GLOBAL WEAK SOLUTION
OF THE VLASOV-POISSON SYSTEM
FOR SMALL ELECTRONS MASS

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Abstract:
We are studying the existence and weak stability of a Vlasov-Poisson system with two types of particles, in which the electrons are supposed to be at thermal equilibrium. This modifies the source term in the Poisson equation, and estimates in the Marcinkiewicz space $M^3$ for the potential are used to get the strong compactness of approximations using a new regularized kernel which preserves an appropriate energy inequality.

Key-words:
Vlasov-Poisson equations — approximated problem — positivity in the Fourier variable — Marcinkiewicz spaces.

Introduction.
We are interested in the evolution of a plasma formed by charged ions (charge $+q$, mass $m_+$) and electrons (charge $-q$, mass $m_-$). Particles are localized in a bounded open set $\Omega$ in $\mathbb{R}^3$.
At time $t$, the number of particles has a density in the phase space $\Omega \times \mathbb{R}^3$ ($x \in \Omega$, $v \in \mathbb{R}^3$); $f_+(t,x,v)$ and $f_-(t,x,v)$ $t \geq 0$. Neglecting the magnetic field, $f_+$ and $f_-$ satisfy the following equations

\[
\begin{cases}
\frac{\partial f_+}{\partial t} + v \cdot \nabla_x f_+ + \frac{q}{m_+} \text{div}_v (Ef_+) = Q_+(f_+) + Q_\pm(f_+, f_-), & f_+ \geq 0, \\
\frac{\partial f_-}{\partial t} + v \cdot \nabla_x f_- - \frac{q}{m_-} \text{div}_v (Ef_-) = Q_-(f_-) + Q_\pm(f_-, f_+), & f_- \geq 0, \\
E = -\nabla_x \phi, & -\varepsilon_0 \Delta_x \phi = \rho_+ - \rho_-, \\
\rho_+ = q \int_{\mathbb{R}^3} f_+ \, dv, & \rho_- = q \int_{\mathbb{R}^3} f_- \, dv.
\end{cases}
\]

The $Q$ operators describe the collisions, $E(t,x)$ is the electric field, $\phi(t,x)$ is the electric potential. These equations are completed by boundary conditions for $\phi$, $f_+$ and $f_-$ and Cauchy data

\[
\begin{cases}
f_+(0,x,v) = f_{+0}(x,v), \\
f_-(0,x,v) = f_{-0}(x,v).
\end{cases}
\]
Considering that \( m_- \ll m_+ \), electrons travel faster than ions after collisions; so they rapidly reach their thermal equilibrium and we obtain the "small mass" approximation

\[
  f_- = D \exp \left( -\frac{m_- |v|^2}{2} - q\phi \right) / kT_-,
\]

where \( T_- \) is the temperature of electrons, \( D \) is a positive constant which can eventually be determined by the conservation of electrons

\[
  \iint_{\mathbb{R}^6} f_-(t, x, v) \, dx \, dv = \iint_{\mathbb{R}^6} f_0(x, v) \, dx \, dv.
\]

So we can consider a system on the density \( f_+ \) only. Notice that if \( E = -\nabla u \) is given, we can write \( \phi(t, x) = u(t, x) + V(t) \) and

\[
  f_- = \frac{\exp \left( -\frac{m_- |v|^2}{2} - qu \right) / kT_-}{\iint \exp \left( -\frac{m_- |v|^2}{2} - qu \right) / kT_- \, dx \, dv} \iint f_0 \, dx \, dv. \tag{1}
\]

In order to simplify the problem, we will now take \( \Omega = \mathbb{R}^3 \), and introduce an external force \( h(x) \) \( (h(x) = -\nabla H(x)) \) which prevents electrons from going to infinity. Then, the equation on \( f_- \) becomes

\[
  \frac{\partial f_-}{\partial t} + v \nabla_x f_- + \frac{1}{m_-} \text{div}_v \left[ (-qE + h) f_- \right] = Q_-(f_-) + Q_+(f_-, f_+), \tag{F}
\]

and the "small mass" approximation (1) is now

\[
  f_-
  = \frac{\exp \left( -\frac{m_- |v|^2}{2} - qu + H \right) / kT_-}{\iint \exp \left( -\frac{m_- |v|^2}{2} - qu + H \right) / kT_- \, dx \, dv} \iint f_0 \, dx \, dv, \tag{2}
\]

\[
  \rho_- = q \iint f_0 \, dx \, dv \frac{de^{\frac{\phi}{H}}}{de^{\frac{\phi}{H}}} \quad d(x) = e^{-\frac{H(x)}{H}}.
\]

Neglecting ions collisions we finally get the system (to be rigorous we should also consider the external force on ions but we will not do so)

\[
  \begin{cases}
    \frac{\partial f_+}{\partial t} + v \nabla_x f_+ + \frac{q}{m_+} \text{div}_v (Ef_+) = 0, \quad f_+ \geq 0, \\
    f_+(0, x, v) = f_{+0}(x, v), \\
    \begin{aligned}
      & E = -\nabla u, \quad -\varepsilon_0 \Delta u = \rho_+ - q \iint f_- \, dx \, dv \frac{de^{\frac{\phi}{H}}}{de^{\frac{\phi}{H}}} \quad d(x) = e^{-\frac{H(x)}{H}}, \\
      & \rho_+ = q \int f_+ \, dv.
    \end{aligned}
  \end{cases} \tag{SMF}
\]

Such an approximation is classical in plasma physics; see M. CASANOVA, O. LARROCHE, J.P. MATTE [5] and the reference therein or the mathematical analysis in G. REIN [18] for a related problem.

If we do not consider the term coming from electrons in (SMF), we find the usual Vlasov-Poisson system, for which we know several results (see A. ARSENEV [1], C. BARDOS, P. DEGOND [2], J. BATT [3], R.J. DI Perna, P.L. Lions [5][9], E. HORST [13], E. HÖRST, R. HUNZE [14], R. Illner, H. Neunzert [15]). In this paper we are interested in the mathematical analysis of the system (SMF) and we prove the global existence of weak solutions, we also introduce an appropriate approximated problem with good energy estimates (i.e. every term of the energy is lower bounded). We will see that \( f |E|^2 \) is replaced
MAIN RESULT

by a positive quadratic form $q$ (see section III). This approach differs from those introduced by J. Bätt. This problem is obtained by regularizing the initial data $f_0$ and the elliptic equation, to be able to use classical methods to find strong solutions of the modified equation. We show that it is possible to pass to the limit in this system. The way to control the non-linear term is based on the simple fact that it is bounded in $L^1$. Actually the elliptic equation leads to a bound in the Marcinkiewicz space $M^3$ for $u$.

