Random walks in random medium on $\mathbb{Z}$ and Lyapunov spectrum

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Abstract

We consider a one-dimensional random walk with bounded steps in a stationary and ergodic random medium. We show that the algebraic structure of the random walk is given by geometrical invariants related to the description of a space of harmonic functions. We then prove a recurrence criterion similar to Key's Theorem [11] in terms of the sign of an intermediate Lyapunov exponent of a random matrix. We show that this exponent is simple and we relate it to the dominant exponents of positive matrices associated to the random walks of left and right records. We also give an algorithm to compute that exponent.

1 Introduction

Random media are frequently introduced in Physics to model properties of statistical homogeneity (see Bernasconi [3]). We consider in this paper a one-dimensional model of random walks with bounded steps in random medium. It corresponds to giving a stationary field of transition laws on $\mathbb{Z}$.

1.1 Model

Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible dynamical system, that is a probability space $(\Omega, \mathcal{F}, \mu)$ with an invertible transformation T, measurable as well as its inverse. We assume the system to be ergodic. The space Ω will be considered as the space of the environments.

We now fix two integers $L \geq 1$ and $R \geq 1$ and introduce the set $\Lambda = \{-L, \cdots, +R\}$ of consecutive integers. Consider then a family $(p_z)_{z\in\Lambda}$ of positive random variables on $(\Omega, \mathcal{F})$, indexed by Λ, satisfying a minoration condition, precisely there exists $\varepsilon > 0$ such that :

$$\forall z \in \Lambda,\ z \neq 0,\ p_z \geq \varepsilon \text{ and } \sum_{z\in\Lambda} p_z = 1,\ \mu - ae. \tag{1}$$

For any fixed environment $\omega \in \Omega$, introduce the Markov chain $(\xi_n(\omega))_{n\geq 0}$ on $\mathbb{Z}$ such that $\xi_0(\omega) = 0$ and with the following transition laws :

$$\forall x \in \mathbb{Z}, \forall z \in \Lambda,\ \mathcal{P}_0^\omega(\xi_{n+1}(\omega) = x + z \mid \xi_n(\omega) = x) := p_z(T^x\omega).$$

We write $(\mathcal{P}_k^\omega)_{k\in\mathbb{Z}}$ for the family of measures on the space of jumps $\Lambda^{\mathbb{N}}$ with such transition laws and conditional to a given departure point $k \in \mathbb{Z}$. The "quenched problem" is to describe the behaviour of the random walk $(\xi_n(\omega))_{n\geq 0}$ with $\mathcal{P}_0^\omega$-probability one, for μ-ae medium ω.

Notations : The dependence in $\omega \in \Omega$ will always be implicit. Any expression of the form $f(T^k\omega)$ will simply be denoted by $T^k f$ or $f(k)$. In the sequel, we write P_k for $\mathcal{P}_k^\omega$, $k \in \mathbb{Z}$.

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1.2 Presentation

We now give an overview on the study of the model, centered on the asymptotic properties of the random walk. We denote by (L, R, erg) the previous model where the environment is a general dynamical system. We also introduce the notation (L, R, iid) for the independent case, corresponding to the situation where Ω is a product space, μ is a product probability measure, T is the left shift on Ω and the $(p_z)_{z\in\Lambda}$ depend only on the first coordinate.

The case $(1, 1, iid)$ has been intensively studied. The first result is due to Solomon [19] in 1975, who showed a recurrence criterion according to the sign of $\int \log(p_{-1}/p_1)\ d\mu$. The proof extends naturally to $(1, 1, erg)$. One may consult Alili [1] for example. For general L and R, the situation is more complex. Key [11] in 1984 proved a recurrence criterion for (L, R, iid), using Oseledets' Ergodic Multiplicative Theorem. The recurrence or transience of the random walk is given by the sign of the sum $\gamma_R(M_K, T^{-1}) + \gamma_{R+1}(M_K, T^{-1})$, involving the R^{th} and $(R+1)^{th}$ Lyapunov exponents with respect to T^{-1} of a random matrix M_K of dimension $(R+L)\times(R+L)$ built with the $(p_z)_{z\in\Lambda}$. The Theorem also indicates that one of those two exponents is always zero.

A first remark is that Key's Theorem extends to (L, R, erg) after a minor modification using conditional expectation in theorem (17), page 539 of Key [11]. It is presented in [4]. The form of the theorem can be simplified as one remarks that $M_K u = u$, where u is the vector in $\mathbb{R}^{R+L}$ whose components are all equal to one. Considering ${}^t(M_K)$ restricted to $u^\perp$ in a particular basis, one can deduce from Key's result a recurrence criterion in terms of the sign of the R^{th} Lyapunov exponent $\gamma_R(M, T)$ of a random matrix M of dimension $(L+R-1)\times(L+R-1)$. This was first noticed by Letchikov [15]. Another proof is given in [6]. The general study by the author in [5] and for the model $(L, 1, erg)$, concerning for example the existence of the absolutely continuous invariant measure for the random walk of the "environments seen from the particle", confirms the role of the matrix M, as well as the present work.

Studies in order to obtain a more "efficient" criterion were developed, first by Letchikov [14] for $(2, 1, erg)$, under an hypothesis of density, and then by Derriennic [8] who suppressed that hypothesis, using the theory of representation of a Markov chain by cycles and weights. The result is a recurrence criterion expressed in terms of the sign of $\int \log f\ d\mu$, where f is the random continued fraction defined by the relation $f = \frac{p_{-1}}{p_1} + \frac{p_{-1}+p_{-2}}{p_1}\frac{1}{T^{-1}f}$. It is checked in [6] that the previous criterion and Key's Theorem are the same, as it is shown that $\int \log f\ d\mu = \gamma_1(M, T)$.

We proved in [5], in the study of $(L, 1, erg)$, a generalization of the previous equality. When $R = 1$ (or $L = 1$, taking M^{-1}), then M has non negative coefficients and one can naturally use the directional contraction properties of M in the positive cone of $\mathbb{R}^L$. It is not necessary, as for the general case (L, R, erg), to understand completely the geometry of the eigenvectors corresponding to the Lyapounov spectrum of M with respect to T. In this case, there exists a unique positive random vector V with a norm equal to 1 and a unique positive random scalar λ, where $\log(\lambda)$ is a bounded function, such that $MV = \lambda TV$ and $\int \log(\lambda)\ d\mu = \gamma_1(M, T)$. The precise behaviour of the random walk can be read on the properties of λ with respect to $(\Omega, \mathcal{F}, \mu, T)$. For example, a characterization of the Law of Large Numbers or the Central Limit Theorem can be given.

1.3 Content of the article

The aim of the present paper is to study the structure of the random walk. It seems to be a prerequisite for the development of the same study as in [5] in the context of the model (L, R, erg). This way, we show that the algebraic structure of the random walk is given by the geometry of a space of harmonic functions. To describe this space, we prove the existence of deterministic and explicit cones in the external powers of order R and L of the underlying space that are invariant by the corresponding external powers of the matrix M. We then show that the central Lyapunov exponent $\gamma_R(M, T)$ of M with respect to T is simple and can be expressed as the difference of the dominant Lyapunov exponents of two nonnegative matrices related to the random walks of the left and right records. We deduce a recurrence criterion according to the sign of $\gamma_R(M, T)$. We then show how numerically, conditionally to a knowledge of the dynamical system, one can compute easily the exponent. We finally mention, as a corollary of [5], that the Law of Large Numbers is always valid for the model (L, R, erg).

2 Harmonic functions and gradient-vectors

We consider an interval of integers $[a,b]$, with $a<b$, and quantities that control the behaviour of the random walk in that interval, conditionally to a departure point k in that interval. In the sequel, we will let a or b become infinite in order to deduce an asymptotic result. For all $k \in [a-L+1, b+R-1]$, we introduce the following probabilities :

$$\begin{cases} P_k\{a,b,+\} = \mathcal{P}_k^{\omega}\{ \text{ reach }]-\infty,a] \cup [b,+\infty[\text{ by the right side } \} \\ P_k\{a,b,-\} = \mathcal{P}_k^{\omega}\{ \text{ reach }]-\infty,a] \cup [b,+\infty[\text{ by the left side } \}. \end{cases}$$

For $\zeta \in \{a-l \mid 0 \le l \le L-1\} \cup \{b+r \mid 0 \le r \le R-1\}$, we also set :

$$P_k\{a,b,\zeta\} = \mathcal{P}_k^{\omega}\{ \text{ reach}]-\infty,a] \cup [b,+\infty[\text{ at the point } \zeta\}.$$

We now introduce the definitions of "difference-vectors" or "gradient-vectors", derived from the previous quantities, and of the matrix M.

Definition 2.1
We set $d = R+L-1$.

Definition 2.2
Let $a \le k < b$ be integers. For any $\zeta \in \{a-l \mid 0 \le l \le L-1\} \cup \{b+r \mid 0 \le r \le R-1\} \cup \{\pm\}$, we write $V_k(a,b,\zeta)$ for the "gradient-vector" in $\mathbb{R}^d$:

$$V_k(a,b,\zeta) = {}^t(g_{k+R-1}(a,b,\zeta), \cdots, g_k(a,b,\zeta), \cdots, g_{k-L+1}(a,b,\zeta)),$$

where we define $g_k(a,b,\zeta) = P_k\{a,b,\zeta\} - P_{k+1}\{a,b,\zeta\}$.

Definition 2.3
We write M for the following random matrix of dimensions $d \times d$, where all entries are equal to 0 except the first line and a sub-diagonal of ones :

$$M = \begin{pmatrix} -a_1 & \cdots & -a_{R-1} & b_L & \cdots & b_1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix}, \tag{2}$$

with $a_i = \left(\frac{p_{R-i}+\cdots+p_R}{p_R}\right)$ and $b_i = \left(\frac{p_{-L+i-1}+\cdots+p_{-L}}{p_R}\right)$.

With begin with a lemma showing how the $V_k(a,b,\zeta)$'s and the matrix M naturally appear.

Lemma 2.4
Let $a<b$ be integers and let $\zeta \in \{a-l \mid 0 \le l \le L-1\} \cup \{b+r \mid 0 \le r \le R-1\} \cup \{\pm\}$. Then for any k such that $a<k<b$, one has :

$$V_k(a,b,\zeta) = M(k)V_{k-1}(a,b,\zeta). \tag{3}$$

Proof of the lemma :
Fix ζ as indicated in the statement of the lemma. We simplify $P_k\{a,b,\zeta\}$ into $f(k)$. Let now $a<k<b$. Using the Markov property, we get :

$$f(k)=\sum_{l=-L}^{R} p_l(k)f(k+l). \tag{4}$$

In factor of the left member $f(k)$, we write $1=\sum_{l=-L}^{R} p_l(k)$. Equality (4) becomes :

$$\sum_{l=1}^{R} p_l(k)(f(k)-f(k+l))=\sum_{l=1}^{L} p_{-l}(k)(f(k-l)-f(k)). \tag{5}$$

Introducing the successive differences of the function f, that is setting $g(k)=f(k)-f(k+1)$, from relation (5) we obtain :

$$\sum_{l=0}^{R-1} g(k+l)(p_{l+1}(k)+\cdots+p_R(k))=\sum_{l=1}^{L} g(k-l)(p_{-l}(k)+\cdots+p_{-L}(k)), \tag{6}$$

which can be rewritten as :

$$g(k+R-1)=-\sum_{l=0}^{R-2} g(k+l)\left(\frac{p_{l+1}(k)+\cdots+p_R(k)}{p_R(k)}\right)+\sum_{l=1}^{L} g(k-l)\left(\frac{p_{-l}(k)+\cdots+p_{-L}(k)}{p_R(k)}\right).$$

Using the matrix M and the definition of $V_k(a,b,\zeta)$, the previous relation is equivalent to :

$$V_k(a,b,\zeta)=M(k)V_{k-1}(a,b,\zeta).$$

This concludes the proof of the lemma.

□

Remark. — The matrix M makes all the previous gradient-vectors $V_k(a,b,\zeta)$ "circulate" on the $\mathbb{Z}$ axis. We now study the linear dependence between those vectors.

