Behaviour of the ergodic sums for non regular continuous functions

Julien Brémont

Université de Rennes 1, 1999

Abstract

We consider a model of random walk on $\mathbb{Z}$ with transition to the closest neighbours when the medium is defined by an irrational rotation T on the circle S^1. We study the invariant measure of the random walk of "the environments seen from the particle". Following [1], the equation for finding such measures is equivalent, in the critical case, to some quasi-invariance equation of the form $T^{-1}\nu = h\nu$ for some function $h > 0$ with $\int \log h \, d\mu = 0$. It is proved in [1] that if $\log h$ has bounded variation then the solution is unique and never atomic. We will give examples showing that it is no longer true if we assume only that $\log h$ is continuous. We will provide similar examples in the context of the 2-odometer. We will then show that for such functions the ergodic sums verify a central limit theorem with an almost linear normalization.

1 introduction

Let S^1 be the circle identified with $[0, 1)$ and $T = T_\alpha$ be an irrational rotation of angle α. We consider the dynamical system(S^1, T, dx), where dx designs Lebesgue Measure. Assume to be given two positive measurable functions p and q on S^1 such that $p + q = 1$ and such that there exists $\varepsilon > 0$ with $\varepsilon < p < 1 - \varepsilon$. Fixing an initial point on S^1, we consider the random walk on S^1 with generator P defined as $Pf(x) = p(x)f(Tx) + q(x)f(T^{-1}x)$. It is the random walk of "the environments seen from the particle" which is associated a random walk on $\mathbb{Z}$ in random medium (see [1]). We consider the invariant Borel probability measures on S^1 for this random walk, that is such that $P\nu = \nu$.

We first recall cocycle notations. For any Borel function $f > 0$ on S^1, we set $f_0 = 1$ and for $n \geq 1$:

$$f_n(x) = \prod_{j=0}^{n-1} f(T^j x) \text{ and } f_{-n}(x) = \prod_{j=-1}^{-n} (1/f)(T^j x).$$

We recall some results of Conze and Guivarc'h [1]. In the critical case, that is $\int \log(p(x)/q(x)) \, dx = 0$, the following lemma exhibits a link with a quasi-invariance equation.

Lemma 1.1
If $\int \log(p/q)(x) \, dx = 0$, the equations $P\nu = \nu$ and $T^{-1}\nu = h\nu$ with $h = p/Tq$ are equivalent.

We now consider the equation $T^{-1}\nu = h\nu$, for some strictly positive continuous function h which satisfies $\int \log h = 0$. Using the separability of the space of continuous functions on S^1 and a diagonal argument it is proved in [1] that there is always a probability measure solution of this equation. If one assumes that $\log(h)$ has bounded variation then one can prove that the solution is unique. In the same context, the measure solution ν has no atom. Indeed, if for some $x \in S^1$ one has $\nu\{x\} > 0$, then $\nu(T^n x) = h_n(x)\nu\{x\}$ and therefore $\sum_{k\in\mathbb{Z}} h_k(x) < +\infty$. One already remarks that reciprocally, if the sum is finite for a certain x, then one builts an atomic measure supported by the orbit of x. Writing $(p_n/q_n)_{n\geq 0}$ for the convergents of α, one can now use the following inequality :

Lemma 1.2 *(Denjoy-Koksma)*
Let $f\ : S^1 \longrightarrow \mathbb{R}$ have bounded variation and such that $\int f(x)\ dx = 0$. Then :

$$\left|\sum_{k=0}^{q_n-1} f(T^k x)\right| \leq 2\ Var\ (f).$$

The previous lemma indicates that there exists a constant $C > 0$ such that $C^{-1} \leq h_{q_n} \leq C$, so the measure ν cannot have any atom.

We will now prove that if we weaken the hypothesis of bounded variation on $\log(h)$, then neither unicity nor the non-atomic character remain valid. This way we will build a continuous function h such that the set of solutions of the equation $T^{-1}\nu = h\nu$ contains atomic measures supported by the orbit of every rational point in S^1.

2 Counter-examples.

We write $(p_n/q_n)_{n\geq 0}$ for the sequence of convergents of the irrational α. If the expansion of α in continued fraction is $\alpha = 1/(a_0 + 1/(a_1 + 1/(\cdots)))$, then p_n/q_n is obtained by truncating at level n. The sequences $(p_{2n}/q_{2n})_{n\geq 0}$ and $(p_{2n+1}/q_{2n+1})_{n\geq 0}$ are adjacent and we have $q_{2n}\alpha - p_{2n} > 0$. Writing $((x))$ the distance from a real x to $\mathbb{Z}$, we have the following result.

Theorem 2.1
For $n \geq 1$, define $a_n = \lfloor\sqrt{\log n}\rfloor$, $b_n = \lfloor((q_{2n}\alpha))^{-1}\theta^{-2n}\rfloor$ with $\theta = (1+\sqrt{5})/2$ and set :

$$\log h(x) := \sum_{n\geq 1} n^{-2} \sin\left(4\pi(a_n!)b_n q_{2n} x\right).$$

Then for all x in $\mathbb{Q}$, $\sum_{k\in\mathbb{Z}} h_k(x) < +\infty$.