The outline of the paper is the following: we first state our main result, then we give in section II some results on the elliptic equation and pass to the limit in section III.

I. Main result.

We take all physical constants equal to 1 in (SMF) and we state our main result.

Theorem 1.

Given

$$d \in L^1(\mathbb{R}^3) \geq 0; \quad d \neq 0,$$

$$f_0 \in L^\infty(\mathbb{R}^6) \geq 0 \quad \text{with} \quad \int_{\mathbb{R}^6} (1 + |x|^2 + |v|^2) f_0(x,v) \, dx \, dv < \infty,$$

there exists $f(t, x, v) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^6)$, $t \geq 0$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ solution to

$$\frac{df}{dt} + v \cdot \nabla_x f + \text{div}_v (Ef) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3), \quad f \geq 0,$$

$$f(0, x, v) = f_0(x,v),$$

$$E = -\nabla u, \quad -\Delta u = \rho - \frac{de^u}{de^u}, \quad \rho = \int_{\mathbb{R}^3} f \, dv,$$

and it satisfies

$$\int_{\mathbb{R}^6} (1 + |x|^2 + |v|^2) f(t,x,v) \, dx \, dv + \int_{\mathbb{R}^3} |E(t,x)|^2 \, dx \in L^\infty(\mathbb{R}_+),$$

$$u \in L^\infty(\mathbb{R}_+, L^6(\mathbb{R}^3)), \quad \text{ess sup } u < \infty,$$

$$V(t) \equiv -\ln \int_{\mathbb{R}^3} e^{u(t,x)} \, dx \in L^\infty(\mathbb{R}_+).$$

Finally $\phi(t, x) \equiv u(t, x) + V(t)$ satisfies

$$(1 + |\phi|)e^\phi \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3).$$

Remark.

Formally, the total energy

$$\mathcal{E}_1(t) \equiv \int_{\mathbb{R}^6} |v|^2 f(t,x,v) \, dx \, dv + \int_{\mathbb{R}^3} |E(t,x)|^2 \, dx + 2 \int_{\mathbb{R}^3} \phi(t,x)d(x)e^{\phi(t,x)} \, dx$$

satisfies $\frac{d\mathcal{E}_1}{dt} = 0$, but we can only prove that $\mathcal{E}_1(t) \leq \mathcal{E}_1(0)$.

Considering the case where the total charge of electrons is no longer fixed, we have the
Theorem 2. Under assumption (3), there exists a solution \( f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3) \) to the equation (4) with
\[
E = -\nabla u, \quad -\Delta u = \rho - de^u, \quad \rho = \int_{\mathbb{R}^3} f \, dv
\]
It satisfies (6), (7) and the energy estimate
\[
\mathcal{E}_2(t) = \int_{\mathbb{R}^3} |v|^2 f(t, x, v) \, dv + \int_{\mathbb{R}^3} |E(t, x)|^2 \, dx + 2 \int_{\mathbb{R}^3} d(x)(u(t, x) - 1)e^{u(t, x)} \, dx \leq \mathcal{E}_2(0). \tag{11}
\]
(In this case we do not use \( \phi \)).

We will prove theorem 2, and give modifications for theorem 1 in section III.4.

Remarks.
1. The problem corresponding to attractive forces seems not to be solved by the method presented here. The energy has no sign, and the equation
\[
-\Delta u = \rho + \frac{de^u}{f \, dv}
\]
cannot be controlled easily (the increase of \( e^u \) prevents us proving the existence for \( \rho \) given).

2. Regularity.

As in the usual Vlasov-Poisson system, the electric field \( E \) and the potential \( u \) in theorem 1 have the following regularity
\[
E \in L^\infty(\mathbb{R}_+, L^p_{-}(\mathbb{R}^3)), \quad \frac{3}{2} < p \leq \frac{15}{4},
\]
\[
u \in L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^3)), \quad 3 < p \leq \infty,
\]
as soon as we assume \( d \in L^{\frac{3}{2}}(\mathbb{R}^3) \). But we could also prove that with appropriate conditions on \( d \),
\[
E(t, \cdot) \in L^p(\mathbb{R}^3), \quad \text{a.e.} \ t > 0, \quad \frac{3}{2} < p < 5;
\]
\[
\frac{\partial}{\partial t} \phi \in L^\infty(\mathbb{R}_+, L^p(\mathbb{R}_+)), \quad \text{a.e.} \ t > 0, \quad \frac{3}{2} < p < \infty.
\]

3. Renormalized solutions.

Suppose \( d \in L^\infty(\mathbb{R}^3) \). We can prove that \( v \equiv \rho - \frac{de^u}{f \, dv} \) satisfies
\[
\partial_{\infty} v \in C^1(\mathbb{R}^3), \quad |v| + |v| \in L^\infty(\mathbb{R}_+, L^1_{\text{loc}}(\mathbb{R}^3)),
\]
and consequently that
\[
\frac{\partial^2 u}{\partial x_j \partial x_k} \in L^\infty(\mathbb{R}_+, L^1_{\text{loc}}(\mathbb{R}^3)).
\]

The transport theory of R.J. DiPerna, P.L. Lions [7] can then be applied to equation (4) and we can deduce that \( f \in C^1(\mathbb{R}^3, L^1(\mathbb{R}^3)) \).
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Using the same idea, we can consider an initial datum \( f_0 \notin L^\infty \) but satisfying

\[
\begin{align*}
(i) & \quad f_0 \geq 0, \quad \int_{\mathbb{R}^3} (1 + |x|^2 + |v|^2 + \ln^+ f_0(x, v)) f_0(x, v) \, dx \, dv < \infty. \\
(ii) & \quad \text{there exists } u_0 \in L^6(\mathbb{R}^3) \text{ with } \nabla u_0 \in L^2, \quad d(1 + |u_0|) e^{u_0} \in L^1 \text{ such that } \\
& \quad -\Delta u_0 = \int_{\mathbb{R}^3} f_0 \, dv - \frac{de^{u_0}}{\int_{\mathbb{R}^3} de^{u_0}}.
\end{align*}
\]

Then we can find a renormalized solution to the same system of equations (R.J. DiPerna, P.L. Lions [8] did it in the linear case).

4. Passing from the system \((P)\) modified by \((F)\) to the "small mass" approximation.

The question is: can we say that when \( m_- \) is small, \( f_+ \simeq f \) solution of the problem \((SMF)\)?

