M(t) < +\infty$ has Hausdorff dimension 0. Assume that \mathcal{F} is $\mathcal{C}^{1+M}(x(t))$, for $M > M(t)$. Let k and then $h > 0$ (chosen later) be such that :

$$t < p_{2k+1}/q_{2k+1} \leq t+h.$$

Consider again the difference $\Delta_t(h)$. The same argument as in the proof of Theorem 1.1 also holds true here. Hence, using successively Lemma 3.1, Proposition 3.1 and Proposition 3.2, we obtain :

$$\begin{aligned} \Delta_t(h) &\geq (\mathcal{F}'_-(x(p_{2k+1}/q_{2k+1})) - \mathcal{F}'_+(x(p_{2k+1}/q_{2k+1}))) |x(p_{2k+1}/q_{2k+1}) - x(t+h)| \\ &\geq \frac{C'}{C} q_{2k+1} 2^{-q_{2k+1}} |p_{2k+1}/q_{2k+1} - (t+h)|. \end{aligned} \quad (24)$$

By hypothesis, for some constant \tilde{C} and a polynomial P of degree at most $1 + \lfloor M \rfloor$:

$$|\Delta_t(h) - P(x(t) - x(t+h))| \leq \tilde{C} |x(t) - x(t+h)|^{1+M}. \quad (25)$$

The previous study gives $|\Delta_t(h)| \leq C_\varepsilon |x(t) - x(t+h)|^{1+M(t)-\varepsilon}$, for any $\varepsilon > 0$. Consequently, the polynomial P has the form $P(X) = \sum_{m=1+\lceil M(t) \rceil}^{1+\lfloor M \rfloor} \alpha_m X^m$. Let M' be the first integer in the interval $[\lceil M(t) \rceil, M]$ such that $\alpha_{1+M'} \neq 0$. We deduce that for another constant C , we have $\Delta_t(h) \leq C |x(t) - x(t+h)|^{1+\min\{M, M'\}}$. Now, using that $u \mapsto x(u)$ is bi-Lipschitz, we finally obtain for some \tilde{C} :

$$\Delta_t(h) \leq \tilde{C} |h|^{1+\min\{M, M'\}}. \quad (26)$$

Combining (24) with (26), there exists a constant C_1 , independent on h , such that :

$$q_{2k+1} 2^{-q_{2k+1}} \frac{|p_{2k+1}/q_{2k+1} - (t+h)|}{h^{1+\min\{M, M'\}}} \leq C_1. \quad (27)$$

Choose $h = 2(p_{2k+1}/q_{2k+1} - t)$, so that $t+h - p_{2k+1}/q_{2k+1} = p_{2k+1}/q_{2k+1} - t$. Since $1/(2q_{2k+1}q_{2k+2}) \leq |p_{2k+1}/q_{2k+1} - t| \leq 1/(q_{2k+1}q_{2k+2})$, inequality (27) becomes, with another constant C_2 :

$$q_{2k+1}^{1+\min\{M, M'\}} 2^{-q_{2k+1}} q_{2k+2}^{\min\{M, M'\}} \leq C_2. \quad (28)$$

This procedure gives the same result for the other half of indices, i.e.

$$q_{2k}^{1+\min\{M, M'\}} 2^{-q_{2k}} q_{2k+1}^{\min\{M, M'\}} \leq C_2. \quad (29)$$

Recall that $M > M(t)$, and $M' \geq M(t)$. We distinguish some cases:

- If $\min(M, M') > M(t)$, then (28) and (29) contradict the definition (23) of $M(t)$. Hence $\mathcal{F} \notin \mathcal{C}^{1+M}(x(t))$, for any $M > M(t)$. This implies that $h_{\mathcal{F}}(x(t)) = 1 + M(t)$.
- If $\min(M, M') = M(t)$, then $M(t) = M' \in \mathbb{N}$. If $M(t) \notin E(t)$, again (28) and (29) contradict (23). If $M(t) \in E(t)$, since M' is the non-zero coefficient of P of smallest degree, (25) gives :

$$\Delta_t(h) \sim \alpha_{1+M(t)} (x(t) - x(t+h))^{1+M(t)}, \text{ as } h \rightarrow 0.$$

- If $M(t)$ is even, then the last equivalence is impossible since $\Delta_t(h)$ is always positive. The proof of the theorem is now complete.
- The remaining case ($M(t) \in E(t)$ and $M(t)$ odd) would require to push further the developments, but has no influence on the result.

□

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