On the recurrence of random walks on $\mathbb{Z}$ in random medium

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Abstract. *We study random walks on $\mathbb{Z}$ in a stationary random medium, defined by an ergodic dynamical system, when the possible jumps are $\{0, \pm 1, \pm 2\}$. We give a simple proof of a recurrence criterion of the same kind as Key's [6]. We show that the intermediate Lyapunov exponent involved in the criterion is always simple.*

Sur la récurrence des marches aléatoires sur $\mathbb{Z}$ en milieu aléatoire

Résumé. *Nous étudions des marches aléatoires sur $\mathbb{Z}$ en milieu aléatoire stationnaire, défini par un système dynamique ergodique, lorsque les sauts possibles sont $\{0, \pm 1, \pm 2\}$. Nous donnons une preuve simple d'un critère de récurrence du même type que celui de Key [6]. Nous montrons que l'exposant de Lyapunov intervenant dans le critère est toujours simple.*

Version française abrégée.

Soit $(\Omega, \mathcal{F}, \mu, T)$ un système dynamique inversible et ergodique. Soient L et R deux entiers supérieurs ou égaux à 1 et notons $\Lambda = \{-L, \cdots, +R\}$, l'ensemble des entiers de $-L$ à $+R$, interprété par la suite comme l'ensemble des "sauts" possibles. Soit $(p_z)_{z \in \Lambda}$ une collection de variables aléatoires strictement positives sur $(\Omega, \mathcal{F})$, indexées par Λ, telles que $\sum_{z \in \Lambda} p_z(\omega) = 1$, $\mu - pp$, et telles qu'il existe une constante $\varepsilon > 0$ vérifiant : $\forall z \in \Lambda$, $z \neq 0$, $(p_z / p_R) \geq \varepsilon$.

A ω fixé, soit $(\xi_n(\omega))_{n \geq 0}$ la chaîne de Markov sur $\mathbb{Z}$, définie par $\xi_0(\omega) = 0$ et les lois de transition : $\forall x \in \mathbb{Z}, \forall z \in \Lambda$, $\mathcal{P}_\omega(\xi_{n+1}(\omega) = x + z | \xi_n(\omega) = x) : = p_z(T^x \omega)$.

En notant $\mathcal{P}_\omega$ la mesure induite par $(\xi_n(\omega))_{n \geq 0}$ sur $\Lambda^{\mathbb{N}}$, nous nous intéressons dans ce travail au comportement asymptotique avec $\mathcal{P}_\omega$-probabilité 1, ω $\mu - pp$ de cette marche aléatoire. Les premiers résultats ont été énoncés dans le cas indépendant par Solomon [10] pour $L = R = 1$, puis par Key [6], qui a prouvé un critère de récurrence pour L et R quelconques, en fonction du signe d'exposants de Lyapunov d'une certaine matrice aléatoire. Dans le cadre ergodique, des résultats pour Λ de la forme $\{-L, \cdots, 0, +1\}$ ont été prouvés par [1], [8], [4] et [2]. Du résultat de Key, on peut déduire un critère à la forme plus simple, faisant intervenir une matrice, notée M par la suite, de dimension inférieure (voir [7]). Cette nouvelle matrice intervient notamment de manière cruciale dans la recherche de la mesure invariante pour la chaîne des "environnements vus de la particule", voir [3] et [2]. Lorsque $R > 1$ et $L > 1$, le prolongement de ces résultats nécessite la compréhension de la géométrie des vecteurs propres d'Oseledets associés à (M, T). Dans ce cadre, nous présentons ici, dans le contexte décrit plus haut et avec $R = L = 2$, une preuve simple d'un critère de récurrence énoncé dans [7] dans le cas indépendant. Pour cela, nous évaluons des expressions du type $\mathcal{P}_\omega\{$partant de k, d'atteindre $]-\infty,-1]\}$, $k \geq 0$, vues comme limites de $\mathcal{P}_\omega\{$partant de k, sortir de $[-1,n]$ par la gauche$\}$, lorsque $n \longrightarrow +\infty$. A cette fin, nous calculons de manière exacte ces dernières quantités, en utilisant la matrice :

$$M := \begin{pmatrix} -a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ où } a = \frac{p_1 + p_2}{p_2},\ b = \frac{p_{-2} + p_{-1}}{p_2} \text{ et } c = \frac{p_{-2}}{p_2}.$$