For example, let us write a Vlasov-Poisson-Fokker-Planck system

\[
(PF) \quad \begin{cases}
\frac{\partial f_+}{\partial t} + v \cdot \nabla_x f_+ + \frac{q}{m_+} \text{div}_v (Ef_+) = 0, \\
\frac{\partial f_-}{\partial t} + v \cdot \nabla_x f_- + \frac{1}{m_-} \text{div}_v [(-qE + h) f_-] = \beta \text{div}_v (vf_-) + \sigma \Delta_v f_-,
\end{cases}
\]

\[
E = -\nabla \phi, \quad -\varepsilon_0 \Delta \phi = \rho_+ - \rho_-.
\]

It is natural to think that \( f_- \) will follow the formula (2) and \( f_+ \) the formula (SMF), where we take \( kT_- = \frac{\sigma}{\beta} m_- \).

For the analysis of this kind of question we refer to K. Dressler [10].

II. Study of the elliptic equation.

This paragraph is dedicated to the uniqueness, existence (for suitable \( \rho \)) and estimates for the solution of the elliptic equation (10) or (5) when the time is fixed. The equation differs from Poisson's equation in the term \( de^u \) in the case of theorem 2 and \( \int de^u \) in the case of theorem 1. All estimates are based on the positivity of \( e^u \).

II.1. Case of theorem 2.

d is given in \( L^0(\mathbb{R}^3) \); with \( d \geq 0 \), and \( \rho \in L^{\frac{6}{5}}(\mathbb{R}^3) \). We first consider the equation

\[
\begin{cases}
-\Delta u = \rho - de^u, \\
u \in L^6(\mathbb{R}^3); \quad \nabla u \in L^2; \quad de^u \in L^1.
\end{cases}
\]

The class where \( u \) is found is justified because if \( u \in \mathcal{C}^\prime(\mathbb{R}^3) \) and \( \nabla u \in L^2(\mathbb{R}^3) \) then there exists a unique constant \( c \) such that

\[
u \in L^{6}(\mathbb{R}^3)
\]

and we have for this \( c \)

\[
||u - c||_{L^6(\mathbb{R}^3)} \leq C_5 ||\nabla u||_{L^2(\mathbb{R}^3)}
\]

(See L. Hörmander [12] theorem 4.5.9.). We can deduce from it that \( B = \{ u \in L^0(\mathbb{R}^3) \mid \nabla u \in L^2 \} \) is a real Hilbert space with inner product

\[
(u, v) = \int_{\mathbb{R}^3} \nabla u \nabla v.
\]
Moreover, if $\varphi \in B \cap L^\infty$ then there exists $(\varphi_n)_n \in C_c^\infty(\mathbb{R}^3)$ with
\[
\begin{cases}
\varphi_n \to \varphi, \\
\|\varphi_n\|_{L^\infty} \leq 1 + \|\varphi\|_{L^\infty}.
\end{cases}
\]

\textbf{Proposition 2.1.}

For every $\rho \in L^\frac{6}{5}(\mathbb{R}^3)$ there exists a unique $u$ solution of (12); and $u$ satisfies
\[
\int_{\mathbb{R}^3} \nabla u \nabla \varphi = \int_{\mathbb{R}^3} \rho \varphi - \int_{\mathbb{R}^3} du \varphi, \quad \varphi \in B \cap L^\infty; \quad (13)
\]
\[
du |u| \in L^1. \quad (14)
\]

Let us begin by demonstrating the uniqueness of $u$.

\textbf{Lemma 2.2.}

Given $u_1$ and $u_2$ solutions of (12) associated with $\rho_1$ and $\rho_2$, we have
\[
\|\nabla (u_1 - u_2)\|_{L^2} \leq C_B \|\rho_1 - \rho_2\|_{L^\frac{6}{5}}.
\]

\textbf{Proof:} Set $w = u_1 - u_2 \in B$. We have
\[
-\Delta w + d(e^{u_1} - e^{u_2}) = \rho_1 - \rho_2
\]
so
\[
\int_{\mathbb{R}^3} \nabla w \nabla \varphi + \int_{\mathbb{R}^3} d(e^{u_1} - e^{u_2}) \varphi = \int_{\mathbb{R}^3} (\rho_1 - \rho_2) \varphi, \quad \varphi \in C_c^\infty.
\]

This identity still remains for $\varphi \in B \cap L^\infty$ because of the density result.

Now we take $\varphi = T_R(w)$ where $T_R$ is the truncating function at height $R$. We get
\[
\int_{\mathbb{R}^3} \mathbb{1}_{|w| \leq R} |\nabla w|^2 + \int_{\mathbb{R}^3} d(e^{u_1} - e^{u_2}) T_R(w) = \int_{\mathbb{R}^3} (\rho_1 - \rho_2) T_R(w).
\]

We have $(e^{u_1} - e^{u_2}) T_R(w) \geq 0$ and using the monotone convergence theorem we obtain when $R \to \infty$
\[
\int_{\mathbb{R}^3} |\nabla w|^2 + \int_{\mathbb{R}^3} d(e^{u_1} - e^{u_2})(u_1 - u_2) = \int_{\mathbb{R}^3} (\rho_1 - \rho_2) w
\]
and
\[
\|\nabla w\|_{L^2}^2 \leq \|\rho_1 - \rho_2\|_{L^\frac{6}{5}} \|w\|_{L^\frac{6}{5}} \leq \|\rho_1 - \rho_2\|_{L^\frac{6}{5}} C_B \|\nabla w\|_{L^2};
\]
\[
\|\nabla w\|_{L^2} \leq C_B \|\rho_1 - \rho_2\|_{L^\frac{6}{5}}.
\]

The uniqueness of $u$ follows easily.

In order to prove the existence of a solution we just notice that the functional
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \rho u + \int_{\mathbb{R}^3} du \in [- \infty; \infty]
\]
is convex and l.s.c., so weakly l.s.c. Thus one can find a sequence $(u_n)_n \in B$ so that $J(u_n) \to \inf J$.

We have $\inf J \leq J(0) < \infty$ (because $d \in L^3$). $J(u_n)$ is upper bounded so $u_n$ is bounded in $B$. After

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an extraction we have \( u_n \in B \) and \( J(u) \leq \lim J(u_n) = \inf J \) so \( u \) is a minimum of \( J \), \( J(u) \in \mathbb{R} \) and \( \int de^u < \infty \) (\( de^u \in L^1 \)). It is classical to deduce that \( u \) satisfies \((12), \) and \((13)\) follows. We are now interested in the regularity of \( u \). \( \varphi = |T_R(u)| \) gives in \((13)\)

\[
\int_{\mathbb{R}^3} (1_{0 < u < R} - 1_{-R < u < 0}) |\nabla u|^2 = \int_{\mathbb{R}^3} \rho |T_R(u)| - \int_{\mathbb{R}^3} de^u |T_R(u)|.
\]

