DUALITY SOLUTIONS FOR PRESSURELESS GASES, MONOTONE SCALAR CONSERVATION LAWS, AND UNIQUENESS

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Abstract
We introduce for the system of pressureless gases a new notion of solution, which consists in interpreting the system as two nonlinearly coupled linear equations. We prove in this setting existence of solutions for the Cauchy problem, as well as uniqueness under optimal conditions on initial data. The proofs rely on the detailed study of the relations between pressureless gases, the dynamics of sticky particles and nonlinear scalar conservation laws with monotone initial data. We prove for the latter problem that monotonicity implies uniqueness, and a generalization of Oleinik’s entropy condition.

Key-words: pressureless gases – duality solutions – sticky particles – scalar conservation laws – Hamilton-Jacobi equations – entropy conditions

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1. Introduction. We consider the following one-dimensional system of conservation laws

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0, \\
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} &= 0,
\end{align*}
\]

in \( ]0, \infty[ \times \mathbb{R} \).
This set of equations is known as the system of pressureless gases, and can of course be viewed as the limit when pressure goes to zero of the usual Euler equations. This operator is involved in several numerical resolution schemes of the Euler equations by splitting methods. It appears also as the macroscopic limit of a Boltzmann equation when the Maxwellian has zero temperature. Finally, it is a model in plasma physics (cold plasmas) and astrophysics (Zeldovich [23]).

Here \( \rho \geq 0 \) denotes the density of the gas, and \( u \) its velocity. Several specific difficulties arise in the mathematical analysis of this system, as pointed out by Bouchut [3]. The first one is that one has to look for \( \rho \) in the class of nonnegative measures, and for \( u \) as a function with bounded variation, so that the product \( \rho u \) is not well-defined. Next, the system (1.1) has to be complemented by entropy conditions to expect uniqueness, and it turns out that the most natural one, namely

\[
\partial_t (\rho S(u)) + \partial_x (\rho u S(u)) \leq 0,
\]

for any convex function \( S \), is not sufficient.

On the other hand, global existence for measure-valued solutions to (1.1) has been obtained independently by Grenier [12] and E, Rykov and Sinai [11]. First they prove that (1.1) describes at a continuous level the behavior of a specific system of interacting particles: the so-called sticky particles. Next, measure-valued solutions to (1.1) can be obtained as limits when the number of particles tends to infinity of the previous discrete solutions. The sticky particles solutions to (1.1), and their limits, satisfy the entropy condition (1.2). Other results have been obtained in a similar way by Wang, Huang and Ding [21] and Wang and Ding [22].

More recently, Brenier and Grenier [8] pointed out several connections between weak solutions to (1.1), sticky particles, and entropy solutions to the scalar conservation law

\[
\partial_t M + \partial_x [f(M)] = 0,
\]

where \( M \) and \( f \) are related to \( \rho \) and \( u \) by

\[
\rho = \partial_x M, \quad \rho u = \partial_x [f(M)].
\]

More precisely, they prove that, if \((\rho, u)\) is a solution to (1.1) corresponding to sticky particles, then there exists \( f \in \text{Lip}(\mathbb{R}) \) and \( M \) entropy solution to (1.3) such that (1.4) holds. Conversely, from monotone entropy solutions to (1.3), they recover weak solutions to (1.1), but without entropy condition.

The aim of this paper is to introduce another notion of solution to (1.1), for which existence and uniqueness hold. We actually interpret (1.1) as a system of two linear equations with discontinuous coefficient \( u \), and unknowns

\[
\partial_t \rho + \partial_x (\rho u) = 0,
\]

\[
\partial_t u + \partial_x u = 0.
\]
We use therefore the so-called duality solutions, introduced by the authors in [4,5] for linear conservation laws with bounded coefficients satisfying a one-sided Lipschitz condition. An intrinsic property of duality solutions is that the products $\rho u$ and $qu$ can be given a precise meaning. Thus we recover the nonlinear coupling between the equations. Duality solutions enjoy also weak stability properties with respect to perturbations of the initial data and the coefficient. This leads to an existence result via an approximation by sticky particles, which turn out to be duality solutions. Duality solutions can also be obtained as limits of solutions to a viscous pressureless model, see Boudin [7].

The proof of uniqueness relies on a careful study of the above mentioned relations between monotone conservation laws and duality solutions. We prove indeed that formula (1.4) gives some kind of equivalence between duality solutions to (1.1) and entropy solutions to (1.3). This has two consequences. On the one hand, uniqueness for entropy solutions to (1.3) leads to uniqueness for duality solutions; on the other hand, the definition of duality solutions gives an analogue of the classical Oleinik entropy condition for monotone scalar conservation laws, without any convexity assumption on the flux $f$. The key lemma here is a uniqueness result for scalar conservation laws with monotone initial data: we prove that, under some natural admissibility conditions on the initial datum, a solution is an entropy solution if and only if it is monotone in $x$.

The sequel of the paper is organized as follows. In Section 2, we recall the definitions and general properties of the duality solutions introduced in [4,5], and give a precise meaning to the same notion for (1.1). We also state and prove the existence and uniqueness results. Section 3 is devoted to the “equivalence” between duality solutions and entropy solutions to (1.3), and states precisely the Oleinik-like entropy condition. Finally, the proof of uniqueness for monotone solutions to (1.3) is given in Section 4. The result follows actually from the study of convex solutions to a Hamilton-Jacobi equation.

Part of these results were announced in [6].