La valeur limite s'obtient en fonction du signe de $\gamma_2(M,T)$, second exposant de Lyapunov de M par rapport à $(\Omega, \mathcal{F}, \mu, T)$. On utilise le fait que M ainsi que M^{-1} sont conjuguées de manière déterministes à des matrices négatives, ce qui implique par exemple que le spectre de Lyapunov de M est simple. On arrive alors au résultat suivant :

– Si $\gamma_2(M,T) > 0$ (resp < 0), la marche est transiente en $-\infty$ (resp $+\infty$).

– Si $\gamma_2(M,T) = 0$, la marche est récurrente.

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1. Introduction and notations.

Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system, with T an invertible measure preserving transformation. We assume that T is ergodic with respect to μ. The space Ω is interpreted as the space of the environments. Fix two integers $(L, R) \geq 1$ and set $\Lambda = \{-L, \cdots, 0, \cdots, +R\}$, the space of jumps. Let $(p_z)_{z\in\Lambda}$ be a collection of strictly positive random variables on $(\Omega, \mathcal{F})$, indexed by Λ, such that $\sum_{z\in\Lambda} p_z(\omega) = 1$, $\mu - ae$ and such that there exists $\varepsilon > 0$ satisfying :

$$\forall z \in \Lambda,\ z \neq 0,\ (p_z/p_R) \geq \varepsilon. \tag{1}$$

Let ω be a fixed environment and let $(\xi_n(\omega))_{n\geq 0}$ be the Markov chain on $\mathbb{Z}$ defined by $\xi_0(\omega) = 0$ and the transition laws: $\forall x \in \mathbb{Z}, \forall z \in \Lambda$, $\mathcal{P}_\omega(\xi_{n+1}(\omega) = x + z | \xi_n(\omega) = x) := p_z(T^x\omega)$.

Writing $\mathcal{P}_\omega$ the measure induced by $(\xi_n(\omega))_{n\geq 0}$ on $\Lambda^{\mathbb{N}}$, we aim here at studying the asymptotic behaviour of this random walk with $\mathcal{P}_\omega$-probability 1, ω $\mu - ae$. The first results were announced in the independent case, by Solomon [10] with $L = R = 1$, and then by Key [6], who proved a recurrence criterion for any L and R, in terms of the sign of some Lyapunov exponents of a certain random matrix. In the ergodic context and for Λ of the form $\{-L, \cdots, 0, 1\}$, results have been proved by [1], [8], [4] and [2]. From Key's result, one can derive a criterion that has a simpler form, involving a matrix, written M, of lower dimension (see [7]). This new matrix appears to be decisive in the research of the invariant measure for the chain of "the environments seen from the particle" for example, see [3] and [2]. When $R > 1$ and $L > 1$, the extension of the preceding results requires the understanding of the geometry of the Oseledets' vectors associated to (M, T). In this frame, we present here, in the context defined above and with $R = L = 2$, a simple proof of a recurrence criterion already enounced in [7] in the independent case. For the sequel, we fix the following notations. Introduce :

$$M := \begin{pmatrix} -a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ with } a = \frac{p_1 + p_2}{p_2},\ b = \frac{p_{-2} + p_{-1}}{p_2} \text{ and } c = \frac{p_{-2}}{p_2}.$$

We will use cocycle notations for scalars and for matrices. For example, for M we set :

$$M_0 = I \text{ and } M_n = T^{n-1}M \cdots M,\ M_{-n} = T^{-n}M^{-1} \cdots T^{-1}M^{-1}, \text{ for } n \geq 1.$$

One easily sees that Oseledets' Multiplicative Theorem [9] can be applied to M. We write $\gamma_1(M,T) \geq \gamma_2(M,T) \geq \gamma_3(M,T)$, the Lyapunov exponents of M with respect to $(\Omega, \mathcal{F}, \mu, T)$. These quantities can be precisely defined by: $\sum_{j=1}^{i} \gamma_j(M,T) = \lim_{n\to+\infty} n^{-1} \| \wedge_i M_n \|$, $\mu - ae$, $1 \leq i \leq 3$.

