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Gibbs measures at temperature zero

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Abstract
Let ν_f be the Gibbs measure associated to a regular function f on a one-sided topologically mixing subshift of finite type. Introducing a parameter λ, we consider the behaviour of the family $(\nu_{\lambda f})$, as $\lambda \to +\infty$. When f depends on p coordinates, we show that the measures $(\nu_{\lambda f})$ converge. Moreover, the limit measure belongs to a finite set dependent only on p and on the subshift. The proof is a consequence of a general statement of Analytic Geometry.

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1. Presentation

A motivation for the present paper is the study of maximizing measures. We briefly recall the context. Let (X, T) be a topological dynamical system, that is a compact metric space X with a continuous and surjective transformation T. Fixing some continuous function f on X, we study the invariant probability measures μ that maximize $\int f \, d\mu$ and their stochastic properties. Using Birkhoff's Ergodic Theorem, this problem is equivalent to finding the points in X that maximize the growth of the ergodic sums of f. We mention the important example of Bousch [1] who has considered the family of functions $x \mapsto \cos(2\pi(x + t))$, where t is a parameter, on $X = [0, 1)$ and with the transformation $Tx = 2x \,\mathrm{Mod}(1)$. The result is that for every t, the maximizing measure is unique and Sturmian. This measure is also periodic for almost every t.

We consider here the construction of maximizing measures. Following Conze–Guivarc'h [4], one way is to proceed by freezing the system. More precisely, when there exists a finite Markov partition as in the previous situation on the circle, we can assume that the problem is given in symbolic dynamics, that is in the context of a subshift of finite type Λ over a finite alphabet. If a function f is regular, we can introduce the Gibbs measure ν_f associated to f, that is the unique measure realizing the maximum in the variational principle. Considering the measures $(\nu_{\lambda f})_{\lambda \to +\infty}$ where λ would be the inverse of the temperature in Statistical Mechanics, one can derive from the variational principle, that the cluster values of $(\nu_{\lambda f})$ are maximizing measures for f and that they have maximal entropy in this set of measures (see Conze–Guivarc'h [4]). If the measures $\nu_{\lambda f}$ were converging as $\lambda \to +\infty$, one would obtain a 'natural' way of constructing a maximizing measure for a regular function on the subshift Λ.

This problem is related to the behaviour of Markov chains with 'rare transitions' on a finite set. The study of such chains was initiated by Freidlin and Wentzell [6] and then extended by Catoni–Cerf [3] or Trouvé [11]. The starting assumption is that the transition rate between two states x and y has order $C(x, y)\exp(-\lambda V(x, y))$, where V is some rate function. Exponential rates come naturally in the discrete modelization of complex phenomena such as the rate of failure of a machine. A preliminary study concerns the limiting behaviour of the invariant measure at low temperature. On that question, an essential tool appears to be the 'matrix tree theorem' and its corollaries such as lemma (3.6) in Catoni–Cerf [3], as it gives a formula with non-negative quantities for the invariant distribution. In the context of a subshift of finite type over a finite alphabet, we cannot use directly the matrix tree theorem, as the transition laws are defined by implicit quantities, whose behaviour at low temperatures is not clear *a priori*.

We will prove that all the transition laws do admit an equivalent of the form $C\exp(-\lambda V)$ and in fact our result is the requirement for the development, in the context of Gibbs measures on subshifts of finite type, of a similar study as the one for Markov chains with rare transitions. We will then obtain that for every locally constant function f, the Gibbs measures $(\nu_{\lambda f})_{\lambda\to+\infty}$ converge. The limit measure is maximizing for f and belongs to a finite set dependent only on the subshift and the rank of the function. The proof we provide relies purely on techniques relevant to Analytic Geometry.