2. Definition and main results. Duality solutions were introduced in [4,5] to solve in the context of measures the linear conservation law

$$\partial_t \mu + \partial_x (a \mu) = 0,$$

where $a \in L^\infty([0,T]\times\mathbb{R})$ satisfies

$$\partial_x a \leq \alpha(t), \quad \alpha \in L^1([0,T]).$$

We refer to [5] for the precise definition and general properties of these solutions. We shall need the following facts.
Theorem 2.1. (Bouchut and James [4,5])

1) Given $\mu^0 \in M_{\text{loc}}(\mathbb{R})$, under the assumptions (2.2), there exists a unique
$\mu \in S_\mathcal{M} \equiv C([0,T];M_{\text{loc}}(\mathbb{R}) - \sigma(M_{\text{loc}},C_c))$, duality solution to (2.1), such
that $\mu(0,.) = \mu^0$.

2) There exists a bounded Borel function $\hat{a}$, called universal representative of $a$, such that $\hat{a} = a$ almost everywhere, and for any duality solution $\mu$,

\begin{equation}
\partial_t \mu + \partial_x(\hat{a}\mu) = 0 \quad \text{in the distributional sense.}
\end{equation}

3) The duality solutions satisfy the entropy inequality

\begin{equation}
\partial_t |\mu| + \partial_x(\hat{a} |\mu|) \leq 0 \quad \text{in the distributional sense.}
\end{equation}

4) Let $(\alpha_n)$ be a bounded sequence in $L^\infty([0,T] \times \mathbb{R})$, such that $\alpha_n \to a$ in $L^\infty([0,T] \times \mathbb{R}) - \text{w*}$. Assume $\partial_x \alpha_n \leq \alpha_n(t)$, where $(\alpha_n)$ is bounded in $L^1([0,T])$, $\partial_x a \leq \alpha \in L^1([0,T])$. Consider a sequence $(\mu_n) \in S_\mathcal{M}$ of duality solutions to

\begin{equation}
\partial_t \mu_n + \partial_x(\alpha_n \mu_n) = 0 \quad \text{in } [0,T] \times \mathbb{R},
\end{equation}

such that $\mu_n(0,.)$ is bounded in $M_{\text{loc}}(\mathbb{R})$, and $\mu_n(0,.) \to \mu^0 \in M_{\text{loc}}(\mathbb{R})$.
Then $\mu_n \to \mu$ in $S_\mathcal{M}$, where $\mu \in S_\mathcal{M}$ is the duality solution to

\begin{equation}
\partial_t \mu + \partial_x(a \mu) = 0 \quad \text{in } [0,T] \times \mathbb{R}, \quad \mu(0,.) = \mu^0.
\end{equation}

Moreover, $\hat{\alpha}_n \mu_n \to \hat{a} \mu$ weakly in $M_{\text{loc}}([0,T] \times \mathbb{R})$.

The set of duality solutions is clearly a vector space, but it has to be noted that a duality solution is not defined as a solution in the sense of distributions. The product $\hat{a} \mu$ is defined \textit{a posteriori}, by the equation itself (assertion 2).

Remark 2.1. Under the same assumptions on $a$, there exists also a notion of
duality solutions for the nonconservative equation

\begin{equation}
\partial_t u + a \partial_x u = 0,
\end{equation}

with $u \in C([0,T];L^1_{\text{loc}}(\mathbb{R})) \cap B([0,T];BV_{\text{loc}}(\mathbb{R}))$. An important point is that, if $\mu$ is a duality solution to (2.1), then there exists a unique (up to a constant) duality solution $u$ to (2.5) such that $\partial_x u = \mu$. Notice also that duality solutions to (2.5) are stable by renormalization: if $u$ is a solution, then so is $S(u)$ for any Lipschitz continuous $S$. See [5] for proofs of these results.

Corollary 2.2. If in (2.2) we only have $\alpha \in L^1_{\text{loc}}([0,T])$, the same existence result holds, and a weakened form of the stability result is valid, up to a subsequence of $(\mu_n)$.

Motivated by these results, we propose the following notion of solution for (1.1).
Definition 2.3. We shall say that a couple \((\rho, q)\), \(\rho, q \in C([0, \infty]; \mathcal{M}_{loc}(\mathbb{R}) - \sigma(\mathcal{M}_{loc}(\mathbb{R}), C_c(\mathbb{R})))\), \(\rho \geq 0\), is a duality solution to (1.1) if there exists \(a \in L^\infty([0, \infty] \times \mathbb{R})\) and \(\alpha \in L^1_{loc}([0, \infty])\) satisfying \(\partial_t a \leq \alpha\) in \([0, \infty] \times \mathbb{R}\), such that

(i) \(\forall\ 0 < t_1 < t_2\), we have in the sense of duality on \([t_1, t_2] \times \mathbb{R}\)

\[
\partial_t \rho + \partial_x (\rho a) = 0, \quad \partial_t q + \partial_x (aq) = 0;
\]

(ii) \(\hat{a}_\rho = q\).

Remark 2.2. Assertion (ii) implies that \(|q| \leq \|a\|_\infty \rho\) in the sense of measures (and actually \(|q(t, \cdot)| \leq \|a\|_\infty \rho(t, \cdot)\) for any \(t \geq 0\) since \(\rho\) and \(q\) are weakly continuous in time), so that we can introduce the effective velocity \(u\) as the Radon-Nikodým quotient \(u \rho = q\). Thus \(u\) is defined \(\rho\)-almost everywhere, and we have indeed \(u = \hat{a} \rho\) \(\rho\)-a.e.

Remark 2.3. It follows from (2.3) that duality solutions are weak solutions to (1.1).

Proposition 2.4 (entropy inequality). Let \((\rho, q)\) be a duality solution to (1.1). Then the entropy inequality (1.2) holds in the sense of distributions for any convex \(S\).