Putting \( R \to \infty \) we get

\[
\int_{\mathbb{R}^3} (1_{u > 0} - 1_{u < 0}) |\nabla u|^2 = \int_{\mathbb{R}^3} \rho |u| - \int_{\mathbb{R}^3} de^u |u|,
\]

\[
\int_{\mathbb{R}^3} de^u |u| \leq \int_{\mathbb{R}^3} |\nabla u|^2 + \|\rho\|_{L^\theta} \|u\|_{L^\theta},
\]

which proves \((14)\). We have also

\[
\int_{\mathbb{R}^3} de^u = \int_{u \leq 1} de^u + \int_{u > 1} de^u
\]

\[
\leq e \int_{\mathbb{R}^3} d + \int_{u > 1} de^u
\]

\[
\leq 2e \int_{\mathbb{R}^3} d + \int_{\mathbb{R}^3} de^u u.
\]

These estimates are consistent since we know that \( u \) is bounded in \( B \) when \( \rho \) is. To prove it take \( \varphi = T_R(u) \) in \((13)\). We obtain

\[
\int_{\mathbb{R}^3} 1_{|u| \leq R} |\nabla u|^2 = \int_{\mathbb{R}^3} \rho T_R(u) - \int_{\mathbb{R}^3} de^u T_R(u)
\]

so when \( R \to \infty \) we get

\[
\int_{\mathbb{R}^3} |\nabla u|^2 = \int_{\mathbb{R}^3} \rho u - \int_{\mathbb{R}^3} de^u u
\]

\[
\leq \|\rho\|_{L^\theta} \|u\|_{L^\theta} + e^{-1} \int_{\mathbb{R}^3} d
\]

\[
\leq \|\rho\|_{L^\theta} C_B \|\nabla u\|_{L^1} + e^{-1} \int_{\mathbb{R}^3} d
\]

and

\[
\|\nabla u\|_{L^1} \leq \frac{1}{2} C_B \|\rho\|_{L^\theta} + \sqrt{e^{-1} \int_{\mathbb{R}^3} d + \frac{1}{4} C_B^2 \|\rho\|_{L^\theta}^2}
\]

\[
\leq C_B \|\rho\|_{L^\theta} + \sqrt{\int_{\mathbb{R}^3} d}.
\]

Notice the connection between equation \((12)\) and its physical expression: \( u \) is the electric potential created by charges distributed in the space. This means that the equation \((12)\) is equivalent to

\[
u = \varepsilon \ast (\rho - de^u)
\]

where \( \varepsilon(x) = \frac{1}{4\pi \|x\|} \).

Proposition 2.3.

If \( \rho \in L^\theta \cap L^p \) with \( \frac{2}{\theta} < p \) then the solution \( u \) of \((12)\) satisfies

\[
\text{ess sup } u \leq C(p) \|\rho\|_{L^\theta}^{1-\theta} \|\rho\|_{L^p}^\theta.
\]

where \( \theta = \frac{1}{5 - \frac{2}{p}} \).
This bound is closely related with the positivity of $de^u$.

Proof: $u = e \ast (\rho - de^u) \leq e \ast \rho; \; e \ast \rho \in C_0$ and

$$\|e \ast \rho\|_{C_0} \leq C(p) \|\rho\|_{L^\theta} \|\rho\|_{L^{\frac{1}{\theta}}}^{1 - \theta}$$

(write $e(x) = e(x)\mathbb{1}_{|x| \leq R} + e(x)\mathbb{1}_{|x| > R}$ and optimize in $R$.) □

Remark: when

$$\rho(x) = \int_{\mathbb{R}^3} f(x, v) \, dv \geq \begin{cases} f \in L^1 \cap L^\infty(\mathbb{R}^3) \\ |v|^2 f \in L^1(\mathbb{R}^3) \end{cases}$$

then we can choose $p = \frac{1}{2}$ and $\theta = \frac{1}{2}$. Indeed the decomposition

$$\rho(x) \leq \frac{1}{R^2} \int |v|^2 f \, dv + \frac{4}{3} \pi R^3 \|f\|_{L^\infty}$$

implies that $\rho \in L^{\frac{3}{2}}$ by choosing $R(x)$ in an optimal way.

II.2. Case of theorem 1.

We now suppose $d \neq 0$. We are interested in the equation ($\rho \in L^1(\mathbb{R}^3)$ is given)

$$\begin{cases} -\Delta u = \rho - \frac{de^u}{\int de^u} , \\
u \in L^6; \; \nabla u \in L^2; \; de^u \in L^1. \end{cases} \tag{15}$$

When $\rho = 0$ and $d = 1$, this equation has been studied by various authors in the context of Vlasov-Poisson: F. Bavaud [4], L. Desvillettes, J. Dolbeault [6], D. Gogny, P.L. Lions [11]. We are just going to give the main technical results that we will use later on. The main difficulty is that $\int de^u$ could be close to 0, thus $\frac{de^u}{\int de^u}$ could be close to a $\delta$ mass. We are going to show that an estimate in $M^3$ is enough to avoid this difficulty.

Proposition 2.4.

For every $\rho \in L^1 \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ there exists a unique solution $u$ to the problem

$$\begin{cases} u \in L^6; \; \nabla u \in L^2; \; de^u \in L^1, \\
-\Delta u = \rho - \frac{de^u}{\int de^u}, \end{cases} \tag{15}$$

and there exists a constant $C(d) > 0$ such that for this $u$

$$\int_{\mathbb{R}^3} de^u \geq Ce^{-\frac{1}{\theta}} \|\rho\|_{L^1}^{\frac{1}{\theta}} \tag{16}$$

and

$$\text{ess sup } u \leq C \|\rho\|_{L^3} \|\rho\|_{L^1}^{\frac{1}{2}}. \tag{17}$$
Lemma 2.5.

Suppose given \( u, v \in L^6(\mathbb{R}^3) \) and \( 0 < \lambda \leq \mu \) satisfying
\[
\begin{align*}
\nabla u, \nabla v & \in L^2, \\
de^u, de^v & \in L^1, \\
-\Delta u + \frac{de^u}{\lambda} & = -\Delta v + \frac{de^v}{\mu};
\end{align*}
\]
then
\[
0 \leq v - u \leq \ln \mu - \ln \lambda.
\]
If in addition \( \frac{1}{\lambda} \int de^u = \frac{1}{\mu} \int de^v \) then \( u = v \).