Lemma 2.5
Let $a<k<b$ be integers. Define the subspaces $E=Vect(V_k(a,b,\zeta)\mid \zeta\in\{a-l\mid l=0,\cdots,L-1\})$ and $F=Vect(V_k(a,b,\zeta)\mid\zeta\in\{b+r\mid r=0,\cdots,R-1\})$. Then :

i) $E+F=\mathbb{R}^d$.

ii) $E\cap F=\mathbb{R}V_k(a,b,+)$, where $V_k(a,b,+)=-V_k(a,b,-)$ is a non-zero vector.

Proof of the lemma :
We check that the intersection of E and F is one-dimensional and that it is given by the direction of the vector $V_k(a,b,+)$. Let $(\alpha_i)_{1\le i\le L}$ and $(\beta_i)_{1\le i\le R}$ be real numbers such that :

$$\alpha_L V_k(a,b,a-L+1)+\cdots+\alpha_1 V_k(a,b,a)+\beta_1 V_k(a,b,b)+\cdots+\beta_R V_k(a,b,b+R-1)=0. \tag{7}$$

Applying to (7), on the one hand the matrices $M(b-1)\cdots M(k+1)$ and on the other hand the matrices $(M(k)\cdots M(a+1))^{-1}$, we get :

$$\begin{cases} \alpha_L V_{b-1}(a,b,a-L+1) + \cdots + \alpha_1 V_{b-1}(a,b,a) + \beta_1 V_{b-1}(a,b,b) + \cdots + \beta_R V_{b-1}(a,b,b+R-1) = 0 \\ \alpha_L V_a(a,b,a-L+1) + \cdots + \alpha_1 V_a(a,b,a) + \beta_1 V_a(a,b,b) + \cdots + \beta_R V_a(a,b,b+R-1) = 0. \end{cases}$$

Consider for example the second equality. Projecting this relation orthogonally on the subspace $\mathrm{Vect}(e_i \mid R+1 \leq i \leq d)$, we obtain :

$$\sum_{r=0}^{L-2} \alpha_{L-r}(e_{d-r} - e_{d-r+1}) - \alpha_1 e_{R+1} = 0, \text{ setting } e_{d+1} = 0.$$

Reordering, we obtain $\alpha_L = \cdots = \alpha_1 =: \alpha_0$. Similarly, we would get $\beta_R = \cdots = \beta_1 =: \beta_0$. Back to relation (7), we arrive at :

$$0 = \alpha_0 V_k(a,b,-) + \beta_0 V_k(a,b,+) = (\alpha_0 - \beta_0) V_k(a,b,-).$$

Remark now that $V_k(a,b,-)$ is not zero, otherwise the function $k \longmapsto P_k\{a,b,-\}$ would be constant on $[a,b]$, whereas it is equal to 1 in a and to 0 in b. Therefore $\alpha_0 = \beta_0$ and the result follows.

□

Remark. — We will see in the sequel that the study of the model relies on the comprehension of the behaviour of $V_k(a,b,+)$, especially in direction, as a and b tend to infinity.

3 Invariant cones

In order to analyse the central vector $V_k(a,b,+)$, we consider particular external powers of the space $\mathbb{R}^d$, namely $\wedge_R \mathbb{R}^d$ and $\wedge_L \mathbb{R}^d$. They respectively contain the R-decomposable vector $V_k(a,b,b+R-1) \wedge \cdots \wedge V_k(a,b,b)$ and the decomposable $L-$vector $V_k(a,b,a-L+1) \wedge \cdots \wedge V_k(a,b,a)$, obtained with the $V_k(a,b,\zeta)$'s, when ζ describes first the right exit points and second the left exit points of the interval $[a,b]$. Recall that from lemma (2.5), the vector $V_k(a,b,+)$ gives the direction of the intersection of the corresponding subspaces.

Focusing on the right exit points, the harmonic functions $k \longmapsto P_k\{a,b,\zeta\}$, for $\zeta \in \{b+r \mid 0 \leq r \leq R-1\}$ induce a particular algebraic structure for the corresponding vectors $V_k(a,b,\zeta)$. We now prove the existence of invariant cones in $\wedge_R \mathbb{R}^d$ for matrices that have the same form as $(-1)^{R-1} \wedge_R M$, where the signature $(-1)^{R-1}$ of a cycle of length R naturally appears.

The space $\wedge_R \mathbb{R}^d$ is equipped with its usual euclidian structure inherited from $\mathbb{R}^d$. We begin with some definitions.

3.1 A few definitions

Definition 3.1
Let $(e_i)_{1 \leq i \leq d}$ be the canonical basis of $\mathbb{R}^d$. For all $i \leq j$, we set $\Sigma_i^j = e_i + \cdots + e_j$. Let $\Phi = \{\varphi\}$ be the set of vectors in $\wedge_R \mathbb{R}^d$ of the form :

$$\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{1+k_2} \wedge \cdots \wedge \Sigma_R^{R+k_R},$$

with $0 \leq k_j \leq L-1$, for $1 \leq j \leq R$, and $i + k_i \neq j + k_j$, if $i \neq j$.

Remark. — We will see later that the above conditions ensure that the elements of Φ are distinct and even non proportional. We now recall some definitions about cones. One may consult Berman-Plemmons [2].

Definition 3.2
A cone in $\mathbb{R}^n$ is a subset stable by non-negative linear combinations. A cone is said to be "polyhedral" if it is generated by finitely many vectors and "solid" if it has non-empty interior.

Definition 3.3
The set of matrices of dimensions $n \times n$ preserving a cone $\mathcal{C} \subset \mathbb{R}^n$ is written $\Pi(\mathcal{C})$. A matrix A in $\Pi(\mathcal{C})$ is "$\mathcal{C}$-positive" if the image by A of $\mathcal{C} - \{0\}$ is contained in the interior of $\mathcal{C}$.

Definition 3.4
Let $\mathcal{C}$ be the polyhedral cone generated by the elements of Φ in $\wedge_R \mathbb{R}^d$:

$$\mathcal{C} = \left\{ \sum_{\varphi \in \Phi} \alpha_\varphi \varphi \mid \alpha_\varphi \geq 0 \right\}.$$

Let $\mathcal{C}^*$ be the dual cone of $\mathcal{C}$:

$$\mathcal{C}^* = \left\{ x \in \bigwedge_R \mathbb{R}^d \mid \langle x, \varphi \rangle \geq 0, \ \forall \varphi \in \Phi \right\}.$$

Remark. — Standard arguments give that $\mathcal{C}^*$ is also polyhedral and that $\mathcal{C}^{**} = \mathcal{C}$. We now introduce the set $\mathcal{M}$ containing the matrices having the same form as the matrix M given in (2).

Definition 3.5
Let $\mathcal{M} = \{M = M(\varepsilon_1, \cdots, \varepsilon_{R-1}, \delta_1, \cdots, \delta_L) \mid \varepsilon_i \geq 0, \ \delta_i \geq 0\}$, where :

$$M = \begin{pmatrix} -a_1 & \cdots & -a_{R-1} & b_L & \cdots & b_1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix}, \text{ with } a_i = 1 + \varepsilon_1 + \cdots + \varepsilon_i \text{ and } b_i = \delta_1 + \cdots + \delta_i.$$

3.2 Properties of the cone $\mathcal{C}$ and its dual $\mathcal{C}^*$

Proposition 3.6
Let $M \in \mathcal{M}$. Then $(-1)^{R-1} \bigwedge_R {}^tM \in \Pi(\mathcal{C})$. Therefore $(-1)^{R-1} \bigwedge_R M \in \Pi(\mathcal{C}^*)$.

Proof of the proposition :
We set $u_i = -\Sigma_i^{R-1}$ for $1 \leq i \leq R-1$ and $v_j = \Sigma_R^{R+L-j}$ for $1 \leq j \leq L$. Let $M \in \mathcal{M}$. The first column vector of $({}^tM)$ can be decomposed in the following way :

$${}^t(-a_1, \cdots, -a_{R-1}, b_L, \cdots, b_1) = u_1 + \sum_{i=1}^{R-1} \varepsilon_i u_i + \sum_{j=1}^{L} \delta_t v_t. \tag{8}$$

Let then $\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \cdots \wedge \Sigma_R^{R+k_R}$, with $0 \leq k_j \leq L-1$, for $1 \leq j \leq R$, and $i + k_i \neq j + k_j$, if $i \neq j$, be an element of Φ. We get :

$$
\begin{aligned}
\bigwedge_R {}^t M\varphi &= \left(u_1 + \sum_{i=1}^{R-1} \varepsilon_i u_i + \sum_{j=1}^{L} \delta_j v_j + \Sigma_1^{k_1}\right) \wedge \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} \\
&= (-1)^{R-1} \sum_{j=1}^{L} \delta_j \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{R+L-j} \qquad (9) \\
&+ \sum_{i=1}^{R-1} \varepsilon_i \left[-\Sigma_i^{R-1}\right] \wedge \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} + K,
\end{aligned}
$$

with :

$$
K = \begin{cases} -\Sigma_{1+k_1}^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R}, \text{ if } k_1 + 1 \leq R-1 \\ (-1)^{R-1} \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{k_1}, \text{ if } k_1 + 1 > R-1. \end{cases}
$$

To conclude, we now show that a term $S := -\Sigma_i^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R}$, with $i \leq R-1$, belongs to $(-1)^{R-1}\Phi$. We consider two cases :

– If $i > 1$, then $S = (-1)^i \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{i-1}^{i-1+k_i} \wedge A$, where :

$$
A = \Sigma_i^{R-1} \wedge \Sigma_i^{i+k_{i+1}} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} = (-1)^{R-i-1} \Sigma_i^{i+k'_i} \wedge \cdots \wedge \Sigma_R^{R+k'_R},
$$

proceeding inductively on the number of terms. We therefore obtain the result.

– If $i = 1$. In S and if $R-1 \geq 1+k_2$, we add the second vector to the first one and we get $S = -\Sigma_{k_2+2}^{R-1} \wedge \Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R}$. We are then back to the previous case. If $R-1 < 1+k_2$, then S is equal to :

$$
S = \Sigma_1^{R-1} \wedge \left(-\Sigma_R^{1+k_2}\right) \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} = (-1)^{R-1} \Sigma_1^{R-1} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^{1+k_2},
$$

which finishes the proof. □

Definition 3.7
We write $e = e_1 \wedge \cdots \wedge e_R$ for the first vector of the canonical basis of $\wedge_R \mathbb{R}^d$.

We now detail some geometrical properties of the cones $\mathcal{C}$ and $\mathcal{C}^*$ and with respect to the class of matrices $\mathcal{M}$. We have the following result.

Proposition 3.8

i) The cone $\mathcal{C}$ is solid, as well as $\mathcal{C}^*$ which contains a neighborhood of e.

ii) The set of extremal vectors of the cone $\mathcal{C}$ is Φ. In particular, two elements of Φ are not proportional.

iii) The cone $\mathcal{C}$ is minimal with respect to the stability by $(-1)^{R-1} \wedge_R {}^t\mathcal{M}$. Moreover, any solid cone stable by $(-1)^{R-1} \wedge_R {}^t\mathcal{M}$ contains $\mathcal{C}$ or $-\mathcal{C}$.

iv) Let $M_1, \cdots, M_R$ be in $\mathcal{M}$. Then the matrix $A := (-1)^{(R-1)R} \wedge_R {}^t(M_1 \cdots M_R) = \wedge_R {}^t(M_1 \cdots M_R)$ is $\mathcal{C}$-positive.