Proof. For any \(k \in \mathbb{R}\), \(\mu \equiv q - k \rho\) is a duality solution to \(\partial_t \mu + \partial_x (a \mu) = 0\) on \([t_1, t_2] \times \mathbb{R}\) for any \(0 < t_1 < t_2\). From (2.4) we have therefore

\[
\partial_t |q - k \rho| + \partial_x ([q - k \rho]|a) \leq 0 \quad \text{in} \quad [0, \infty] \times \mathbb{R}.
\]

Since \(q = \rho u\), \(\rho \geq 0\) and \(\hat{a} = u \rho\) \(\rho\)-a.e., this can be rewritten

\[
\partial_t (\rho |u - k|) + \partial_x (\rho u |u - k|) \leq 0 \quad \text{in} \quad [0, \infty] \times \mathbb{R}.
\]

Thus the inequality holds for the entropies \(S_k = |\cdot - k|\), and the general result follows. \(\square\)

Theorem 2.5. (stability). Let \((\rho_n, q_n)\) be a sequence of duality solutions to (1.1), with corresponding \(a_n\) and \(\alpha_n\). Assume that

- \(\alpha_n \to a\) in \(L^\infty - w^*\), with \(\partial_x a \leq \alpha(t)\), \(\alpha \in L^1_{loc}([0, \infty])\);
- \(\alpha_n\) is bounded in \(L^1_{loc}([0, \infty])\);
- \(\rho_n(0, \cdot)\) is bounded in \(\mathcal{M}_{loc}(\mathbb{R})\).

Then, up to a subsequence, \(\rho_n \to \rho\) and \(q_n \to q\) in \(C([0, \infty]; \mathcal{M}_{loc}(\mathbb{R}) - \sigma(\mathcal{M}_{loc}(\mathbb{R}), C_c(\mathbb{R})))\), and \((\rho, q)\) is a duality solution to (1.1) with velocity \(a\). Moreover, \(\rho_n^2 \to \rho^2\) in the sense of distributions in \([0, \infty] \times \mathbb{R}\).

Proof. The bound on \(\rho_n(0, \cdot)\) gives a bound on \(\rho_n\) and \(q_n\) since \(a_n\) is bounded in \(L^\infty\). The compactness of the sequence \((\rho_n, q_n)\) follows from the equations in the distributional sense, and assertion 4) in Theorem 2.1 allows to pass to the limit. \(\square\)
Remark 2.4. Another notion of solution to (2.1) was defined by Poupaud and Rascle [18] in the multidimensional case, through generalized characteristics in the sense of Filippov. This might be used also for pressureless gases.

We turn now to the main results concerning pressureless gases, namely global existence and uniqueness for duality solutions. We begin by existence.

**Theorem 2.6.** (existence of duality solutions). Let \( \rho^0, q^0 \in \mathcal{M}_{loc}(\mathbb{R}) \), with \( \rho^0 \geq 0 \) and \( |q^0| \leq K \rho^0 \) for some \( K \geq 0 \). Then there exists at least one duality solution to (1.1), with initial data \( \rho^0, q^0 \), with

\[
\|a\|_\infty \leq K \quad \text{and} \quad \alpha(t) = 1/t.
\]

The proof of this theorem is an easy consequence, via the stability result stated in Theorem 2.5, of the following lemma.

**Lemma 2.7.** Consider a system of \( N \) particles with mass \( m_i > 0 \), initial velocity \( v_i^0 \in \mathbb{R} \), and initial position \( x_i^0 \in \mathbb{R} \), for \( 1 \leq i \leq N \), with \( x_1^0 < \ldots < x_N^0 \). Let

\[
\begin{align*}
\rho^0(x) &= \sum_{i=1}^{N} m_i \delta(x - x_i^0), \quad q^0(x) = \sum_{i=1}^{N} m_i v_i^0 \delta(x - x_i^0), \\
\end{align*}
\]

Assume that the system follows the sticky particles dynamics, which defines for \( t \geq 0 \) the trajectories \( x_i(t) \), and their velocities \( v_i(t) = \dot{x}_i(t) \), \( 1 \leq i \leq N \), with \( v_i \) continuous on the right. Let

\[
\begin{align*}
\rho(t, x) &= \sum_{i=1}^{N} m_i \delta(x - x_i(t)), \quad q(t, x) = \sum_{i=1}^{N} m_i v_i(t) \delta(x - x_i(t)).
\end{align*}
\]

Then \( (\rho, q) \) is a duality solution to (1.1).

**Proof.** Recall that, in the sticky particles dynamics, particles are moving along straight lines until they collide, and then they stick together, forming a bigger particle with mass and velocity satisfying the conservation of mass and momentum. Notice that the notations in (2.7) imply that two particles after a collision have different indices. Since there is a finite number of particles, there is a finite number of collision times, say \( t_1, \ldots, t_p \).

Firstable, thanks to the conservation of mass and momentum, \( \rho \) and \( q \) are obviously weakly continuous with respect to time. The effective velocity \( u \) is defined here \( \rho \) a.e., that is on the trajectories \( x_i(t) \), by the Radon-Nikodym quotient \( q = \rho u \). For \( t \geq 0 \) and \( 1 \leq i \leq N \),

\[
\begin{align*}
u(t, x_i(t)) &= \frac{\sum_{j \neq i} m_j v_j(t)}{\sum_{j} m_j},
\end{align*}
\]
the sums being taken on the j’s such that $x_j(t) = x_i(t)$.

We prove now that $(\rho, q)$ satisfies the conditions of Definition 2.3.

(i) Definition of $a$. We interpolate $u$ to the whole space in the following way: $a(t, \cdot)$ is the bounded continuous function of $x$ defined for each $t \geq 0$ by

$$a(t, x) = \begin{cases} 
    u(t, x) & \text{if } x \in \{x_i(t); 1 \leq i \leq N\}, \\
    \text{affine interpolation} & \text{if } x \in [x_i(t), x_{i+1}(t)], \\
    \text{constant} & \text{if } x \leq x_1(t) \text{ or } x \geq x_N(t).
\end{cases}$$

Notice that $|a| \leq \sup_j |v_j^2|$.