2. Introduction and reduction of the problem

We fix some integer $d \geqslant 2$ and introduce $\Omega = \{1, \dots, d\}^{\mathbb{N}}$ with its usual distance. For $x \in \Omega$ and any integer $p \geqslant 1$, let $C_p(x)$ be the cylinder of size p associated to x, i.e.

$$C_p(x) = \{y \in \Omega \mid y_i = x_i, 0 \leqslant i \leqslant p-1\}.$$

We write $\mathcal{B}_p$ for the finite Boole algebra generated by the cylinders of size p and $\mathcal{B}$ for the σ-algebra generated by all the cylinder sets. Let $\Lambda = (\Omega, T, A)$ be a one-sided topologically mixing subshift of finite type, where T is the shift on Ω and A is a transition matrix (with entries equal to zero or one) giving the allowed transitions and such that A^{N_0} has strictly positive entries for some integer $N_0 \geqslant 1$. Given a locally constant function f on Λ, we write ν_f for the corresponding Gibbs measure. We will prove the following.

Theorem 2.1. *Let Λ be a one-sided topologically mixing subshift of finite type.*

(1) Let f be $\mathcal{B}_p$-measurable. Then, the measures $(\nu_{\lambda f})$ converge weakly as $\lambda \to +\infty$ to a Markovian measure written $\nu_{\infty,f}$.
(2) The set $\mathcal{M}(\Lambda, p) := \{\nu_{\infty,f} \mid f$ is $\mathcal{B}_p$-measurable$\}$ is finite.

Remark. The second part of theorem 2.1 will follow directly from the proof, but without control on $\mathcal{M}(\Lambda, p)$. Anyway, the cardinal of $\mathcal{M}(\Lambda, p)$ seems to grow very fast with p, as suggested by the example of the next section.

First step. We recall a general frame for the construction of the Gibbs measure ν_f in the particular case when f is locally constant (see, e.g. Bowen [2] or Denker–Grillenberg–Sigmund [5]). Then, let f be $\mathcal{B}_p$-measurable for some integer $p \geqslant 1$. Up to conjugating the system by increasing the alphabet (see [5]), we suppose that f depends on two coordinates. We then introduce the transfer matrix M_f associated to f and which is defined by

$$(M_f)(i,j) = \begin{cases} e^{f(j,i)}, & \text{if } A(j,i) = 1, \\ 0, & \text{if } A(j,i) = 0. \end{cases}$$

As the matrix A^{N_0} has strictly positive entries for some $N_0 \geqslant 1$, it is also the case for $(M_f)^{N_0}$. Therefore, M_f admits a unique strictly positive eigenvector $\tilde{H}_f$ with $\|\tilde{H}_f\|_1 = 1$ and a strictly positive eigenvalue $e^{s(f)}$. With $D = \operatorname{diag}(\tilde{H}_f)$, we then consider the stochastic matrix,

$$Q_f = e^{-s(f)} \left(D^{-1} M_f D\right).$$

Writing H_f for the strictly positive eigenvector of ${}^tQ_f$ such that $\|H_f\|_1 = 1$ and $(X_i)_{i\geqslant 0}$ for the coordinate applications, we have that ν_f is the Markovian measure on Λ such that for any $(x_i)_{0\leqslant i\leqslant n-1}$ with $A(x_{l-1}, x_l) = 1$, $1 \leqslant l \leqslant n-1$,

$$\nu_f\{X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\} = H_f(x_{n-1}) Q_f(x_{n-1}, x_{n-2}) \cdots Q_f(x_1, x_0).$$

Second step. We now reduce theorem 2.1 to a rather general statement on analytic functions, independent of the above problem. We consider the set $S \subset M_d(\mathbb{R})$ of positive matrices B with the same non-zero elements as $({}^tA)$, that is such that $B(i,j) > 0$ if $A(j,i) = 1$ and $B(i,j) = 0$ if $A(j,i) = 0$. This is also the form of the previous matrix M_f.