2. Characteristic probabilities of the Random Walk.

For integers $a < b$ and k, write $P_k\{a,b,l\} = \mathcal{P}_\omega\{\text{starting from } k, \text{ leave } [a,b] \text{ at the point } l\}$, where $l \in \{a-1, a, b, b+1\}$. Set also $P_k\{a,b,+\} = P_k\{a,b,b\} + P_k\{a,b,b+1\}$ and similarly $P_k\{a,b,-\} = 1 - P_k\{a,b,+\}$. For $n > -1$, we have:

Lemma 1

If $k \leq -1$, $P_k\{-1,n,+\} = 0$. If $k \geq n$, $P_k\{-1,n,+\} = 1$. If $-1 < k < n$, then:

$$P_k\{-1,n,+\} = \frac{\sum_{p=-1}^{k-1} d(-1,n,p)}{\sum_{p=-1}^{n-1} d(-1,n,p)}, \text{ with } d(-1,n,p) = \begin{vmatrix} (M_n)_{11} & (M_n)_{12} \\ (M_{p+1})_{21} & (M_{p+1})_{22} \end{vmatrix}.$$

Proof of the lemma. Take $-1 < k < n$ and consider any function $P_k\{-1,n,l\}$, $l \in \{-2,-1,n,n+1\}$, writing it $f_l(k)$. Using the Markov property, we get a harmonic type recurrence relation:

$$f_l(k) = p_0(k) f_l(k) + p_1(k) f_l(k+1) + p_2(k) f_l(k+2) + p_{-1}(k) f_l(k-1) + p_{-2}(k) f_l(k-2).$$

To suppress $p_0(k)$, we introduce $g_l(k) := f_l(k) - f_l(k+1)$. Substituting and reordering, we obtain:

$$g_l(k)[p_2(k) + p_1(k)] + g_l(k+1) p_2(k) = g_l(k-1)[p_{-1}(k) + p_{-2}(k)] + g_l(k-2) p_{-2}(k).$$

Using matrix M and setting $V_l(k) := {}^t(g_l(k+1), g_l(k), g_l(k-1))$, the previous relation can be written as $V_l(k) = T^k M V_l(k-1)$ and we get $V_l(k) = M_{k+1} V_l(-1)$. Taking $f_+(k) = P_k\{-1,b,+\}$, writing g_+, V_+ the corresponding quantities and summing from k to $n-1$ and from 0 to $n-1$, we are led to:

$$\begin{cases} f_+(k) - 1 = \sum_{p=k}^{n-1} {}^t e_2 M_{p+1} V_+(-1) \\ \\ -1 = {}^t e_2 \left[I + \sum_{p=0}^{n-1} M_{p+1} \right] V_+(-1). \end{cases} \tag{2}$$

For $l \in \{n, n+1\}$, we now remark that $V_l(-1) = {}^t(g_l(0), g_l(-1), 0)$ has zero as third coordinate. Since, for such l's, $V_l(n-1) = M_n V_l(-1)$ and $< V_n(n-1) \wedge V_{n+1}(n-1), e_1 \wedge e_2 >= -f_-(n-1) < 0$, we deduce that the sub-matrix $(M_n)_{2,2}$ is invertible. As $V_+(n-1) = {}^t(0, g_+(n-1), g_+(n-2)) = M_n V_+(-1)$ with $V_+(-1) = {}^t(g_+(0), g_+(-1), 0)$, we get:

$$\begin{pmatrix} g_+(0) \\ g_+(-1) \end{pmatrix} = [(M_n)_{2,2}]^{-1} \begin{pmatrix} 0 \\ g_+(n-1) \end{pmatrix}.$$

Using the previous relations (2), we deduce: $f_+(k) = 1 - \dfrac{\sum_{p=k}^{n-1} {}^t e_2 (M_{p+1})_{2,2} [(M_n)_{2,2}]^{-1} e_2}{\sum_{p=-1}^{n-1} {}^t e_2 (M_{p+1})_{2,2} [(M_n)_{2,2}]^{-1} e_2}$.