Proof of the proposition :
i) Consider integers $1 \leq i_1 < \cdots < i_R \leq d$. If $u \in \wedge_R \mathbb{R}^d$ is orthogonal to all the elements of Φ, it is in particular orthogonal to $\Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \cdots \wedge \Sigma_R^{i_R}$ and to $\Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \cdots \wedge \Sigma_R^{i_R-1}$, that is by substraction to $\Sigma_1^{i_1} \wedge \Sigma_2^{i_2} \wedge \cdots \wedge e_{i_R}$. Proceeding inductively, u is orthogonal to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_R}$. Thus $u = 0$. The statement concerning $\mathcal{C}^*$ is direct, as for all $\varphi \in \Phi$, we have $\langle \varphi, e \rangle = 1$.

ii) We first show that two vectors of Φ are not proportional. From the previous remark, if two vectors φ_1 and φ_2 in Φ are proportional, we have $\varphi_1 = \varphi_2$, that is an equality of the form :

$$\Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \cdots \wedge \Sigma_R^{R+k_R} = \Sigma_1^{1+k_1'} \wedge \Sigma_2^{2+k_2'} \wedge \cdots \wedge \Sigma_R^{R+k_R'},$$

with $(i + k_i)_{1 \leq i \leq R} \neq (i + k_i')_{1 \leq i \leq R}$ and $0 \leq k_i \leq L - 1$, $0 \leq k_i' \leq L - 1$. Let then t be the greatest index such that $k_t \neq k_t'$. If $t > 1$, we make the matrix $(-1)^{R-1} \wedge_R {}^t M(0, \cdots, 0, \delta_1, 0, \cdots, 0)$ act on the two members of the previous equality. Using equality (9) and letting δ_1 tend to $+\infty$, we obtain that the limit in direction is :

$$\Sigma_1^{1+k_2} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R} \wedge \Sigma_R^d = \Sigma_1^{1+k_2'} \wedge \cdots \wedge \Sigma_{R-1}^{R-1+k_R'} \wedge \Sigma_R^d,$$

Repeating this operation, there exists $0 \leq (r_i)_{2 \leq i \leq R} \leq L - 1$ such that we arrive to a relation of the following form :

$$\Sigma_1^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R} = \Sigma_1^{1+k'} \wedge \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R},$$

with only the first vector that is different in the above decomposable $R-$vectors, that is $k \neq k'$. Suppose that $k > k'$, then :

$$0 = \Sigma_{k'+2}^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R}$$

and also :

$$0 = \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_{k'+2}^{k'+2+r_{k'+2}} \wedge \Sigma_{k'+2}^{1+k} \wedge \cdots \wedge \Sigma_R^{R+r_R}.$$

We consider the two central terms. Comparing $k' + 2 + r_{k'+2}$ to $1 + k$, we subtract the "shortest" term to the "longest" one and then we shift the result to the right. One never gets 0 when substracting, since all the endings are distinct. Finally, we arrive to a relation of the form $0 = \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R} \wedge \Sigma_s^{s+t_s}$, with $s > R$ and $t_s \geq 0$, which is impossible.

Suppose now that there exists $\varphi \in \Phi$ such that $\varphi = \sum_{\psi \in \Psi} \lambda_\psi \psi$, with $\lambda_\psi > 0$, for a certain subset $\Psi \subset \Phi$. Taking the scalar product with e, we obtain $\sum_{\psi \in \Psi} \lambda_\psi = 1$ and then we can suppose that $\varphi \notin \Psi$. Consider then the greatest t such that the t^{th} ending of the decomposable R-vectors φ and all ψ in Ψ is not the same for all these R-vectors. Applying a sufficient number of times $(-1)^{R-1} \wedge_R {}^t M(0, \cdots, 0, \delta_1, 0 \cdots, 0)$ and letting δ_1 tend to $+\infty$ as above, we are reduced to the following situation, with some $1 \leq (r_j)_{2 \leq j \leq R} \leq L - 1$:

$$\Sigma_1^{1+k} \wedge \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R} = \sum_{\psi \in \Psi} \lambda_\psi \Sigma_1^{1+k_\psi} \wedge \Sigma_2^{2+r_2} \wedge \cdots \wedge \Sigma_R^{R+r_R}, \tag{10}$$

where $\mathrm{card}\{k, k_\psi \mid \psi \in \Psi\} \geq 2$. Using the fact that $\sum \lambda_\psi = 1$, we consider the situation where k is distinct from all the k_ψ. We also suppose that card $\Psi \geq 2$, otherwise we are back to the case treated above. Write now :

$$\Sigma_1^{1+k} - \sum_{\psi \in \Psi} \lambda_\psi \Sigma_1^{1+k_\psi} = \sum_{j=1}^{J} \lambda_j B_j, \ J \geq 1, \ B_j = \Sigma_{t_{j-1}+1}^{t_j} \text{ and } \lambda_j \neq 0, \ \forall 1 \leq j \leq J.$$

We note that the (t_j) are in the set of endings and are distinct from all the $(j + r_j)_{2 \leq j \leq R}$. Substracting the right member of (10) to the left member and using the previous equality, we repeat as above the procedure consisting of substracting successively and shifting to the right. Similarly, we get at the end a non-zero R-vector, whose last vector begins with some e_s with $s > R$, using the fact that at each step, the substraction is not 0. We thus obtain the result.

iii) If a subcone $\mathcal{C}_1 \subset \mathcal{C}$ is $(-1)^{R-1} \wedge_R {}^t\mathcal{M}$-stable, we show that $\Phi \subset \mathcal{C}_1$. Fix :

$$\varphi = \Sigma_1^{1+k_1} \wedge \Sigma_2^{2+k_2} \wedge \cdots \wedge \Sigma_R^{R+k_R} \in \Phi.$$

Take now $x \in \mathcal{C}_1$, of the form $x = \sum_{\psi \in \Psi} \delta_\psi \psi$, with $\delta_\psi > 0$, for $\psi \in \Psi \subset \Phi$. We then apply successively to this equality $(-1)^{R-1} \wedge_R {}^tM(0, \cdots, 0, \cdots, 0, \delta_{L-k_j}, 0 \cdots, 0)$, for $j = 1$, etc, $j = R$. Letting then each δ_{j+k_j} tend to $+\infty$, the limit direction is φ.

Consider now a solid cone $\mathcal{C}_1$ stable by $(-1)^{R-1} \wedge_R {}^t\mathcal{M}$. Let x be a point interior to $\mathcal{C}_1$. As Φ generates the whole space, one can write in a non necessarily unique way $x = \sum_{\varphi \in \Phi} \alpha_\varphi \varphi$. We suppose that $c := \sum_{\varphi \in \Phi} \alpha_\varphi \neq 0$, up to adding an element of $\mathcal{C}$ by perturbation. One can then make again the previous reasoning and we obtain that $c\varphi$ is in $\mathcal{C}_1$, for all $\varphi \in \Phi$.

iv) From (9), the image of any non-zero vector is of the form $\sum_{\varphi \in \Phi} \alpha_\varphi \varphi$, with $\alpha_\varphi > 0$ for all φ. We conclude using the fact that Φ generates the whole space.

□

Remark 1— We consider the question of the cardinal of Φ. When $L \geq R$, elementary calculations furnish the following formula :

$$\text{card } \Phi = \sum_{t=0}^{R} \sum_{1 \leq i_1 < \cdots < i_t \leq R, i_0 = 0} \left[\prod_{j=1}^{t} (R - i_j - (t-j))(L - R + j)^{i_j - i_{j-1} - 1} \right] (L - R + t + 1)^{R - i_t}.$$

An important remark is the following one. We recall that in [4], when $L = R = 2$, we had obtained that the matrix M is deterministically conjugated to a non-positive matrix. If R is fixed and $L \longrightarrow +\infty$, the previous formula gives that card $\Phi \sim L^R$, whereas dim $(\wedge_R \mathbb{R}^d) \sim L^R/R!$. Consequently in the general case, using only the cone $\mathcal{C}$, there is no deterministic change of basis such that the class of the matrices $(-1)^{R-1} \wedge_R {}^tM$ becomes a class of matrices with non-negative coefficients. If $R > L$, one can produce a more complicated formula for the cardinal of Φ, involving the euclidian division of R by L.

Remark 2— If $L \leq 2$, then one can check that $\mathcal{C} \subset \mathcal{C}^*$. In fact, if $L = 1$, then $\mathcal{C} = \mathcal{C}^* = \{e\}$. If $L = 2$, then two elements φ_1 and φ_2 in Φ can be written in the form :

$$\begin{cases} \varphi_1 = e_1 \wedge \cdots \wedge e_t \wedge (e_{t+1} + e_{t+2}) \wedge \cdots \wedge (e_R + e_{R+1}) \\ \\ \varphi_2 = e_1 \wedge \cdots \wedge e_s \wedge (e_{s+1} + e_{s+2}) \wedge \cdots \wedge (e_R + e_{R+1}). \end{cases}$$

One then checks that $\langle \varphi_1, \varphi_2 \rangle = R - (s \wedge t)$. The previous inclusion between the two cones $\mathcal{C}$ and $\mathcal{C}^*$ is not true any more for larger values of L. Indeed, if $L = 4$ and $R = 2$, with $\varphi_1 = (e_1 + e_2) \wedge (e_2 + e_3 + e_4)$ and $\varphi_2 = (e_1 + e_2 + e_3 + e_4) \wedge (e_2)$, we get $\langle \varphi_1, \varphi_2 \rangle = -1$.

4 Lyapunov spectrum and simplicity

We introduce the Lyapunov exponents $(\gamma_i(M,T))_{1 \leq i \leq d}$, in non-increasing order, of the matrix M with respect to the dynamical system $(\Omega, \mathcal{F}, T, \mu)$.

4.1 Definitions and preliminary study

Definition 4.1
The Lyapunov exponents $(\gamma_i(M,T))_{1\le i\le d}$ *of* M *with respect to* $(\Omega,\mathcal{F},T,\mu)$ *can be recursively defined by the equalities :*

$$\gamma_1(M,T)+\cdots+\gamma_i(M,T)=\lim\frac{1}{n}\log\|\wedge_i T^{n-1}M\cdots M\|,\ \ \mu-ae,\quad \textit{for all } 1\le i\le d.$$

Definition 4.2
Let $(V_i)_{1\le i\le d}$ *be a measurable family of Oseledets' vectors associated to the Lyapunov spectrum* $(\gamma_i(M,T))_{1\le i\le d}$ *and such that* $\|V_i\|=1$*. They satisfy :*

$$\lim_{n\to+\infty}\frac{1}{|n|}\log\|T^{n-1}M\cdots MV_i\|=-\lim_{n\to+\infty}\frac{1}{|n|}\log\|T^{-n}M^{-1}\cdots T^{-1}M^{-1}V_i\|=\gamma_i(M,T).$$

Remark. — Recall that if an exponent $\gamma_i(M,T)$ is simple, then the corresponding Oseledets' vector V_i is uniquely determined in direction.

We will see in the sequel that the asymptotic properties of the model depend on the sign of the central exponent $\gamma_R(M,T)$. The next proposition shows that the other exponents of M with respect to T have fixed signs. It relies on ideas of Key [11] and [5].

Proposition 4.3
The following inequalities hold :

$$\gamma_1(M,T)\ge\cdots\ge\gamma_{R-1}(M,T)>0>\gamma_{R+1}(M,T)\ge\cdots\ge\gamma_d(M,T).$$

Proof of the proposition :
We remark that for $\zeta\in\{-1,\cdots,-L\}$ and all $k\ge 0$:

$$V_k(-1,+\infty,\zeta)=T^kM\cdots MV_{-1}(-1,+\infty,\zeta).$$

We then notice that the $V_{-1}(-1,+\infty,\zeta)$ span a subspace of $\mathbb{R}^d$ of dimension at least $L-1$. As the $V_k(-1,+\infty,\zeta)$ are bounded, the exponent of $V_{-1}(-1,+\infty,\zeta)$ with respect to (M,T) is ≤ 0. Therefore $\gamma_{R+1}(M,T)\le 0$ and similarly, we would get $\gamma_{R-1}(M,T)\ge 0$. To prove that the inequalities are strict, we now introduce $A(r)=\mathrm{diag}(1,r,\cdots,r^{d-1})$, for some real r. Writing $M=M(a_1,\cdots,a_{R-1},b_L,\cdots,b_1)$, we observe that :

$$A(r)MA(r)^{-1}=rM(a'_1,\cdots,a'_{R-1},b'_L,\cdots,b'_1),\text{ with } a'_i=\frac{a_i}{r^i},\ b'_i=\frac{b_i}{r^{R+L-i}}.$$

There exists $\varepsilon>0$, such that for $r\in(1-\varepsilon,1+\varepsilon)$, the matrix $M(r):=M(a'_1,\cdots,a'_{R-1},b'_L,\cdots,b'_1)$ is also the matrix associated to a random walk, as the transition probabilities verify the minoration condition (1). We then have $\gamma_{R+1}(M(r),T)\le 0$ and $\gamma_{R-1}(M(r),T)\ge 0$. The exponents of M are then deduced by translation of $\log r$. This concludes the proof.