Proof of the theorem :
Set $f = \log h$. Then one has $f(x) = -f(-x)$ and $\int f(x)\ dx = 0$. For any integer $M \geq 1$, write $(S_M f)(x) = \sum_{k=0}^{M-1} f(x + k\alpha)$. Writing $m_n = (a_n!)b_n q_{2n}$, one gets for the point 0 :

$$(S_M f)(0) = \sum_{n\geq 1}(1/n^2)\left(\frac{\sin(2\pi m_n M\alpha)\sin\left(2\pi m_n(M+1)\alpha\right)}{\sin(2\pi m_n\alpha)}\right).$$

One can now develop $\sin\left(2\pi m_n(M+1)\alpha\right) = \sin\left(2\pi m_n M\alpha\right)\cos\left(2\pi m_n\alpha\right) + \sin\left(2\pi m_n\alpha\right)\cos\left(2\pi m_n M\alpha\right)$ and use the fact that $\cos\left(2\pi m_n\alpha\right) = 1 + O((m_n\alpha))^2$. Therefore we have :

$$(S_M f)(0) = \sum_{n\geq 1}(1/n^2)\frac{\sin^2\left(2\pi m_n M\alpha\right)}{\sin\left(2\pi m_n\alpha\right)} + O(1).$$

One then remarks that $m_n\alpha = ((q_{2n}\alpha))(a_n!)b_n + p_{2n}(a_n!)b_n$ and that we have the majoration :

$$((q_{2n}\alpha))(a_n!)b_n \leq ((q_{2n}\alpha))\theta^{-2n}\left(e^{\sqrt{\log n}}\right)^{\sqrt{\log n}}((q_{2n}\alpha))^{-1} \leq n\theta^{-2n},$$

which is $< (1/2)$ for n large enough. Thus :

$$(S_M f)(0) \geq \frac{8}{\pi}\sum_{((m_n\alpha))\leq(1/4M)}(1/n^2)M^2((m_n\alpha)) + O(1).$$

As the sequence $\{((m_n))\}_{n\geq 1}$ tends to 0, we now write n_0 an index such that $((m_{n_0}\alpha)) \leq (1/4M) < ((m_{n_0-1}\alpha))$. Writing $C > 0$ a generic constant, we have :

$$(S_M f)(0) \geq CM(1/n_0^2)\frac{((m_{n_0}\alpha))}{((m_{n_0-1}\alpha))} + O(1).$$

As for any $n \geq 1$, one has $(1/4)(a_n!)\theta^{-2n} \leq ((m_n\alpha)) \leq (a_n!)\theta^{-2n}$, we get that :

$$((m_{n_0}\alpha))/((m_{n_0-1}\alpha)) \geq C_1 > 0 \text{ and } n_0 \geq C_2 \log M \text{ with } C_2 > 0.$$

Finally, we obtain :

$$(S_M f)(0) \geq C\frac{M}{(\log M)^2} + O(1).$$

For any x, one has the formula :

$$(S_M f)(x) = \sum_{n\geq 1}(1/n^2)\frac{\sin(2\pi m_n(M+1)\alpha)\sin(2\pi m_n(M\alpha+2x))}{\sin(2\pi m_n\alpha)}. \tag{1}$$

If $x \in \mathbb{Q}$, as $m_n = q_{2n}(a_n!)b_n$ with $a_n \to +\infty$, one observes that in (1), $(S_M f)(x)$ and $(S_M f)(0)$ will differ by finitely many terms, independently on M. The result follows.

□

Remark. — If α has a bounded expansion in continued fraction, then the function $f(x) = \sum_{n\geq 1}(1/n^2)\sin(4\pi q_{2n}x)$ gives the same minoration with the points 0 and 1/2. The fact that $\int f(x)\, dx = 0$ implies the recurrence of the ergodic sums $(S_M f)(x)$ for almost every x. We thus have an example where the rationals don't satisfy this property. A calculus in L^2- norm confirms an almost linear growth of the ergodic sums. Proceeding as above, one can obtain :

$$C_1\frac{M}{(\log M)^2} \leq ||S_M f||_{L^2} \leq C_2 M.$$

As for every $x \in \mathbb{Q}$, one creates an atomistic extremal measure supported by the orbit of x, one gets a set indexed by $\mathbb{Q}$ of distinct measures, solutions of the equation $T^{-1}\nu = h\nu$.

Corollary 2.2
Let h be defined as in theorem (2.1), then the equation $T^{-1}\nu = h\nu$ admits infinitely many solutions. Similarly if positive p and q on S^1 satisfy $p+q=1$ and $\log(p/q) = h$, the equation $T^{-1}\nu = (p/Tq)\nu$ admits infinitely many solutions. Therefore it is also the case for the equation $P\nu = \nu$ with $Pf = pTf + qT^{-1}f$.

3 Similar examples in the context of the 2−odometer.

For further study, we now consider the model of the "adding machine". Let $(G, +)$ where G is the compact group $\{0,1\}^{\mathbb{N}}$ equipped with the product of the discrete topologies and " + " is the usual addition in base 2.
In a first step, we search the characters of this group. Let $\varphi : G \longrightarrow \mathbb{C}$ be a character. One has then for any $n \geq 0$: $\varphi(x_0, \cdots, x_n, 0, \cdots) = e^{\theta\sum_{i=0}^{n} 2^i x_i}$, having set $\varphi(1, 0, \cdots) = e^{2i\pi\theta}$. As we must have $e^{2i\pi\theta 2^n} \longrightarrow 1$, when $n \to +\infty$, it is necessary that θ can be written as $\theta = k/2^m$ for integers $k \geq 0$ and $m \geq 0$, which defines a character $\varphi_{k,m}$.

