Davenport series and almost-sure convergence

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Abstract

We consider Davenport-like series with coefficients in l^2 and discuss L^2-convergence as well as almost-everywhere convergence. We give an example where both fail to hold. We next improve former sufficient conditions under which these convergences are true.

1 Introduction

Let $\mathbb{R}\backslash\mathbb{Z}$ be the Circle and L^2 be the restriction of the space $L^2(\mathbb{R}\backslash\mathbb{Z} \to \mathbb{R})$ to odd functions. For a real parameter $\lambda > 1/2$, we introduce the map :

$$g_\lambda(x) = \sum_{m\geq 1} \frac{\sin(2\pi m x)}{m^\lambda}.$$

This function is defined everywhere on $\mathbb{R}\backslash\mathbb{Z}$. It is continuous, except at 0 when $1/2 < \lambda \leq 1$, and belongs to L^2. For real sequences $(a_n) \in l^2$, we consider expansions based on the dilated functions system $\{g_\lambda(nx)\}_{n\geq 1}$ of the following form :

$$\sum a_n g_\lambda(nx), \qquad (1)$$

where we write $\sum$ for the summation $\sum_{n=1}^{+\infty}$. We are interested in the questions of L^2-convergence and Lebesgue almost-everywhere (a.e) convergence of such series.

This is a natural problem which can be formulated with g_λ replaced by a general $g \in L^2(\mathbb{R}\backslash\mathbb{Z})$. A reason for focusing on odd functions is that sin series in general better converge than cos series. When $g(x) = \sin(2\pi x)$, the L^2-convergence of $\sum a_n g(nx)$ follows from the fact that the $\{g(nx)\}_{n\geq 1}$ are orthonormal. A.e-convergence in this case is the difficult theorem of Carleson [4]. For a different g such that the $\{g(nx)\}_{n\geq 1}$ are complete in L^2, the $\{g(nx)\}_{n\geq 1}$ are not orthogonal, see Bourgin and Mendel [2], and the question of L^2-convergence is already not clear. The case of $g = g_\lambda$ was introduced by Wintner in [17]. A special motivation comes Arithmetics and the case $\lambda = 1$, corresponding to the first Bernoulli polynomial or "sawtooth function" $\{x\} := x - [x] - 1/2$, where $[x]$ is the integer part of $x \in \mathbb{R}$. Indeed :

$$\{x\} \quad = \quad -\frac{1}{\pi} \sum_{m\geq 1} \frac{\sin(2\pi m x)}{m}.$$

Series of the form $\sum a_n\{nx\}$ appear since long ago in the litterature, at the interface of Number Theory and Analysis. We recommend the very detailed presentation of such series by Jaffard in [12], where they are called Davenport series, due to Davenport's initial systematic study [5, 6]. In

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this article and for simplicity we call D_λ-series a series of the form (1), the case of Davenport series corresponding to $\lambda = 1$.

Beginning with a discussion in L^2, Wintner [17] established that the family $\{g_\lambda(nx)\}_{n\geq 1}$ is complete in L^2 for any $\lambda > 1/2$. We now consider the L^2-convergence of D_λ-series with $(a_n) \in l^2$. According to work by Wintner [17] and next Hedenmalm, Lindqvist and Seip [11], the $\{g_\lambda(nx)\}_{n\geq 1}$ form a Riesz basis when $\lambda > 1$. By a "Riesz basis" we mean a complete sequence (ξ_n) in L^2 such that for some constant $C > 0$:

$$C^{-1}\sum a_n^2 \leq \|\sum a_n \xi_n\|^2 \leq C\sum a_n^2,\ \forall (a_n) \in l^2.$$

Here and in the whole article we denote by $\|\ \|$ the usual L^2-norm, with scalar product $\langle , \rangle$. Lindqvist and Seip [15] provide the inequalities :

$$\frac{\zeta(2\lambda)}{\zeta(\lambda)^2}\sum a_n^2 \leq \|\sum a_n g_\lambda(nx)\|^2 \leq \frac{\zeta(\lambda)^2}{\zeta(2\lambda)}\sum a_n^2,$$

where $\zeta(s) = \sum_{n\geq 1} n^{-s}$, for $s > 1$, is the Riemann Zeta function. Constants are optimal. This settles the question of L^2-convergence when $\lambda > 1$. In this case, the a.e-convergence of D_λ-series for $(a_n) \in l^2$ follows from Carleson's theorem [4] on the a.e-convergence of Fourier series. Indeed :

$$\sum_{n=1}^{N} a_n g_\lambda(nx) \quad = \quad \sum_{m\geq 1} m^{-\lambda} \sum_{n=1}^{N} a_n \sin(2\pi m n x). \tag{2}$$

For each $m \geq 1$, $\sum_{n=1}^{N} a_n \sin(2\pi mnx)$ converges a.e, as $N \to +\infty$, by Carleson's theorem. Next :

$$|\sum_{n=1}^{N} a_n \sin(2\pi mnx)| \leq \sup_{K\geq 1} |\sum_{n=1}^{K} a_n \sin(2\pi mnx)| =: M(mx).$$

By classical work on the maximal operator, $\|M\|_{L^1(\mathbb{R}\setminus\mathbb{Z})} \leq C\sum a_n^2$ (cf for instance Fefferman [7]). Thus $\sum_{m\geq 1} m^{-\lambda} M(mx)$ is integrable and in particular a.e finite. One can now a.e apply the Lebesgue dominated convergence theorem in (2) to conclude. Of course this argument does not work when $1/2 < \lambda \leq 1$. Mention also that Carleson's theorem uses properties of the Fourier basis. There exists orthonormal bases of L^2 for which L^2-convergence does not imply a.e-convergence, see Rademacher [16].

For the rest of the article we suppose that $1/2 < \lambda \leq 1$. As a consequence of an analysis by Wintner [17] of some Dirichlet series associated to D_λ-series, for any $1/2 < \lambda \leq 1$ there exists $(a_n) \in l^2$ such that $\sum a_n g_\lambda(nx)$ is L^2-divergent. In particular for $1/2 < \lambda \leq 1$, the $\{g_\lambda(nx)\}_{n\geq 1}$ do not form a Riesz basis of L^2. We now detail known sufficient conditions for L^2 and a.e-convergence. We are essentially aware of results concerning Davenport series. Wintner [17] proved that $\sum a_n\{nx\}$ converges in L^2 whenever $a_n = O(n^{-\kappa})$, with $\kappa > 1/2$. Extending this result, Jaffard [12] showed that for $(a_n) \in l^2$ the sum $\sum a_n\{nx\}$ converges in a Sobolev space very close to L^2. About a.e-convergence, Davenport in his fundamental papers [5, 6] gave non-trivial arithmetical examples where a.e-convergence is true, such as :

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n}\{nx\},\ \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n}\{nx\},\ \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^2}\{n^2 x\}, \tag{3}$$

where $\lambda(n)$, $\Lambda(n)$ and $\mu(n)$ are respectively Liouville's function, Von Mangolt's function and Mobius' function. When the a_n are slowly varying, the a.e-convergence of $\sum a_n\{nx\}$ follows, via an Abel transform, from estimates on $\sum_{n<N}\{nx\}$, cf Lang [14]. Jaffard [12] deduced the a.e-convergence of $\sum a_n\{nx\}$, whenever $a_n = O(\log n)^{-\alpha}$ and $a_{n+1} - a_n = O(n^{-1}(\log n)^{-(1+\alpha)})$ for some $\alpha > 2$. In particular, Hecke series :

$$\mathcal{H}_s(x) = \sum_{n=1}^{+\infty} \frac{\{nx\}}{n^s} \tag{4}$$

converge a.e for $\mathrm{Re}(s) > 0$, a result already shown by Hardy and Littlewood [8]. For general sequences, Hartman proved in [9] the a.e-convergence of $\sum a_n\{nx\}$ when $a_n = O(n^{-\kappa})$, with $\kappa > 2/3$. Mention finally some results going further than a.e-convergence. Using P-summation techniques, de la Bretêche and Tenenbaum [3] proved that (4) is convergent when $s = 1$ outside a set of Hausdorff dimension zero that they describe. Also the second series in (3) is everywhere convergent.