□

4.2 Simplicity of the central exponent

Theorem 4.4
The exponent $\gamma_R(M,T)$ *is simple.*

Corollary 4.5
There exists a random vector V_R*, with* $\|V_R\|_1 = 1$*, uniquely determined in direction, and a random scalar* λ_R*, such that* $\log|\lambda_R|$ *is bounded, verifying :*

$$MV_R = \lambda_R T V_R \text{ and } \int \log|\lambda_R| \, d\mu = \gamma_R(M,T).$$

The proof of theorem (4.4) relies on the following theorem about the simplicity of the dominant exponent for random matrices which are positive with respect to a solid polyhedral cone.

Theorem 4.6
In $\mathbb{R}^n$*,* $n \geq 1$*, let* $\mathcal{C}$ *be a solid and polyhedral cone. Assume that the dual cone* $\mathcal{C}^*$ *has the same properties. Let* $A \in GL_n(\mathbb{R})$ *be a random matrix with respect to* $(\Omega, \mathcal{F}, \mu, T)$ *such that* $A \in \Pi(\mathcal{C})$ *and the random variables* $\log\|A\|$ *and* $\log\|A^{-1}\|$ *are bounded. If :*

$$\mu\{\exists n \geq 0,\ T^{n-1}A \cdots A \text{ is } \mathcal{C} - \text{positive}\} > 0,$$

then we have :

i) The dominant exponent $\gamma_{\max}(A,T)$ *is simple.*

ii) There exists a vector $V \in \mathcal{C}$*, uniquely determined in direction, satisfying* $AV = \lambda_V TV$*, for some random scalar* λ_V *such that* $\log|\lambda_V|$ *is bounded and* $\int \log|\lambda_V| \, d\mu = \gamma_{\max}(A,T)$*.*

iii) All non-zero vectors in $\mathcal{C}$ *have maximal exponent with respect to* A *and* T*. If there exists* $W \in \mathcal{C}$ *and a random scalar* λ_W *such that* $AW = \lambda_W TW$*, then* V *and* W *have the same direction.*

Proof of theorem (4.4) *:*
We consider the matrix $A := (-1)^{R-1} \wedge_R M$ on $\mathbb{R}^d$. From proposition (3.8), $A \in \Pi(\mathcal{C})$, where $\mathcal{C}$ is the polyhedral and solid cone defined in definition (3.4) and $T^{-R+1}({}^tA) \cdots T^{-1}({}^tA)({}^tA)$ is $\mathcal{C}-$positive. As :

$$\gamma_{\max}({}^tA, T^{-1}) = \gamma_{\max}(A,T) = \gamma_{\max}(\wedge_R M, T).$$

From theorem (4.6), $\gamma_{\max}(\wedge_R M, T)$ is simple. Since $\gamma_{\max}(\wedge_R M, T) = \sum_{i=1}^{R} \gamma_i(M,T)$ and the second exponent of (A,T) is equal to $\sum_{i=1}^{R-1} \gamma_i(M,T) + \gamma_{R+1}(M,T)$, we obtain :

$$\gamma_R(M,T) > \gamma_{R+1}(M,T).$$

A symmetric study, using $((-1)^{L-1} \wedge_L M^{-1}, T^{-1})$, would give $\gamma_{R-1}(M,T) > \gamma_R(M,T)$. This concludes the proof.

□

4.3 Proof of theorem (4.6)

We assume that the context is the one of theorem (4.6). Let Φ be the set of extremal vectors of $\mathcal{C}$ and Ψ be the set of extremal vectors of $\mathcal{C}^*$, all these vectors having a norm equal to 1 with respect to the usual euclidian norm on R^n. We adapt a proof due to Hennion [10] on the simplicity of the dominant exponent of a non-negative random matrix with positive iterates. We assume first that A is any matrix in $\Pi(\mathcal{C})$.

Definition 4.7
Let $\| \ \|_\Psi$ *be the following norm on* $\mathbb{R}^n$ *:*

$$\|x\|_\Psi = \sum_{\psi \in \Psi} |\langle x, \psi \rangle|.$$

Definition 4.8
Set $\bar{B} = \mathcal{C} \cap \{x \mid \|x\|_\Psi = 1\}$. *If* x *and* y *are in* $\bar{B}$, *we define :*

$$m(x,y) = \sup\{s \geq 0 \mid s\langle y, \psi \rangle \leq \langle x, \psi \rangle, \ \forall \psi \in \Psi\} = \min_{\psi \in \Psi} \left\{ \frac{\langle x, \psi \rangle}{\langle y, \psi \rangle} \mid \langle y, \psi \rangle > 0 \right\}.$$

As $\|x\|_\Psi = \|y\|_\Psi = 1$, *we have* $0 \leq m(x,y) \leq 1$. *We finally set :*

$$d(x,y) = \theta(m(x,y)m(y,x)), \text{ with } \theta(s) = \frac{1-s}{1+s}, \ s \in [0,1].$$

We will see that d is a distance on $\bar{B}$ for which a matrix preserving $\mathcal{C}$ acts as a contraction. It is related to Hilbert's distance d_H on a cone by $d = \tanh(d_H/2)$, but it is bounded.

Lemma 4.9
i)– If x, y *and* z *are in* $\bar{B}$, *then* $m(x,z)m(z,y) \leq m(x,y)$.
– *If* x *and* y *are in* $\bar{B}$, *then* $m(x,y)m(y,x) = 1$ *if and only if* $x = y$.
– *For* x *and* y *in* $\bar{B}$, *we have* $m(x,y) = 0$ *if and only if* $\exists \psi \in \Psi$ *such that* $\langle x, \psi \rangle = 0$ *and* $\langle y, \psi \rangle > 0$.
ii) The map d *is a distance on* $\bar{B}$.

Proof of the lemma :
The point *i)* is a consequence of the previous definitions. Concerning *ii)*, we first notice that $\theta'(s) = -2/(1+s)^2$ and then θ is non-increasing on $[0,1]$. The map $F(s) = \theta(s) + \theta(t) - \theta(st)$ verifies $F(1) = 1$ and :

$$F'(s) = -\frac{2(1-t)}{(1+s)^2(1+st)^2}(1-s^2t).$$

Thus for all s, t in $[0,1]$, we have $\theta(st) \leq \theta(s) + \theta(t)$. We then use the point *i)*.

□

Lemma 4.10
For all x *and* y *in* $\bar{B}$, $x \neq y$, *we set :*

$$\begin{aligned} a &= (1-\lambda_1)x + \lambda_1 y, \ \lambda_1 = \inf\{\lambda \mid (1-\lambda)x + \lambda y \in \bar{B} \ \} \\ b &= (1-\lambda_2)x + \lambda_2 y, \ \lambda_2 = \sup\{\lambda \mid (1-\lambda)x + \lambda y \in \bar{B} \ \} \end{aligned}$$

Setting $x = u_1 a + u_2 b$ *and* $y = v_1 a + v_2 b$, *we then have :*

$$d(x,y) = \frac{|u_1 v_2 - u_2 v_1|}{u_1 v_2 + u_2 v_1}.$$

Proof of the lemma :
As there exists extremal vectors ψ_1 and ψ_2 such that $\langle x,\psi_1\rangle > \langle y,\psi_1\rangle$ and $\langle x,\psi_2\rangle < \langle y,\psi_2\rangle$, we observe that λ_1 and λ_2 are finite quantities. Let then $I=\{\psi \mid \langle a,\psi\rangle>0\}$ and $J=\{\psi \mid \langle b,\psi\rangle>0\}$. There is not inclusion between I and J, in particular none is equal to Ψ, otherwise there would exist $\varepsilon>0$ such that $(1+\varepsilon)a-\varepsilon b\in\bar{B}$, which would be a contradiction.

Let us check the formula if $x=a$. We then have $m(x,y)=0$ and $d(x,y)=1$, since $a\neq y$, which corresponds to the announced formula with $u_1=1$, $u_2=0$ and $v_2\neq 0$. The case $y=b$ is symmetric. Suppose then that $x\neq a$, $y\neq b$ and set $r=\min\{u_i/v_i \mid i=1,2\}$. We have :

$$r\langle y,\psi\rangle = rv_1\langle a,\psi\rangle + rv_2\langle b,\psi\rangle \leq u_1\langle a,\psi\rangle + u_2\langle b,\psi\rangle \leq \langle x,\psi\rangle.$$

Thus $m(x,y)\geq r$. For $\psi\in\Psi$, we also have the inequality :

$$m(x,y)(v_1\langle a,\psi\rangle + v_2\langle b,\psi\rangle) \leq u_1\langle a,\psi\rangle + u_2\langle b,\psi\rangle.$$

Taking $\psi_1\in J/I$ and $\psi_2\in I/J$, we obtain $m(x,y)\leq u_1/v_1$ and $\leq u_2/v_2$. Finally $m(x,y)=r$. As $x\neq y$, we have $m(x,y)m(y,x)<1$ and then :

$$m(x,y)m(y,x)=\min\left\{\frac{u_1v_2}{v_1u_2},\ \frac{u_2v_1}{v_2u_1}\right\},$$

which provides the formula.

□

Lemma 4.11
i) For all x and y in $\bar{B}$, we have $d(x,y)\geq \frac{1}{2}\|x-y\|_\Psi$.

ii) Let d_1 be the distance induced by $\|\ \|_\Psi$. Then $(int(B),d)$ and $(int(B),d_1)$ are homeomorphic.

Proof of the lemma :
i) Using lemma (4.10) and the fact that $u_1+u_2=v_1+v_2=1$, we have :

$$|u_1v_2-u_2v_1| = |u_2(1-v_1)-(1-u_1)v_1| = |u_1-v_1|.$$

However $\|x-y\|_\Psi = \sum_{\psi\in\Psi}|\langle x-y,\psi\rangle| \leq |u_1-v_1|\sum|\langle a,\psi\rangle| + |u_2-v_2|\sum|\langle b,\psi\rangle| \leq 2|u_1-v_1|$. Moreover :

$$0<u_1v_2+u_2v_1 \leq (u_1^2+u_2^2)^{1/2}(v_1^2+v_2^2)^{1/2} \leq (u_1+u_2)^{1/2}(v_1+v_2)^{1/2}=1.$$

ii) If $d_1(x^{(n)},x)\longrightarrow_n 0$, then :

$$m(y,x^{(n)})=\min_{\psi\in\Psi}\left\{\frac{\langle y,\psi\rangle}{\langle x^{(n)},\psi\rangle}\mid \langle x^{(n)},\psi\rangle>0\right\}=\min_{\psi\in\Psi}\left\{\frac{\langle y,\psi\rangle}{\langle x^{(n)},\psi\rangle}\mid \langle x,\psi\rangle>0\right\},$$

if n is large enough. We then deduce that $d(x^{(n)},x)\longrightarrow 0$.

□

Remark. — The second point of lemma (4.11) is false for $(\bar{B},d)$ and $(\bar{B},d_1)$, as $(\bar{B},d)$ is not connected.

Lemma 4.12
Let $g\in GL_n(\mathbb{R})$ be in $\Pi(\mathcal{C})$. Let $c(g)=\sup\{d(g.x,g.y)\mid x,y\in\bar{B}\}$, where $g.x=gx/\|gx\|_\Psi$. Then :

i) For all $x,y\in\bar{B}$, $d(g.x,g.y)\leq c(g)d(x,y)$.

ii) If g and g' are invertible and in $\Pi(\mathcal{C})$, then $c(gg')\leq c(g)c(g')$.

iii) We have $c(g)\leq 1$. Moreover $c(g)<1$ if and only if g is $\mathcal{C}$-positive.