Proof: It is a weak maximum principle. We clearly have
\[
\int_{\mathbb{R}^3} (v - u) \nabla \varphi = \int_{\mathbb{R}^3} \left( \frac{de^u}{\lambda} - \frac{de^v}{\mu} \right) \varphi, \quad \varphi \in C_c(\mathbb{R}^3).
\]
Taking \( \varphi = h(v - u) \) with \( h(t) = \begin{cases} 1 & \text{if } t \leq -1 \\ -t & \text{if } -1 \leq t \leq 0 \\ 0 & \text{if } t \geq 0 \end{cases} \), \( \nabla \varphi = -1_{-1 \leq v - u \leq 0} \nabla (v - u) \) so we obtain
\[
-\int_{\mathbb{R}^3} |\nabla [h(v - u)]|^2 = \int_{\mathbb{R}^3} \left( \frac{de^u}{\lambda} - \frac{de^v}{\mu} \right) h(v - u).
\]
But \( h \geq 0 \) and \( 1_{-1 \leq u \leq 0} \left( \frac{de^u}{\lambda} - \frac{de^v}{\mu} \right) \geq 0 \) because \( \lambda \leq \mu \) so \( \nabla [h(v - u)] = 0 \) and \( h(v - u) \) is a constant.
Knowing \( h(v - u) \in L^6 \) we conclude that \( h(v - u) = 0 \) and then that \( v - u \geq 0 \).

Similarly, taking \( \varphi = k(v - u) \) with \( k(t) = \begin{cases} 0 & \text{if } t \leq \ln \frac{\mu}{\lambda} \\ t - \ln \frac{\mu}{\lambda} & \text{if } \ln \frac{\mu}{\lambda} \leq t \leq 1 + \ln \frac{\mu}{\lambda} \\ 1 & \text{if } 1 + \ln \frac{\mu}{\lambda} \leq t \end{cases} \) we get \( k(v - u) = 0 \) and \( v - u \leq \ln \frac{\mu}{\lambda} \).

If now \( \frac{1}{\lambda} \int de^u = \frac{1}{\mu} \int de^v \) then \( \frac{1}{\lambda} de^u = \frac{1}{\mu} de^v \) (because \( \frac{de^v}{\mu} - \frac{de^u}{\lambda} \leq 0 \)). We deduce from this that
\[
-\Delta (v - u) = 0 \quad \text{and then that } v - u = 0.
\]

A straightforward consequence of lemma 2.5. is that, given \( \rho \in \mathcal{D}'(\mathbb{R}^3) \), there exists at the most one solution \( u \) to problem (15).

We are now interested in an a priori estimate which will be decisive to get a lower bound on \( \int de^u \).
When \( \rho \in L^1 \) and \( -\Delta u = \rho - \frac{de^u}{\int de^u} \) we know that \( -\Delta u \in L^1 \) so \( u \in M^p(\mathbb{R}^3) \). (See L. Hörmander [12] lemma 4.5.7.) The Marcinkiewicz space \( M^p(\mathbb{R}^3) \) is defined for \( 0 < p < \infty \) by
\[
M^p(\mathbb{R}^3) = \left\{ u : \mathbb{R}^3 \to \mathbb{R} \text{ measurable, defined a.e. such that } \|u\|_{M^p} = \sup_{r > 0} m(\{|u| > r\})^{\frac{1}{p}} \right\}
\]
and \( \| \cdot \|_{M^p(\mathbb{R}^3)} \) is a pseudo-norm on \( M^p \).
So it is interesting to estimate \( \int de^u \) in terms of \( \|u\|_{M^p} \).
The result is based on the lemma

**Lemma 2.6.**

Let \( g \in L^1(\mathbb{R}^3) \geq 0 \) with \( \int g = 1 \) and \( g \leq M \); \( u \in M^3(\mathbb{R}^3) \) with \( ||u||_{M^3} \leq N \), for some constants \( M, N > 0 \). Then the inequality holds

\[
\int_{\mathbb{R}^3} ge^{-|u|} \geq \frac{1}{27} e^{-3NM^{\frac{1}{2}}}.
\]

**Proof:** set \( d\mu = g \, dx \). \( \mu \) is a positive measure and \( \mu(\mathbb{R}^3) = 1 \).

The estimate is a variant of Jensen inequality. If \( u \) were in \( L^3(\mathbb{R}^3) \) we could write with \( \varphi(t) = e^{-t^{\frac{3}{2}}} \)

\[
e^{-\|u\|_3 M^{\frac{3}{2}}} \leq \varphi \left( \int_{\mathbb{R}^3} |u|^3 \, d\mu \right) \leq \int_{\mathbb{R}^3} \varphi(|u|^3) \, d\mu = \int_{\mathbb{R}^3} ge^{-|u|} \, dx.
\]

Now in the case \( u \in M^3 \); for \( r, s > 0 \) we can write

\[
I = \int_{\mathbb{R}^3} ge^{-|u|} \geq \int_{|u| \geq e^{-s}} ge^{-|u|} \geq e^{-s} \mu(|u| \leq s),
\]
and by convexity of \( t \mapsto e^{-t} \),

\[
e^{-s} \mu(|u| \leq s) + e^{-r} \mu(|u| > s) \geq \exp \left[ s \mu(|u| \leq s) + r \mu(|u| > s) \right] \\
\geq \exp \left[ s + r M m(|u| > s) \right] \\
\geq \exp \left[ s + \frac{rMN^3}{s^3} \right],
\]

so \( I \geq e^{-s + \left( \frac{rMN^3}{s^3} \right)} - e^{-r} \). Take \( r = \frac{s^4}{MN^3} \). we obtain \( I \geq e^{-2s} - e^{-\frac{s^4}{MN^3}} \). Now take \( s = 3^{\frac{1}{2}} NM^{\frac{1}{2}} \).

\[
I \geq e^{-2 \cdot 3^{\frac{1}{2}}NM^{\frac{1}{2}}} - e^{-3NM^{\frac{1}{2}}} \\
\geq (3 - 2 \cdot 3^{\frac{1}{2}})NM^{\frac{1}{2}}e^{-3NM^{\frac{1}{2}}} \\
\geq \frac{1}{3} NM^{\frac{1}{2}} e^{-3NM^{\frac{1}{2}}}
\]

If \( NM^{\frac{1}{2}} \geq \frac{1}{2} \) we obtain the inequality.

If \( NM^{\frac{1}{2}} \leq \frac{1}{2} \) we can increase \( NM^{\frac{1}{2}} \) till \( \frac{1}{2} \) so \( I \geq \frac{1}{9} e^{-1} \geq \frac{1}{27} \). \( \Box \).

**Proposition 2.7.**

There exists a constant \( C(d) > 0 \) such that

\[
\int_{\mathbb{R}^3} de^{-|u|} \geq Ce^{-\frac{1}{2}||u||_{M^3}}, \quad u \in M^3(\mathbb{R}^3).
\]

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Proof: \( d \neq 0 \) so there exists \( R > 0 \) such that \( \int_{1 \leq R} d > 0 \). Take

\[
\begin{align*}
g &= \frac{1_{d \leq R}}{\int 1_{d \leq R} d} \quad \text{and} \quad M = \frac{R}{\int 1_{d \leq R} d},
\end{align*}
\]

Lemma 2.6. gives the inequality with

\[
C = \min \left( \frac{1}{27} \int 1_{d \leq R} d, \frac{1}{3M^4} \right).
\]

Corollary 2.8.