(ii) We can take $\alpha(t) = 1/t$. Indeed a geometrical lemma proved by Grenier in [12] states that, denoting by $y_k(t)$ the $x_i(t)$ strictly ordered, for any $t > 0$

$$\frac{u(t, y_{k+1}(t)) - u(t, y_k(t))}{y_{k+1}(t) - y_k(t)} \leq \frac{1}{t}.$$ 

This result is obtained mainly by noticing that if two particles have different locations, they must never have met in the past. Since the derivative of $a$ is constant on $[y_k(t), y_{k+1}(t)]$, and zero if $x < x_1(t)$ or $x > x_N(t)$, we have indeed $\partial_x(a(t, \cdot)) \leq 1/t$.

(iii) Choice of $\tilde{a}$. One can take $\tilde{a} = a$, which is admissible in the sense of [4,5]. We have in particular for $1 \leq i \leq N$

$$\tilde{a}(t, x_i(t)) = a(t, x_i(t)) = u(t, x_i(t)).$$

(iv) $(\rho, q)$ is a duality solution. We use here the characterization of duality solutions when the coefficient is piecewise continuous, here with only discontinuity lines defined by $t = t_k$, $1 \leq k \leq p$ (see Theorem 4.3.7. in [5]). It suffices to prove that $\rho$ and $q$ are solutions in the sense of distributions. Since $\tilde{a} = u \rho$ a.e., it is equivalent to state that $\rho$ and $q$ are weak solutions, and this can be checked locally on the trajectories and at collisions, the computations are worked out in [3].

(v) Finally, we have $\tilde{a} \rho = u \rho = q$ by (2.10).

Remark 2.5. It is worth noticing here that $\alpha$ can be chosen integrable at 0 (actually bounded), since by (2.9) $\partial_x a(0, \cdot)$ is bounded.

We turn now to uniqueness. Unfortunately, there is no general result in the class of duality solutions, in the sense that we have to specify some additional conditions on the initial data to ensure uniqueness.

Theorem 2.8. (uniqueness for duality solutions) Let $\rho^0, q^0$ be as in Theorem 2.6.

(i) If $\rho^0$ is nonatomic, then there exists a unique duality solution with initial data $\rho^0, q^0$.

(ii) In the general case, there exists at most one duality solution such that $\alpha$ is integrable at 0.
This result deserves a few comments. First, one can say that there are two classes of initial data which lead to uniqueness: “smooth” (case (i) when \( \rho^0 \) is nonatomic), and “entropy” (when there exists a solution with \( \int_0^t \alpha < \infty \)) initial data. Next, we emphasize the fact that both classes are not stable with respect to weak perturbations. This is illustrated by the following examples.

**Example 2.1.** For \( \rho^0 = 2\delta \), \( q^0 \equiv 0 \), one can construct at least three duality solutions:

1. \( \rho(t, x) = 2\delta(x), q(t, x) \equiv 0, \)
2. \( \rho(t, x) = \delta(x - t) + \delta(x + t), q(t, x) = \delta(x - t) - \delta(x + t), \)
3. \( \rho(t, x) = t^{-1} \mathbb{I}_{|x| < t}, q(t, x) = t^{-2} x \mathbb{I}_{|x| < t}. \)

This is the case of “entropy” initial data, the only solution satisfying \( \alpha \in L^1([0,1]) \) is the first one, and corresponds to a single sticky particle. The non stability is clearly evidenced, since by approximating \( \rho^0 \) by \( \delta(x-\varepsilon) + \delta(x+\varepsilon) \) and \( q^0 \) by \( \delta(x-\varepsilon) - \delta(x+\varepsilon), \varepsilon \to 0 \), we get the second solution. Notice finally that these solutions are very similar to those obtained by Majda, Majda and Zheng [16] for the Vlasov equation.

**Example 2.2.** For \( \rho^0 \equiv 1 \) and \( q^0(x) = \text{sgn} x \), there is uniqueness (case (i)), and the solution is given by \( \rho(t, x) = \mathbb{I}_{|x| > t} \) and \( q(t, x) = \mathbb{I}_{|x| > t} \text{sgn} x \). However, \( \alpha(t) = 1/t \) is optimal, the initial data are not “entropic”.

The proof of Theorem 2.8 relies on some kind of equivalence between duality solutions and monotone solutions to (1.3). Namely, we have the following theorem, which will be proved in the next section.

**Theorem 2.9.** (i) Let \((\rho, q)\) be a duality solution to (1.1). Then there exists a Lipschitz continuous function \( f \) and an entropy solution \( M \) to (1.3) such that (1.4) holds and such that \( M(0, \cdot) = M^0 \), any given function satisfying \( d_x M^0 = \rho(0, \cdot) \).

(ii) Moreover, if \( \alpha \) is integrable at 0, one can choose \( f \) depending only on the initial data \( \rho^0, q^0 \) (and on \( M^0 \)).

(iii) If \( \rho^0 \) is nonatomic, \( f \) is uniquely determined up to an additive constant on \( |M^0(-\infty), M^0(+\infty)| \) by \( \rho^0 \) and \( q^0 \).

**Remark 2.6.** A similar argument was used by Serre [19], who considers a model of planar waves in electromagnetism, which leads to transport equations with discontinuous coefficients. He solves this problem by replacing these equations by an infinite set of nonlinear conservation laws, which are obtained in a similar way as in Theorem 2.9.

**Proof of Theorem 2.8.** The result follows easily from assertions (ii) and (iii) of Theorem 2.9. Indeed in both cases the flux \( f \) can be chosen depending only on initial data. Thus the entropy solution \( M \) is uniquely determined by \( \rho^0, q^0 \). Since \((\rho, q)\) satisfies (1.4), it is also uniquely determined. \( \square \)

### 3. Duality solutions and conservation laws

This section is devoted to the study of relationships between duality solutions to pressureless gases and entropy solutions to conservation laws with monotone initial data. Actually,
we generalize the results obtained by Brenier and Grenier in [8]. On the one hand, they proved that the sticky particles dynamics is described by entropy solutions to (1.3)-(1.4). Theorem 2.9, which will be proved below, generalizes this to any duality solution. As stated before, this leads to uniqueness for duality solutions to (1.1).