Proof of the lemma :
Take points in $\bar{B}$ such that $x \neq y$ and then $g(x) \neq g(y)$. We denote by a, b and a_1, b_1 the extremal points in $\bar{B}$ obtained as the intersections with the line passing by (x, y) and $(g.x, g.y)$. With respect to these bases, the g has a matrix of the form :

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ with } \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \delta \geq 0.$$

We remark that $\alpha\delta + \beta\gamma > 0$, otherwise the above matrix would be a line matrix (whereas the images are distinct) or a column matrix (one of the images would be zero). We have $x = u_1 a + u_2 b$ and $y = v_1 a + v_2 b$. Thus :

$$\begin{aligned} d(g.x, g.y) &= \frac{|(\alpha u_1 + \beta u_2)(\gamma v_1 + \delta v_2) - (\gamma u_1 + \delta u_2)(\alpha v_1 + \beta v_2)|}{(\alpha u_1 + \beta u_2)(\gamma v_1 + \delta v_2) + (\gamma u_1 + \delta u_2)(\alpha v_1 + \beta v_2)} \\ &= \frac{|\alpha\delta - \beta\gamma||u_1 v_2 - u_2 v_1|}{2\alpha\gamma u_1 v_1 + (\alpha\delta + \beta\gamma)(u_1 v_2 + u_2 v_1) + 2\beta\delta u_2 v_2} \\ &\leq \frac{|\alpha\delta - \beta\gamma|}{\alpha\delta + \beta\gamma} \frac{|u_1 v_2 - u_2 v_1|}{u_1 v_2 + u_2 v_1} \leq d(g.a, g.b) d(x, y) \leq c(g) d(x, y). \end{aligned}$$

This proves $i)$. The second point is direct. Let us show $iii)$. If g is $\mathcal{C}$-positive, we have $g.\bar{B} \subset \mathrm{int}(B)$, thus $g.\bar{B}$ is a compact of $(\mathrm{int}(B), d_1)$ and then of $(\mathrm{int}(B), d)$ by the previous lemma (4.11). Therefore, there exists x_0 and y_0 such that $c(g) = d(g.x_0, g.y_0) < 1$.

□

Lemma 4.13
Let $g \in GL_n(\mathbb{R})$ and $\mathcal{C}$−positive. Then :

$$\begin{aligned} c(g) &= \max_{(\varphi,\varphi')\in\Phi} d(g.\varphi, g.\varphi') = \max_{(\psi,\psi')\in\Psi,\ (\varphi,\varphi')\in\Phi} \frac{|\langle g\varphi', \psi\rangle\langle g\varphi, \psi'\rangle - \langle g\varphi, \psi\rangle\langle g\varphi', \psi'\rangle|}{\langle g\varphi', \psi\rangle\langle g\varphi, \psi'\rangle + \langle g\varphi, \psi\rangle\langle g\varphi', \psi'\rangle} \\ &= c({}^t g). \end{aligned}$$

Proof of the lemma :
Let x and y be in $g.\bar{B} \subset \mathrm{int}(B)$. We write $x = \sum_{\varphi\in\Phi} \alpha_\varphi g.\varphi$, $y = \sum_{\varphi\in\Phi} \beta_\varphi g.\varphi$, where the $(\alpha_\varphi)'$s and the $(\beta_\varphi)'$s are ≥ 0 and $\|\sum \alpha_\varphi \varphi\|_\Psi = \|\sum \beta_\varphi \varphi\|_\Psi = 1$. We have :

$$m(x, y)m(y, x) = \min_{\psi,\psi'} \frac{\langle x, \psi\rangle}{\langle y, \psi\rangle} \frac{\langle y, \psi'\rangle}{\langle x, \psi'\rangle}.$$

However :

$$\frac{\langle x, \psi\rangle}{\langle x, \psi'\rangle} \frac{\langle y, \psi'\rangle}{\langle y, \psi\rangle} \geq \min_{\varphi\in\Phi} \frac{\langle g\varphi, \psi\rangle}{\langle g\varphi, \psi'\rangle} \min_{\varphi'\in\Phi} \frac{\langle g\varphi', \psi'\rangle}{\langle g\varphi', \psi\rangle} = \min_{\varphi,\varphi'} \frac{\langle g\varphi, \psi\rangle}{\langle g\varphi, \psi'\rangle} \frac{\langle g\varphi', \psi'\rangle}{\langle g\varphi', \psi\rangle}.$$

We also have :

$$\min_{\psi,\psi'} \frac{\langle g\varphi, \psi\rangle}{\langle g\varphi, \psi'\rangle} \frac{\langle g\varphi', \psi'\rangle}{\langle g\varphi', \psi\rangle} = m(g.\varphi, g.\varphi')m(g.\varphi', g.\varphi).$$

We deduce the first equality as θ is non-increasing. We therefore obtain :

$$c(g) = \max_{(\psi,\psi')\in\Psi,\ (\varphi,\varphi')\in\Phi} \frac{\langle g\varphi', \psi\rangle\langle g\varphi, \psi'\rangle - \langle g\varphi, \psi\rangle\langle g\varphi', \psi'\rangle}{\langle g\varphi', \psi\rangle\langle g\varphi, \psi'\rangle + \langle g\varphi, \psi\rangle\langle g\varphi', \psi'\rangle},$$

which gives the announced formula.

□

In the context of theorem (4.6), we introduce $\tau(\omega) = \inf\{n \geq 1 \mid (T^{n-1}A \cdots A)(\omega) \text{ is } \mathcal{C}-\text{positive}\}$. The assumption is that $\mu\{\tau < \infty\} > 0$.

Proposition 4.14
i) We have $\mu\{\tau < \infty\} = 1$, $\int \tau \, d\mu < \infty$ *and for* $n \geq \tau(\omega)$, $(T^{n-1}A \cdots A)(\omega)$ *is* $\mathcal{C}$*-positive.*

ii) We call "contraction coefficient" the following number $0 \leq \kappa < 1$ *:*

$$\log \kappa = \lim_n \frac{1}{n} \int \log c(T^{-1}A \cdots T^{-n}A) \, d\mu = \inf_n \frac{1}{n} \int \log c(T^{-1}A \cdots T^{-n}A) \, d\mu < 0.$$

Moreover $\lim_n c(T^{-1}A \cdots T^{-n}A)^{1/n} = \kappa$, $\mu - ae$, *and* $\lim_n c(T^{n-1}A \cdots A)^{1/n} = \kappa$, $\mu - ae$.

iii) There exists a unique measurable $V \in \bar{B}$ *such that* $A.V = TV$. *Then* V *belongs to int*(B).

iv) The vector V *has maximal exponent with respect to* A *and* T.

Proof of the proposition :
i) From the hypothesis, there exists $N \geq 0$ such that $\mu\{T^{N-1}A \cdots A \text{ is } \mathcal{C} - \text{positive}\} > 0$. Set then $\tau'(\omega) = \inf\{n \geq 1 \mid (T^{N-1}A \cdots A)(T^n(\omega)) \text{ is } \mathcal{C} - \text{positive}\}$. From Kac's lemma, we have $\mu(\{\tau' < \infty\}) = 1$ and $\int \tau' \, d\mu < \infty$. Now, if two invertible matrices g and g' are in $\Pi(\mathcal{C})$ and if g' is $\mathcal{C}$-positive, then $g'g$ and gg' are also $\mathcal{C}$-positive. Thus $\tau \leq \tau' + N$ and the result follows.

ii) We have the following inequality of sub-additivity, for $n \geq 0$, $m \geq 0$:

$$\log c(T^{-1}A \cdots T^{-n-m}A) \leq \log c(T^{-1}A \cdots T^{-m}A) + \log c(T^{-1}A \cdots T^{-n}A) \circ T^{-m}.$$

Moreover, each term in the above inequality is ≤ 0. We then use Kingman's sub-additive Ergodic Theorem. We obtain $\kappa < 1$, as there exists $n \geq 1$ such that $\int \log c(T^{-1}A \cdots T^{-n}A) < 0$. The sequence $(\log c(T^{n-1}A \cdots A))_n$ is also sub-additive and the last point follows from the invariance of the measure μ.

iii) We have $c(T^{-1}A \cdots T^{-n}A) \longrightarrow 0$, $\mu - ae$. Since :

$$\text{diam}((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B})) \leq c((T^{-1}A \cdots T^{-n}A)(\omega)),$$

the sequence of compact sets $((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B}))_n$ decreases to a unique element, $\mu - ae$. We set $\{V(\omega)\} = \cap_n (T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B})$. Then one has $A.V = TV$. The unicity of V follows also from the fact that $((T^{-1}A \cdots T^{-n}A)(\omega).(\bar{B}))_n$ decreases to $V(\omega)$.

iv) To prove that V has maximal exponent, as $\text{int}(\mathcal{C})$ generates the whole space $\mathbb{R}^n$, it is enough to show that all the vectors of $\text{int}(\mathcal{C})$ have the same exponent with respect to A and T. For $x \in \mathbb{R}^n$, we set $\|x\|' = \langle x, x\rangle^{1/2}$. If $x \in \text{int}(\mathcal{C})$, then $\langle x, \psi\rangle > 0$, for all $\psi \in \Psi$. For every $k \geq 0$, one has :

$$\|T^{k-1}A \cdots Ax\|_\Psi \leq \|x\|' \sum_{\psi \in \Psi} \|({}^tA) \cdots (T^{k-1}\, {}^tA)\psi\|' \tag{11}$$

Now, if $y \in \mathcal{C}^*$ and $\|y\|_\Psi = 1$, then one can write $y = \sum_{\psi \in \Psi} \alpha_\psi \psi$, with $\alpha_\psi \geq 0$. Thus :

$$\langle x, y\rangle \geq \left(\sum_{\psi \in \Psi} \alpha_\psi\right) \min_{\psi \in \Psi} \{\langle x, \psi\rangle\} \text{ and } 1 \leq \left(\sum_{\psi \in \Psi} \alpha_\psi\right) \max_{\psi \in \Psi} \{\|\psi\|_\Psi\}.$$

Consequently $\langle x, y\rangle \geq \min_{\psi \in \Psi} \{\langle x, \psi\rangle\} (\max_{\psi \in \Psi} \{\|\psi\|_\Psi\})^{-1}$, if $\|y\|_\Psi = 1$. Therefore, for $k \geq 0$:

$$\begin{aligned} \|T^{k-1}A \cdots Ax\|_\Psi &= \sum_{\psi \in \Psi} |\langle x, ({}^tA) \cdots (T^{k-1}\, {}^tA)\psi / \|({}^tA) \cdots (T^{k-1}\, {}^tA)\psi\|_\Psi\rangle| \|({}^tA) \cdots (T^{k-1}\, {}^tA)\psi\|_\Psi \\ &\geq \left(\frac{\min_{\psi \in \Psi} \{\langle x, \psi\rangle\}}{\max_{\psi \in \Psi} \{\|\psi\|_\Psi\}}\right) \sum_{\psi \in \Psi} \|({}^tA) \cdots (T^{k-1}\, {}^tA)\psi\|_\Psi. \end{aligned} \tag{12}$$

The conclusion follows from the equivalence of the norms on $\mathbb{R}^n$ and the inequalities (11) and (12).

□

Proposition 4.15
Let $0 \leq \kappa < 1$ be the contraction coefficient introduced in proposition (4.14). Denote by $\gamma_2(A,T)$ the second Lyapunov exponent of A with respect to T. Then :

$$\gamma_2(A,T) \leq \gamma_{\max}(A,T) + \log\kappa.$$

Proof of the proposition :
We introduce $\wedge_2\mathbb{R}^n$ equipped with the euclidian structure inherited from $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, set $||x||' = \langle x, x\rangle^{1/2}$. Let d_a be the angular distance on $\mathbb{R}^n$:

$$d_a(x,y) = \frac{||x \wedge y||'}{||x||'||y||'}.$$

Writing $A_n = T^{n-1}A \cdots A$, for $n \geq 0$, we first have :

$$\gamma_1(A,T) + \gamma_2(A,T) = \lim_{n\to+\infty} \frac{1}{n} \log || \wedge_2 A_n||,\ \ \mu - ae.$$

For any $y \in \bar{B}$, we have $1 = \sum_{\psi\in\Psi} |\langle y, \psi\rangle| \leq \left(\sum_{\psi\in\Psi} ||\psi||'\right) ||y||' = \mathrm{card}(\Psi)||y||'$, as $||\psi||' = 1$ for $\psi \in \Psi$. Then for x and y in $\bar{B}$:

$$||x \wedge y||' = ||x \wedge (y-x)||' \leq ||x||'||y-x||' \leq ||x||'||y-x||'\mathrm{card}(\Psi)||y||'.$$

Using the equivalence of the norms on $\mathbb{R}^n$ and lemma (4.11), we obtain the existence of a constant $C > 0$ such that :

$$d_a(x,y) \leq \mathrm{card}(\Psi)||y-x||' \leq C||y-x||_\Psi \leq 2Cd(x,y).$$

For x and y in $\bar{B}$, setting $A_n = T^{n-1}A\cdots A$, we get :

$$d_a(A_n.x, A_n.y) \leq 2Cd(A_n.x, A_n.y) \leq 2Cc(A_n).$$

Consequently :

$$\log ||A_n x \wedge A_n y||' - \log ||A_n x||' - \log ||A_n y||' \leq \log(2C) + \log c(A_n).$$

From proposition (4.14), we deduce that for all x all y in $\bar{B}$:

$$\limsup_{n\to+\infty} \frac{1}{n} \log ||A_n x \wedge A_n y||' \leq 2\gamma_{\max}(A,T) + \log\kappa.$$

As the cone $\mathcal{C}$ is solid, this inequality is true for all x and y in $\mathbb{R}^n$, that is :

$$\gamma_{\max}(A,T) + \gamma_2(A,T) \leq 2\gamma_{\max}(A,T) + \log\kappa,$$

which is equivalent to the announced formula.