Detailing $[(M_n)_{2,2}]^{-1}$ leads to the announced formula. □

3. Study of the Lyapunov spectrum.

Lemma 2
1)The Lyapunov spectrum of M is simple : $\gamma_1(M,T) > \gamma_2(M,T) > \gamma_3(M,T)$.
2) We have :

$$\gamma_2(M,T) = \gamma_{\max}\left[\begin{pmatrix} p_{-1} & p_{-2} & 0 \\ p_{-1}+p_1+p_2 & 0 & p_{-2} \\ p_{-1}+p_1 & 0 & p_{-2} \end{pmatrix}, T\right] - \gamma_{\max}\left[\begin{pmatrix} p_1 & p_2 & 0 \\ p_1+p_{-1}+p_{-2} & 0 & p_2 \\ p_1+p_{-1} & 0 & p_2 \end{pmatrix}, T^{-1}\right].$$

3) The first and third exponents have fixed signs : $\gamma_1(M,T) > 0$ *and* $\gamma_3(M,T) < 0$.

Proof of the lemma. This lemma is a consequence of the existence of underlying positive matrices. One can check that the following decompositions hold :

$$M = -\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{p_1}{p_2} & \frac{p_1+p_{-1}+p_{-2}}{p_2} & \frac{p_1+p_{-1}}{p_2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (3)$$

$$M^{-1} = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{p_{-1}}{p_{-2}} & \frac{p_{-1}+p_1+p_2}{p_{-2}} & \frac{p_{-1}+p_1}{p_{-2}} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The above positive matrices have simple dominant exponent with respect to T and T^{-1} (apply corollary (2), page 1548, of [5] for example). Thus $\gamma_1(M,T) > \gamma_2(M,T)$ and $\gamma_1(M^{-1},T^{-1}) > \gamma_2(M^{-1},T^{-1})$, that is $\gamma_2(M,T) > \gamma_3(M,T)$ and we obtain 1). The point 2) follows from the previous decompositions and simple computations. Consider now 3). In the context of the previous section, we let $n \longrightarrow +\infty$. Using the same notations for the limit vectors, we have for $l \in \{-2,-1\}$: $V_l(k-1) = M_k V_l(-1)$, $\forall k \geq 0$. Since the left side remains bounded (as $k \longrightarrow +\infty$) and the subspace generated by the vectors $(V_{-2}(-1), V_{-1}(-1))$ has at least rank 1 (their first component is ± 1), we get $\gamma_3(M,T) \leq 0$. Symmetrically, one gets $\gamma_1(M,T) \geq 0$. Introducing $K_r = \text{diag}(1,r,r^2)$, we observe the following relation :

$$K_r\ M\ K_r^{-1} = rM', \text{ with } M' = \begin{pmatrix} -a/r & b/r^2 & c/r^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For r close to 1, using condition (1), there exists random transition probabilities $(p'_z)_{z\in\Lambda}$ such that the associated matrix is M'. Since the exponents of (M,T) are deduced from the previous ones by translation by $\log(r)$, we deduce at last that $\gamma_1(M,T) > 0$ and $\gamma_3(M,T) < 0$. □

Thus there exists a random Oseledets' basis (Φ_1, Φ_2, Φ_3), each Φ_i having a unique random direction, such that for all $1 \leq i \leq 3$, there exists a random scalar λ_i, with $\log(\lambda_i)$ bounded (since condition (1) holds) and : $M\Phi_i = \lambda_i T\Phi_i$, $\int \log|\lambda_i|\ d\mu = \gamma_i(M,T)$ and $||\Phi_i|| = 1$. We write $\Phi = [\Phi_1, \Phi_2, \Phi_3]$ and then $M = T\Phi D\Phi^{-1}$, with $D = \text{diag}\ (\lambda_1, \lambda_2, \lambda_3)$.

Lemma 3
One has $< \wedge_2 \Phi^{-1} e_1 \wedge e_2, e_1 \wedge e_2 > \neq 0$ *and there exists a constant* $\alpha > 0$ *such that* $|\Phi_{11}| \geq \alpha$.

Proof of the lemma:
In the basis $(e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$, one can check the following relation:

$$\bigwedge_2 {}^t M = - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b-c & b-c+a & b-c+a-1 \\ c & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $\bigwedge_2 {}^t M = \bigwedge_2 {}^t \Phi^{-1} \bigwedge_2 {}^t D \bigwedge_2 {}^t T\Phi$, the vector of maximal exponent of an Oseledets' basis associated to $\bigwedge_2 {}^t M$, up to a non-zero scalar, has the form ${}^t(u+v+w, v+w, w)$, with $u > 0, v > 0, w > 0$. Thus $< \bigwedge_2 {}^t \Phi^{-1} e_1 \wedge e_2, e_1 \wedge e_2 > \neq 0$. The second point is a simple consequence of (3) and condition (1).