There exists a constant \( C(d) > 0 \) such that if \( \rho \in L^1(\mathbb{R}^3); \lambda > 0; \ u \in L^6 \) with \( \nabla u \in L^2 \) and \( d^u \in L^1 \)

\[
-\Delta u = \rho - \frac{1}{\lambda} d^u,
\]

then

\[
\int_{\mathbb{R}^3} d^{e^{u}} \geq e^{-\frac{C}{27} (\|\rho\|_{L^1} + \frac{1}{\lambda} \int d^u}).
\]

Proof: \( u = e \ast (\rho - \frac{1}{\lambda} d^u) \) so \( u \in M^3 \) and

\[
\|u\|_{M^3} \leq C_d \left\| \rho - \frac{1}{\lambda} d^u \right\|_{L^1}.
\]

(See L. Hörmander [12] lemma 4.5.7.) Proposition 2.7. gives the result.

We now come back to proposition 2.4. We could prove it using the functional

\[
u \mapsto \int_{\mathbb{R}^3} |\nabla v|^2 - \int_{\mathbb{R}^3} \rho v + \ln \int_{\mathbb{R}^3} d^v,
\]

but we here use the results of case 1.

Proof of proposition 2.4. The uniqueness has already been treated. For the existence proof, consider

\[
\begin{align*}
f : [0; \infty[ & \longrightarrow [0; \infty[, \\
\lambda & \longmapsto \int_{\mathbb{R}^3} d^{u^\lambda},
\end{align*}
\]

where \( u^\lambda \) is solution of

\[
\begin{align*}
&\left\{ u^\lambda \in L^6; \nabla u^\lambda \in L^2; \ d^{u^\lambda} \in L^1; \\
&-\Delta u^\lambda = \rho - \frac{1}{\lambda} d^{u^\lambda}
\end{align*}
\]

(See case 1). We want to prove that \( f \) has a fixed point (we already know that it is unique).

Suppose \( \lambda \leq \mu \). By lemma 2.5. we get

\[
0 \leq u_\mu - u_\lambda \leq \ln \mu - \ln \lambda
\]

so

\[
\begin{align*}
&\left\{ e^{u^\lambda} \leq e^{u_\mu}, \\
&\frac{e^{u^\mu}}{\lambda} \leq \frac{e^{u^\lambda}}{\lambda},
\end{align*}
\]

and

\[
\begin{align*}
&f(\lambda) \leq f(\mu), \\
&\frac{f(\mu)}{\lambda} \leq \frac{f(\lambda)}{\lambda}.
\end{align*}
\]

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We conclude from this that
\[ \lambda \mapsto f(\lambda) \] is non-decreasing,
\[ \lambda \mapsto \frac{f(\lambda)}{\lambda} \] is non-increasing.

We can write for \( \lambda \leq \mu \)
\[ 0 \leq f(\mu) - f(\lambda) \leq \frac{f(\lambda)}{\lambda} (\mu - \lambda). \]

\( \lambda \mapsto \frac{f(\lambda)}{\lambda} \) is non-increasing, so is locally bounded and consequently \( f \) is locally Lipschitzian, and therefore continuous. Moreover,
\[ f(\lambda) = \int_{\mathbb{R}^3} e^{u_\lambda} \geq \int_{\mathbb{R}^3} e^{-|u_\lambda|} \geq C e^{-\frac{1}{2} (\|\rho\|_4 + \frac{\|\rho\|_{\frac{4}{3}}}{\lambda})}. \] (see 2.8.)

If \( \frac{f(\lambda)}{\lambda} \) were bounded, we could write \( f(\lambda) \geq \alpha > 0 \) and then \( \frac{f(\lambda)}{\lambda} \geq \frac{\alpha}{\lambda} \) which is not bounded. We conclude from this that \( \frac{f(\lambda)}{\lambda} \) is not bounded; and
\[ \frac{f(\lambda)}{\lambda} \xrightarrow[\lambda \to 0]{} \infty. \]

But \( \rho \in L^\frac{3}{2} \) so by the upper bound on \( u_\lambda \) in proposition 2.3.,
\[ f(\lambda) = \int_{\mathbb{R}^3} e^{u_\lambda} \leq \left( \int d \right) e^{C \|\rho\|_4 \frac{1}{2} \|\rho\|_{\frac{4}{3}} L^{\frac{3}{2}}}, \]
and \( f \) is bounded; \( \frac{f(\lambda)}{\lambda} \xrightarrow[\lambda \to 0]{} 0 \). Using the continuity, we find \( \lambda_0 \) such that \( \frac{f(\lambda_0)}{\lambda_0} = 1 \). \( \square \)

Remark.
We can deduce estimates on \( u \) from those in II.1, using the definition of \( u_{\lambda_0} \).

III. Approximate system and weak limits.

We first introduce an appropriate regularization, which can give estimates on the energy; and we prove preliminary results. Our aim is to introduce a regularization of the kernel \( \epsilon(x) = \frac{1}{4\pi|x|} \) which still gives lower bounded terms in the related energy formula.

III.1. Preliminary results.

Lemma 3.1.

There exists \( \zeta_1 \in C^\infty_c(\mathbb{R}^3) \) even and non-negative with \( \int \zeta_1 = 1 \) and satisfying
\[ \zeta_1 \text{ is non-negative}. \]

Proof: first get \( \rho \in C^\infty_c(\mathbb{R}^3) \) even, non-negative with \( \int \rho = 1 \). Then \( \zeta_1 = \rho * \rho \) suits. \( \square \)

Define for \( \sigma > 0 \), \( \zeta_{\sigma}(x) = \frac{1}{\sigma^3} \zeta_1 \left( \frac{x}{\sigma} \right) \).
In order to simplify the notation we fix \( \sigma > 0 \) and \( \zeta_{\sigma} \) is only denoted by \( \zeta \).

Set \( a = \zeta * \epsilon \) with \( \epsilon(x) = \frac{1}{4\pi|x|} \) \( a \in C^\infty(R^3) \cap C_0(R^3) \) satisfies \( \partial^\alpha a \in C_0 \) (every \( \alpha \)), and
\[ \tilde{a}(\xi) = (2\pi)^\frac{3}{2} \xi(\xi) \xi(\xi) = \frac{\xi(\xi)}{\xi^2} \geq 0. \]
Lemma 3.2.