□

5 Matrices of the random walks of left and right records

We present another way of estimating the exit probabilities of an interval. For example, if the exit is on the right side, we consider the successive records on the right of the random walk.

Definition 5.1
Let $a < k < b$, where $-\infty \leq a$. We define a matrix of size $R \times R$:

$$D_k(a,b) = \begin{pmatrix} P_{k+R-1}\{a,b,b+R-1\} & \cdots & P_{k+R-1}\{a,b,b\} \\ \vdots & & \vdots \\ P_k\{a,b,b+R-1\} & \cdots & P_k\{a,b,b\} \end{pmatrix}.$$

We also introduce :

$$D = \begin{pmatrix} 0 & 1 & \cdots \\ \cdots & \cdots & 1 \\ P_0\{-\infty,1,R\} & \cdots & P_0\{-\infty,1,1\} \end{pmatrix}.$$

Remark. — One has $D = D_0(-\infty,1)$ and for any $a < k$, the matrix $D_k(a,k+1)$ has a determinant equal to $(-1)^{R-1}P_k\{a,k+1,k+R\}$. The following lemma is directly checked.

Lemma 5.2
Let $a < k < \beta \leq b$ and $\zeta \in \{b,\cdots,b+R-1,+\}$.

i) Making vary the departure point, we have the following matricial relation :

$$\begin{aligned} \begin{pmatrix} P_{k+R-1}\{a,b,\zeta\} \\ \vdots \\ P_k\{a,b,\zeta\} \end{pmatrix} &= D_k(a,\beta) \begin{pmatrix} P_{\beta+R-1}\{a,b,\zeta\} \\ \vdots \\ P_\beta\{a,b,\zeta\} \end{pmatrix} \\ &= D_k(a,k+1)\cdots D_{b-1}(a,b)e_{R-r}, \text{ if } \zeta = b+r. \end{aligned}$$

ii) Making vary the exit points on the right side, we obtain :

$$\begin{aligned} \begin{pmatrix} P_k\{a,b,b+R-1\} \\ \vdots \\ P_k\{a,b,b\} \end{pmatrix} &= {}^tD_\beta(a,b) \begin{pmatrix} P_k\{a,\beta,\beta+R-1\} \\ \vdots \\ P_k\{a,\beta,\beta\} \end{pmatrix} \\ &= {}^tD_{b-1}(a,b)\cdots {}^tD_{k+1}(a,k+2) \begin{pmatrix} P_k\{a,k+1,k+R\} \\ \vdots \\ P_k\{a,k+1,k+1\} \end{pmatrix}. \end{aligned}$$

iii) We have the equality :

$$(V_k(a,b,b+R-1)\cdots V_k(a,b,b)) = (V_k(a,\beta,\beta+R-1)\cdots V_k(a,\beta,\beta))D_\beta(a,b).$$

Consequently :

$$V_k(a,b,b+R-1)\wedge\cdots\wedge V_k(a,b,b) = \det\left(D_\beta(a,b)\right) V_k(a,\beta,\beta+R-1)\wedge\cdots\wedge V_k(a,\beta,\beta).$$

We now repeat the same study for the exit points on the left side.

Definition 5.3
Let $a < k < b$, where $b \leq +\infty$. We define a matrix of size $L \times L$:

$$G_k(a,b) = \begin{pmatrix} P_k\{a,b,a\} & \cdots & P_k\{a,b,a-L+1\} \\ \vdots & & \vdots \\ P_{k-L+1}\{a,b,a\} & \cdots & P_{k-L+1}\{a,b,a-L+1\} \end{pmatrix}.$$

We also introduce :

$$G = \begin{pmatrix} P_0\{-1,+\infty,-1\} & \cdots & P_0\{-1,+\infty,-L\} \\ 1 & \cdots & \cdots \\ \cdots & 1 & 0 \end{pmatrix}.$$

Remark. — One has $G = G_0(-1,+\infty)$ and for any $k < b$, the matrix $G_k(k-1,b)$ has a determinant equal to $(-1)^{L-1}P_k\{k-1,b,k-L\}$. The following lemma follows by direct calculations.

Lemma 5.4
Let $a \leq \alpha < k < b$ and $\zeta \in \{a,\cdots,a-L+1,-\}$.

i) Making vary the departure point, we have the following matricial relation :

$$\begin{aligned} \begin{pmatrix} P_k\{a,b,\zeta\} \\ \vdots \\ P_{k-L+1}\{a,b,\zeta\} \end{pmatrix} &= G_k(\alpha,b) \begin{pmatrix} P_\alpha\{a,b,\zeta\} \\ \vdots \\ P_{\alpha-L+1}\{a,b,\zeta\} \end{pmatrix} \\ &= G_k(k-1,b)\cdots G_{a+1}(a,b)e_{1+l}, \text{ if } \zeta = a-l. \end{aligned} \tag{13}$$

ii) Making vary the exit points on the left side, we obtain :

$$\begin{aligned} \begin{pmatrix} P_k\{a,b,a\} \\ \vdots \\ P_k\{a,b,a-L+1\} \end{pmatrix} &= {}^tG_\alpha(a,b) \begin{pmatrix} P_k\{\alpha,b,\alpha\} \\ \vdots \\ P_k\{\alpha,b,\alpha-L+1\} \end{pmatrix} \\ &= {}^tG_{a+1}(a,b)\cdots {}^tG_{k-1}(k-2,b) \begin{pmatrix} P_k\{k-1,b,k-1\} \\ \vdots \\ P_k\{k-1,b,k-L\} \end{pmatrix}. \end{aligned}$$

iii) We have the equality :

$$(V_k(a,b,a-L+1)\cdots V_k(a,b,a)) = (V_k(\alpha,b,\alpha-L+1)\cdots V_k(\alpha,b,\alpha))G_\alpha(a,b).$$

Consequently :

$$V_k(a,b,a-L+1) \wedge \cdots \wedge V_k(a,b,a) = \det\left(G_\alpha(a,b)\right) V_k(\alpha,b,\alpha-L+1) \wedge \cdots \wedge V_k(\alpha,b,\alpha).$$

Remark. — The matrices G and D can be easily interpreted. Concerning D for example, consider the random walk on $\mathbb{Z}$ deduced from $(\xi_n(\omega))_{n\geq 0}$, when the transitions to the L left neighbors are suppressed and the transitions to the R right neighbors at the point k are changed into $(T^kP_0\{-\infty,1,r\}(\omega))_{1\leq r\leq R}$, the probability to stay definitely in k being $T^kP_0\{-\infty,1,-\}(\omega)$. This random walk is the random walk of the successive records on the right for the initial random walk $(\xi_n(\omega))_{n\geq 0}$. Similarly, the matrix G is related to the records on the left side. We will see later that the central exponent $\gamma_R(M,T)$ compares the influences of these two random walks.

6 An expression of the central exponent $\gamma_R(M,T)$

In a first step, we build the main eigenvector of $(-1)^{R-1}\wedge_R M$. From the symmetric construction for $(-1)^{L-1}\wedge_L M^{-1}$, we deduce a formula for the central exponent $\gamma_R(M,T)$ in terms of the maximal exponents of G and D.

Definition 6.1
Let $a<k<b$. With the gradient-vectors associated to the exit points of $[a,b]$ on the right side and on the left side, we introduce the following vectors :

$$\begin{cases} \mathcal{R}_k(a,b) = V_k(a,b,b+R-1)\wedge\cdots\wedge V_k(a,b,b)\in\wedge_R\mathbb{R}^d, \\ \\ \mathcal{R}_k^*(a,b) = V_k(a,b,b+R-1)\wedge\cdots\wedge V_k(a,b,b+1)\in\wedge_{R-1}\mathbb{R}^d, \end{cases}$$

and :

$$\begin{cases} \mathcal{L}_k(a,b) = V_k(a,b,a-L+1)\wedge\cdots\wedge V_k(a,b,a)\in\wedge_L\mathbb{R}^d, \\ \\ \mathcal{L}_k^*(a,b) = V_k(a,b,a-L+1)\wedge\cdots\wedge V_k(a,b,a-1)\in\wedge_{L-1}\mathbb{R}^d, \end{cases}$$

Proposition 6.2
i) For any $a<k<b$, we have $\mathcal{R}_k(a,k+1)\in(-1)^R\ int(\mathcal{C}^*)$.
ii) The following convergence holds :

$$\frac{\mathcal{R}_{-1}(-n,1)}{P_{-1}\{-n,1,-\}}\longrightarrow V,\ \ as\ n\longrightarrow+\infty,\mu-ae,$$

where $V\neq 0$ is such that $\log\|V\|$ is bounded.
iii) Moreover, V satisfies :

$$\left[(-1)^{R-1}\underset{R}{\wedge}M\right]V=\lambda TV,$$

where, $\mu-ae$:

$$\lambda=\frac{1}{P_0\{-\infty,1,R\}}\lim_{n\to+\infty}\frac{P_0\{-n,1,-\}}{P_{-1}\{-n,0,-\}} \tag{14}$$

and V has maximal exponent with respect to $(-1)^{R-1}\wedge_R M$ and T.

Proof of the proposition :
i) We first notice that $(-1)^R\mathcal{R}_k(a,k+1)=[-V_k(a,k+1,k+R)]\wedge\cdots\wedge[-V_k(a,k+1,k+1)]$. We write X for this vector. Let now $\varphi=\Sigma_1^{1+k_1}\wedge\cdots\wedge\Sigma_R^{1+k_R}$ be an extremal vector of $\mathcal{C}$. Setting $f_r^s=P_{k+R-r-k_r}\{a,k+1,k+s\}$, for $1\le r\le R$ and $1\le s\le R$, we obtain :

$$\langle X,\varphi\rangle=\begin{vmatrix} 1-f_1^R & -f_2^R & \cdots & -f_R^R \\ -f_1^{R-1} & 1-f_2^{R-1} & \cdots & -f_R^{R-1} \\ \vdots & \vdots & \cdots & \vdots \\ -f_1^1 & \cdots & \cdots & 1-f_R^1 \end{vmatrix}. \tag{15}$$

We remark that for all $1\le r\le R$, we have :

$$\sum_{1\le s\le R, s\neq R-r+1} f_r^s = \left(1-f_r^{R-r+1}\right) - P_{k+R-r-k_r}\{a,k+1,-\}.$$

Therefore the matrix appearing in (15) has a strictly dominating diagonal. Introducing parameters outside the diagonal and letting them decrease from 1 to 0, the determinant does not pass to zero. For a constant $C>0$, we obtain :

$$\langle X,\varphi\rangle = C\prod_{r=1}^{R}\left(1-f_r^{R-r+1}\right) > 0.$$

Consequently, $\langle X,\varphi\rangle > 0$, for all $\varphi\in\Phi$, and then $(-1)^R\mathcal{R}_k(a,k+1)\in \operatorname{int}(\mathcal{C}^*)$.

ii) and iii) We first have, for all $n\geq 1$:

$$\mathcal{R}_0(-n,1) = \underset{R}{\wedge} M\mathcal{R}_{-1}(-n,1) = \left[(-1)^{R-1}\underset{R}{\wedge} M\right]\mathcal{R}_{-1}(-n,0)\ P_0\{-n,1,R\}.$$