We now detail the content of the article. We discuss the L^2 and a.e-convergence of D_λ-series of the form (1) for general $(a_n) \in l^2$. We first complete the L^2-divergence result of Wintner [17] by an a.e-divergence result. We next improve former sufficient conditions for L^2 and a.e-convergence.

Theorem 1.1
Assume that $1/2 < \lambda \le 1$.

i) There exists $(a_n) \in l^2$ *such that* $\sum a_n g_\lambda(nx)$ *is simultaneously* L^2*-divergent and a.e-divergent.*

ii) Suppose that for some $\varepsilon > 0$ *:*

$$\begin{cases} \sum a_n^2 n^{\frac{(1+\varepsilon)(\log n)^{-(2\lambda-1)}}{2(1-\lambda)\log\log n}} < \infty, \text{ when } 1/2 < \lambda < 1, \\ \\ \sum a_n^2 (\log n)^3 (\log\log n)^{2+\varepsilon} < \infty, \text{ when } \lambda = 1. \end{cases}$$

Then $\sum a_n g_\lambda(nx)$ *converges in* L^2 *and a.e.*

In particular, the latter conditions are verified if $\sum a_n^2 n^\varepsilon < \infty$, for some $\varepsilon > 0$. For example, the following series converge in L^2 and a.e when $s > 1/2$:

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s}\{nx\}, \quad \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s}\{nx\} \text{ and } \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{2s}}\{n^2x\}.$$

We note that whereas Wintner's approach is abstract we build here an explicit example. The fact that $(a_n) \in l^2$ does not imply the a.e-convergence of D_λ-series is not that surprising, since this condition is not the correct one for L^2-convergence. One can make the second moment explode and develop a probabilistic argument based on the Central Limit Theorem. The true question, more difficult, is whether L^2-convergence implies a.e-convergence. A weak formulation is as follows :

Question. If $1/2 < \lambda \le 1$ and $\sum_{k\ge1}\Big(\sum_{n\ge1} n^{-\lambda}|a_{kn}|\Big)^2 < +\infty$, does $\sum a_n g_\lambda(nx)$ converge a.e ?

The above condition is strictly stronger than $(a_n) \in l^2$, when $1/2 < \lambda \le 1$. As detailed below, it ensures the L^2-convergence of $\sum a_n g_\lambda(nx)$ and is necessary when the a_n have constant sign.

We next consider three classical situations, for instance Hadamard lacunarity, where we can prove L^2-convergence, but a.e-convergence only under stronger conditions. We define the support $\mathrm{supp}(n)$ of an integer n as its set of prime divisors and write $|\mathrm{supp}(n)|$ for the cardinal of this set.

Theorem 1.2
Suppose that $1/2 < \lambda \le 1$.

i) Let (n_k) *check* $n_{k+1}/n_k \ge c$*, where* $c > 1$*. Then* $\sum a_k g_\lambda(n_k x)$ *converges in* L^2 *whenever* $(a_k) \in l^2$ *and more precisely :*

$$C_1 \sum a_k^2 \le \|\sum a_k g_\lambda(n_k x)\|^2 \le C_2 \sum a_k^2,$$

where :

$$C_1 = (1-1/e)\frac{\zeta(4\lambda)}{2}\left(\frac{2\lambda-1}{2\lambda}\right)\left(\frac{\ln(c^{2\lambda-1})}{1+\ln(c^{2\lambda-1})}\right)^2 \quad \textit{and } C_2 = \frac{\zeta(2\lambda)}{2}\left(\frac{c^\lambda+1}{c^\lambda-1}\right).$$

If the stronger condition $\sum a_k^2(\log k)^2 < \infty$ is verified, then $\sum a_k g_\lambda(n_k x)$ is also a.e-convergent.

ii) If $(a_n) \in l^2$ and $\{|supp(n)|,\ a_n \neq 0\}$ is finite, then $\sum a_n g_\lambda(nx)$ converges in L^2. In fact, for $N \geq 1$ there exists $C(\lambda, N) > 0$ such that for $(a_n) \in l^2$ with $a_n = 0$ if $|supp(n)| > N$, then :

$$C^{-1}(\lambda, N)\ \sum a_n^2 \leq \|\sum a_n g_\lambda(nx)\|^2 \leq C(\lambda, N)\ \sum a_n^2.$$

If moreover $\sum a_n^2(\log n)^2 < \infty$, then $\sum a_n g_\lambda(nx)$ is also a.e-convergent.

iii) Let $a_n = O(b_n)$, where $(b_n)_{n\geq 1} \in l^2 \cap (\mathbb{R}_+)^{\mathbb{N}}$ satisfies $b_{nm} = b_n b_m$ whenever n and m are relatively prime. Then $\sum a_n g_\lambda(nx)$ converges in L^2.

A word on the method. The study of the convergence of Davenport series often starts with trying to write $\sum a_n\{nx\}$ as a Fourier series $\sum c_m \sin 2\pi mx$. It was indeed remarked by Davenport [5] that formally the (c_m) are explicitly given in terms of the (a_n) and vice-versa. An alternative approach, developed here, when considering L^2-convergence is to orthonormalize the $\{g_\lambda(nx)\}_{n\geq 1}$. The Gram-Schmidt orthonormalisation procedure is explicit and a consequence of Carlitz's lemma on the reduction of quadratic forms. We provide details for simplicity. This furnishes a rather simple characterization of L^2-convergence. Theorem 1.2 follows via more or less standard computations. Concerning a.e-convergence, the orthonormalisation approach allows to adapt a technique of Rademacher [16] initially developed for the pointwise convergence of series built with general orthonormal systems.

A few notations. We write $i \wedge j$ and $i \vee j$ respectively for the greatest common divisor and the smallest common multiple of integers i and j. The set of primes is $\mathcal{P} = \{p_n, n \geq 1\}$.

2 Orthonormalization

Recall that $1/2 < \lambda \leq 1$. We first study correlations. The following computation is already contained in Lindqvist and Seip [15].

Lemma 2.1

Let $i, j \geq 1$. Then $\langle g_\lambda(i.), g_\lambda(j.)\rangle = \dfrac{\zeta(2\lambda)}{2}\left(\dfrac{i\wedge j}{i \vee j}\right)^\lambda$.

Proof of the lemma :

Let $i' = i/i\wedge j$, $j' = j/i \wedge j$. Since Lebesgue measure on $\mathbb{R}\backslash\mathbb{Z}$ is invariant by $x \longmapsto px$ for any integer p, we have $\langle g_\lambda(i.), g_\lambda(j.)\rangle = \langle g_\lambda(i'.), g_\lambda(j'.)\rangle$. Using the Fourier expansion of g_λ :

$$\langle g_\lambda(i'.), g_\lambda(j'.)\rangle = \sum_{k,l\geq 1}\int_0^1 \frac{\sin(2\pi k i' x)}{k^\lambda}\frac{\sin(2\pi l j' x)}{l^\lambda}\,dx = \frac{1}{2}\sum_{m\geq 1}\frac{1}{(m^2 i' j')^\lambda} = \frac{\zeta(2\lambda)}{2}(i'j')^{-\lambda},$$

since a relation $ki' = lj'$ reduces to $k = j'm$ and $l = i'm$ for some integer m. This concludes the proof of the lemma. □

Remark. — The correlations being positive, if $\sum b_n g_\lambda(nx)$ is L^2-convergent with $(b_n) \in (\mathbb{R}_+)^{\mathbb{N}}$ and if $a_n = O(b_n)$, then $\sum a_n g_\lambda(nx)$ also converges in L^2.

We turn to the orthonormalization of the $\{g_\lambda(nx)\}_{n\geq 1}$. The following proposition is an application of Carlitz's lemma, cf for instance Haukkanen, Wang and Sillanp [10].

Recall that the Möbius function μ on the integers is defined by $\mu(1)=1$, $\mu(p_{i_1}\cdots p_{i_k})=(-1)^k$ and $\mu(n)=0$ if n is not square-free. If f and g are real maps defined on the integers related by $f(n)=\sum_{k|n} g(k)$, then (Möbius inversion formula) we have $g(n)=\sum_{k|n}\mu(n/k)f(k)$.