We now replace the angle α of the previous section by 1. The sequence $(q_n)_{n\geq 0}$ is replaced by the sequence $(2^n)_{n\geq 0}$. We consider functions of the following type :

$$g(x) = \text{Im}\left(\sum_{n\geq 1} n^{-2}\varphi_{2,n}(x)\right) = \sum_{n\geq 1} n^{-2}\sin\left(\frac{4\pi}{2^n}\sum_{i=0}^{+\infty} 2^i x_i\right), \text{ with } x = (x_i)_{i\geq 0}.$$

The function g has 0-integral with respect to Haar measure and verifies $g(-x) = -g(x)$. The same calculus as in the preceding section shows that the ergodic sums $(S_M g)(0)$ at the point 0 are also minored a quantity in $O(M(\log M)^{-2})$. We will now consider a slightly different function that will still satisfy a minoration of the same kind, but for which one can prove a law convergence.

Theorem 3.1
Fix $1 < \delta < 2$, write $\varphi(n) = \lfloor \delta^n \rfloor$ for $n \geq 0$ and consider $f(x) = \sum_{n\geq 1} n^{-2} \sin\left(\frac{4\pi}{2^{\varphi(n)}} \sum_{i=0}^{+\infty} 2^i x_i\right)$. Then there exists a constant $C > 0$ such that for M large enough :

$$(S_M f)(0) \geq C \frac{M^{(2-\delta)}}{(\log\log M)^2}.$$

Proof of the theorem :
As in the proof of theorem (2.1), neglecting quantities of order $\left(\sum 1/n^2\right)$, we have :

$$(S_M f)(0) = \sum_{n\geq 1} (1/n^2) \frac{\sin^2\left(\frac{2\pi}{2^{\varphi(n)}} M\right)}{\sin\left(\frac{2\pi}{2^{\varphi(n)}}\right)} + O(1).$$

Writing n_0 the index such that $2^{\varphi(n_0-1)} \leq 4M < 2^{\varphi(n_0)}$, we have with $C > 0$ designing a generic constant :

$$\begin{aligned}(S_M f)(0) &\geq (1/n_0^2) \frac{\sin^2\left(\frac{2\pi}{2^{\varphi(n_0)}} M\right)}{\sin\left(\frac{2\pi}{2^{\varphi(n_0)}}\right)} + O(1) \\ &\geq C(1/n_0^2) M^2\ 2^{-\varphi(n_0)} + O(1) \\ &\geq C \frac{M^{(2-\delta)}}{(\log\log M)^2} + O(1),\end{aligned}$$

as we have $2^{\varphi(n_0)} \leq 2^{\delta+\delta\varphi(n_0-1)} \leq 2^\delta (4M)^\delta$.

□

We will now prove a law convergence for a function of the previous form.

Theorem 3.2
Fix $\delta > 1$, write $\varphi(n) = \lfloor \delta^n \rfloor$ for $n \geq 0$ and set $f(x) = \sum_{n\geq 1} n^{-2} \sin\left(\frac{4\pi}{2^{\varphi(n)}} \sum_{i=0}^{+\infty} 2^i x_i\right)$. Then the ergodic sums $(S_M f)_{M\geq 0}$ satisfy the following law convergence :

$$\frac{(\log\log M)^{3/2}}{M} (S_M f) \rightharpoonup \mathcal{N}\left(0, \frac{(\log\delta)^3}{6}\right).$$

Proof of the theorem :
Recall that as in the previous section one has :

$$(S_M f)(x) = \sum_{n\leq l-1} (1/n^2) \sin\left[\frac{2\pi}{2^{\varphi(n)}}\left(M + \sum_{i=0}^{\varphi(n)-2} 2^{i+1} x_i\right)\right] \frac{\sin\left(\frac{2\pi M}{2^{\varphi(n)}}\right)}{\sin\left(\frac{2\pi}{2^{\varphi(n)}}\right)} + O(1). \tag{2}$$

Let $l = l(M)$ be such that $2^{\varphi(l-1)} < M \leq 2^{\varphi(l)}$ and decompose relation (2) into $A(x) = \sum_{n\leq l-1}$ and $B(x) = \sum_{n\geq l}$. Writing $C > 0$ for a generic constant, we have :

$$\begin{aligned}|A(x)| &= \left|\sum_{n\leq l-1} (1/n^2) \sin\left[\frac{2\pi}{2^{\varphi(n)}}\left(M + \sum_{i=0}^{\varphi(n)-2} 2^{i+1} x_i\right)\right] \frac{\sin\left(\frac{2\pi M}{2^{\varphi(n)}}\right)}{\sin\left(\frac{2\pi}{2^{\varphi(n)}}\right)}\right| \\ &\leq C\left(\sum_{n\leq l-1} (1/n^2) 2^{\varphi(n)}\right) \leq CM\left(\sum_{n\leq l-1} (1/n^2) 2^{\varphi(n)-\varphi(l-1)}\right) \\ &\leq C(M/l^2), \text{ since } \left(2^{\varphi(n)-\varphi(l-1)}\right) \text{ decreases geometrically as } n \to 0, \\ &\leq CM(\log\log M)^{-2}, \text{ as } l \sim (\log\delta)^{-1}(\log\log M).\end{aligned}$$

Since the forthcoming normalization will be in $M(\log\log M)^{-3/2}$, one can neglect $A(x)$. We now consider the rest $B(x)$. We use the fact that :

$$\frac{\sin\left(2\pi M/2^{\varphi(n)}\right)}{\sin\left(2\pi/2^{\varphi(n)}\right)} = M\left(1+\frac{x^2}{6}(1-M^2)+o(M^2x^2)\right), \text{ with } x = 2\pi/2^{\varphi(n)}.$$

Considering the terms of second order, one has :

$$M\sum_{n\geq l}(1/n^2)\frac{M^2}{2^{2\varphi(n)}} \leq C(M/l^2) \leq CM(\log\log M)^{-2}.$$

