Entropy and maximizing measures of generic continuous functions

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Abstract

In the natural context of ergodic optimization, we provide a short proof of the assertion that the maximizing measure of a generic continuous function has zero entropy.

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Dans le cadre usuel de l'étude des mesures maximisantes, nous donnons une preuve courte du fait que la mesure maximisante d'une fonction continue générique est d'entropie nulle.

1 Introduction

Let (X,T) be a topological dynamical system, where X is a compact metric space with a continuous transformation $T: X \longmapsto X$. Introduce the set $\mathcal{M}_T$ of Borel T-invariant probability measures on X, endowed with the compact and metrizable weak-$*$ topology. We assume that measures supported by a periodic orbit are dense in $\mathcal{M}_T$ and that the map $\mu \longmapsto h(\mu)$ is upper-semi-continuous (usc) on $\mathcal{M}_T$. These assumptions are for instance verified if (X,T) satisfies expansiveness and specification (cf Denker-Grillenberger-Sigmund [5]).

Fixing a continuous $f : X \to \mathbb{R}$, "ergodic optimization" (see Jenkinson [6] and references therein) is concerned with the following variational problem :

$$\beta(f) = \sup\{\mu(f) \mid \mu \in \mathcal{M}_T\} \text{ and } \mathcal{M}ax(f) = \{\mu \in \mathcal{M}_T \mid \beta(f) = \mu(f)\},$$

where $\mu(f)$ is for $\int f d\mu$. The aim is to describe the set $\mathcal{M}ax(f)$ of *maximizing measures* for f, which is always a non-empty compact and convex subset of $\mathcal{M}_T$. Notice also that any measure in the ergodic decomposition of a maximizing measure is a maximizing measure. We consider here genericity results in functional spaces. Recall that a set is *residual* if it contains a dense G_δ-set. A property defining a residual set is *generic*. An element in a residual set is declared *generic*.

The regularity of f plays a crucial role. In a Hölder or Lipschitz functional space, the Conze-Guivarc'h-Mañé lemma (see [6] for instance) gives a characterization of the maximizing measures via their support. The analysis is fairly delicate and difficult conjectures about periodic measures remain open (cf [6], [3] and references therein). The analysis of the case of the space $C(X)$ of real-valued continuous functions on X (endowed with the supremum norm) is completely different. The Conze-Guivarc'h-Mañé lemma is not valid any more, but duality arguments are available. Bousch and Jenkinson [1, 2] showed that for a generic f in $C(X)$ the situation is somehow pathological.

Theorem 1.1 *(Bousch-Jenkinson)*
A generic function in $C(X)$ has a unique maximizing measure and it has full support.

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In a recent article on a closely related problem, Jenkinson and Morris [7] considered the entropy of "Lyapunov maximizing measures" for C^1-expanding maps of the Circle. Certainly, their method allows to complete the picture in the following way, in some sense restricting the "pathology" :

Theorem 1.2 *(Jenkinson-Morris)*
The maximizing measure of a generic function in $C(X)$ has zero entropy.

The purpose of this note is to give a short and rather elegant proof of the latter result. Let us also mention that in the particular case of a symbolic setup (such as the shift T on some product space $X = \{0, \cdots, m-1\}^{\mathbb{Z}}$), theorem (1.2) can be proved elementarily using the density in $C(X)$ of locally constant functions, cf Conze-Guivarc'h [4].

2 Proof of theorem 1.2

Define a non-negative map $\varphi : f \longmapsto \sup_{\mu \in \mathcal{M}ax(f)} h(\mu)$ on $C(X)$. Let us check that it is usc. Indeed, since $\mu \longmapsto h(\mu)$ is usc and $\mathcal{M}ax(f)$ is compact, $\varphi(f) = h(\mu_f)$ for some $\mu_f \in \mathcal{M}ax(f)$. If now $f_n \to f$, then up to extraction μ_{f_n} weakly converges to some μ in $\mathcal{M}ax(f)$ and thus $\varphi(f) \geq h(\mu) \geq \limsup h(\mu_{f_n})$. This proves the assertion.

As a result, φ is continuous on a residual set R. We will show that φ in restriction to R is equal to zero. This latter fact will be a corollary from the following claim, of independent interest.

Proposition 2.1 *On a dense set D in $C(X)$, the maximizing measure is unique and supported by a periodic orbit.*

Assuming this result, let $f \in R$ and $f_n \to f$ with f_n in D. Since $\varphi(f_n) = 0$ and φ is continuous at f, we get $\varphi(f) = 0$. Thus $\varphi_{|R}$ is zero, as announced. This gives theorem (1.2).

To prove the latter proposition, first notice that it is enough to show that densely in $C(X)$ there is a periodic maximizing measure. Indeed, if g has a maximizing measure μ supported by some periodic orbit $Orb(x_0)$, introduce for $\eta_0 > 0$ the map $\eta(x) = -\eta_0 \operatorname{dist}(x, Orb(x_0))$, $\forall x \in X$. Then for $\nu \in \mathcal{M}_T$, one has $\nu(g + \eta) = \nu(g) + \nu(\eta)$ and $\nu(g) \leq \beta(g)$ and $\nu(\eta) \leq 0$, with both equalities simultaneously if and only if $\nu = \mu$. We therefore obtain $\mathcal{M}ax(g + \eta) = \{\mu\}$ and this gives the result since $\|\eta\|_\infty \to 0$ as $\eta_0 \to 0$.