Proposition 2.2
i) Let $f_{n,\lambda}(x)=n^{-\lambda}\sum_{k|n} k^{\lambda}\mu(n/k)\; g_{\lambda}(kx)$, $n\geq 1$. *Then* $\{f_{n,\lambda}\}_{n\geq1}$ *is an orthogonal family with :*

$$\|f_{n,\lambda}\|^2=\frac{\zeta(2\lambda)}{2}\prod_{p|n,p\in\mathcal{P}}(1-p^{-2\lambda})\in\left(\frac{1}{2},\frac{\zeta(2\lambda)}{2}\right).$$

The $\{f_{n,\lambda}\}_{n\geq1}$ *form an orthogonal Riesz basis of* L^2*, with :*

$$2\zeta(2\lambda)^{-1}\sum_{n\geq1}\langle f_{n,\lambda},h\rangle^2\leq\|h\|^2\leq 2\sum_{n\geq1}\langle f_{n,\lambda},h\rangle^2,\ \forall h\in L^2.$$

ii) An equality $\sum_{i=1}^{n}a_i g_{\lambda}(i.)=\sum_{i=1}^{n}b_i f_{i,\lambda}$ *holds if and only if* $b_i=\sum_{k=1}^{[n/i]}k^{-\lambda}a_{ki},\ 1\leq i\leq n$. *These equalities are reversed into* $a_i=\sum_{k=1}^{[n/i]}k^{-\lambda}\mu(k)b_{ki},\ 1\leq i\leq n$.

Proof of the proposition :
Let $n\geq1$. We introduce n-square matrices D and T, where D is diagonal and T is upper-triangular. Set $T=(t_{ij})$ with $t_{ij}=j^{-\lambda}1_{i|j}$ and write $D=\mathrm{diag}(d_i)$, where the (d_i) are defined below. First :

$$({}^tTDT)_{ij}=\sum_{1\leq k\leq n}t_{ki}d_k t_{kj}=(ij)^{-\lambda}\sum_{k|i\wedge j}d_k.$$

We choose D so that $\sum_{k|m}d_k=(\zeta(2\lambda)/2)m^{2\lambda}$, which by the Möbius inversion formula corresponds to setting $d_k=(\zeta(2\lambda)/2)\sum_{l|k}\mu(k/l)l^{2\lambda}$. Lemma 2.1 gives $({}^tTDT)_{ij}=\langle g_{\lambda}(i.),g_{\lambda}(j.)\rangle$.

Next, the inverse of T is given by $T^{-1}_{ij}=1_{i|j}i^{\lambda}\mu(j/i)$, since for any $i\leq j$:

$$\sum_{k=1}^{n}1_{i|k}i^{\lambda}\mu(k/i)j^{-\lambda}1_{k|j}=1_{i|j}i^{\lambda}j^{-\lambda}\sum_{l|j/i}\mu(l)=1_{i=j},$$

using that $\sum_{k|m}\mu(k)=0$, if $m\geq2$. Observe that $f_{i,\lambda}=i^{-\lambda}\sum_{1\leq k\leq n}({}^tT^{-1})_{ik}g_{\lambda}(k.)$, for $1\leq i\leq n$. The $\{f_{i,\lambda}\}_{i\geq1}$ are therefore orthogonal in L^2, with $\|f_{i,\lambda}\|^2=d_i i^{-2\lambda}$. They also form a complete family, since it is the case for the $\{g_{\lambda}(nx)\}$, cf Wintner [17]. Decomposing $i=p_{l_1}^{\alpha_1}\cdots p_{l_k}^{\alpha_k}$ in prime factors, we have :

$$\|f_{i,\lambda}\|^2=\frac{\zeta(2\lambda)}{2}\sum_{d|i}d^{-2\lambda}\mu(d)=\frac{\zeta(2\lambda)}{2}\prod_{j=1}^{k}\sum_{d|p_{l_j}^{\alpha_j}}d^{-2\lambda}\mu(d)=\frac{\zeta(2\lambda)}{2}\prod_{p|i,\ p\in\mathcal{P}}(1-p^{-2\lambda}).$$

Finally :

$$\sum_{i=1}^{n}a_i g_{\lambda}(i.)=\sum_{i=1}^{n}a_i\sum_{k=1}^{n}({}^tT)_{ik}k^{\lambda}f_{k,\lambda}=\sum_{i=1}^{n}a_i\sum_{k=1}^{n}i^{-\lambda}1_{k|i}k^{\lambda}f_{k,\lambda}=\sum_{k=1}^{n}f_{k,\lambda}\sum_{i=1}^{[n/k]}i^{-\lambda}a_{ki}.$$

The reversed formula is proved in a similar way. □

We deduce the following characterization of L^2-convergence.

Corollary 2.3
i) The series $\sum a_n g_\lambda(nx)$ *converges in* L^2 *if and only if the numerical series* $\sum_{k\geq 1} k^{-\lambda} a_{ki}$ *converge for all* $i \geq 1$*, together with the uniformity condition :*

$$\sum_{i\geq 1}\left(\sum_{k>[n/i]} k^{-\lambda} a_{ki}\right)^2 \to 0, \ \ \text{as } n \to \infty.$$

ii) A sufficient condition for $\sum a_n g_\lambda(nx)$ *to be* L^2*-convergent is :*

$$\sum_{i\geq 1}\left(\sum_{k\geq 1} k^{-\lambda}|a_{ki}|\right)^2 < +\infty.$$

This condition is necessary when $(a_n) \in (\mathbb{R}_+)^{\mathbb{N}}$*.*

Proof of the corollary :
If $\sum a_n g_\lambda(nx)$ converges in L^2, by proposition 2.2 the component $\sum_{k=1}^{[n/i]} k^{-\lambda} a_{ki}$ with respect to each $f_{i,\lambda}$ converges as $n \to +\infty$. The L^2-limit then has to be $\sum_{i\geq 1} f_{i,\lambda}(\sum_{k\geq 1} k^{-\lambda} a_{ki})$. The uniformity condition is a consequence from the fact that the norm of $f_{i,\lambda}$ belongs to $(1/2, \zeta(2\lambda)/2)$. The first assertion of the second item is an application of the Lebesgue Dominated Convergence Theorem, whereas the second one follows from the first item.

□

Remark. — Corollary 2.3 can also be obtained when considering directly the Fourier expansion of $\sum a_n g_\lambda(nx)$ given by g_λ. In the sequel, the orthonormalization point of view has the practical advantage to keep finite all partial sums.

3 A l^2-example of a L^2-divergent and a.e-divergent series

We prove theorem 1.1 *i*), using that for $1/2 < \lambda \leq 1$ the series $\sum_{p\in\mathcal{P}} p^{-\lambda}$ is divergent. For each integer $K \geq 1$, we choose a finite set $\mathcal{P}_K = \{p_{j,K}\}_{1\leq j\leq l_K}$ of consecutive primes satisfying :

$$\sum_{j=1}^{l_K} (p_{j,K})^{-\lambda} \geq K. \tag{5}$$

We fix $m_K \geq 2$ so that $\left(\frac{m_K - 1}{m_K}\right)^{l_K} \geq 1/2$. Introduce sets :

$$\begin{cases} F_{1,K} = \left\{p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}} \ , 1 \leq u_1, \cdots, u_{l_K} \leq m_K\right\} \\ \\ F'_{1,K} = \left\{p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}} \ , 1 \leq u_1, \cdots, u_{l_K} \leq m_K - 1\right\}. \end{cases}$$

We have $|F_{1,K}| = (m_K)^{l_K}$ and $|F'_{1,K}| = (m_K - 1)^{l_K}$. Let $q_{1,K} = 1$ and take next infinitely many primes $q_{2,K} < \cdots < q_{n,K} < \cdots$, larger than $p_{l_K,K}$ and subject to the condition :

$$(p_{1,K} \cdots p_{l_K,K})^{m_K}\left(1 + \frac{(m_K)^{l_K/2}}{K}\right)\sum_{r=2}^{+\infty}\frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \leq \frac{1}{K}. \tag{6}$$