Conversely, from (1.3)-(1.4), Brenier and Grenier derive weak solutions to (1.1), but without entropy conditions, except for piecewise smooth solutions to the conservation law. We state here a more precise result, which is somewhat a converse to Theorem 2.9.

**Theorem 3.1.** Let $f$ be Lipschitz continuous, $M$ an entropy solution to (1.3) with nondecreasing initial datum $M^0$. Then $(\rho, q) \equiv (\partial_x M, \partial_x [f(M)])$ is a duality solution to (1.1), with some $a$ such that $\|a\|_\infty \leq \text{Lip}(f)$.

An interesting consequence of Theorem 3.1 is a generalization of the classical Oleinik entropy condition [17]. Recall that, if $M$ is an entropy solution to (1.3), with a $C^1$ convex flux $f$, then

\begin{equation}
\partial_x [f'(M(t,x))] \leq 1/t \quad \text{in } [0, \infty) \times \mathbb{R}.
\end{equation}

This optimal estimate has been obtained by Hoff [13]. Notice that in both papers, uniqueness is obtained through a duality argument. We prove the following result, which is in some sense dual to the preceding one: the convexity of $f$ is replaced by the monotonicity of $M$.

**Theorem 3.2.** Consider an entropy solution to (1.3), with nondecreasing initial datum $M^0$, and where $f$ is any Lipschitz continuous function. Then,

(i) there exists $a \in L^\infty([0, \infty) \times \mathbb{R}), \|a\|_\infty \leq \text{Lip}(f)$, satisfying

\begin{equation}
\partial_x a \leq 1/t,
\end{equation}

such that its universal representative $\widehat{a}$ satisfies

\begin{equation}
\partial_x [f(M)] = \widehat{a} \partial_x M.
\end{equation}

(ii) If moreover $f$ is of class $C^1$, then one can choose $a = f'(M)$.

**Remark 3.1.** Relation (3.3) actually implies that the equation

\begin{equation}
\partial_t M + \widehat{a} \partial_x M = 0
\end{equation}

holds in the sense of distributions (and also in the duality sense).

**3.1. Proof of Theorem 2.9.** The proof relies on the following two lemmas.

**Lemma 3.3.** Let $M \in C([0, \infty[; L^\infty_{\text{loc}}(\mathbb{R}) - w^*)$ be a weak solution to (1.3) with nondecreasing initial datum $M^0$, where $f \in \text{Lip}(\mathbb{R})$. Assume that, for any discontinuity point $x_0$ of $M^0$, $f$ lies above its chord between $M^0(x_0^-)$ and $M^0(x_0^+)$.
$M^\circ(x_0+)$. Then $M$ is an entropy solution if and only if it is nondecreasing in $x$.

The proof of this lemma is independent of the remaining of the paper. It follows actually from a result on convex solutions to Hamilton-Jacobi equations. We postpone it until Section 4.

**Lemma 3.4.** Let $Q$ and $M$ be two functions on $\mathbb{R}$ with locally bounded variation, such that $M' \geq 0$ and $|Q'| \leq KM'$ for some $K \geq 0$. Then there exists a Lipschitz continuous function $f$, with $\text{Lip}(f) \leq K$, such that $Q = f(M)$. Moreover, $f$ can be chosen affine on the jump intervals of $M$.

The proof of this lemma is trivial if $M$ is smooth and strictly increasing, and follows from standard regularization arguments in the general case.

**Proof of Theorem 2.9.** We use Remark 2.1: there exist two functions $M$ and $Q$ such that $\partial_x M = \rho$ and $\partial_x Q = q$, which satisfy in the sense of duality on any subinterval $[t_1, t_2] \subset [0, \infty[$

\[ \partial_t M + a \partial_x M = 0, \quad \partial_t Q + a \partial_x Q = 0. \]

Moreover, we can choose $M$ such that $M(0, \cdot) = M^\circ$. Thanks to Remark 2.2, for a given $\varepsilon > 0$, we can apply Lemma 3.4 to $M(\varepsilon, \cdot)$ and $Q(\varepsilon, \cdot)$ with $K = ||a||_\infty$. Thus we obtain a Lipschitz continuous function $f_\varepsilon$ which by construction satisfies the chord condition of Lemma 3.3. By the renormalization properties of Remark 2.1, $f_\varepsilon(M)$ is also a duality solution. Since $Q(\varepsilon, \cdot) = f_\varepsilon(M(\varepsilon, \cdot))$, the uniqueness of duality solutions ensures that $Q(t, \cdot) = f_\varepsilon(M(t, \cdot))$ for any $t \geq \varepsilon$.

Now, thanks to relation (ii) in Definition 2.3, $\partial_x f_\varepsilon M = q = \partial_x Q = \partial_x [f_\varepsilon(M)]$, so that $M$ is a weak solution to (1.3) with flux $f_\varepsilon$ on $[\varepsilon, \infty[$. Since it is nondecreasing, it is the entropy solution by Lemma 3.3. Letting $\varepsilon$ go to 0, we obtain (up to a subsequence in $\varepsilon$) a Lipschitz continuous function $f$ with $\text{Lip}(f) \leq ||a||_\infty$, and $M$ is an entropy solution to (1.3), with flux $f$, on $[0, \infty[$. This proves (i). Notice that since $f$ is defined via a subsequence, it is not determined by the initial data.

To prove (ii), if $\alpha \in L^1([0,1])$, we can choose directly $\varepsilon = 0$ in the preceding construction. Indeed, in this case we can take $t_1 = 0$ in Definition 2.3(i). Thus $f$ can be chosen depending only on the initial data, via Lemma 3.4. Finally, case (iii) follows directly from the fact that, if $\rho^\circ$ is nonatomic, then $f$ is determined up to an additive constant by $\partial_x [f(M^\circ)] = q^\circ$, since $M^\circ$ is continuous and nondecreasing.