To conclude, take any f and a measure $\mu \in \mathcal{M}_T$ supported by a periodic orbit with small $\beta(f) - \mu(f)$. By the next proposition, f can be perturbed into g with a maximizing measure ν such that μ and ν are not mutually singular (taking $\varepsilon = 1/2$ in the statement of the proposition). As μ is ergodic, it appears in the ergodic decomposition of ν and thus $\mu \in \mathcal{M}ax(g)$.

□

The next proposition comes from the classical proof of the Bishop-Phelps theorem. It is adapted from a preliminary version of Pollicott and Sharp [8].

Proposition 2.2 *Let $f \in C(X)$ and $\mu \in \mathcal{M}_T$. Write $\beta(f) - \mu(f) = \varepsilon\delta$, with $\varepsilon \geq 0, \delta \geq 0$. Then there exist $g \in C(X)$ and $\nu \in \mathcal{M}ax(g)$ such that $\|f - g\|_\infty \leq \delta$ and $\|\mu - \nu\|_{C(X)} \leq \varepsilon$.*

Proof of the proposition : From homogeneity and the fact that $\mathcal{M}ax(g) = \mathcal{M}ax(\lambda g)$ for $\lambda > 0$, it is enough to suppose that $\delta = 1$. Clearly we can also assume that $\varepsilon > 0$. Define $\Phi(u) = \beta(u) - \mu(u)$ on $C(X)$ and let, for $v \in C(X)$:

$$A(v) = \{u \in C(X) \mid \Phi(u) \leq \Phi(v) - \varepsilon \|v - u\|_\infty\}.$$

By the triangular inequality, observe that $A(u) \subset A(v)$ if $u \in A(v)$. Let now $f_0 = f$, with $\Phi(f_0) = \varepsilon$, and for $n \geq 0$, choose $f_{n+1} \in A(f_n)$ such that $\Phi(f_{n+1}) \leq 2^{-n-1}\varepsilon + \inf\{\Phi(u) \mid u \in A(f_n)\}$. Then $(A(f_n))$ is decreasing and one has for $n \geq 0$ and any $u \in A(f_n)$:

$$\Phi(f_n) - \varepsilon 2^{-n} \leq \Phi(u) \leq \Phi(f_n) - \varepsilon \|f_n - u\|_\infty.$$

As a result $\|f_n - u\|_\infty \leq 2^{-n}$. Thus (f_n) is a Cauchy sequence converging to some g and $\mathrm{diam}(A(f_n)) \leq 2^{-n+1}$. Therefore $\|f - g\|_\infty \leq 1$ and $A(g) = \{g\}$.

By this last property of g, the open convex set $\{(u, y) \in C(X) \times \mathbb{R} \mid y < \Phi(g) - \varepsilon\|g - u\|_\infty\}$ and the convex set $\{(u, y) \in C(X) \times \mathbb{R} \mid y \geq \Phi(u)\}$ are disjoint. From the Hahn-Banach separation theorem (cf Ruelle [9], Appendix A.3.3 (a)), there is a linear form $L(u, y) = y - \tilde{\mu}(u)$, with a signed Borel measure $\tilde{\mu}$, and $t \in \mathbb{R}$ such that for all $u \in C(X)$:

$$\Phi(g) - \varepsilon\|g - u\|_\infty - \tilde{\mu}(u) \leq t \leq \Phi(u) - \tilde{\mu}(u).$$

Taking $u = g$ gives $t = \Phi(g) - \tilde{\mu}(g)$. Thus for all $u \in C(X)$, we have $\Phi(g) - \tilde{\mu}(g - u) \leq \Phi(u)$ and $\tilde{\mu}(g - u) \leq \varepsilon\|g - u\|_\infty$, which can be rewritten as $\beta(g) + (\mu + \tilde{\mu})(u) \leq \beta(g + u)$ and $|\tilde{\mu}(u)| \leq \varepsilon\|u\|_\infty$.

Consequently and by definition, $\nu = \mu + \tilde{\mu}$ is a tangent functional for β at g (cf Ruelle [9], Appendix A.3.6). As detailed in the next lemma, it is thus a maximizing measure for g.

□

Lemma 2.3 *Let $f \in C(X)$ and a signed Borel measure ν be such that $\beta(f) + \nu(g) \leq \beta(f + g)$, for all $g \in C(X)$. Then ν is an invariant probability measure and it belongs to $\mathcal{M}ax(f)$.*

Proof of the lemma : Let $g \geq 0$ in $C(X)$. Since $\beta(f) \geq \beta(f - g)$, we get $\nu(g) \geq \beta(f) - \beta(f - g) \geq 0$. Thus ν is positive. Also for any real constant a, we have $\beta(f + a) = \beta(f) + a$, giving $\nu(a) \leq a$. Therefore $\nu(1) = 1$ and ν is a probability measure.

Let $g \in C(X)$. Since $\beta(f + g - g \circ T) = \beta(f)$, we have $\nu(g - g \circ T) \leq 0$. Taking $-g$, we get equality. Thus ν is T-invariant. Next, as $\beta(0) = 0$, when taking $g = -f$ we obtain $\beta(f) - \nu(f) \leq 0$. This shows that $\nu \in \mathcal{M}ax(f)$ and concludes the proof of the lemma.

□

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