Define the random variable :

$$X_{1,K} = \frac{1}{K|F_{1,K}|^{1/2}}\sum_{n\in F_{1,K}} g_\lambda(nx).$$

It has zero mean and belongs to L^2. We write $\sigma_K^2 = \int_0^1 (X_{1,K})^2(x)\, dx$ for its variance and choose an integer $T_K \geq K$ so that :

$$\int_0^1 (X_{1,K})^2(x) 1_{\{|X_{1,K}|>(T_K)^{1/12}\sigma_K\}}\, dx \leq \frac{1}{K}. \tag{7}$$

We define another collection of sets :

$$\begin{cases} F_{2,K} = q_{2,K} F_{1,K} \\ \\ F'_{2,K} = q_{2,K} F'_{1,K} \end{cases} \cdots \begin{cases} F_{T_K,K} = q_{T_K,K} F_{1,K} \\ \\ F'_{T_K,K} = q_{T_K,K} F'_{1,K}. \end{cases}$$

Grouping sets, we define :

$$E_K = \bigcup_{r=1}^{T_K} F_{r,K} \text{ and } E'_K = \bigcup_{r=1}^{T_K} F'_{r,K}.$$

We have $|E_K| = T_K |F_{1,K}|$ and $|E'_K| = T_K |F'_{1,K}|$. In particular :

$$\frac{|E'_K|}{|E_K|} = \frac{|F'_{1,K}|}{|F_{1,K}|} = \left(\frac{m_K - 1}{m_K}\right)^{l_K} \geq \frac{1}{2}. \tag{8}$$

When considering the next integer (ie $K+1$) we start with $p_{1,K+1} > q_{T_K,K}(p_{1,K} \cdots p_{l_K,K})^{m_K}$. Observe that all the $(F_{r,K})_{K\geq 1, 1\leq r\leq T_K}$, are pairwise disjoint and in particular the $(E_K)_{K\geq 1}$, which furthermore are consecutive. We finally set :

$$a_n = \begin{cases} \frac{1}{K|E_K|^{1/2}} & \text{, when } n \in E_K \text{, for some } K \geq 1, \\ \\ 0 & \text{, otherwise.} \end{cases}$$

This completes the definition of the sequence (a_n). Formally $\sum a_n g_\lambda(nx) = \sum_{K\geq 1} Z_K$, with :

$$Z_K = \frac{1}{\sqrt{T_K}} \sum_{r=1}^{T_K} X_{r,K}(x) \text{ and } X_{r,K}(x) = \frac{1}{K|F_{1,K}|^{1/2}} \sum_{n\in F_{r,K}} g_\lambda(nx). \tag{9}$$

From the previous construction, observe that a partial sum $\sum_{K=1}^N Z_K(x)$ corresponds to a partial sum of $\sum a_n g_\lambda(nx)$. We now proceed to verifications.

i) The sequence (a_n) belongs to l^2. Indeed :

$$\sum_{n\geq 1} a_n^2 = \sum_{K\geq 1} \sum_{n\in E_K} \frac{1}{K^2|E_K|} = \sum_{K\geq 1} \frac{1}{K^2} < \infty.$$

ii) The series $\sum a_n g_\lambda(nx)$ is L^2-divergent. Indeed, using (5) and (8) :

$$\begin{aligned} \sum_{n\geq 1} \left(\sum_{k\geq 1} k^{-\lambda} a_{kn}\right)^2 &\geq \sum_{K\geq 1} \sum_{n\in E'_K} \left(\sum_{k\geq 1} k^{-\lambda} a_{kn}\right)^2 \geq \sum_{K\geq 1} \sum_{n\in E'_K} \left(\sum_{k\in \mathcal{P}_K} k^{-\lambda} a_{kn}\right)^2 \\ &\geq \sum_{K\geq 1} \sum_{n\in E'_K} \frac{1}{K^2|E_K|} \left(\sum_{k\in\mathcal{P}_K} k^{-\lambda}\right)^2 \geq \sum_{K\geq 1} |E'_K| \frac{K^2}{K^2|E_K|} \geq \sum_{K\geq 1} \frac{1}{2} = +\infty. \end{aligned}$$

Since the a_n are positive, the conclusion comes from corollary 2.3.

$iii)$ The series $\sum a_n g_\lambda(nx)$ is a.e-divergent. This requires longer computations. For a fixed $K \geq 1$, all $(X_{r,K})_{1\leq r\leq T_K}$ have the same law, due to the invariance of Lebesgue measure on $\mathbb{R}\backslash\mathbb{Z}$ under multiplication by an integer. They do not form a stationary process, but are nearly independent. Under our hypotheses, it is routine to check that the law of $(\sigma_K^2 T_K)^{-1/2}\sum_{r=1}^{T_K} X_{r,K}$ is asymptotically normal.

In a first step, we compute the variance σ_K^2 and verify that it grows rapidly to infinity, as suggested by $ii)$. Via lemma 2.1, we have :

$$\begin{aligned}
\sigma_K^2 = \mathrm{Var}(X_{1,K}) &= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2(m_K)^{l_K}}\sum_{n,m\in F_{1,K}}\left(\frac{n\wedge m}{n\vee m}\right)^\lambda \\
&= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2(m_K)^{l_K}}\sum_{1\leq a_j,b_j\leq m_K, 1\leq j\leq l_K}\prod_{j=1}^{l_K}(p_{j,K})^{-\lambda|a_j-b_j|} \\
&= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2(m_K)^{l_K}}\prod_{j=1}^{l_K}\left(\sum_{a,b=1}^{m_K}(p_{j,K})^{-\lambda|a-b|}\right) \\
&= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2(m_K)^{l_K}}\prod_{j=1}^{l_K}\left(m_K + 2(p_{j,K})^{-\lambda(m_K-1)}\sum_{k=1}^{m_K-1}k(p_{j,K})^{\lambda(k-1)}\right).
\end{aligned}$$

For $x>1$, we have $\sum_{k=1}^{n-1}kx^{k-1} = ((n-1)x^n - nx^{n-1}+1)/(x-1)^2 = nx^{n-2}(1+o(1))$, when x and n are large. Inserting this in the previous calculations, we obtain, with a uniform $o(1)$:

$$\begin{aligned}
\sigma_K^2 &= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2(m_K)^{l_K}}\prod_{j=1}^{l_K}\left(m_K + 2m_K(p_{j,K})^{-\lambda}(1+o(1))\right) \\
&= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2}e^{\sum_{j=1}^{l_K}\log\left(1+2(p_{j,K})^{-\lambda}(1+o(1))\right)} \\
&= \frac{\zeta(2\lambda)}{2}\frac{1}{K^2}e^{2\left(\sum_{j=1}^{l_K}(p_{j,K})^{-\lambda}\right)(1+o(1))} \geq e^K, \qquad (10)
\end{aligned}$$

for large K, using (5).

We now establish the convergence :

$$(\sigma_K^2 T_K)^{-1/2}\sum_{r=1}^{T_K} X_{r,K} \to \mathcal{N}(0,1), \text{ in law.} \qquad (11)$$

We write E for the expectation under Lebesgue measure on $\mathbb{R}\backslash\mathbb{Z}$. Set $Y_{r,K} = (\sigma_K^2 T_K)^{-1/2}X_{r,K}$ and $S_N = \sum_{r=1}^N Y_{r,K}$. For $1\leq r\leq T_K$, introduce the finite partitions :

$$\mathcal{F}_r = \{[k/q_{r,K},(k+1)/q_{r,K}),\ 0\leq k< q_{r,K}\}.$$

Each $Y_{r,K}$ being $(1/q_{r,K})$-periodic, for a bounded measurable f :

$$E(f(Y_{r,K})) = E(f(Y_{r,K})|\mathcal{F}_r). \qquad (12)$$

For $t\in\mathbb{R}$ and $2\leq N\leq T_K$, we have :

$$\begin{aligned}
E(e^{itS_N}) &= E(e^{itS_{N-1}}e^{itY_{N,K}}) \\
&= E(E(e^{itS_{N-1}}|\mathcal{F}_N)e^{itY_{N,K}}) + E((e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N))e^{itY_{N,K}}) \\
&= \quad A \quad + \quad B.
\end{aligned}$$