If $B \in S$, then, let $\tilde{H}(B)$, with $\|\tilde{H}(B)\|_1 = 1$, be the unique positive eigenvector of B and write $e^{s(B)}$ for the corresponding eigenvalue. Similarly, we define $Q(B) = e^{-s(B)} \operatorname{diag}(\tilde{H}(B))^{-1} B \operatorname{diag}(\tilde{H}(B))$ and $H(B)$ as the unique strictly positive eigenvector of ${}^tQ(B)$, with $\|H(B)\|_1 = 1$. We then introduce on S the application ζ into $M_d(\mathbb{R}) \times \mathbb{R}^d$:

$$\zeta : B \mapsto (Q(B), H(B)). \tag{2}$$

We extend ζ on $M_d(\mathbb{R})$ by 0. The picture is as follows: there exists a stratified structure determined by ζ, in the set of positive matrices with the same non-zero entries as $({}^tA)$. This will be given by the 'preparation theorem' for subanalytic functions applied to ζ. We will give a version of that theorem in section 4, and we postpone the required definitions to that section as well.

Remark. Although the needed class here is the semialgebraic one, we provide a statement available for the general subanalytic class, as the techniques are similar.

Lemma 2.2. *The map ζ defined in (2) is semialgebraic and thus subanalytic.*

Theorem 2.3. *Let $\zeta : \mathbb{R}^{n+1} \to \mathbb{R}^m$ be a subanalytic application and $\eta : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map $(x, \epsilon = (\epsilon_i)) \mapsto (x^{\epsilon_i})$. Then, there exists a finite set $\tilde{S}$ of real numbers such that for every $\epsilon \in \mathbb{R}^{n+1}$ and as $x \to 0$, each coordinate of $(\zeta \circ \eta)(x, \epsilon)$ is either ultimately 0 or admits an equivalent of the form ux^v, with $u \in \tilde{S}$ and $v = \sum p_k/q_k \epsilon_k$, where all p_k/q_k are rationals and in $\tilde{S}$.*

Proof of theorem 2.1. Let $n+1$ be the number of non-zero entries of any matrix in S. We order them and denote them by $(u_k(B))_{1\leqslant k\leqslant n+1}$ for any matrix $B \in S$, independently on $B \in S$. Now, let f be defined on Λ and let it depend on two coordinates. Fixing $\lambda > 0$, each matrix $M_{\lambda f}$ is in S and any non-zero coordinate of $M_{\lambda f}$ has the form $u_k(M_{\lambda f}) = \exp(\lambda f(j,i))$. We then set $x = \exp(-\lambda)$ and $\epsilon_k = -f(j,i)$. The path of $(M_{\lambda f})_{\lambda\to+\infty}$ in S is identified to the path $(x^{\epsilon_k})_{x\to 0^+}$ in $\mathbb{R}^{n+1}$.

We then apply lemma 2.2 and theorem 2.3 to ζ and $(x, (\epsilon_k))$. As a consequence, we obtain that the entries of $Q_{\lambda f}$ and $H_{\lambda f}$ converge and that the limit values belong to a finite set independent of f, as the number of u and (p_k/q_k) is finite. This concludes the proof of the theorem. □

3. An example

Before developing the analytic tools we will use, we consider the example of the full shift on $\{1,2\}^{\mathbb{N}}$ with a function f depending on two coordinates. In this case, the convergence of $(\nu_{\lambda f})$ can be proved by direct calculations. In each case, we indicate the limit $Q_{\infty,f}$ of the stochastic matrices $Q_{\lambda f}$ and the limit $H_{\infty,f}$ of the initial distributions $H_{\lambda f}$. For convenience, we write f_{ij} in place of $f(i,j)$ for $1 \leqslant (i,j) \leqslant 2$ and set $\phi = (1+\sqrt{5})/2$. We obtain the following.

(1) If $f_{11} = f_{22} = \frac{1}{2}(f_{12}+f_{21})$, then, $Q_{\infty,f} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

(2) If $f_{11} = f_{22} > \frac{1}{2}(f_{12}+f_{21})$, then, $Q_{\infty,f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

(3) If $f_{11} < f_{22} = \frac{1}{2}(f_{12}+f_{21})$, then, $Q_{\infty,f} = \begin{pmatrix} 0 & 1 \\ 1/\phi^2 & 1/\phi \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} 1/(1+\phi^2) \\ \phi^2/(1+\phi^2) \end{pmatrix}$.