With the above notations, we define for \( u, v \in L^1(\mathbb{R}^3) \)
\[
[u, v] = \int_{\mathbb{R}^3} u(x) a * v(x) \, dx,
\]
and
\[
q(u) = [u, u].
\]

[ ] is a scalar product on \( L^1(\mathbb{R}^3) \) and has the following properties:
\[
[u, v] \leq ||a||_{C_0} ||u||_{L^1} ||v||_{L^1}, \quad u, v \in L^1; \tag{19}
\]
\[
||a * v||_{C_0} \leq ||a||_{C_0}^{\frac{1}{2}} q(v)^{\frac{1}{2}}, \quad v \in L^1; \tag{20}
\]
\[
||a * w||_{L^\frac{1}{2}} \leq C_1 ||w||_{L^\frac{1}{2}}, \quad w \in L^1 \cap L^\frac{1}{2}; \tag{21}
\]
\[
q(w)^{\frac{1}{2}} \leq \sqrt{C_1} ||w||_{L^\frac{1}{2}}, \quad w \in L^1 \cap L^\frac{1}{2}; \tag{22}
\]
\[
||a * v||_{L^\frac{3}{2}} \leq \sqrt{C_1} q(v)^{\frac{1}{2}}, \quad v \in L^1; \tag{23}
\]
and \( C_1 \) does not depend on \( \sigma \).

Proof: We can write
\[
[u, v] = \int_{\mathbb{R}^3} u(x) a * v(x) \, dx
\]
\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} u(x) a(x - y) v(y) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^3} \tilde{u}(\xi) a * v(\xi) \, d\xi
\]
\[
= \int_{\mathbb{R}^3} \frac{\tilde{u}(\xi)(2\pi)^{\frac{3}{2}}}{\tilde{a}(\xi) \tilde{v}(\xi)} \, d\xi.
\]
\( \tilde{\zeta} \) is analytic and non-identically zero, from which we deduce that \( \tilde{\zeta}(\xi) \neq 0 \) for a.e. \( \xi \in \mathbb{R}^3 \), which proves that [ ] is a scalar product. (19) is clear.

Now \( q(u)^{\frac{1}{2}} \leq ||a||_{C_0} ||u||_{L^1} \) and \( ||u, v|| \leq q(u)^{\frac{1}{2}} q(v)^{\frac{1}{2}} \leq q(v)^{\frac{1}{2}} ||a||_{C_0}^{\frac{1}{2}} ||v||_{L^1}, \) and (20) is proved.

For \( w \in L^1 \cap L^\frac{1}{2} \) we have \( a * w = (\zeta * c) * w = \zeta * (c * w) \) so \( ||a * w||_{L^\frac{1}{2}} \leq ||\zeta||_{L^1} ||c * w||_{L^\frac{1}{2}} \leq C_1 ||w||_{L^\frac{1}{2}}. \)

For \( w_1, w_2 \in L^1 \cap L^\frac{1}{2}, \) \( ||w_1, w_2|| \leq ||w_1||_{L^\frac{1}{2}} ||a * w_1||_{L^\frac{1}{2}} \leq C_1 ||w_1||_{L^\frac{1}{2}} ||w_2||_{L^\frac{1}{2}} \) so \( q(w)^{\frac{1}{2}} \leq \sqrt{C_1} ||w||_{L^\frac{1}{2}}. \)

(21) and (22) are proved. Moreover, \( \int w a * v | = ||w, v|| \leq q(w)^{\frac{1}{2}} q(v)^{\frac{1}{2}} \leq \sqrt{C_1} q(v)^{\frac{1}{2}} ||w||_{L^\frac{1}{2}}. \)

By a duality argument, the above formula implies that for every \( v \in L^1 \); the convolution \( a * v \in L^6 \) with the estimate (23).

We will see in proposition A.2. that (23) is basic to get bounds on \( u \) independant from \( n \).

III.2. Approximate system.

We want to approximate the system given in theorem 2.
Proposition 3.3.

Assume that $d$ and $f_0$ fulfill the conditions (3). Then one can find a sequence $f_{n0} \in C^1_c(\mathbb{R}^d) \geq 0$ such that $f_{n0} \rightarrow f_0$ in any $L^p$, $1 \leq p < \infty$; and for any sequence $\sigma_n \rightarrow 0$, $\sigma_n \rightarrow 0$; a solution $f_n \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ to the modified system

\begin{align*}
\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n + \text{div}_v(E_n f_n) &= 0, \quad f_n \geq 0, \quad (24) \\
f_n(0, x, v) &= f_{n0}(x, v), \\
E_n &= -\nabla u_n, \quad u_n \equiv a_n * (\rho_n - d\bar{e}^u), \quad \rho_n = \int_{\mathbb{R}^d} f_n \, dv; \quad (25)
\end{align*}

with the regularity

$f_n(t, .) \in C^1_c(\mathbb{R}^d)$, with controlled support when $t$ is bounded;

$u_n \in C^1(\mathbb{R}_+, C_0(\mathbb{R}^d))$;

$E_n \in C(\mathbb{R}_+, C_0(\mathbb{R}^d))$;

$\nabla E_n \in C(\mathbb{R}_+, C_0(\mathbb{R}^d))$.

Moreover, the total energy defined by

$$
E_n(t) = \int_{\mathbb{R}^d} |v|^2 f_n \, dx \, dv + q_n(\rho_n - d\bar{e}^u) + 2 \int_{\mathbb{R}^d} d(u_n - 1) e^u
$$

is a constant.

Here we set $a_n = \zeta_{\sigma_n} \ast \frac{1}{4\pi|\cdot|}$, and $q_n$ is defined as in the preliminary results.

Due to the regularity of the kernel $a_n$ (instead of $\epsilon$), it is classical to find a $C^1_c$ solution $f_n$ to the system (24), (25). We only need to know that the approximated elliptic equation (25) has a solution $u_n$, for a given $\rho_n$. We just refer to the appendix for more details, and to J. Batt [3] for the first approximation of this type.

We easily compute that

$$
||f_n(t, \cdot)||_{L_p} = ||f_{n0}||_{L_p}, \quad t \geq 0, \quad 1 \leq p \leq \infty;
$$

$$
\frac{\partial \rho_n}{\partial t} + \text{div}_v j_n = 0.
$$

Multiplying (24) by $|v|^2$ and integrating for $v \in \mathbb{R}^d$ we find the energy estimate as follows

$$
\frac{d}{dt} \int_{\mathbb{R}^d} |v|^2 f_n \, dx \, dv = - \int_{\mathbb{R}^d} E_n \, dx \int_{\mathbb{R}^d} |v|^2 \nabla_v f_n \, dv
$$