First of all, taking conditional expectation and using (12) :

$$\begin{aligned} A = E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}}|\mathcal{F}_N)) &= E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}})) \\ &= E(e^{itS_{N-1}})E(e^{itY_{N,K}}). \end{aligned} \tag{13}$$

Next :

$$|B| \leq E(|e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)|). \tag{14}$$

The map $x \longmapsto e^{itx}$ is $|t|$-Lipschitz. On each piece of $\mathcal{F}_N$ which contains no discontinuity of S_{N-1}, when counting the oscillation we have :

$$\begin{aligned} |e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)| &\leq \frac{|t|}{q_{N,K}} \frac{(\sigma_K^2 T_K)^{-1/2}}{(K(m_K)^{l_K}))^{1/2}} \sum_{r=1}^{N-1} (m_K)^{l_K} (p_{1,K} \cdots p_{l_K,K})^{m_K} q_{r,K} \\ &\leq |t| \left(\frac{(m_K)^{l_K/2}}{K} (p_{1,K} \cdots p_{l_K,K})^{m_K} \right) \frac{(N-1)q_{N-1,K}}{q_{N,K}}, \end{aligned} \tag{15}$$

since $T_K \geq K$ and $\sigma_K \geq 1$ for large K, by (10). Next, S_{N-1} is continuous on the interior of each segment of the partition whose step^{-1} is $q_{1,K} \cdots q_{N-1,K}(p_{1,K} \cdots p_{l_K,K})^{m_K}$. The total measure of the pieces of $\mathcal{F}_N$ which may contain a discontinuity of S_{N-1} is bounded from above by :

$$q_{1,K} \cdots q_{N-1,K}(p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{1}{q_{N,K}} \leq (p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \tag{16}$$

From (14), (15) and (16), we deduce that :

$$|B| \leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left(1 + \frac{(m_K)^{l_K/2}}{K}\right) \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \tag{17}$$

Using that for all $1 \leq N \leq T_K$, we have $|E(e^{itY_{N,K}})| \leq 1$, when iterating the procedure with (13) and (14), we obtain via (6) :

$$\begin{aligned} |E(e^{itS_{T_K}}) - \prod_{r=1}^{T_K} E(e^{itY_{r,K}})| &\leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left(1 + \frac{(m_K)^{l_K/2}}{K}\right) \sum_{r=2}^{T_K} \frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \\ &\leq 2(1+|t|)\frac{1}{K}. \end{aligned} \tag{18}$$

As a consequence of (18), in order to show (11) we only need to focus on :

$$\prod_{r=1}^{T_K} E(e^{itY_{r,K}}) = E(e^{itY_{1,K}})^{T_K}. \tag{19}$$

We now use the fact that for all $t \in \mathbb{R}$:

$$|e^{it} - (1 + it - t^2/2)| \leq \min\{|t|^3/6, |t|^2\}, \tag{20}$$

which comes from $e^{it} - (1 + it - t^2/2) = i^3/2 \int_0^t (t-s)^2 e^{is} \, ds = i^2 \int_0^t (t-s)(e^{is} - 1) \, ds$. Via (20) and the property that $X_{1,K}$ has zero mean, we now deduce the following inequalities :

$$
\begin{aligned}
\left|E\left(e^{itY_{1,K}}\right)-\left(1-\frac{t^2}{2T_K}\right)\right| &\le \left|E\left(e^{itY_{1,K}}-\left(1+itY_{1,K}-\frac{t^2}{2}Y_{1,K}^2\right)\right)\right|\\
&\le E\left|\left(e^{itY_{1,K}}-\left(1+itY_{1,K}-\frac{t^2}{2}Y_{1,K}^2\right)\right)\right|\\
&\le E\left(\min\{|tY_{1,K}|^3/6, |tY_{1,K}|^2\}\right).
\end{aligned}
$$

With $\varepsilon = (T_K)^{-5/12}$ and using (7), as well as $T_K \ge K$ and $\sigma_K \ge 1$ for large K :

$$
\begin{aligned}
\left|E\left(e^{itY_{1,K}}\right)-\left(1-\frac{t^2}{2T_K}\right)\right| &\le \frac{|t|^3}{6}E\left(|Y_{1,K}|^3 1_{|Y_{1,K}|\le\varepsilon}\right)+|t|^2E\left(|Y_{1,K}|^2 1_{|Y_{1,K}|>\varepsilon}\right)\\
&\le \frac{|t|^3}{6(T_K)^{5/4}}+\frac{|t|^2}{\sigma_K^2 T_K}E\left(|X_{1,K}|^2 1_{|X_{1,K}|>(T_K)^{1/12}\sigma_K}\right)\\
&\le \frac{1}{T_K}\left(\frac{|t|^3}{6(T_K)^{1/4}}+\frac{|t|^2}{K\sigma_K^2}\right)\le\frac{1}{T_K}\left(\frac{|t|^3}{6K^{1/4}}+\frac{|t|^2}{K}\right). \qquad (21)
\end{aligned}
$$

Since $T_K \to +\infty$, as $K \to +\infty$, we deduce from (18), (19) and (21) that $E(e^{itS_{T_K}}) \to e^{-t^2/2}$, as $K \to +\infty$, for all $t \in \mathbb{R}$. This proves (11).

To conclude, for all $L \ge 1$ we choose $K_L \ge L$ so that :

$$
P(|S_{T_{K_L}}| \le 1/L^2) \le 2\int_{|t|\le 1/L^2} d\mathcal{N}(0,1)(t) =: \delta_L.
$$

Clearly $\sum_{L\ge1}\delta_L < \infty$, so by Borel-Cantelli's lemma, for a.e x when L is large enough we have $|S_{T_{K_L}}| \ge 1/L^2$. For such a x, using (10) and when L is large enough :

$$
|Z_{K_L}| = \left|\frac{1}{\sqrt{T_{K_L}}}\sum_{r=1}^{T_{K_L}} X_{r,K_L}(x)\right| = \sigma_{K_L}|S_{T_{K_L}}| \ge \frac{e^{K_L/2}}{L^2} \ge \frac{e^{L/2}}{L^2}.
$$

Since partial sums $\sum_{K=1}^N Z_K(x)$ are partial sums of $\sum a_n g_\lambda(nx)$, this prevents $\sum a_n g_\lambda(nx)$ from converging at x. This completes the proof of item $i)$ of theorem 1.1.

4 Sufficient conditions for L^2 and a.e-convergence

We take a finite sequence (a_n) and write $\sum a_n g_\lambda(nx) = \sum b_n f_{n,\lambda}(x)$, where (b_n) is also finite. By proposition 2.2 :

$$
\|\sum a_n g_\lambda(nx)\|^2 = \|\sum b_n f_{n,\lambda}\|^2 \le \frac{\zeta(2\lambda)}{2}\sum b_n^2 \le \frac{\zeta(2\lambda)}{2}\sum_{n\ge1}\left(\sum_{k\ge1}k^{-\lambda}|a_{kn}|\right)^2.
$$

Set $\psi_\lambda(k) = k^{1-\lambda}(\log k)^2$ if $1/2 < \lambda < 1$ and $\psi_1(k) = \log k(\log\log k)^{1+\varepsilon}$, for some $\varepsilon > 0$. For simplicity we write $\log(x)$ for $\max\{1, \log(x)\}$. Using Cauchy-Schwarz's inequality :

$$\begin{aligned}\sum_{n\geq 1}\left(\sum_{k\geq 1}k^{-\lambda}|a_{kn}|\right)^2 &= \sum_{k,k'\geq 1}(kk')^{-\lambda}\sum_{n\geq 1}|a_{nk}a_{nk'}| \leq \sum_{k,k'\geq 1}(kk')^{-\lambda}\left(\sum_{n\geq 1}a_{nk}^2\right)^{1/2}\left(\sum_{n\geq 1}a_{nk'}^2\right)^{1/2} \\ &\leq \left[\sum_{k\geq 1}k^{-\lambda}\left(\sum_{n\geq 1}a_{nk}^2\right)^{1/2}\right]^2 \leq \left(\sum_{k\geq 1}k^{-\lambda}\frac{1}{\psi_\lambda(k)}\right)\left(\sum_{k\geq 1}k^{-\lambda}\psi_\lambda(k)\sum_{n\geq 1}a_{nk}^2\right) \\ &\leq C_\varepsilon\sum_{n\geq 1}a_n^2\sum_{k|n}k^{-\lambda}\psi_\lambda(k). \qquad (22)\end{aligned}$$