Therefore :

$$(-1)^{R-1}\underset{R}{\wedge} M\left[\frac{\mathcal{R}_{-1}(-n,0)}{P_{-1}\{-n,0,-\}}\right] = \frac{P_0\{-n,1,-\}}{P_{-1}\{-n,0,-\}P_0\{-n,1,R\}}\left[\frac{\mathcal{R}_0(-n,1)}{P_0\{-n,1,-\}}\right]. \tag{16}$$

To study the above vector $\mathcal{R}_0(-n,1)/P_0\{-n,1,-\}$, we set for $0\le l\le L-1$ and $0\le r\le R-1$:

$$f_{-l}^{R-r}(-n) = \sum_{s=0}^{r}\left(P_{-l}\{-n,1,R-s\} - P_{-l+1}\{-n,1,R-s\}\right).$$

We observe that :

$$\mathcal{R}_0(-n,1) = \begin{pmatrix} -1\\ 0\\ \vdots\\ 0\\ f_0^R(-n)\\ \vdots\\ f_{-L+1}^R(-n)\end{pmatrix} \wedge \begin{pmatrix} 0\\ -1\\ \vdots\\ 0\\ f_0^{R-1}(-n)\\ \vdots\\ f_{-L+1}^{R-1}(-n)\end{pmatrix} \wedge\cdots\wedge \begin{pmatrix} 0\\ \vdots\\ \vdots\\ 0\\ f_0^1(-n)\\ \vdots\\ f_{-L+1}^1(-n)\end{pmatrix}.$$

We remark that $f_{-l}^1(-n) = -P_{-l}\{-n,1,-\}+P_{-l}\{-n,1,-\}$ and $f_0^1(-n) = -P_0\{-n,1,-\}$. The first $R-1$ vectors in the above decomposable R-vector converge as $n\to+\infty$ to linearly independent vectors. To prove ii), we show that the last vector divided by $P_0\{-n,1,-\}$ converges to a bounded vector. Let then U be the matrix of dimensions $L\times L$ with a diagonal of (-1) and a sub-diagonal of 1. We have :

$$\begin{pmatrix} f_0^1(-n)\\ \vdots\\ f_{-L+1}^1(-n)\end{pmatrix} = U\begin{pmatrix} P_0\{-n,1,-\}\\ \vdots\\ P_{-L+1}\{-n,1,-\}\end{pmatrix} = UG_0(-1,1)\cdots G_{-n+1}(-n,1)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

We then notice that the product of any L matrices of the form $G_r(s,t)$ is a positive matrix. From the condition of minoration of the transition probabilities and the value of the directional contraction coefficient given in lemma (6.3) and applied to matrices of the form $G_k(k-1,b)$ in the cone $\mathbb{R}^L$, we obtain that the limit direction of $G_0(-1,1)\cdots G_{-n+1}(-n,1)^t(1,\cdots,1)$ is strictly within the cone $\mathbb{R}^L$, uniformly in ω. Dividing by $P_0\{-n,1,-\}$, the limit vector has the logarithm of its norm that is bounded. This proves ii). The first part of *iii)* follows then from (16). To see that V is the main eigenvector with respect to $(-1)^{R-1} \wedge_R M$ and T, we notice that from i), the vector $(-1)^R V$ is in $\mathcal{C}^*$. Then theorem (4.6) indicates that the main eigenvector is the unique eigenvector in $\mathcal{C}^*$.

□

Remark 1 — If $R = 1$, the first point of the previous proposition reduces to the fact that the map $k \longmapsto P_k\{a, b, +\}$ is non-decreasing. In the general case, there is a condition of geometrical type.

We now give an expression of the central exponent of M with respect to T. Recall that $D = D_0(-\infty, 1)$ and $G = G_0(-1, +\infty)$. The proof of the next theorem consists in adding the equalities of the next proposition.

Theorem 6.3
We have the equality :

$$\gamma_R(M,T) = \gamma_{\max}(G,T) - \gamma_{\max}(D,T^{-1}).$$

Proposition 6.4
We have the following equalities :
i)

$$\begin{cases} \gamma_1(M,T) + \cdots + \gamma_R(M,T) = \gamma_{\max}(G,T) - \int \log P_0\{-\infty, 1, R\}\ d\mu \\ \gamma_R(M,T) + \cdots + \gamma_d(M,T) = -\gamma_{\max}(D,T^{-1}) + \int \log P_0\{-1, +\infty, -L\}\ d\mu. \end{cases}$$

ii)

$$\int \log \left(\frac{p_{-L}}{p_R} \frac{P_0\{-\infty, 1, R\}}{P_0\{-1, +\infty, -L\}} \right)\ d\mu = 0.$$

iii)

$$\gamma_1(M,T) + \cdots + \gamma_d(M,T) = \int \log(p_{-L}/p_R)\ d\mu.$$

Proof of the lemma :
The formula in *iii)* is the application of a classical and general result. We now prove the first equality in i). The proof of the second one is similar. Using (14) and the T-invariance of the measure μ, we obtain :

$$(\gamma_1 + \cdots + \gamma_R)(M,T) = \lim_{n \to +\infty} \int \log \frac{P_0\{-n, 1, -\}}{P_0\{-n+1, 1, -\}}\ d\mu - \int \log P_0\{-\infty, 1, R\}\ d\mu. \qquad (17)$$

Considering the first term of the right member of (17), we have :

$$P_0\{-n, 1, -\} = \langle e_1, G_0(-1,1) \cdots G_{-n+1}(-n,1)u \rangle, \text{ with } u = {}^t(1, \cdots, 1).$$

Set now $v_n = {}^tG_0(-1, n-1) \cdots {}^tG_{n-2}(n-3, n-1)e_1$. We then get :

$$\frac{P_0\{-n, 1, -\}}{P_0\{-n+1, 1, -\}} = T^{-n+2} \left(\frac{\langle G_{-1}(-2, n-1)u, v_n \rangle}{\langle u, v_n \rangle} \right),$$

We then observe that v_n converges in direction to the positive vector W, with $\|W\| = 1$, verifying ${}^tG_{-1}(-2,+\infty)W = \rho T^{-1}W$, with $\int \log\rho \; d\mu = \gamma_{\max}({}^tG, T^{-1}) = \gamma_{\max}(G,T)$, where $\log\rho$ is bounded. Moreover for $n \geq L$, v_n is strictly within the positive cone, uniformly in ω, as the product of any L matrices of the form $G_k(k-1,b)$ has positive entries that are minored by a positive constant. We therefore obtain :

$$\lim_{n\to+\infty} \int \log \frac{P_0\{-n,1,-\}}{P_0\{-n+1,1,-\}} \; d\mu = \int \log \frac{\langle G_{-1}(-2,+\infty)u, W\rangle}{\langle u, W\rangle} \; d\mu = \int \log\rho \; d\mu = \gamma_{\max}(G,T).$$

This proves the first formula of i).

ii) Let $a < k < b$. A first remark is that $|\mathcal{L}_k(a,b) \wedge \mathcal{R}_k^*(a,b)| = |\mathcal{L}_k^*(a,b) \wedge \mathcal{R}_k(a,b)|$. Using the relation (3) and the determinant of the matrix M, we have the equality :

$$|\mathcal{L}_{a+1}(a,b) \wedge \mathcal{R}_{a+1}^*(a,b)| \times \prod_{k=a+2}^{b-1} \frac{p_{-L}(k)}{p_R(k)} = |\mathcal{L}_{b-1}(a,b) \wedge \mathcal{R}_{b-1}^*(a,b)|. \tag{18}$$

Notice that for all $0 \leq l \leq L-1$, we have :

$$V_k(a,b,a-l) = V_k(a,b-1,a-l) + V_k(a,b-1,b-1)P_{b-1}\{a,b,a-l\}.$$

We then get :

$$\begin{aligned} |\mathcal{L}_{a+1}^*(a,b) \wedge \mathcal{R}_{a+1}(a,b)| &= |\mathcal{L}_{a+1}^*(a,b) \wedge \mathcal{R}_{a+1}(a,b-1)| \; P_{a+1}\{a,b,b+R-1\} \\ &= |\mathcal{L}_{a+1}^*(a,b-1) \wedge \mathcal{R}_{a+1}(a,b-1)| \; P_{b-1}\{a,b,b+R-1\}. \end{aligned}$$

We thus deduce :

$$|\mathcal{L}_{a+1}^*(a,b) \wedge \mathcal{R}_{a+1}(a,b)| = |\mathcal{L}_{a+1}^*(a,a+2) \wedge \mathcal{R}_{a+1}(a,a+2)| \prod_{k=a+2}^{b-1} P_k\{a,k+1,k+R\}. \tag{19}$$

Symmetrically, we also have :

$$|\mathcal{L}_{b-1}(a,b) \wedge \mathcal{R}_{b-1}^*(a,b)| = |\mathcal{L}_{b-1}(b-2,b) \wedge \mathcal{R}_{b-1}^*(b-2,b)| \prod_{k=a+1}^{b-2} P_k\{k-1,b,k-L\}. \tag{20}$$

However, for any k, $|\mathcal{L}_k(k-1,k+1) \wedge \mathcal{R}_k^*(k-1,k+1)| = p_{-L}(k)$. Using relations (18), (19) and (20), we finally obtain with $a = -n-1$ and $b = n+1$:

$$2n \int \log\left(\frac{p_{-L}}{p_R}\right) \; d\mu + \sum_{k=-2n-1}^{-2} \int \log P_0\{k,1,R\} \; d\mu = \sum_{k=2}^{2n+1} \int \log P_0\{-1,k,-L\} \; d\mu.$$

Dividing by $2n$ the two members of the previous equality and using the monotone convergence theorem, we deduce the announced formula.

□

7 Asymptotic behaviour of the model

7.1 A recurrence criterion

We first notice that the sum on the rows of the matrix G and of the matrix D is ≤ 1. Thus these matrices don't increase the norm $\| \ \|_\infty$ and then $\gamma_{\max}(G,T) \leq 0$ and $\gamma_{\max}(D,T^{-1}) \leq 0$. Remark also that the set $\{P_0\{-\infty,1,+\} < 1\}$ is T-invariant. Therefore from the ergodicity of μ with respect to T, we have $P_0\{-\infty,1,+\} < 1$, $\mu - ae$, or $P_0\{-\infty,1,+\} = 1$, $\mu - ae$.

Lemma 7.1
The following statements are equivalent :

i) $\gamma_{\max}(D,T^{-1}) < 0$.

ii) $P_0\{-\infty,1,+\} < 1$, $\mu - ae$.

iii) $\sup_{n\geq 0} \xi_n(\omega) < +\infty$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$, *that is* $\xi_n(\omega) \longrightarrow -\infty$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$.

Proof of the lemma :
i) $\Rightarrow$ *iii)* We have $P_0\{-\infty,n,+\} = \langle e_R, D \cdots T^{n-1}Du\rangle$, with $u := \sum_{i=1}^R e_i$. Thus :

$$\sum_{n=0}^{+\infty} P_0\{-\infty,n,+\} = \sum_{n=0}^{+\infty} \langle T^{n-1}\,({}^tD)\cdots({}^tD)\,e_R, u\rangle < +\infty,\ \mu - ae,$$

as $\gamma_{\max}(D,T^{-1}) = \gamma_{\max}({}^tD,T)$. The conclusion follows from Borel-Cantelli's lemma.

iii) $\Rightarrow$ *ii)* We have, $\mu - ae$, $\exists N \geq 1$ such that $P_0\{-\infty,N,-\} > 0$. Considering the first position after the last visit in $\{1,\cdots,N\}$, there exists $-L+1 \leq x \leq 0$ with $P_x\{-\infty,1,-\} > 0$. Thus we obtain $P_0\{-\infty,1,-\} > 0$.

ii) $\Rightarrow$ *i)* We have :

$$DTD\cdots T^{R-1}D = \begin{pmatrix} P_{R-1}\{-\infty,R,R+R-1\} & \cdots & P_{R-1}\{-\infty,R,R\} \\ \vdots & \vdots & \vdots \\ P_0\{-\infty,R,R+R-1\} & \cdots & P_0\{-\infty,R,R\} \end{pmatrix}.$$

Let $\| \ \|_\infty$ be the norm subordinated to the infinite norm. Then :

$$\|DTD\cdots T^{R-1}D\|_\infty \leq \max_{0\leq l\leq R-1} P_l\{-\infty,R,+\} =: \eta < 1,\ \mu - ae.$$

Take $N > 1$ such that $U := \{\eta < 1 - 1/N\}$ verifies $\mu(U) > 0$. Denote by $(\tau_n(\omega))_{n\geq 1}$ the passage times in U. Kac's lemma implies that $\tau_n/n \longrightarrow 1/\mu(U)$, $\mu - ae$. For any $n \geq 1$, we choose $p = p(n)$ such that $R - 1 + R\tau_p < n \leq R - 1 + R\tau_{p+1}$. Then, using the fact that $\|D\|_\infty \leq 1$, we get :

$$\limsup_n \frac{1}{n} \log \|D\cdots T^{n-1}D\|_\infty \leq \limsup_p \frac{1}{R\tau_p} \log(1-1/N)^p \leq \frac{\mu(U)}{R}\log(1-1/N) < 0.$$

This concludes the proof. □

Theorem 7.2 *(Recurrence criterion)*

i) If $\gamma_R(M,T) > 0$, *then* $\xi_n(\omega) \longrightarrow -\infty$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$.

ii) If $\gamma_R(M,T) = 0$, *then* $-\infty = \liminf \xi_n(\omega) < \limsup \xi_n(\omega) = +\infty$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$.

iii) If $\gamma_R(M,T) < 0$, *then* $\xi_n(\omega) \longrightarrow +\infty$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$.