(4) If $f_{22} > \max\left\{f_{11}, \frac{1}{2}(f_{12}+f_{21})\right\}$, then, $Q_{\infty,f} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(5) If $f_{22} < f_{11} = \frac{1}{2}(f_{12}+f_{21})$, then, $Q_{\infty,f} = \begin{pmatrix} 1/\phi & 1/\phi^2 \\ 1 & 0 \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} \phi^2/(1+\phi^2) \\ 1/(1+\phi^2) \end{pmatrix}$.

(6) If $f_{11} > \max\left\{f_{22},\ \frac{1}{2}(f_{12}+f_{21})\right\}$, then, $Q_{\infty,f} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(7) If $\max\{f_{11},\ f_{22}\} < \frac{1}{2}(f_{12}+f_{21})$, then, $Q_{\infty,f} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H_{\infty,f} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

Remark. We check that there are finitely many possible limit measures and that their expressions do not depend on f. We also observe that the limit may have non-zero entropy, as in the case (1) which corresponds to the measure of Parry–Renyi ($f = 0$). It can also be periodic or be a strict barycentre of periodic measures and thus not ergodic.

4. The theorem of 'preparation' of subanalytic functions

We will now prove lemma 2.2 and theorem 2.3. To begin the study, we introduce the general class of subanalytic functions. We will then need some form of implicit function theorem adapted to such a class of functions and this will be the theorem of preparation.

As references, we mention Gabrielov [7], Lion [9] and Parusiński [10]. We begin with a few definitions. First we embed algebraically $\mathbb{R}$ into the compact $\mathbb{P}_1$ by the map $x \mapsto [x:1]$ and see $\mathbb{R}^n$ as a subset of $(\mathbb{P}_1)^n$. Any real analytic function will simply be called analytic.

Definition 4.1.

(1) A subset $X \subset \mathbb{R}^n$ is semi-analytic if for all a in $(\mathbb{P}_1)^n$, there exists a neighbourhood U of a, and a finite number of analytic functions $f_{i,j}$, $g_{i,j}$, $(i,j) \in I \times J$ such that

$$X \cap U = \bigcup_{i \in I} \left(\bigcap_{j \in J} \{x \in U \mid f_{i,j}(x) = 0, g_{i,j}(x) > 0\} \right).$$

A subset $X \subset \mathbb{R}^n$ is semi-algebraic if we assume that the functions $f_{i,j}, g_{i,j}$ are polynomials.

(2) *A subset $E \subset \mathbb{R}^n$ is global subanalytic if it is the image of a semi-analytic set of $\mathbb{R}^{n+m}$ by the canonical projection of $\mathbb{R}^n \times \mathbb{R}^m$ on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$, is global subanalytic if its graph is global subanalytic in $\mathbb{R}^n \times \mathbb{R}^m$.*

Now, let $\mathcal{A}$ be any family of functions from $\mathbb{R}^n$ into $\mathbb{R}$.

(3) *A set $E \subset \mathbb{R}^n$ is an $\mathcal{A}$-set if there exists $(f_{i,j}, g_{i,j})$ in $\mathcal{A}$, with $(i, j) \in I \times J$ where I and J are finite, such that*

$$E = \bigcup_I \left(\bigcap_J \{f_{i,j} = 0,\ g_{i,j} > 0\} \right).$$

(4) *An $\mathcal{A}$-cylinder $C \subset \mathbb{R}^{n+1}$ is a set such that there exists an $\mathcal{A}$-set $B \subset \mathbb{R}^n$ called the basis, φ and possibly ψ in $\mathcal{A}$ such that C has one of the following forms:*

$$C = \{(x, y) \mid x \in B,\ \varphi(x) < y < \psi(x)\},\ (\text{then } \varphi < \psi \text{ on } B)$$

$$C = \{(x, y) \mid x \in B,\ y < \varphi(x)\},$$

$$C = \{(x, y) \mid x \in B,\ y > \varphi(x)\},$$

$$C = \{(x, y) \mid x \in B,\ y = \varphi(x)\}.$$

Remark 1. A theorem of Gabrielov on the complementary [7] says that the global subanalytic sets are stable by complementation and thus form a Boole algebra. It is also stable by projection. The semi-algebraic sets have the same properties by a theorem of Tarski–Seidenberg, which states stability by projection, whereas the semi-analytic sets are not stable by projection.