Finally, with a $O(M(\log\log M)^{-2})$ uniform in x, we deduce :

$$(S_M f)(x) = M\sum_{n\geq l}(1/n^2)\sin\left[\frac{2\pi}{2^{\varphi(n)}}\left(M+\sum_{i=0}^{\varphi(n)-2}2^{i+1}x_i\right)\right] + O(M(\log\log M)^{-2})).$$

Set now $y_k = \sum_{i=\varphi(k-1)}^{\varphi(k)-1} 2^{i+1}x_i$. Since the Haar measure on G is translation invariant, one just needs to consider :

$$D(x) := M\sum_{n\geq l}(1/n^2)\sin\left(\frac{2\pi}{2^{\varphi(n)}}(y_0+\cdots+y_n)\right).$$

Setting $z_n := y_0+\cdots+y_{n-1}$ for $n\geq l$, one has :

$$\sin\left(2\pi 2^{-\varphi(n)}(z_n+y_n)\right) = \sin\left(2\pi 2^{-\varphi(n)}z_n\right)\cos\left(2\pi 2^{-\varphi(n)}y_n\right)+\sin\left(2\pi 2^{-\varphi(n)}y_n\right)\cos\left(2\pi 2^{-\varphi(n)}z_n\right).$$

Moreover :

$$\cos\left(2\pi 2^{-\varphi(n)}z_n\right) = 1+O\left(2^{2\varphi(n-1)-2\varphi(n)}\right) \text{ and } \sin\left(2\pi 2^{-\varphi(n)}z_n\right) = O\left(2^{\varphi(n-1)-\varphi(n)}\right), \quad (3)$$

Since $\varphi(n)-\varphi(n-1)\geq \delta^{n-1}(\delta-1)-1$, we deduce that the quantities that appear on the right sides of (3) decrease exponentially fast as $n\to+\infty$. Therefore we obtain :

$$D(x) := M\sum_{n\geq l}(1/n^2)\sin\left(2\pi 2^{-\varphi(n)}y_n\right) + O(M(\log\log M)^{-2})). \quad (4)$$

One now observes that the $(y_n 2^{-\varphi(n)})$ are independent. Also, the law of $(y_n 2^{-\varphi(n)})$ converges to the uniform law on $[0,1]$. We set $u_n := \sin\left(2\pi 2^{-\varphi(n)}y_n\right)$. We will proceed to a uniform calculus, when computing the characteristic function of u_n. For t in $\mathbb{R}$, we have :

$$\chi_n(t) := \mathbb{E}\left[e^{itu_n}\right] = 2^{\varphi(n-1)-\varphi(n)}\sum_{k=0}^{2^{\varphi(n)-\varphi(n-1)}} e^{it\sin\left(2\pi k 2^{\varphi(n-1)-\varphi(n)}\right)}.$$

Since $(d/dx)\, e^{it\sin(2\pi x)} = 2\pi i t e^{it\sin(2\pi x)}\cos(2\pi x)$, we deduce that with unform constants :

$$\chi_n(t) = \int_0^1 e^{it\sin(2\pi x)}\, dx + O\left(t2^{\varphi(n-1)-\varphi(n)}\right) = 1-\frac{t^2}{4}+O(t^3)+O\left(t2^{\varphi(n-1)-\varphi(n)}\right).$$

Consequently, considering the characteristic function of D (see (4)) for some fixed t in $\mathbb{R}$:

$$\begin{aligned}\mathbb{E}\left[e^{it(l^{3/2}/M)D}\right] &= \prod_{n\geq l}\chi_n\left(tl^{\frac{3}{2}}/n^2\right) = \prod_{n\geq l}\left(1-t^2l^3/(4n^4)+o(l^3/n^4)\right)\\ &= e^{-\sum_{n\geq l}t^2l^3/(4n^4)+o(1)} \longrightarrow e^{-t^2/12}, \text{as } l\longrightarrow+\infty,\end{aligned}$$

since $l^3 \sum_{n \geq l}(1/n^4) \longrightarrow 1/3$ as $l \longrightarrow +\infty$. As $l \sim (\log\delta)^{-1}(\log\log M)$, this finishes the proof. $\square$

Remark. — Concerning the last part of the proof, one can apply directly to equation (4) a modified version of Brown's Theorem on the CLT for martingale differences. It will be enounced and proved in the next section (see theorem 4.2).

4 A law convergence theorem for some ergodic sums on the circle.

We now come back to the initial model of the circle equipped with an irrational rotation $T = T_\alpha$ and Lebesgue measure. Writing $(p_n/q_n)_{n\geq 0}$ the convergents of α, recall that $(p_{2n}/q_{2n})_{n\geq 0}$ and $(p_{2n+1}/q_{2n+1})_{n\geq 0}$ are adjacent and that we have $q_{2n}\alpha - p_{2n} > 0$ (see beginning of section (2)).

Theorem 4.1
For $n \geq 1$, define $r_n = (\lfloor\sqrt{\log n}\rfloor!)\,\lfloor((q_n\alpha))^{-1}\theta^{-n}\rfloor$ with $\theta = (1+\sqrt{5})/2$. For $\delta > 1$ and $n \geq 0$, set $\varphi(n) = 2\lfloor\delta^n\rfloor$ and introduce

$$\log h_\delta(x) := \sum_{n\geq 1} n^{-2} \sin\left(4\pi r_{\varphi(n)} q_{\varphi(n)} x\right).$$

Then we have :
1) Assume that $1 < \delta < 2$. For $x \in \mathbb{Q}$, there exists a constant $C(x) > 0$ such that :

$$S_M(\log h_\delta)(x) \geq C(x)\frac{M^{(2-\delta)}}{(\log\log M)^{2+\delta}}.$$

Therefore for all x in $\mathbb{Q}$, $\sum_{k\in\mathbb{Z}}(h_\delta)_k(x) < +\infty$.