We first consider the case $1/2<\lambda<1$. Remark that $0<2\lambda-1<1$. Using a classical upper-bound for $\sum_{k|n}k^{2\lambda-1}$, cf Krätzel [13], we have for any $\delta>0$:

$$\begin{aligned}\sum_{k|n}k^{-\lambda}\psi_\lambda(k) &= \sum_{k|n}k^{1-2\lambda}(\log k)^2 \leq (\log n)^2\sum_{k|n}k^{1-2\lambda} \leq (\log n)^2 n^{1-2\lambda}\sum_{k|n}k^{2\lambda-1} \\ &\leq (\log n)^2 n^{1-2\lambda}C_\delta n^{2\lambda-1}e^{\frac{(1+\delta)(\log n)^{1-(2\lambda-1)}}{(1-(2\lambda-1))\log\log n}} \leq C'_\delta n^{\frac{(1+2\delta)(\log n)^{1-2\lambda}}{2(1-\lambda)\log\log n}}.\end{aligned}$$

In the situation when $\lambda=1$, we have :

$$\sum_{k|n}k^{-1}\psi_1(k)\leq \psi_1(n)\sum_{k|n}k^{-1}=\psi_1(n)n^{-1}\sum_{k|n}k.$$

We use this time the inequality $\sum_{k|n}k\leq Cn\log\log n$, see again [13]. As a result, for any $\varepsilon>0$ there is a constant $C_\varepsilon>0$ such that for any sequence (a_n) :

$$\begin{cases}\|\sum a_n g_\lambda(nx)\|^2\leq C_\varepsilon\sum a_n^2 n^{\frac{(1+\varepsilon)(\log n)^{-(2\lambda-1)}}{2(1-\lambda)\log\log n}}, \text{ when } 1/2<\lambda<1, \\ \|\sum a_n\{nx\}\|^2\leq C_\varepsilon\sum a_n^2\log n(\log\log n)^{2+\varepsilon}, \text{ when } \lambda=1.\end{cases} \qquad (23)$$

These properties imply the L^2-convergence of D_λ-series under the assumptions of theorem 1.1.

We turn to the question of the a.e-convergence of D_λ-series. The second item of theorem 1.1 is a consequence of inequalities (23) and of the following proposition. The latter is an adaptation of a method due to Rademacher [16] for the study of series based on a general orthonormal family.

Proposition 4.1
Let $(a_n)_{n\geq 1}$ and $(\varphi(n))_{n\geq 1}$ be such that $\sum a_n^2\varphi(n)(\log n)^2<\infty$ and for any $M\leq N$:

$$\left\|\sum_{n=M}^N a_n g_\lambda(nx)\right\|^2\leq\sum_{n=M}^N a_n^2\varphi(n).$$

Then $\sum a_n g_\lambda(nx)$ converges a.e.

Proof of the proposition :
We can suppose that log is the logarithm in base 2. Let $S(n)(x)=\sum_{1\leq k\leq n}a_k g_\lambda(kx)$. For $m<n$, introduce the notations :

$$S(m,n)(x)=\sum_{m\leq k<n}a_k g_\lambda(nx) \text{ and } \sigma_l(m,n)=\sum_{m\leq k<n}a_k^2\varphi(k)(\log k)^l, \text{ for } l\in\{0,1,2\}.$$

Step 1. We show that $(S(2^n)(x))$ converges for a.e x. Let $0 < N < n$. We have :

$$\begin{aligned}\int_0^1 \sum_{N\le r<n} S(2^r,2^n)^2(x)dx &\le \sum_{N\le r<n}\sigma_0(2^r,2^n) = \sum_{N\le r<n}\sum_{s=r}^{n-1}\sigma_0(2^s,2^{s+1})\\ &\le \sum_{s=N}^{n-1}(s-N+1)\sigma_0(2^s,2^{s+1})\\ &\le \sum_{s=N}^{n-1} s\sigma_0(2^s,2^{s+1}) \le \sigma_1(2^N,2^n).\end{aligned}$$

By Markov's inequality, $\sum_{N\le r<n} S(2^r,2^n)^2(x) \le \sigma_1(2^N,2^n)^{2/3}$ for all x in a Borel set $E_{N,n}$ with :

$$\lambda(E_{N,n}) \ge 1-\sigma_1(2^N,2^n)^{1/3} \ge 1-\sigma_1(2^N,\infty)^{1/3}.$$

In particular, for $x \in E_{N,n}$ and all $N \le r < n$, we have $S(2^r,2^n)(x) \le \sigma_1(2^N,2^n)^{1/3}$. Define a set $E'_{N,n}$ by the condition that for all $N \le r \le r' < n$:

$$S(2^r,2^{r'})(x) \le 2\sigma_1(2^N,\infty)^{1/3}.$$

Since $E_{N,n} \subset E'_{N,n}$, we have $\lambda(E'_{N,n}) \ge 1-\sigma_1(2^N,\infty)^{1/3}$. Fixing N, the $E'_{N,n}$ are monotonic in n.

The set D_N defined by the condition that for all $N \le r \le r'$, $S(2^r,2^{r'})(x) \le 2\sigma_1(2^N,\infty)^{1/3}$ has therefore a measure $\lambda(D_N) \ge 1-\sigma_1(2^N,\infty)^{1/3}$. Since $\lambda(D_N) \to 1$, as $N \to +\infty$, we deduce that $\lambda(\limsup D_N) = 1$. If $x \in \limsup D_N$, the sequence $(S(2^n)(x))$ clearly satisfies the Cauchy criterion, so converges. This concludes *step 1.*

Step 2. To complete the proof, we show that a.e $\sup_{2^r<n<2^{r+1}} |S(2^r,n)(x)| \to 0$, as $r \to +\infty$. Let $2^r < n < 2^{r+1}$ and decompose n in base 2 :

$$n = 2^r + \sum_{l=1}^{r}\theta_l 2^{r-l}, \text{ with } \theta_l \in \{0,1\}.$$

Then :

$$S(2^r,n) = \sum_{l=1}^{r} S\left(2^r + \sum_{m=1}^{l-1}\theta_m 2^{r-m}, 2^r + \sum_{m=1}^{l}\theta_m 2^{r-m}\right).$$

By convexity :

$$\begin{aligned}S(2^r,n)^2 &\le r\sum_{l=1}^{r} S\left(2^r + \sum_{m=1}^{l-1}\theta_m 2^{r-m}, 2^r + \sum_{m=1}^{l}\theta_m 2^{r-m}\right)^2\\ &\le r\sum_{l=1}^{r}\sum_{h=0}^{2^{r-l}-1} S\left(2^r+h2^l, 2^r+h2^l+2^{l-1}\right)^2 =: T(r).\end{aligned}$$

The quantity $T(r)$ is independent on $2^r < n < 2^{r+1}$. Next :

$$\begin{aligned}\int_0^1 T(r)(x)\, dx &= r\sum_{l=1}^{r}\sum_{h=0}^{2^{r-l}-1}\int_0^1 S\left(2^r+h2^l, 2^r+h2^l+2^{l-1}\right)^2(x)\, dx \\ &\leq r\sum_{l=1}^{r}\sum_{h=0}^{2^{r-l}-1}\sigma_0(2^r+h2^l, 2^r+h2^l+2^{l-1}) \\ &\leq r\sum_{l=1}^{r}\sigma_0(2^r,2^{r+1}) = r^2\sigma_0(2^r,2^{r+1}) \leq \sigma_2(2^r,2^{r+1}).\end{aligned}$$

Fix N and let $r \geq N$. By Markov's inequality, for x in a Borel set $F_r(N)$ of Lebesgue measure $\lambda(F_r(N)) \geq 1 - \sigma_2(2^r,2^{r+1})/\sigma_2(2^N,\infty)^{2/3}$, we have :

$$\sup_{2^r<n<2^{r+1}} S(2^r,n)^2(x) \leq T(r) \leq \sigma_2(2^N,\infty)^{2/3}.$$

Let $G_N = \cap_{r\geq N} F_r(N)$. Then $\lambda(G_N) \geq 1 - \sum_{r\geq N}\sigma_2(2^r,2^{r+1})/\sigma_2(2^N,\infty)^{2/3} = 1-\sigma_2(2^N,\infty)^{1/3}$. For $x \in G_N$:

$$\forall r \geq N, \quad \sup_{2^r<n<2^{r+1}} |S(2^r,n)(x)| \leq \sigma_2(2^N,\infty)^{1/3}.$$

As $\lambda(G_N) \to 1$, we get $\lambda(\limsup G_N) = 1$. If $x \in \limsup G_N$, then $\sup_{2^r<n<2^{r+1}} |S(2^r,n)(x)|$ tends to 0, as $r \to \infty$. This concludes *step 2* and the proof of the proposition.