$\square$

4. Recurrence criterion.

Theorem 1
The asymptotic behaviour of the Random Walk is the following:

(i) If $\gamma_2(M,T) < 0$, then: $\xi_n(\omega) \longrightarrow +\infty$, $\mathcal{P}_\omega - ae$, $\mu - ae$.
(ii) If $\gamma_2(M,T) = 0$, then: $(\xi_n(\omega))$ is recurrent, $\mathcal{P}_\omega - ae$, $\mu - ae$.
(iii) If $\gamma_2(M,T) > 0$, then: $\xi_n(\omega) \longrightarrow -\infty$, $\mathcal{P}_\omega - ae$, $\mu - ae$.

Proof of the theorem:
We will evaluate the quantities $P_k\{l, +\infty, \pm\}$ and $P_k\{-\infty, l, \pm\}$, for any k and l in $\mathbb{Z}$. By symmetry and stationarity, we just need to consider the expressions $P_k\{-1, +\infty, -\}$, for $k \geq 0$. Furthermore, since for all $(i,j) \geq 0$, $P_i\{\text{reach } j, \text{ staying in } [0, +\infty[\} > 0$, either $P_k\{-1, +\infty, -\} = 1$, $\forall k \geq 0$ or $P_k\{-1, +\infty, -\} < 1$, $\forall k \geq 0$. In the first case the Random Walk visits $-\infty$, in the second case it does not visit $-\infty$ and then is transient to $+\infty$.

1) Case when $\gamma_2(M,T) > 0$. Consider the interval $[-1, +\infty[$. As in the previous section, for $l \in \{-2, -1\}$, we have $V_l(k) = M_{k+1} V_l(-1)$, $\forall k \geq 0$. The vectors $V_{-2}(-1)$ and $V_{-1}(-1)$ therefore have strictly negative exponent and thus are linked. From their form, one deduces that $V_{-2}(-1) + V_{-1}(-1) = 0$, which gives $P_0\{-1, +\infty, -\} = 1$, $\mu - ae$.
2) Case when $\gamma_2(M,T) = 0$. The following idea is taken from [6]. For $n \geq 0$ and $0 \leq j \leq 1$:

$$P_{2n+j}\{-1, +\infty, -\} = \sum_{l=0}^{1} P_{2n+j}\{2n-1, +\infty, 2(n-1)+l\} \times P_{2(n-1)+l}\{-1, +\infty, -\}.$$

Therefore: $\max_{0 \leq j \leq 1} P_{2n+j}\{-1, +\infty, -\} \leq \max_{0 \leq j \leq 1} P_{2n+j}\{2n-1, +\infty, -\} \times \max_{0 \leq l \leq 1} P_{2(n-1)+l}\{-1, +\infty, -\}$ and:

$$\begin{aligned} \log \max_{0 \leq j \leq 1} P_{2n+j}\{-1, +\infty, -\} &\leq \sum_{t=0}^{n} \log \max_{0 \leq j \leq 1} P_{2(n-t)+j}\{2(n-t)-1, +\infty, -\} \\ &= \sum_{t=0}^{n} \log \max_{0 \leq j \leq 1} P_j\{-1, +\infty, -\}(T^{2t}\omega). \end{aligned}$$

We introduce the function $\varphi := \log \max_{0 \leq j \leq 1} P_j\{-1, +\infty, -\}$ and $\mathcal{I}_2$, the σ-algebra generated by the T^2-invariant sets. Since $\varphi \leq 0$, we have $\varphi = 0$, $\mu - ae$, on the set $\{\mathbb{E}[\ \varphi \mid \mathcal{I}_2] = 0\}$ and then for the previous $\omega' s$, there exists $j(\omega) \in \{0, 1\}$ such that $P_{j(\omega)}\{-1, +\infty, -\} = 1$.