**3.2. Proof of Theorem 3.1.** We first prove the result for initial data $M^\circ$ and fluxes $f$ satisfying

- $M^\circ$ is nondecreasing, with a finite number of values

\[ M_0^\circ < M_1^\circ < \ldots < M_N^\circ. \]

We denote by $x_1^\circ < \ldots < x_N^\circ$ its points of discontinuity.
• $f$ is continuous, affine on $[M^0_k, M^0_{k+1}]$, constant on $]-\infty, M^0_0]$ and $[M^0_N, +\infty[$.

We set

$$
\rho^0(x) = \sum_{i=1}^{N} m_i \delta(x - x^0_i) = dx M^0, \quad m_i = M^0_i - M^0_{i-1},
$$

$$
q^0(x) = \sum_{i=1}^{N} m_i v^0_i \delta(x - x^0_i), \quad v^0_i = \frac{f(M^0_i) - f(M^0_{i-1})}{M^0_i - M^0_{i-1}} = f'|M^0_{i-1},M^0_i |.
$$

We notice that by construction $\text{Lip}(f) = \sup |v^0_i|$, and

$$
d_x[f(M^0)] = \sum_{i=1}^{N} \delta(x - x^0_i)[f(M^0_i) - f(M^0_{i-1})] = \sum_{i=1}^{N} \delta(x - x^0_i) m_i v^0_i = q^0.
$$

Therefore $Q^0 \equiv f(M^0)$ satisfies $d_x Q^0 = q^0$. We consider $(\rho, q)$ solution of the sticky particles with initial data $(\rho^0, q^0)$. By Lemma 2.7, $(\rho, q)$ is a duality solution. By Theorem 2.9, we find $\tilde{M}$ entropy solution to (1.3) such that $\partial_x \tilde{M} = \rho$, $\partial_x [f(\tilde{M})] = q$, and $\tilde{M}(0, \cdot) = M^0$. Since $(\rho, q)$ is built by sticky particles, by Remark 2.5 $\alpha$ is integrable at 0, so that $f$ can be chosen depending only on $M^0$ and $Q^0$. We have only to check that one can choose $f = f$ in Lemma 3.4., which is obvious by construction. Finally, uniqueness of entropy solutions to (1.3) implies that $M = \tilde{M}$, so that $\rho = \partial_x M, q = \partial_x [f(M)]$ and we are done.

The result for any initial data follows by approximating $M^0, f$ by $M^0_n, f_n$ satisfying the preceding assumptions, with $\text{Lip}(f_n) \leq \text{Lip}(f)$, in the same spirit as Dafermos [9] and LeVeque [15], and using the stability of duality solutions. The function $M^0_n$ has to be chosen in such a way that the jumps tend uniformly to 0 when $n \to \infty$, and $f_n$ can be built by affine interpolation of $f$ at these points. \square

**Remark 3.2.** Notice that the above proof gives that $(\rho, q)$ are weak limits of sticky particles. Combining this with Theorem 2.9, we obtain that any duality solution to pressureless gases can be obtained as weak limit of sticky particles.

### 3.3. Entropy condition for monotone solutions

We turn now to the **Proof of Theorem 3.3.2.** Assertion (i) is an immediate consequence of Theorem 3.1. Indeed from equation (1.3), we proceed to the pressureless gases, and the functions $a$ and $\hat{a}$ of the theorem are those of the definition of duality solutions. Remark 3.2 ensures that $\partial_x a \leq 1/t$.

The proof of assertion (ii) is slightly more technical. We first notice that we have on the one hand, using the formulation by pressureless gases, $\partial_x [f(M)] = \hat{a} \partial_x M$. On the other hand, Vol’pert’s calculus on $BV$ functions
[20] asserts that there exists a function \( a_V \) such that \( \partial_x [f(M)] = a_V \partial_x M \). Namely,

\[
a_V(t, x) = \begin{cases} 
  f'(M(t, x)) & \text{if } M(t, \cdot) \text{ is continuous at } x, \\
  \frac{f(M(t, x+)) - f(M(t, x-))}{M(t, x+) - M(t, x-)} & \text{if not.}
\end{cases}
\]

Notice that in particular \( a_V = f'(M) \) almost everywhere. From the two expressions of \( \partial_x [f(M)] \), and since \( \rho = \partial_x M \), we have that \( \widehat{a} = a_V \rho \)-almost everywhere. Since \( a = \widehat{a} \) a.e., we are done if we prove \( \widehat{a} = a_V \) a.e. for the Lebesgue measure, and thus it is enough to prove that \( \rho \) is locally bounded from below.

Assume moreover that \( f \) and \( M^0 \) are smooth and satisfy \( \rho^0 \equiv d_x M^0 \geq \nu \) for some \( \nu > 0 \). Then the entropy solution \( M \) to (1.3) is also smooth for small times, and satisfies \( \partial_x M = \rho \geq \nu' > 0 \), for \( t \leq t_0, t_0 \) small enough. For \( t > t_0 \), we consider \( \rho = \partial_x M \) as the duality solution to \( \partial_t \rho + \partial_x (a \rho) = 0 \) with \( \rho(t_0, \cdot) \) as initial datum. Since \( \rho(t_0, \cdot) \geq \nu' > 0 \) and \( \partial_x a \leq 1/t \) for \( t > t_0 \), by the lower estimate stated in [5] (Proposition 4.2.9), we have for any \( t > t_0 \)

\[
\rho(t, \cdot) \geq \nu' t_0 / t.
\]

Hence for such initial data, the result holds. For general initial data, we approximate \( \rho^0 \) by \( \rho_\varepsilon^0 = \rho^0 \ast \zeta_\varepsilon + \varepsilon \) and \( f \) by \( f_\varepsilon = f \ast \zeta_\varepsilon \), where \( \zeta_\varepsilon \) is a sequence of standard mollifiers, and \( \varepsilon > 0 \). Assertion (ii) follows by stability. \( \square \)

4. Convex solutions to Hamilton-Jacobi equations. This section is devoted to the proof of Lemma 3.3, which is a consequence of the following theorem.