□

5 Particular classes where L^2-convergence is true

We consider the proof of theorem 1.2.

5.1 Proof of $i)$

Let (n_k) be lacunary in the sense that $n_{k+1}/n_k \geq c > 1$ and $(a_k) \in l^2$. We first consider the upper bound. We can assume the sequence (a_k) finite. By lemma 2.1, the L^2-norm of $(2/\zeta(2\lambda))^{1/2}\sum a_k g_\lambda(n_k x)$ is given by :

$$\sum_{k,l\geq 1} a_k a_l \left(\frac{n_k \wedge n_l}{n_k \vee n_l}\right)^\lambda = \sum a_k^2 + 2\sum_{k<l} a_k a_l \frac{(n_k\wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda}.$$

Using Cauchy-Schwarz's inequality, the second term is bounded by :

$$\begin{aligned}\sum_{k<l}|a_k a_l|\frac{(n_k\wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} \leq \sum_{k<l}|a_k a_l|\frac{n_k^{2\lambda}}{(n_k n_l)^\lambda} &\leq \sum_{k<l}|a_k a_l| c^{-\lambda(l-k)} \leq \sum_{k\geq 1}|a_k|\sum_{l\geq 1} c^{-\lambda l}|a_{k+l}| \\ &\leq \left(\sum_{n\geq 1} a_k^2\right)^{1/2}\left(\sum_{k\geq 1}\left(\sum_{l\geq 1} c^{-\lambda l}|a_{k+l}|\right)^2\right)^{1/2}.\end{aligned}$$

Next, still via Cauchy-Schwarz's inequality :

$$
\begin{aligned}
\sum_{k\geq 1}\left(\sum_{l\geq 1} c^{-\lambda l}|a_{k+l}|\right)^2 &= \sum_{l,l'\geq 1} c^{-\lambda(l+l')}\sum_{k\geq 1}|a_{k+l}a_{k+l'}| \\
&\leq \sum_{l,l'\geq 1} c^{-\lambda(l+l')}\left(\sum_{k\geq 1} a_{k+l}^2\right)^{1/2}\left(\sum_{k\geq 1} a_{k+l'}^2\right)^{1/2} \\
&\leq \left(\sum_{l\geq 1} c^{-\lambda l}\left(\sum_{k\geq 1} a_{k+l}^2\right)^{1/2}\right)^2 \leq \frac{1}{(c^\lambda-1)^2}\sum_{k\geq 1} a_k^2.
\end{aligned}
$$

As a consequence :

$$
\sum_{k<l}|a_k a_l|\frac{(n_k\wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} \leq \frac{1}{(c^\lambda-1)}\sum_{k\geq 1} a_k^2.
$$

Since $1+2/(c^\lambda-1) = (c^\lambda+1)/(c^\lambda-1)$, this completes the proof of the upper-bound.

For the lower bound, one can also suppose that (a_k) is finite. We have $\sum a_k g_\lambda(n_k x) = \sum b_k f_{k,\lambda}$, where (b_k) is also finite. By proposition 2.2 :

$$
\|\sum a_k g_\lambda(n_k x)\|^2 = \|\sum b_k f_{k,\lambda}\|^2 \geq \frac{1}{2}\sum b_k^2.
$$

Fixing $0<\varepsilon<1-(2\lambda)^{-1}$, giving $2\lambda(1-\varepsilon)>1$:

$$
\begin{aligned}
\sum a_k^2 = \sum_{k\geq 1}\left(\sum_{l\geq 1} l^{-\lambda}\mu(l) b_{ln_k}\right)^2 &\leq \sum_{k\geq 1}\left(\sum_{l\geq 1} l^{-2\lambda(1-\varepsilon)}\mu(l)^2\right)\left(\sum_{l\geq 1} l^{-2\lambda\varepsilon} b_{ln_k}^2\right) \\
&\leq \prod_{i\geq 1}\left(1+p_i^{-2\lambda(1-\varepsilon)}\right)\left(\sum_{l\geq 1} b_l^2\sum_{k, n_k|l}\left(\frac{n_k}{l}\right)^{2\lambda\varepsilon}\right) \\
&\leq \prod_{i\geq 1}\left(\frac{1-p_i^{-4\lambda(1-\varepsilon)}}{1-p_i^{-2\lambda(1-\varepsilon)}}\right)\left(\sum_{m\geq 0} c^{-2\lambda\varepsilon m}\right)\sum_{l\geq 1} b_l^2 \\
&\leq \frac{\zeta(2\lambda(1-\varepsilon))}{\zeta(4\lambda(1-\varepsilon))}(1-c^{-2\lambda\varepsilon})^{-1}\sum_{l\geq 1} b_l^2.
\end{aligned}
$$

To complete the proof, we first use that $\zeta(4\lambda(1-\varepsilon)) \geq \zeta(4\lambda)$ and $\zeta(2\lambda(1-\varepsilon)) \leq 1+1/(2\lambda(1-\varepsilon)-1)$. Set $\varepsilon = \rho(1-1/(2\lambda))$, with $0<\rho<1$. We have :

$$
\left(1+\frac{1}{2\lambda(1-\varepsilon)-1}\right)\frac{1}{1-c^{-2\lambda\varepsilon}} \leq \frac{2\lambda}{2\lambda-1}\ \frac{1}{(1-\rho)(1-c^{-(2\lambda-1)\rho})}.
$$

Minimizing in ρ, we take $\rho = 1/(1+\ln c^{2\lambda-1})$. We finally use the inequality $1-e^{-x} \geq (1-1/e)x$, for $0\leq x\leq 1$, giving $(1-c^{-(2\lambda-1)\rho}) \geq (1-1/e)(1-\rho)$.

Concerning a.e-convergence, we can now apply proposition 4.1 with $\varphi = 1$.

□

5.2 Proof of *ii*)

We start from (22). For n with $|\mathrm{supp}(n)| \le N$ and any $0<\delta<2\lambda-1$, we have :

$$\sum_{k|n} k^{-\lambda}\psi_\lambda(k) \le C_\delta \sum_{k|n} k^{1-2\lambda+\delta} \le C_\delta \prod_{p|n, p\in\mathcal{P}} (1-p^{1-2\lambda+\delta})^{-1} \le C_\delta \prod_{i=1}^{N} (1-p_i^{1-2\lambda+\delta})^{-1}.$$

For the lower bound, we use $\|\sum b_n f_{n,\lambda}\|^2 \ge (1/2)\sum b_n^2$, by proposition 2.2. Next :

$$\sum a_k^2 = \sum_{k\ge 1}\left(\sum_{l\ge 1} l^{-\lambda}\mu(l) b_{lk}\right)^2 \le \sum_{k\ge 1}\left(\sum_{l\ge 1} l^{-\lambda}|b_{lk}|\right)^2 \le C_\varepsilon \sum_{n\ge 1} b_n^2 \sum_{k|n} k^{-\lambda}\psi_\lambda(k),$$

when proceeding in the same way as for (22). We then conclude as above, using the fact that $b_n = 0$ when $|\mathrm{supp}(n)| > N$. For a.e-convergence, we apply proposition 4.1 with $\varphi = 1$.