2)For any $\delta > 1$, the following convergence holds as $M \longrightarrow +\infty$:

$$\left(\frac{(\log\log M)^{3/2}}{M} S_M(\log h_\delta)([Mt])\right)_{t\in[0,1]} \rightharpoonup \mathcal{W}\left(0, \frac{(\log\delta)^3}{6}\right).$$

Proof of the theorem :
1) Fix $1 < \delta < 2$ and set $f = \log h_\delta$. As in the proof of theorem (2.1), one remarks that $f(x) = -f(-x)$ and $\int f(x)\,dx = 0$. Writing $m_n = r_{\varphi(n)}q_{\varphi(n)}$, one gets :

$$(S_M f)(0) = \sum_{n\geq 1}(1/n^2)\left(\frac{\sin(2\pi m_n M\alpha)\sin(2\pi m_n(M+1)\alpha)}{\sin(2\pi m_n\alpha)}\right).$$

Developping $\sin(2\pi m_n(M+1)\alpha) = \sin(2\pi m_n M\alpha)\cos(2\pi m_n\alpha) + \sin(2\pi m_n\alpha)\cos(2\pi m_n M\alpha)$ and using the fact that $\cos(2\pi m_n\alpha) = 1 + O((m_n\alpha))^2$, where $((m_n\alpha))$ tends to 0 exponentially fast (see further estimate (5)), we obtain :

$$(S_M f)(0) = \sum_{n\geq 1}(1/n^2)\frac{\sin^2(2\pi m_n M\alpha)}{\sin(2\pi m_n\alpha)} + O(1).$$

As $\varphi(n)$ is even, one then remarks that $m_n\alpha = ((q_{\varphi(n)}\alpha))r_{\varphi(n)} + p_{\varphi(n)}r_{\varphi(n)}$ and that we have the majoration :

$$((q_{\varphi(n)}\alpha))r_{\varphi(n)} \leq ((q_{\varphi(n)}\alpha))\theta^{-\varphi(n)}\left(e^{\sqrt{\log\varphi(n)}}\right)^{\sqrt{\log\varphi(n)}}((q_{\varphi(n)}\alpha))^{-1} \leq \varphi(n)\theta^{-\varphi(n)}, \qquad (5)$$

which is $< (1/2)$ for n large enough. Thus :

$$(S_M f)(0) \geq \frac{8}{\pi}\sum_{((m_n\alpha))\leq(1/4M)}(1/n^2)M^2((m_n\alpha)) + O(1).$$

As the sequence $\{((m_n))\}_{n\geq 1}$ tends to 0, we write n_0 an index such that $((m_{n_0}\alpha)) \leq (1/4M) < ((m_{n_0-1}\alpha))$. Writing $C > 0$ a generic constant, we have :

$$(S_M f)(0) \geq C(M^2/n_0^2)((m_{n_0}\alpha)) + O(1).$$

For any $n \geq 1$, one has $(1/4)(\lfloor\sqrt{\log\varphi(n)}\rfloor!)\theta^{-\varphi(n)} \leq ((m_n\alpha)) \leq (\lfloor\sqrt{\log\varphi(n)}\rfloor!)\theta^{-\varphi(n)}$. We therefore get the following estimates :

$$\begin{aligned} M^\delta((m_{n_0}\alpha)) \geq C((m_{n_0}\alpha))/((m_{n_0-1}\alpha))^\delta &\geq C\frac{\lfloor\sqrt{\log\varphi(n_0)}\rfloor!}{(\lfloor\sqrt{\log\varphi(n_0-1)}\rfloor!)^\delta}\theta^{\delta\varphi(n_0-1)-\varphi(n_0)} \\ &\geq C(n_0)^{-\delta}. \end{aligned}$$

Finally, as $n_0 \sim (\log\delta)^{-1}(\log\log M)$, we obtain :

$$(S_M f)(0) \geq C\frac{M^{(2-\delta)}}{(\log\log M)^{2+\delta}} + O(1).$$

For any x, one has the formula :

$$(S_M f)(x) = \sum_{n\geq 1}(1/n^2)\frac{\sin(2\pi m_n(M+1)\alpha)\sin(2\pi m_n(M\alpha+2x))}{\sin(2\pi m_n\alpha)}. \tag{6}$$

If $x \in \mathbb{Q}$, as any integer divides m_n for n large enough, we see that in (6), $(S_M f)(x)$ and $(S_M f)(0)$ will differ by finitely many terms, independently on M. The result follows.

2) From formula (6), one has :

$$(S_M f)(x) = \sum_{n\geq 1}(1/n^2)\sin(2\pi m_n(M\alpha+2x))\frac{\sin(2\pi m_n M\alpha)}{\sin(2\pi m_n\alpha)} + O(1).$$

Let $l = l(M)$ be such that $((m_l\alpha)) \leq (1/4M) < ((m_{l-1}\alpha))$ and decompose relation (2) into $A(x) = \sum_{n\leq l-1}$ and $B(x) = \sum_{n\geq l}$. Writing $C > 0$ for a generic constant, we have :

$$\begin{aligned} |A(x)| &= \left|\sum_{n\leq l-1}(1/n^2)\sin(2\pi m_n(M\alpha+2x))\frac{\sin(2\pi m_n M\alpha)}{\sin(2\pi m_n\alpha)}\right| \\ &\leq C\left(\sum_{n\leq l-1}(1/n^2)((m_n\alpha))^{-1}\right) \leq CM\left(\sum_{n\leq l-1}(1/n^2)((m_{l-1}\alpha))/((m_n\alpha))\right) \\ &\leq C(M/l^2), \text{ since } ((m_{l-1}\alpha))/((m_n\alpha)) \text{ tends geometrically to 0 as } n \to 0, \\ &\leq CM(\log\log M)^{-2}, \text{ as } l \sim (\log\delta)^{-1}(\log\log M). \end{aligned}$$