Remark 2. Subanalytic sets are an example of o-minimal family of subsets of $\mathbb{R}^n$, i.e. one for which 'good' topological and geometric results can be proved. For global subanalytic functions, Parusiński [10] has shown a theorem of preparation. As in the implicit functions theorem, considering any function $\varphi = \varphi(x_1, \dots, x_n, y)$ and the equation $\varphi = 0$ in y, to 'prepare' φ in the variables $(x_1, \dots, x_n, y)$ means to decompose $\mathbb{R}^{n+1}$ in cylinders on which φ admits a principal part of the same class as φ. To make this statement precise, we now introduce the set of 'reduced functions'.

Definition 4.2. *We define inductively on $n \geqslant 0$, the set $\mathcal{R}_n$ of 'reduced functions' from $\mathbb{R}^n$ into $\mathbb{R}$:*

(a) *The set $\mathcal{R}_0$ is the set of real constants.*

(b) *Assume that $\mathcal{R}_n$ has been defined. Then, $f \in \mathcal{R}_{n+1}$ if the following two conditions hold:*

(1) *There exists a finite partition of $\mathbb{R}^{n+1}$ in $\mathcal{R}_n$-cylinders and each cylinder of the partition is contained in $\mathbb{R}^n \times [-1, 1]$ or in $\mathbb{R}^n \times \{|y| \geqslant 1\}$, calling y the last variable of $\mathbb{R}^{n+1}$.*

(2) *On each cylinder C of the previous partition, the function f or $1/f$ is equal to*

$$\alpha^{q/p} A U(\psi), \qquad \text{with } \psi = \left(\phi_1, \dots, \phi_s, \alpha^{1/p}, \beta^{1/p}\right), \text{ where} \tag{3}$$

- *the functions α and β are bounded on C and have the respective forms $(y' - \theta)/a$ and $b/(y' - \theta)$ or, on the contrary, the functions $A, a, b, \theta, (\phi_i)_{1 \leqslant i \leqslant s}$ belong to $\mathcal{R}_n$ and are bounded on the basis B of the cylinder C. One has $y' = y$ if $C \subset \mathbb{R}^n \times [-1, 1]$ and $y' = 1/y$ if $C \subset \mathbb{R}^n \times \{|y| \geqslant 1\}$. The function θ is either identically 0 on C or there exists a constant $c > 0$ such that $|y'|/c \leqslant |\theta(x)| \leqslant c|y'|$ on the whole cylinder C.*

- *the quantities p and q are strictly positive integers.*
- *the function U is analytic without zero on a neighbourhood of the compact $\overline{\psi(C)}$ in $(\mathbb{P}_1)^{s+2}$.*

Such a reduction is adapted for recursive proofs as we will see later. We use the following result, which is due to Parusiński [10]. For the form presented here and for a proof, we refer to [9].

Theorem 4.3. *A real global subanalytic function is reduced.*

5. Proof of lemma 2.2 and theorem 2.3

Proof of lemma 2.2. We consider, for example, the map $B \mapsto H(B)$ on the cylinder S and call it ξ. Recall that any unitary polynomial of degree d is seen as an element of $\mathbb{R}^d$ and a matrix of size $d \times d$ as an element of $\mathbb{R}^{d^2}$. We have to show that $\{(B, \xi(B)) \mid B \in S\}$ is semialgebraic. Consider first the subset D_1 of $\mathbb{R}^{d+1}$ defined by

$$D_1 := \{(P, r)\text{ , such that } (C)\}. \tag{4}$$

where (C) is the condition 'P is a real unitary polynom of degree d and r is the greatest real root of P'. We prove that this set is semialgebraic. This way, introduce for $1 \leqslant l \leqslant d$,

$$D_{1,l} := \{(P, u_1, \ldots, u_l) \mid P(u_1) = \cdots = P(u_l) = 0, u_1 < \cdots < u_l\}.$$