Proof of the theorem :
Recall that $\gamma_{\max}(D, T^{-1}) \leq 0$ and $\gamma_{\max}(G, T) \leq 0$. Suppose that $\gamma_R(M, T) > 0$. From theorem (6.3), we get $\gamma_{\max}(D, T^{-1}) < 0$. Then lemma (7.1) implies that the random walk is transient to $-\infty$. This proves *i*). The point *iii*) is similar. Consider *ii*), that is $\gamma_R(M, T) = 0$. If $\gamma_{\max}(D, T^{-1}) < 0$ and $\gamma_{\max}(G, T) < 0$, the random walk would be transient to $-\infty$ and to $+\infty$, which is impossible. Thus $\gamma_{\max}(D, T^{-1}) = \gamma_{\max}(G, T) = 0$ and the random walk visits $-\infty$ and $+\infty$, thus is recurrent.

□

Remark. — One can establish in a different way the above recurrence criterion, by calculating explicitly the exit probabilities of an interval with the same method as in [4]. It is proved in Letchikov [13], or in [6], that the above theorem is equivalent to Key's Theorem [11].

7.2 Algorithm of calculus of the central exponent $\gamma_R(M, T)$

Conditionally to a numerical knowledge of the dynamical system $(\Omega, \mathcal{F}, \mu, T)$, the recurrence criterion (7.2) can be handled easily, using the directional contraction properties of the matrices $(-1)^{R-1} \wedge_R M$ and $(-1)^{L-1} \wedge_L M^{-1}$. An algorithm is the following. For every ω, one evaluates the following decomposable $R-$vector in $\wedge_R \mathbb{R}^d$, with a norm equal to one :

$$V_N(\omega) := \left[(-1)^{R-1} \wedge_R M(T^{-1}\omega)\right] \cdots \left[(-1)^{R-1} \wedge_R M(T^{-N}\omega)\right] .(e_1 \wedge \cdots \wedge e_R).$$

The integer N is taken large enough, in function of the directional contraction constant of the matrix $(-1)^{R-1} \wedge_R M$ in the cone $\mathcal{C}$. Recall that the convergence is exponential and uniform. Then V_N is an approximation of the vector V of maximal exponent with respect to $(-1)^{R-1} \wedge_R M$ and T. Therefore :

$$\gamma_1(M, T) + \cdots + \gamma_R(M, T) \simeq \int \log \| \wedge_R M V_N \| \, d\mu.$$

Repeating this procedure with the decomposable $L-$vector $e_R \wedge \cdots \wedge e_d \in \wedge_L \mathbb{R}^d$ and with the matrix $(-1)^{L-1} \wedge_L M^{-1}$, one gets an approximation of $\gamma_R(M, T) + \cdots + \gamma_d(M, T)$. As :

$$\gamma_R(M, T) + \cdots + \gamma_d(M, T) = \int \log(p_{-L}/p_R) \, d\mu,$$

we finally deduce an approximation of the central exponent $\gamma_R(M, T)$, whose sign determines the asymptotic behaviour of the random walk.

8 On the form of the central vector V_R

We consider the vector V_R defined in corollary (4.5). Recall that this vector is uniquely determined in direction, has a norm equal to 1 and verifies $MV_R = \lambda_R T V_R$, with $\int \log |\lambda_R| \, d\mu = \gamma_R(M, T)$. We first have the following proposition :

Proposition 8.1
The vector $V_{-1}(-m, n, +)$ converges in direction to V_R, as $(m, n) \longrightarrow +\infty$.

Proof of the proposition :
From proposition (6.2), we know that $\mathcal{R}_{-1}(-n, 0)$ converges in direction to the main eigenvector of the matrix $(-1)^{R-1} \wedge_R M$ with respect to T, which is $V_1 \wedge \cdots \wedge V_R$. In the same way, one would show that $\mathcal{L}_{-1}(0, n)$ converges in direction to $V_R \wedge \cdots \wedge V_d$. However, for any $(m, n) \geq 0$, $\mathcal{R}_{-1}(-m, 0)$ and $\mathcal{R}_{-1}(-m, n)$ on the one side and $\mathcal{L}_{-1}(0, n)$ and $\mathcal{L}_{-1}(-m, n)$ on the other side have the same direction. As the subspaces corresponding to $\mathcal{R}_{-1}(-m, n)$ and $\mathcal{L}_{-1}(-m, n)$ intersect in the direction of $V_{-1}(-m, n, +)$ and since $\text{Vect}\{V_j \mid 1 \leq j \leq R\} \cap \text{Vect}\{V_j \mid R \leq j \leq d\}$ is a one-dimensional subspace, we deduce that $V_{-1}(-m, n, +)$ converges in direction to V_R.

□

We now give an expression of V_R. We reduce our study to the transient cases, as if $\gamma_R(M,T)=0$, then the matrix $M(r) := (1/r)A(r)MA(r)^{-1}$, with $A(r) = \mathrm{diag}(1,r,\cdots,r^{d-1})$, is the matrix associated to another random walk if r is close to 1 and $\gamma_R(M(r),T) = \gamma_R(M,T) - \log r$.

Definition 8.2
Let $a \leq k$ and set $f^r_{k-L+1}(a) = P_{k-L+1}\{a, k-L+2, k-L+1+r\}$, for $1 \leq r \leq R$. We introduce a random matrix $\mathcal{D}_k(a)$ of dimensions $d \times d$:

$$\mathcal{D}_k(a) = \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & 0 & f^R_{k-L+1}(a) & \cdots & f^1_{k-L+1}(a) \end{pmatrix}.$$

Definition 8.3
Let W be the unique positive vector in $\mathbb{R}^R$ with $\langle W, e_R\rangle = 1$ and ρ the unique positive scalar ρ such that $DTW = \rho W$. The map ρ is bounded.

Proposition 8.4
Assume that $\gamma_{\max}(D,T^{-1}) < 0$. Then there exists $\tilde{V}_R$ and $\tilde{\lambda}_R$ such that $M\tilde{V}_R = \tilde{\lambda}_R T\tilde{V}_R$, $\tilde{V}_R$ has the direction of V_R and :

$$\tilde{V}_R = T^{-L+1} \begin{pmatrix} (1/T\rho\cdots T^{d-2}\rho)(1-1/T^{d-1}\rho) \\ \vdots \\ (1-1/T\rho) \\ (\rho-1) \end{pmatrix} \quad \textit{and } \tilde{\lambda}_R = 1/T^{-L+2}\rho.$$

Proof of the proposition :
We first remark that for all $m \geq 1$ and all $n \geq 1$, we have :

$$V_{-1}(-m,n,+) = (\mathcal{D}_{-1}(-m) - I)\mathcal{D}_0(-m)\cdots\mathcal{D}_{n+L-2}(-m)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

From proposition (8.1), we know that $V_{-1}(-m,n,+)$ converges in direction to V_R. Notice that if $\gamma_{\max}(D,T^{-1}) < 0$, the matrix $(\mathcal{D}_{-1}(-\infty) - I)$ is invertible. Furthermore we observe that the vector $\mathcal{D}_0(-m)\cdots\mathcal{D}_{n+L-2}(-m)^t(1,\cdots,1)$ converges in direction as m and n tend to $+\infty$ to $T^{-L+1}\left({}^t(1/(\rho\cdots T^{d-2}\rho),\cdots,1/\rho,1\right)$. This gives the result.

□

Remark. — A similar expression is available if $\gamma_{\max}(G,T) < 0$. It is affirmed in Letchikov [14] that V_R has positive components but the proof is false.

9 On the Law of Large Numbers

We mention in this section a result concerning the Law of Large Numbers contained in [5] and that applies directly to the model that we study here. It relies on the study of the random walks of the left and right records together with the formalism introduced by Kozlov [12] for the absolutely continuous invariant measure for the random walk of "the environments seen from the particle".

For integers $a < k < b$, we denote by $E_k\{a,b\}$ the expectation under the measure P_k of the time to reach $(-\infty,a] \cup [b,+\infty)$.

Proposition 9.1
1) If $\int E_0\{-\infty,1\}\, d\mu = +\infty$*, then we have* $\limsup \xi_n(\omega)/n \leq 0$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$. *If on the contrary* $\int E_0\{-\infty,1\}\, d\mu < +\infty$*, then :*

$$\frac{1}{n}\xi_n(\omega) \longrightarrow c = \frac{\int \left(\sum_{r=1}^{R} r P_0\{-\infty,1,r\}\right) \pi_1\, d\mu}{\int E_0\{-\infty,1\}\pi_1\, d\mu} > 0,\ \mathcal{P}_0^\omega - ae,\ \mu - ae \tag{21}$$

where $\pi_1 = \langle \Pi_1, e_1\rangle$ *and* Π_1 *is the positive eigenvector such that* $T\Pi_1 = {}^tD\Pi_1$ *and* $\|\Pi_1\|_1 = 1$.

2) If $\int E_0\{-1,+\infty\}\, d\mu = +\infty$*, then we have* $\liminf \xi_n(\omega)/n \geq 0$, $\mathcal{P}_0^\omega - ae$, $\mu - ae$. *If on the contrary* $\int E_0\{-1,+\infty\}\, d\mu < +\infty$*, then :*

$$\frac{1}{n}\xi_n(\omega) \longrightarrow c = -\frac{\int \left(\sum_{l=1}^{L} l P_0\{-1,+\infty,-l\}\right) \pi_2\, d\mu}{\int E_0\{-1,+\infty\}\pi_2\, d\mu} < 0,\ \mathcal{P}_0^\omega - ae,\ \mu - ae \tag{22}$$

where $\pi_2 = \langle \Pi_2, e_1\rangle$ *and* Π_2 *is the positive eigenvector such that* $T^{-1}\Pi_2 = {}^tG\Pi_2$ *and* $\|\Pi_2\|_1 = 1$.

Corollary 9.2
There always exists a real constant c such that :

$$\frac{1}{n}\xi_n(\omega) \longrightarrow c,\ \mathcal{P}_0^\omega - ae,\ \mu - ae.$$

Remark. — According to the study in [5] when $R = 1$, the value of the previous constant c will depend on the sign of the central exponent $\gamma_R(M,T)$ and integrability properties of the central eigenvalue λ_R.

10 Concluding remarks

The next step in the study of the model (L, R, erg) seems to be the description of the geometry of the eigenvectors associated to the Lyapunov spectrum of M. In particular V_R and the properties of the homological class of the corresponding eigenvalue λ_R may be related to the precise behavior of the model, as suggested by [5] when $R = 1$. In this direction, a complete study of the space of harmonic functions is required. One can compute for example the extremal vectors of the cone $\mathcal{C}^*$, but surprinsingly other completely determined cones are stable by the class of matrices $\wedge_R\mathcal{M}$. Their interpretation in terms of random walks is to be precised.

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