Since the forthcoming normalization will be in $M(\log\log M)^{-3/2}$, one can neglect $A(x)$. We now consider the rest $B(x)$. We use the fact that :

$$\frac{\sin(2\pi M m_n\alpha)}{\sin(2\pi m_n\alpha)} = M\left(1+\frac{x^2}{6}(1-M^2)+o(M^2x^2)\right), \text{ with } x = 2\pi((m_n\alpha)).$$

Considering the terms of second order, one has :

$$M\sum_{n\geq l}(1/n^2)M^2((m_n\alpha))^2 \leq CM\sum_{n\geq l}(1/n^2)((m_n\alpha))^2/((m_l\alpha))^2 \leq C(M/l^2) \leq CM(\log\log M)^{-2}.$$

Finally, with a $O(M(\log\log M)^{-2})$ uniform in x, we deduce :

$$(S_M f)(x) = M \sum_{n\geq l} (1/n^2) \sin\left(2\pi m_n (M\alpha + 2x)\right) + O(M(\log\log M)^{-2})).$$

Since Lebesgue measure is translation invariant, one just needs to consider :

$$D(x) := M \sum_{n\geq l} (1/n^2) \sin\left(4\pi m_n x\right).$$

We shall show that the previous expression can be written as the rest of a sum of martingale differences with a small error. We choose first the sub $\sigma-$algebras of the Borel $\sigma-$algebra. For $n \geq 1$, let $\mathcal{F}_n$ be the $\sigma-$ algebra generated by the atoms $[r/2^{k_n}, (r+1)/2^{k_n})$, $0 \leq r \leq 2^{k_n} - 1$, where k_n is the integer satisfying $2^{k_n} \leq (m_n m_{n+1})^{1/2} < 2^{k_n+1}$. Writing $(x_k)_{k\geq 1}$ the expansion of $2x$ in base 2, that is $2x = \sum_{k\geq 1} 2^{-k} x_k$, we have :

$$\left| \sum_{n\geq l} (1/n^2) \sin(4\pi m_n x) - \sum_{n\geq l} (1/n^2) \sin\left(2\pi m_n \sum_{k=1}^{k_n} 2^{-k} x_k\right) \right| \leq C \sum_{n\geq l} (1/n^2) \left(m_n/m_{n+1}\right)^{1/2}.$$

We now recall that for any $p \geq 0$, one has $1/2 \geq q_{p+1}((q_p\alpha)) \leq 1$ and that there exists a constant $C > 0$ such that for all $n \geq 0$ and $p \geq 0$, one has $q_{n+p} \geq C\theta^n q_p$. Therefore :

$$m_n/m_{n+1} \leq C \frac{q_{\varphi(n)+1}\theta^{-\varphi(n)} q_{\varphi(n)}}{q_{\varphi(n+1)+1}\theta^{-\varphi(n+1)} q_{\varphi(n+1)}} \leq C\theta^{-\varphi(n+1)+\varphi(n)},$$

which tends to 0 geometrically fast. Thus, with $X_n := \sin\left(2\pi m_n \sum_{k=1}^{k_n} 2^{-k} x_k\right)$, we deduce that :

$$D(x) = M \sum_{n\geq l} (1/n^2) X_n + O(M(\log\log M)^{-2}).$$

Consequently :

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}](x) = 2^{k_{n-1}-k_n} \sum_{k=0}^{2^{k_n-k_{n-1}}-1} \sin\left[2\pi m_n \left(\sum_{p=1}^{k_{n-1}} 2^{-p} x_p + 2^{-k_{n-1}} \frac{k}{2^{k_n-k_{n-1}}}\right)\right]$$

and also with $t_n(x) := m_n \sum_{p=1}^{k_{n-1}} 2^{-p} x_p$, we obtain :

$$\begin{aligned}
\left| \mathbb{E}[X_n|\mathcal{F}_{n-1}] - \int_0^1 \sin 2\pi\left(t_n(x) + m_n 2^{-k_{n-1}} t\right) \, dt \right| &\leq C(m_n 2^{-k_{n-1}})/(2^{k_n-k_{n-1}}) \\
&\leq C(m_n/m_{n+1})^{1/2}.
\end{aligned}$$

Moreover :

$$\left| \int_0^1 \sin 2\pi\left(t_n(x) + m_n 2^{-k_{n-1}} t\right) \, dt \right| \leq C 2^{k_{n-1}}/m_n \leq C(m_{n-1}/m_n)^{1/2}.$$

Finally :

$$|\mathbb{E}[X_n|\mathcal{F}_{n-1}]| \leq C\theta^{-(\delta-1)\delta^{n-1}}.$$

Considering now $X'_n = X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}]$, we get :

$$D(x) = M \sum_{n\geq l} (1/n^2) X'_n + O(M(\log\log M)^{-2}).$$

We then check that we can apply to (X'_n/n) a modified version of Brown's Theorem [2], presented just below. With the notations of that theorem, we have $r_l^2 \sim (1/2)\sum_{n\geq l}(1/n^4)$. Similarly, estimating $\mathbb{E}[(X'_n)^2|\mathcal{F}_{n-1}]$ by replacing X'_n and then by proceeding as above for $\mathbb{E}[(X_n)^2|\mathcal{F}_{n-1}]$, one gets $1/2$ plus a geometrical error. Thus $V_l^2 \sim (1/2)\sum_{n\geq l}(1/n^4)$. We also have $|X'_n|/n^2 \leq (2/n^2)$. Since $r_l \sim (1/\sqrt{6})l^{-3/2}$, for every $\varepsilon > 0$, one has $1_{\{|X'_n|/n^2 \geq \varepsilon r_l\}} = 0$ for l large enough. The conclusion of the theorem follows.