The set $D_{1,l}$ is the set of real unitary polynoms P of degree d together with l distinct real roots of P. It is semialgebraic. By projection, the set

$$D_{2,l} := \{(P \mid \exists (u_i)_{1\leqslant i\leqslant l}, (P, u_1, \ldots, u_l) \in D_{1,l}\}$$

is also semialgebraic by the theorem of Tarski–Seidenberg. Then, define $D_{3,l} := D_{2,l} - D_{2,l+1}$. Then, $D_{3,l}$ is also semialgebraic. It is the set of polynoms that have exactly l distinct real roots, without considering their multiplicity. Finally, define

$$\begin{aligned} D_{4,l} := \{(P, u_1, \ldots, u_{l-1}, r) \mid\ & P \in D_{3,l}, u_1 < \cdots < u_{l-1} < r, \\ & P(u_1) = \cdots = P(u_{l-1}) = P(r) = 0\} \end{aligned}$$

and by projection

$$D_{5,l} = \{(P, r) \mid \exists (u_i)_{1\leqslant i\leqslant l-1},\ (P, u_1, \ldots, u_{l-1}, r) \in D_{4,l}\}$$

is semialgebraic. It is, therefore, also the case for the set D_1 defined in (4), as $D_1 = \cup_{1\leqslant l\leqslant d} D_{5,l}$.

Consequently, writing $\chi(B)$ for the characteristic polynomial of the matrix B, we deduce that the set $D_2 = \{(B, \rho) \mid (\chi(B), \rho) \in D_1\}$ is semialgebraic. Hence, it is also the case for the set $\{(B, \xi(B)) \mid B \in S\}$ as it is a projection of

$$\begin{aligned} \{(B, U, V, \rho) \mid\ & B \in S, (B, \rho) \in D_2,\ BU = \rho U,\ U > 0, \|U\|_1 = 1, {}^t Q(B)V = \rho V, \\ & V > 0, \|V\|_1 = 1\}. \end{aligned}$$

The other coordinate applications of ζ can be treated similarly. □

Proof of theorem 2.3. Let ξ be a coordinate of the application ζ. From the preparation theorem 4.3, there exists a finite partition of $\mathbb{R}^{n+1}$ in $\mathcal{R}_n$-cylinders such that on each cylinder C, ξ or $1/\xi$ has the form described in (3). Then, set $E(n + 1) = \{\xi\}$ and let $E(\xi, n + 1, n)$ be the finite set of all the functions in $\mathcal{R}_n$, which intervene in the decomposition of ξ or $1/\xi$

on each cylinder and of all the functions of $\mathcal{R}_n$ used to define those cylinders. Recursively for $0 \leqslant p \leqslant n$, we set

$$E(p) = \bigcup_{\varphi \in E(p+1)} E(\varphi, p+1, p),$$

where $E(\varphi, p+1, p)$ is defined in the same way as $E(\xi, n+1, n)$.

We remark that $E(0)$ is a finite set of real constants. We will prove by increasing induction on $0 \leqslant p \leqslant n+1$, the following claim $\mathcal{H}(p)$:

'There exists a finite set $\mathcal{L}(p)$ of linking relations over $\mathbb{Q}$, a finite set $A(p)$ of non-zero real constants and a finite constant $M_p > 0$, all independent of the (ε_i), such that if $(\varepsilon_1, \dots, \varepsilon_p) \notin \mathcal{L}(p)$, then, for all φ in $E(p)$ and as $x \to 0$, $\varphi(x^{\varepsilon_1}, \dots, x^{\varepsilon_p})$ is either stationary at 0 or equivalent to a quantity ux^v, where $u \in A(p)$ and $v = (1/q) \sum_{l=1}^{p} p_l \varepsilon_l$, where $|p_l|, |q| \leqslant M_p$, $q \neq 0$.'