□

5.3 Proof of *iii*)

Using the remark after lemma 2.1 we only need to focus on (b_n). Set $b_{i,n} = b_{p_i^n}$. Multiplicativity implies that the $(b_{i,n})_{i,n\ge 1}$ entirely determine the sequence (b_n). Via corollary (2.3), we show that :

$$\sum_{n\ge 1}\left(\sum_{k\ge 1} k^{-\lambda} b_{kn}\right)^2 < +\infty. \tag{24}$$

Each term in this series is finite, by Cauchy-Schwarz's inequality. We first claim the equivalence $(b_n)\in l^2 \Leftrightarrow \sum_{i\ge 1}\sum_{n\ge 1} b_{i,n}^2 < +\infty$. Indeed, using that $b_1 = 1$, we have :

$$\sum_{n\ge 1} b_n^2 = 1 + \sum_{k\ge 1}\sum_{1\le i_1<\cdots<i_k}\ \sum_{u_1\ge 1,\cdots,u_k\ge 1} b_{i_1,u_1}^2\cdots b_{i_k,u_k}^2 = \prod_{i\ge 1}\left(1+\sum_{n\ge 1} b_{i,n}^2\right). \tag{25}$$

This proves the claim.

For technical reasons, up to considering $\tilde{b}_{i,n} = b_{i,n} + 1/(in)$, $(i,n)\ge 1$, and the corresponding multiplicative sequence $(\tilde{b}_n)$, which satisfies $(\tilde{b}_n)\in l^2 \Leftrightarrow (b_n)\in l^2$, we assume that $b_{i,n}>0$ for all indices $(i,n)\ge 1$. Decomposing in prime factors $n = p_{i_1}^{u_1}\cdots p_{i_k}^{u_k}$, with $k=0$ if $n=1$, and using multiplicativity :

$$\begin{aligned}\sum_{n\ge 1}\left(\sum_{l\ge 1} l^{-\lambda} b_{ln}\right)^2 &= \sum_{k\ge 0}\sum_{1\le i_1<\cdots<i_k}\ \sum_{u_1\ge 1,\cdots,u_k\ge 1}\\ &\prod_{j\notin\{i_1,\cdots,i_k\}}\left(1+\sum_{m\ge 1} p_j^{-\lambda m} b_{j,m}\right)\times\prod_{l=1,\cdots,k}\left(\sum_{v_l\ge 0} p_{i_l}^{-\lambda v_l} b_{i_l,u_l+v_l}\right).\end{aligned}$$

The first product term is uniformly bounded from above since for a constant C :

$$\sum_{j,m\ge 1} p_j^{-\lambda m} b_{j,m} \le \left(\sum_{j,m\ge 1} p_j^{-2\lambda m}\right)^{1/2}\left(\sum_{j,m\ge 1} b_{j,m}^2\right)^{1/2} \le C\left(\sum_{j\ge 1} p_j^{-2\lambda}\right)^{1/2}\left(\sum_{j,m\ge 1} b_{j,m}^2\right)^{1/2} < +\infty.$$

To prove (24), it remains to check the finiteness of :

$$\sum_{k\geq 0}\sum_{1\leq i_1<\cdots<i_k}\prod_{l=1}^{k}\left[\sum_{u_l\geq 1}\left(\sum_{v_l\geq 0}p_{i_l}^{-\lambda v_l}b_{i_l,u_l+v_l}\right)^2\right]=\prod_{i\geq 1}\left[1+\sum_{u\geq 1}\left(\sum_{v\geq 0}p_i^{-\lambda v}b_{i,u+v}\right)^2\right].$$

It is equivalent to showing :

$$\sum_{i\geq 1,u\geq 1}\left(\sum_{v\geq 0}p_i^{-\lambda v}b_{i,u+v}\right)^2<+\infty.$$

Set $c_{i,n}=p_i^{\lambda n}\sum_{v\geq n}p_i^{-\lambda v}b_{i,v}$, which is finite by Cauchy-Schwarz's inequality. We thus verify that $\sum_{i\geq 1,n\geq 1}c_{i,n}^2<+\infty$. Fixing $0<\varepsilon<1-2^{-\lambda}$, we prove below that for all $i\geq 1$:

$$\begin{aligned}\sum_{n\geq 1}c_{i,n}^2 &\leq \frac{1}{\varepsilon^2(1-(1-\varepsilon)^{-2}p_i^{-2\lambda})}\sum_{n\geq 1}c_{i,n}^2\left(1-\frac{c_{i,n+1}}{p_i^{\lambda}c_{i,n}}\right)^2 \\ &\leq \frac{1}{\varepsilon^2(1-(1-\varepsilon)^{-2}2^{-2\lambda})}\sum_{n\geq 1}b_{i,n}^2.\end{aligned}\qquad(26)$$

Since $\sum_{i,n\geq 1}b_{i,n}^2<\infty$, this brings the conclusion.

Fix $i\geq 1$ and introduce $\mathcal{C}=\{n\geq 1,\ |1-c_{i,n+1}/(p_i^{\lambda}c_{i,n})|<\varepsilon\}$. We claim that if $\mathcal{C}$ is infinite, then it does not contain all large integers. Indeed, if $n\in\mathcal{C}$, then $c_{i,n+1}\geq(1-\varepsilon)p_i^{\lambda}c_{i,n}$, since $c_{i,n+1}<p_i^{\lambda}c_{i,n}$. If $\mathcal{C}$ contains some interval $[n_0,+\infty)$, then for $n\geq n_0$:

$$\sum_{v\geq 0}b_{i,v+n}p_i^{-\lambda v}=p_i^{\lambda n}\sum_{v\geq n}b_{i,v}p_i^{-\lambda v}\geq p_i^{\lambda n}(1-\varepsilon)^{n-n_0}\sum_{v\geq n_0}b_{i,v}p_i^{-\lambda v}.$$

However $\sum_{v\geq n_0}b_{i,v}p_i^{-\lambda v}$ is fixed and >0, since $b_{i,v}>0$. As $p_i^{\lambda}(1-\varepsilon)>1$, a contradiction is given by Cauchy-Schwarz's inequality, because the left-hand side is bounded from above by :

$$\left(\sum_{v\geq 0}p_i^{-2\lambda v}\right)^{1/2}\left(\sum_{v\geq 0}b_{i,n+v}^2\right)^{1/2}\leq\left(\sum_{v\geq 0}p_i^{-2\lambda v}\right)^{1/2}\left(\sum_{v\geq 0}b_{i,v}^2\right)^{1/2}<+\infty.$$

Decompose now into disjoint intervals $\mathcal{C}=\cup_{k\geq 1}[a_k,b_k]$ and write in a disjoint union :

$$\{n\geq 1\}=[a_1,b_1]\cup[a_1',b_1']\cup\cdots\cup[a_k,b_k]\cup[a_k',b_k']\cup\cdots.$$

Notice that the first interval $[a_1,b_1]$ may be empty, whereas the other ones are not, and that the collection of $([a_k,b_k],[a_k',b_k'])_k$ may be finite. We have :

$$\sum_{n\notin\mathcal{C},n\notin\{a_k',\ k\geq 1\}}c_{i,n}^2\left(1-\frac{c_{i,n+1}}{p_i^{\lambda}c_{i,n}}\right)^2\geq\varepsilon^2\sum_{n\notin\mathcal{C},n\notin\{a_k',\ k\geq 1\}}c_{i,n}^2.\qquad(27)$$

Also :

$$\sum_{k\geq 1}\sum_{l=a_k}^{a_k'}c_{i,l}^2\left(1-\frac{c_{i,l+1}}{p_i^{\lambda}c_{i,l}}\right)^2\geq\varepsilon^2\sum_{k\geq 1}c_{i,a_k'}^2.\qquad(28)$$

Observe finally that :

$$\sum_{l=a_k}^{a'_k} c_{i,l}^2 \leq c_{i,a'_k}^2 \sum_{m\geq 0} (1-\varepsilon)^{-2m} p_i^{-2\lambda m} \leq (1-(1-\varepsilon)^{-2} p_i^{-2\lambda})^{-1} c_{i,a'_k}^2. \quad (29)$$

Combining (27), (28) and (29), we get (26). This completes the proof of this item.

□

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