□

For the following result, we also refer to [3].

Theorem 4.2
On a probability space, let $(X_n)_{n\geq 0}$ be a sequence of martingale differences with respect to a non decreasing filtration $(\mathcal{F}_n)_{n\geq 0}$ with $|X_n| \leq a_n$ and $\sum_{n\geq 0} a_n < +\infty$. Set $R_l := \sum_{n\geq l} X_n$, $r_l^2 := \mathbb{E}[R_l^2]$ and $V_l^2 = \sum_{n\geq l}\mathbb{E}[X_n^2|\mathcal{F}_{n-1}]$. If $(V_l^2/r_l^2) \longrightarrow 1$ in probability and if

$$\forall \varepsilon > 0,\ (r_l)^{-2}\sum_{j\geq l}^{l}\mathbb{E}\left[X_j^2 1_{\{|X_j|>\varepsilon r_l\}}\right] \underset{l\to+\infty}{\longrightarrow} 0, \tag{7}$$

then an invariance principle holds :

$$\left(\frac{R_{[lt]}}{r_l}\right)_{t\in[0,1]} \rightharpoonup \mathcal{W}(0,1).$$

Proof of the theorem :
We will just prove the validity of the Central Limit Theorem. We will follow the same scheme as in [2]. We write $\mathbb{E}_j[\,]$ the conditional expectation with respect to the $\sigma-$algebra $\mathcal{F}_j$. We introduce $V_{l,k}^2 := \sum_{l\leq n\leq k}\mathbb{E}_{n-1}[X_n^2]$ and in a similar way $R_{l,k}$. Let $C > 1$ be a constant. Fixing $l \geq 1$, we define $X_k^*(l) = X_k 1_{\{V_{l,k}^2 \leq C r_l^2\}}$ for $k \geq l$. It is directly checked that $(X_k^*(l))_{k\geq l}$ is a sequence of martingale differences. One also has :

$$\mathbb{P}\left[\exists k \geq l,\ X_k^*(l) \neq X_k\right] \leq \mathbb{P}\left[V_l^2 \geq C r_l^2\right] \longrightarrow 0.$$

We can then replace X_k by $X_k^*(l)$ and moreover we have :

$$\forall k, \mathbb{P}\left[\sum_{j=l}^{k}\mathbb{E}_{j-1}[X_k^{*2}(l)] \leq C r_l^2\right] = 1.$$

For convenience, we return to the notation X_k. We first have :

$$\mathbb{E}\left|e^{(t^2/2)(V_l^2/r_l^2)} - e^{(t^2/2)}\right| \longrightarrow 0.$$

It is then enough to show that $\mathbb{E}\left[e^{it(R_l/r_l)+(t^2V_l^2/2r_l^2)} - 1\right] \longrightarrow 0$. Then for $m > l$:

$$1-e^{itR_{l,m}/r_l+(t^2V_{l,m}^2)/(2r_l^2)} = \sum_{k=l}^{m-1}\left(e^{itR_{l,k}/r_l+(t^2V_{l,k}^2)/(2r_l^2)} - e^{itR_{l,k+1}/r_l+(t^2V_{l,k+1}^2)/(2r_l^2)}\right) =: \sum_{k=l}^{m-1}(-Z_{k+1}).$$

Define functions Q, M and Z in the following way :

$$e^{ix} = 1 + ix + (1/2)x^2 + (1/2)x^2Q(x)$$

and $M(x) = \min(x/3, 2)$. We have $|1 - Q(x)| \leq 1$ and $|Q(x)| \leq M(x)$. We also introduce $Z(x) = e^{-x} - 1 + x$ for $x \geq 0$, which satisfies $Z(x) \leq x^2/2$. We can then write :

$$\begin{aligned} Z_k &= e^{itR_{l,k-1}/r_l+(t^2V_{l,k}^2)/(2r_l^2)}\left(e^{itX_k/r_l} - e^{-t^2\sigma_k^2/(2r_l^2)}\right), \text{ with } \sigma_k^2 = \mathbb{E}_{k-1}[X_k^2] \\ &= e^{itR_{l,k-1}/r_l+(t^2V_{l,k}^2)/(2r_l^2)}\left[itX_k/r_l - (t^2X_k^2)/(2r_l^2)\left(1 - Q\left(tX_k/r_l\right)\right) + (t^2\sigma_k^2)/(2r_l^2) - Z\left(t^2\sigma_k^2)/(2r_l^2\right)\right)\right]. \end{aligned}$$

Using the property of martingale differences :

$$\begin{aligned} |\mathbb{E}_{k-1}[Z_k]| &\leq e^{t^2C/2}\left|(1/2)\mathbb{E}_{k-1}\left[t^2(X_k^2/r_l^2)Q\left(tX_k/r_l\right)\right] - Z\left(t^2\sigma_k^2/(2r_l^2)\right)\right| \\ &\leq (1/2)e^{t^2C/2}\left[\mathbb{E}_{k-1}\left[t^2X_k^2/r_l^2 M\left(tX_k/r_l\right)\right] + t^4\sigma_k^4/(4r_l^4)\right]. \end{aligned}$$