**Theorem 4.1.** Let \( v \in \text{Lip}_{loc}([0, \infty) \times \mathbb{R}) \) be a weak solution to

\[
\partial_t v + f(\partial_x v) = 0 \quad \text{in } [0, \infty) \times \mathbb{R},
\]

where \( f \) is a Lipschitz continuous function. Assume that \( v^0 = v(0, \cdot) \) is convex and that \( f \) satisfies the chord condition

\[
\text{if } x_0 \text{ is a discontinuity point of } dv^0 / dx, \text{ } f \text{ lies above its chord between } dv^0 / dx(x_0-) \text{ and } dv^0 / dx(x_0+).
\]

Then \( v \) is a viscosity solution if and only if \( v \) is convex in \( x \).

Lemma 3.3 follows from this theorem by integrating (1.3) in \( x \), and using the fact that, if \( v \) is a viscosity solution to (4.1), then \( u = \partial_x v \) is an entropy solution to (1.3).

Theorem 4.1 makes more precise some results obtained by Barles [2] for any space dimension. It is known that the viscosity solution is convex. The proof of the converse consists in showing that any convex solution to (4.1) must coincide with the function

\[
V(t, x) = \sup_{z \in \mathbb{R}} \inf_{y \in \mathbb{R}} \{ -tf(z) + (x - y)z + v^0(y) \}.
\]
This function was introduced in [14] by Hopf, who proved that $V$ is a generalized solution to (4.1). Our approach gives an alternative proof of this fact. Moreover, since the viscosity solution is convex, it coincides also with $V$ (this was proved by Bardi and Evans [1]).

Notice that, if $v^0$ is $C^1$, condition (4.2) is trivially satisfied. Otherwise, (4.2) expresses that the discontinuities of $dv^0/dx$ are “entropic”. Condition (4.2) is encountered in the resolution of the Riemann problem with non convex fluxes.

**Lemma 4.2.** Let $v$ be a weak solution to (4.1), convex in $x$. Then $v$ is convex with respect to $(t,x)$.

**Proof.** It relies on the $BV$ calculus, as defined by Vol’pert [20], and generalized by Dal Maso, LeFloch, Murat [10], and is closely related to the proof in [8] giving that monotone entropy solutions to (1.3) lead to weak solutions to (1.1). Let $W \in BV_{loc}([0,\infty[ \times \mathbb{R})$, and $f$ be Lipschitz continuous. Then there exists a scalar measurable function $a_V(t,x)$, bounded, which satisfies in the sense of distributions

$$\partial_t[f(W)] = a_V \partial_t W, \quad \partial_x[f(W)] = a_V \partial_x W.$$

Let $W = \partial_x v$. Since $v$ is convex in $x$, $W$ is nondecreasing with respect to $x$. Since $W$ is bounded, we deduce that $W \in B([0,\infty[\cap BV_{loc}(\mathbb{R}))$. Therefore, by differentiating (4.1) in $x$, we obtain $\partial_t W = -\partial_x[f(W)]$, so $\partial_t W \in L^\infty_{loc}([0,\infty[,\mathcal{M}_{loc}(\mathbb{R}))$. We can therefore take in (4.4) $W = \partial_x v \in BV_{loc}$. We differentiate equation (4.1) with respect to $t$, then with respect to $x$. By using again (4.1), we easily obtain the Hessian matrix of $v$

$$D^2_{tx} v = \begin{pmatrix} \frac{a_V}{a_V} & -\frac{a_V}{1} \\ -\frac{a_V}{a_V} & 1 \end{pmatrix} \partial^2_{xx} v.$$

The result follows since $\partial^2_{xx} v$ is a non-negative measure. □

This result enables to use the following convexity inequalities,

$$\begin{align*}
\partial_t v(t,x) &\leq v^0(y) + t\partial_t v(t,x) + (x-y)\partial_x v(t,x), \\
\partial_x v(t,x) &\geq v^0(y) + tp_x + (x-y)p_x,
\end{align*}$$

where $\partial v(0,y)$ stands for the subdifferential of $v$ at $(0,y)$. This set has to be precised before we can conclude, and it is the purpose of the next two lemmas.

**Lemma 4.3.** If $x_0$ is a continuity point of $u^0 = dv^0/dx$, then when $x \to x_0$ and $t \to 0$,

$$v(t,x) = v^0(x_0) + (x-x_0)u^0(x_0) - tf(u^0(x_0)) + o(|x-x_0| + t).$$

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Remark 4.1. This lemma actually states that $v$ is differentiable at $(t = 0, x_0)$, and that $\partial_t v(0, x_0) = -f(\partial_x v(0, x_0))$, which means that (4.1) holds true at $t = 0$. It has to be compared to Lemma I.1 in [2], where a similar result is obtained in the more general setting of supersolutions to (4.1). Indeed, Barles proves that if $x_0$ is a continuity point of $u^0$, then for all $(t, x) \in ]0, \infty[ \times \mathbb{R}$,

$$v(t, x) \geq v^0(x_0) + (x - x_0)u^0(x_0) - tf(u^0(x_0)).$$

This means that in some sense the inequality defining supersolutions is satisfied at $t = 0$.

Proof of Lemma 4.3 - 1st step. Set $u(t, x) = \partial_x v(t, x)$, interpolated in a monotone way at discontinuity points. We first prove that $u(t, x) \to u^0(x_0)$ when $x \to x_0$ and $t \to 0$. Since $u \in B([0, \infty[; L^1_{\text{loc}}(\mathbb{R}))$ by the proof of Lemma 4.2, and $u = \partial_x v \in C([0, \infty[, D')$, we have $u \in C([0, \infty[, L^1_{\text{loc}}(\mathbb{R}))$.