Using now the Ergodic Theorem, $\mu-ae$ on the set $\{\mathbb{E}[\,\varphi \mid \mathcal{I}_2] < 0\}$, the sequence $(P_k\{-1,+\infty,-\})_{k\geq 0}$ tends to 0 with an exponential rate. Since for $l \in \{-2,-1\}$, one has $V_l(k) = M_{k+1}V_l(-1)$, $\forall k \geq 0$, we get that $V_{-2}(-1)$ and $V_{-1}(-1)$ have strictly negative exponent. Since $\gamma_2(M,T) = 0$, these vectors are linked and, as above, one gets $P_0\{-1,+\infty,-\} = 1$.

3) Case when $\gamma_2(M,T) < 0$.
From lemma (1), we have $P_k\{-1,n,-\} = S(k,n)/S(-1,n)$, with $S(l,n) = \sum_{p=l}^{n-1} d(-1,n,p)$. Fixing k, we will give an equivalent of $S(k,n)$ as $n \to +\infty$. Using the form of $M_l = T^l \Phi D_l \Phi^{-1}$, for $l \geq 0$, we decompose $(\underset{2}{\wedge}\Phi^{-1}\, e_1 \wedge e_2)$ into the canonical basis of $\underset{2}{\wedge}\mathbb{R}^3$:

$$S(k,n) = \sum_{1\leq i_1<i_2\leq 3} \begin{vmatrix} (\lambda_{i_1})_n T^n \Phi_{1i_1} & (\lambda_{i_2})_n T^n \Phi_{1i_2} \\ \sum_{p=k}^{n-1}(\lambda_{i_1})_{p+1}T^{p+1}\Phi_{2i_1} & \sum_{p=k}^{n-1}(\lambda_{i_2})_{p+1}T^{p+1}\Phi_{2i_2} \end{vmatrix} \times < (\underset{2}{\Lambda}\Phi^{-1})e_1\wedge e_2, e_{i_1}\wedge e_{i_2} > .$$

Set $R(k) : = \sum_{j=2,3} < (\underset{2}{\Lambda}\Phi^{-1})e_1 \wedge e_2, e_1 \wedge e_j > \sum_{p=k}^{+\infty}(\lambda_j)_{p+1}T^{p+1}\Phi_{2j} \neq 0$. If $R(k) \neq 0$, then, since $\gamma_2(M,T) < 0$ and $T^n\Phi_{11}$ is not small (cf lemma (3)), we obtain that:

$$S(k,n) \sim (\lambda_1)_n T^n \Phi_{11} R(k),\ n \longrightarrow +\infty.$$

We now prove that $R(k) \neq 0$. If for every $k \geq 0$, $R(k) = 0$, then from the fact that $\gamma_2(M,T) > \gamma_3(M,T)$ and from lemma (3), we would get that $T^n\Phi_{22} \longrightarrow 0$, as $n \longrightarrow +\infty$. From the form of M, one would get $T^n\Phi_2 \longrightarrow 0$, contradicting the fact that $\|\Phi_2\| = 1$. Thus there exists $k_0 \geq 0$, such that $R(k_0) \neq 0$. Evaluating $P_{k_0}\{-1,n,-\}$ gives that $R(-1) \neq 0$, otherwise the numerator would have greater order than the denominator. Thus $S(-1,n) \sim (\lambda_1)_n T^n \Phi_{11} R(-1)$. Since for all $k \geq 0$, $P_k\{-1,+\infty,-\} > 0$, we get that $R(k) \neq 0$. Finally, we obtain:

$$\forall k \geq 0,\ P_k\{-1,+\infty,-\} = \frac{\sum_{j=2,3} < (\underset{2}{\Lambda}\Phi^{-1})e_1 \wedge e_2, e_1 \wedge e_j > \sum_{p=k}^{+\infty}(\lambda_j)_{p+1}T^{p+1}\Phi_{2j}}{\sum_{j=2,3} < (\underset{2}{\Lambda}\Phi^{-1})e_1 \wedge e_2, e_1 \wedge e_j > \sum_{p=-1}^{+\infty}(\lambda_j)_{p+1}T^{p+1}\Phi_{2j}},$$

with numerator and denominator both $\neq 0$. Thus, $P_k\{-1,+\infty,-\} \longrightarrow 0$, as $k \longrightarrow +\infty$. Consequently $P_k\{-1,+\infty,-\} < 1$, $\forall k \geq 0$, which concludes the proof of the theorem.

□

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