We remark that $\mathcal{H}(0)$ is verified with $A(0) = E(0)/\{0\}$, $M_0 = 0$ and $\mathcal{L}(0)$ empty. Assuming $\mathcal{H}(p)$, let us prove $\mathcal{H}(p+1)$. Then, let $\varphi \in E(p+1)$ and consider its decomposition in $\mathcal{R}_p$-cylinders. We will first build a finite set $\mathcal{L}'(p+1)$ of linking relations over $\mathbb{Q}$ for $(\varepsilon_i)_{1 \leqslant i \leqslant p+1}$ such that the path $(x^{\varepsilon_1}, \dots, x^{\varepsilon_p}, x^{\varepsilon_{p+1}})$ ends in a unique cylinder, as $x \to 0$. Indeed, each cylinder C in $\mathbb{R}^{p+1}$ and involved in the decomposition of φ has, for example, the following form, where we write (z, y) for the coordinates in $\mathbb{R}^p \times \mathbb{R}$:

$$C = \{(z, y) \mid z \in B,\ h_1(z) < y < h_2(z)\}, \qquad h_1,\ h_2 \in E(p)$$

$$B = \bigcup_I \bigcap_J \{z \mid f_{i,j}(z) = 0,\ g_{i,j}(z) > 0\},\ f_{i,j},\ g_{i,j} \in E(p), \qquad \text{with } I \text{ and } J \text{ finite.}$$

Consider the case of the basis B. Using $\mathcal{H}(p)$, any function $f_{i,j}$ is either stationary at 0 or equivalent to a quantity $u_{i,j} x^{v_{i,j}}$ with $u_{i,j} \in A(p)$ and $v_{i,j} = (1/q) \sum_{l=1}^{p} p_l \varepsilon_l$, $|p_l|, |q| \leqslant M_p$, $q \neq 0$. The same remark holds for the $(g_{i,j})$s. Therefore, if x is small enough, the conditions defining B are either always true or always false for the path $(x^{\varepsilon_1}, \dots, x^{\varepsilon_p})$. Similarly, for the condition

$$h_1(x^{\varepsilon_1}, \dots, x^{\varepsilon_p}) < x^{\varepsilon_{p+1}} < h_2(x^{\varepsilon_1}, \dots, x^{\varepsilon_p}),$$

the functions h_i are either stationary at 0 or equivalent to some $u_i x^{v_i}$. To ensure that such a condition is ultimately true or false, we impose on ε_{p+1} the conditions $\varepsilon_{p+1} \notin \{0,\ v_1,\ v_2\}$. We then define the finite set $\mathcal{L}'(p+1)$ as the union $\mathcal{L}(p)$ and the conditions $\varepsilon_{p+1} \notin \{0,\ v_1,\ v_2\}$, for all the h_i involved in the decomposition in $\mathcal{R}_p$-cylinders of φ and for all φ in $E(p+1)$. Consequently, for some fixed $\varphi \in E(p+1)$ and if $(\varepsilon_1, \dots, \varepsilon_p, \varepsilon_{p+1}) \notin \mathcal{L}'(p+1)$, the path $(x^{\varepsilon_1}, \dots, x^{\varepsilon_p}, x^{\varepsilon_{p+1}})$ ends in a unique cylinder. We call it C and set $z = (x^{\varepsilon_1}, \dots, x^{\varepsilon_p})$ and $y = x^{\varepsilon_{p+1}}$. From the preparation theorem (4.3), we have that for x small enough, the function φ can be written as

$$(\varphi(z, y))^{\pm 1} = \alpha^{r/s}(z, y')\ A(z)\ U(\phi_1(z), \dots, \phi_s(z), \alpha^{1/s},\ \beta^{1/s}) \tag{5}$$

where α and β are, respectively, $(y' - \theta(z))/a(z)$ and $b(z)/(y' - \theta(z))$ or the contrary, and y', A, $(\phi_i)_{1 \leqslant i \leqslant s}$, θ, a, b, r and s are as indicated in definition (4.2). Using $\mathcal{H}(p)$, we get that the function θ is either stationary at 0 or equivalent to some $u_\theta x^{v_\theta}$ with u_θ and v_θ as stated in $\mathcal{H}(p)$. We then define $\mathcal{L}(p+1)$ as the union of $\mathcal{L}'(p+1)$ and the conditions $\{\varepsilon_{p+1} = \pm v_\theta\}$ for all the θ involved in the decomposition of φ in $\mathcal{R}_p$-cylinders, for all φ in $E(p+1)$.