Therefore, we can find a sequence $t_n \to 0$ such that $u(t_n, x) \to u^0(x)$ a.e. Let $A = \{x \in \mathbb{R} \text{ s.t. convergence holds}\}$. Take $\varepsilon > 0$, since $u^0$ is continuous at $x_0$, one can find $\eta > 0$ such that $|x - x_0| \leq 2\eta$ implies $|u^0(x) - u^0(x_0)| \leq \varepsilon/2$.

Now, if we choose $x_1 \in [x_0 - 2\eta, x_0 - \eta]\cap A$ and $x_2 \in [x_0 + \eta, x_0 + 2\eta]\cap A$, then $\exists N > 0$ such that, for $n \geq N$, we have $|u(t_n, x_j) - u^0(x_j)| \leq \varepsilon/2$, $j = 1, 2$.

Putting things together and using the monotonicity of $u$, we obtain, if $n \geq N$ and $|x - x_0| \leq \eta$,

$$u^0(x_0) - \varepsilon \leq u^0(x_1) - \frac{\varepsilon}{2} \leq u(t_n, x_1) \leq u(t_n, x),$$

$$u(t_n, x) \leq u(t_n, x_2) \leq u^0(x_2) + \frac{\varepsilon}{2} \leq u^0(x_0) + \varepsilon.$$

In other words, $\forall \varepsilon > 0$, $\exists \eta > 0$, $\exists N > 0$, such that $|x - x_0| \leq \eta$ and $n \geq N \Rightarrow |u(t_n, x) - u^0(x_0)| \leq \varepsilon$. Since for any sequence of times that tend to 0 we can find a subsequence such that the convergence holds, we are done.

2nd step. Let $\varphi(t, x) = v(t, x) - v^0(x_0) - (x - x_0)u^0(x_0) + tf(u^0(x_0))$. This is a Lipschitz continuous function, and $\varphi(0, x_0) = 0$. We can write

$$|\varphi(t, x)| \leq (t + |x - x_0|) \sup_{s, y \in [0, t] \times [x_0, x]} |\nabla \varphi(s, y)|.$$

But we have almost everywhere

$$\partial_x \varphi(s, y) = \partial_x v(s, y) - u^0(x_0), \quad \partial_t \varphi(s, y) = -f(\partial_x v(s, y)) + f(u^0(x_0)),$$

so that both terms are controlled by the first step and the Lipschitz constant of $f$. \(\square\)
Lemma 4.4. If \( x_0 \) is a discontinuity point of \( u^0 \), then
\[
\left\{ (p_t,p_x) \in \mathbb{R}^2 : p_x \in \partial v^0(x_0), \ p_t = -\left( \frac{p_x - u^0_+}{u^0_+ - u^0_-} f(u^0_+) + \frac{u^0_+ - p_x}{u^0_+ - u^0_-} f(u^0_-) \right) \right\} 
\subset \partial v(0,x_0),
\]
where \( \partial v^0(x_0) \) is the subdifferential of \( v^0 \) at \( x_0 \), and \( u^0_\pm = u^0(x_0 \pm) \).

Proof. We only have to prove the inclusion for \((u^0_\pm,-f(u^0_\pm))\), since \( p_t \) is an affine function of \( p_x \) and the inequalities are affine in \( p_t,p_x \). For instance, consider \( u^0_+ \): since the set of discontinuity points is countable, we can approximate \( x_0 \) by a sequence \((x_n)_{n \geq 0}, x_n \geq x_0 \), and \( x_n \) continuity point of \( u^0 \). From Lemma 4.3, \( v \) is differentiable at \((0,x_n)\), so we have the following convexity inequality, for any given \((t,x)\),
\[
v(t,x) \geq v^0(x_n) + (x - x_n) u^0(x_n) - tf(u^0(x_n)).
\]
Letting \( n \) go to \( \infty \), \( u^0(x_n) \to u^0_+ \), so that
\[
v(t,x) \geq v^0(x_0) + (x - x_0) u^0_+ - tf(u^0_+),
\]
which means exactly that \((u^0_+,-f(u^0_+)) \in \partial v(0,x_0) \).

We turn now to the Proof of Theorem 4.1. First, the inequality (4.5) implies obviously that \( v \leq V \) a.e.. To prove the converse inequality, we first rewrite (4.6) for the pairs \((p_t,p_x)\) defined in Lemma 4.4 (which also holds true if \( x_0 \) is a continuity point), then use condition (4.2) on \( f \). Thus we obtain
\[
\forall t \geq 0, \ \forall x, \ \forall y, \ \forall p_x \in \partial v^0(y), \quad v(t,x) \geq v^0(y) + (x - y) p_x - tf(p_x).
\]
Next we notice that if \( z \notin \overline{\partial v^0(\mathbb{R})} \), we have
\[
\inf_{y \in \mathbb{R}} \{ -y z + v^0(y) \} \equiv -v^0*(z) = -\infty.
\]
Thus the Hopf solution \( V \) given by (4.3) can be rewritten
\[
V(t,x) = \sup_{z \in \partial v^0(\mathbb{R})} \{ -tf(z) + xz - v^0*(z) \}
= \sup_{z \in \partial v^0(\mathbb{R})} \{ -tf(z) + xz - v^0*(z) \}.
\]
But \( z \in \partial v^0(\mathbb{R}) \) precisely means that there exists \( y_z \in \mathbb{R} \) such that \( z \in \partial v^0(y_z) \), and thus \( v^0*(z) = y_z z - v^0( y_z) \). From this remark and (4.8), it follows that \( v(t,x) \geq V(t,x) \), which is the desired result. \( \square \)
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