Summing from l to $m-1$ and setting $b_l = \max_{n\geq l} \mathbb{E}[X_n^2]$, we obtain :

$$\begin{aligned} \left|\mathbb{E}\left[e^{itR_{l,m}/r_l + t^2V_{l,m}^2/(2r_l^2)} - 1\right]\right| &\leq \mathbb{E}\left[\sum_{k=l}^{m-1}|\mathbb{E}_{k-1}[Z_k]|\right] \\ &\leq (t^2/2)e^{t^2C/2}\mathbb{E}\left[\sum_{k=l}^{m-1}\mathbb{E}_{k-1}\left[t^2X_k^2/(r_l^2)M\left(tX_k/r_l\right)\right] + (t^2/4)b_l\sum_{k=l}^{m-1}(\sigma_k^2)/(r_l^2)\right]. \end{aligned}$$

Letting m tend to $+\infty$:

$$\left|\mathbb{E}\left[e^{itR_l/r_l + t^2V_l^2/(2r_l^2)} - 1\right]\right| \leq (t^2/2)e^{t^2C/2}\mathbb{E}\left[\sum_{k=l}^{+\infty}\mathbb{E}_{k-1}\left[t^2X_k^2/r_l^2 M\left(tX_k/r_l\right)\right] + (t^2/4)b_l\sum_{k=l}^{+\infty}(\sigma_k^2/r_l^2)\right].$$

The conclusion of the theorem follows then from the next two lemmas.

□

Lemma 4.3
The quantity (V_l^2/r_l^2) converges to 1 in probability if and only if $\mathbb{E}\left|R_l^2/r_l^2 - 1\right| \longrightarrow 0$.

Proof of the lemma :
One sense is obvious. Let us that if the convergence holds in probability then it isalso the case in L^1. Using the dominated convergence theorem, we will have $\mathbb{E}\left(R_l^2/r_l^2 - 1\right)_- \longrightarrow 0$. However, from the definition of R_l and r_l, we have $\mathbb{E}\left(R_l^2/r_l^2 - 1\right)_- = \mathbb{E}\left(R_l^2/r_l^2 - 1\right)_+$, which concludes the proof.
□

As in [2], we introduce $g(n,\varepsilon) = (V_n)^{-2}\sum_{j=n}^{+\infty}\mathbb{E}_{j-1}\left[X_j^2 1_{\{|X_j|\geq\varepsilon r_l\}}\right]$, $G(n,\varepsilon) = (V_n^2/r_n^2)g(n,\varepsilon)$. Similarly, if U is a positive function on $\mathbb{R}^+$ with finite variation such that $U(0) = 0$ and such that $U(x)$ tends to some constant > 0 as $x \to +\infty$ (the function M introduced above is an example of such a function), set $h(n,\varepsilon) = (V_n)^{-2}\sum_{j=n}^{+\infty}\mathbb{E}_{j-1}\left[X_j^2 U(|X_j|\varepsilon^{-1}r_l^{-1})\right]$ and $H(n,\varepsilon) = (V_n^2/r_n^2)h(n,\varepsilon)$. Then we have the following result.

Lemma 4.4
1) If (V_l^2/r_l^2) converges to 1 in probability, then the Lindeberg like condition (7) is equivalent to the convergence to 0, either in probability or in L^1, of the functions g, G, h, H, for all $\varepsilon > 0$.
2) With $b_l = \max_{n\geq l}\mathbb{E}[X_n^2]$, one has $b_l/r_l \to 0$, as $l \to +\infty$.

Sketch of proof of the lemma :
1) We refer to [2] for the essential. The key points are first that there exists constants $a, b > 0$ such that $b1_{\{x\geq a\}} \leq U(x)$, $\forall x \geq 0$. One can then write :

$$\begin{aligned} h(n,\varepsilon) = \int_0^{+\infty} g(n,\varepsilon y)\, dU(y) &= (V_n)^{-2}\sum_{j=n}^{+\infty}\mathbb{E}_{j-1}\left[X_j^2\int_0^{+\infty}1_{\{|X_j|\geq\varepsilon y r_n\}}\, dU(y)\right] \\ &\leq \int_0^{\delta} dU(y) + Kg(n,\varepsilon\delta) \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[H(n,\varepsilon)] &= \mathbb{E}[(1+\delta_n)h(n,\varepsilon)] \\ &= \mathbb{E}[h(n,\varepsilon)] + \mathbb{E}[\delta_n h(n,\varepsilon)].\end{aligned}$$

One has $\mathbb{E}[\delta_n h(n,\varepsilon)] \le K\mathbb{E}[|\delta_n|] \to 0$, as $n \to +\infty$, using lemma (4.3). The rest is the same.

2) One can write :

$$\begin{aligned}(r_l)^{-2}\mathbb{E}[X_n^2] &\le \varepsilon^2 + (r_l)^{-2}\mathbb{E}[X_n^2 1_{\{|X_n| \ge \varepsilon r_l\}}] \\ &\le \varepsilon^2 + \mathbb{E}[G(l,\varepsilon)].\end{aligned}$$

The conclusion follows.

□

References

[1] J-P. Conze and Y. Guivarc'h, *Marches en milieu aléatoire et mesures quasi-invariants pour un système dynamique*, Colloq. Math. **84/85** (2000), no. , part 2, 457–480, Dedicated to the memory of Anzelm Iwanik.

[2] B. M. Brown, *Martingale central limit theorems*, Ann. Math. Statist. **42** (1971), 59–66.

[3] P. Hall and C. C. Heyde, *Martingale limit theory and its application*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Probability and Mathematical Statistics.