Consequently, if $(\varepsilon_1, \dots, \varepsilon_p, \varepsilon_{p+1}) \notin \mathcal{L}(p+1)$, then θ is 0 on C since it cannot have the same order as y' when $x \to 0$. Hence, in (5), the last two arguments of U, since they are bounded on C, converge to bounded quantities (δ_1, δ_2), as $x \to 0$. Moreover, as a and b are also stationary at 0 or equivalent to some quantities $u_a x^{v_a}$ and $u_b x^{v_b}$, we get that for $i \in \{1,\ 2\}$, $\delta_i \in \{0,\ u_a^{-1/s},\ u_b^{1/s}\}$. Similarly, for $1 \leqslant i \leqslant s$, ϕ_i is either stationary at 0 or equivalent to some $u_{\phi_i} x^{v_{\phi_i}}$, with u_{ϕ_i} and v_{ϕ_i} as in $\mathcal{H}(p)$ and $v_{\phi_i} \geqslant 0$ as ϕ_i is bounded on the basis B. We also have that A is either stationary at 0 or equivalent to some $u_A x^{v_A}$. We now observe that $\varphi(z, y)$ may be stationary at 0 if A, a or b are stationary at 0. Finally, if $\varphi(z, y)$ is not stationary at 0, since U is continuous and between two strictly positive constants on the cylinder C, we obtain that, as $x \to 0$

$$(\varphi(z, y))^{\pm 1} \sim \gamma \ u_A x^{v_A} \ U\,(l_1, \ldots,\ l_s, \delta_1,\ \delta_2)\,,$$

where

$$\gamma = \left(\frac{x^{\pm\varepsilon_{p+1}}}{u_a x^{v_a}}\right)^{r/s} \text{ or } \left(\frac{u_b x^{v_b}}{x^{\pm\varepsilon_{p+1}}}\right)^{r/s},\ l_i = 0 \text{ or } u_{\phi_i} \text{ for } 1 \leqslant i \leqslant s \text{ and } \delta_i \in \left\{0,\ u_a^{-1/s},\ u_b^{1/s}\right\}.$$

The result then follows and the assertion $\mathcal{H}(p + 1)$ is proved. Consequently, the theorem is shown for the $(\varepsilon_i)_{1\leqslant i\leqslant n+1}$ that do not verify the finite number of linking relations over $\mathbb{Q}$ contained in $\mathcal{L}(n + 1)$. Then, take one relation and assume, for example, that it has the form $\varepsilon_{n+1} = (1/q) \sum_{l=1}^{n} p_l \varepsilon_l$ for some integers $(p_l)_{1\leqslant l\leqslant n}$, $q \neq 0$. Then, replace $\xi(x^{\varepsilon_1}, \ldots,\ x^{\varepsilon_n},\ x^{\varepsilon_{n+1}})$ by $\tilde{\xi}\left(x^{\varepsilon'_1}, \ldots,\ x^{\varepsilon'_n}\right)$, where $\varepsilon'_j = \varepsilon_j/q$ and $\tilde{\xi} = \xi \circ \eta$ with

$$\eta(y_1, \ldots, y_n) = \left(y_1^q, \ldots,\ y_n^q,\ \left(y_1^{p_1}, \ldots, y_n^{p_n}\right)\right).$$

The function $\tilde{\xi}$ is also global subanalytic. We can now make the same proof as above with $\tilde{\xi}$ and the path $(x^{\varepsilon'_1}, \ldots, x^{\varepsilon'_n})$. As there is one dimension less, the result follows recursively. □

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