$$
= - \int_{\mathbb{R}^d} E_n \, dx \int_{\mathbb{R}^d} -2v f_n \, dv
$$

$$
= 2 \int_{\mathbb{R}^d} E_n j_n \, dx
$$

$$
= 2 \int_{\mathbb{R}^d} u_n \, \text{div}_v j_n \, dx
$$

$$
= -2 \int_{\mathbb{R}^d} u_n \frac{\partial \rho_n}{\partial t} \, dx.
$$

But we know that $q_n(\rho_n - d\bar{e}^u) \in C^1(\mathbb{R}_+)$ and

$$
\frac{d}{dt} q_n(\rho_n - d\bar{e}^u) = 2 \left[ \rho_n - d\bar{e}^u, \frac{\partial \rho_n}{\partial t} - d\bar{e}^u \frac{\partial u_n}{\partial t} \right]_n
$$

$$
= 2 \int_{\mathbb{R}^d} u_n \left( \frac{\partial \rho_n}{\partial t} - d\bar{e}^u \frac{\partial u_n}{\partial t} \right)
$$

$$
= 2 \int_{\mathbb{R}^d} u_n \frac{\partial \rho_n}{\partial t} - 2 \int_{\mathbb{R}^d} u_n d\bar{e}^u \frac{\partial u_n}{\partial t}
$$

$$
= 2 \int_{\mathbb{R}^d} u_n \frac{\partial \rho_n}{\partial t} - 2 \int_{\mathbb{R}^d} d(u_n - 1) e^u.
$$
APPROXIMATE SYSTEM AND WEAK LIMITS

Therefore $\mathcal{E}_n(t) \in C^1(\mathbb{R}^+)$ satisfies thanks to the preceding calculation

$$\frac{d\mathcal{E}_n}{dt} = 0.$$ 

We have here followed the proof of the Vlasov-Poisson case (see especially E. Horst, R. Hunze [14]).

III.3. Passage to the limit.

We are going to get $n \to \infty$ in the approximate system (24), (25). The same ideas could be used to prove weak stability for the system considered in theorems 1 and 2.

Lemma 3.4.

The following quantities are bounded, independently from $n$ and $t$

$$\int |v|^2 f_n; \|\rho_n\|_{L^\frac{3}{2}}; \|\rho_n\|_{L^1}; \rho_n(\rho_n - de^{u_n}); \|u_n\|_{L^2}; \text{ ess sup } u_n.$$ 

Moreover, $\int |x|^2 f_n$ is also bounded when $t$ is bounded.

Proof: $\rho_{n0}$ is bounded in $L^1 \cap L^\frac{3}{2}$ so $g_n(\rho_{n0} - de^{u_{n0}}); \|u_{n0}\|_{L^2}; \int d(1 + |u_{n0}|)e^{u_{n0}}$ are bounded by proposition A.2., and thus $\mathcal{E}_n(0)$ is bounded. We have

$$\begin{align*}
\int|v|^2 f_n & \leq \mathcal{E}_n(t) - 2 \int d(u_n - 1)e^{u_n} \\
& \leq \mathcal{E}_n(0) + 2 \int d. 
\end{align*}$$

(because $(z - 1)e^z \geq -1$)

$||f_n||_{L^\infty} = ||f_{n0}||_{L^\infty}; ||f_n||_{L^1} = ||f_{n0}||_{L^1}$ are bounded. $\rho_n$ is bounded in $L^\frac{3}{2}$ and in $L^1$. Proposition A.2. gives

$$\text{ess sup } u_n \leq \text{ess sup } a_n + \rho_n \leq ||a_n + \rho_n||_{L^\infty} \leq ||\rho||_{L^\infty} \leq C||\rho_n||_{L^\frac{3}{2}}^{\frac{5}{3}}.$$ 

The bound on $\int |x|^2 f_n$ is classical, as is the way to prove the existence of a non-negative function $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3))$ such that after an extraction we have for every $\varphi \in L^1 \cup L^\infty(\mathbb{R}^3)$

$$\int f_n(t, \cdot ) \varphi \xrightarrow{n \to \infty} \int f(t, \cdot ) \varphi$$ uniformly when $t$ is bounded,

and the function $f$ satisfies

$$||f(t, \cdot)||_{L^1(R^3)} \leq ||f_0||_{L^1(R^3)}, \quad t \geq 0,$$

$$||f(t, \cdot)||_{L^\infty(R^3)} \leq ||f_0||_{L^\infty(R^3)}, \quad t \geq 0,$$

$$\int_{R^3} |v|^2 f \in L^\infty(\mathbb{R}_+),$$

$$\int_{R^3} |x|^2 f \in L^\infty(\mathbb{R}_+),$$

$$\rho \equiv \int_{R^3} f dv \in C(R_+, L^1(R^3)) \cap L^\infty(R_+, L^1) \cap L^\infty(R_+, L^\frac{3}{2})$$

and $\rho_n(t) \xrightarrow{L^w} \rho(t), \quad 1 \leq p \leq \frac{5}{3}, \quad t \geq 0.$
We now want to give a convergence result about the elliptic equation. A classical argument is the regularizing effect of \(-\Delta\) which gives local compactness. But the main point of the proof is the way that we get an \(L^2\) bound on \(E\). Actually, we do not dispose of any estimate on \(\|E_n\|_{L^2}\). But we know that \(q_n(\rho_n - de^{u_n})\) is bounded. We will see that it approximately represents \(\|E\|_{L^2}\) when \(n\) is large. Here the positivity of the quadratic form \(q_n\) is essential. A regularization by a classical kernel, (as it is done in the Vlasov-Poisson case) would not have led to this limiting bound on \(\|E\|_{L^2}\).

We obtain the following result.

**Lemma 3.5.**

For every \(t \geq 0\) we have

\[
\begin{align*}
    u_n(t) &\xrightarrow{L^1_{loc} (\mathbb{R}^3)} u(t) \quad \text{solution of problem (12)}, \\
    de^{u_n(t)} &\xrightarrow{L^1} de^{u(t)}.
\end{align*}
\]

**Proof:** Let \(v_n \equiv \rho_n(t) - de^{u_n(t)}\) be weakly relatively compact in \(L^1_\text{loc}(\mathbb{R}^3)\). After an extraction we get \(v_n(t) \xrightarrow{L^1_{loc}} v(t) \in L^1\) and \(\zeta_n \ast v_n(t) \xrightarrow{a.e.} v(t)\). But the map \(\omega \mapsto e \ast \omega\), defined on a bounded subset of \(L^1\), is continuous for the topologies \(D' \xrightarrow{a.e.} L^1_{loc}\) (by the regularizing effect of \(-\Delta\)). \(\zeta_n \ast v_n\) is bounded in \(L^1\) so \(u_n(t) = a_n \ast v_n(t) = e \ast (\zeta_n \ast v_n(t)) \xrightarrow{a.e.} e \ast v(t)\) and then \(u_n(t) \xrightarrow{a.e.} u(t) \equiv e \ast v(t) \in L^1_{loc}(\mathbb{R}^3)\).

We are now going to prove that \(u(t)\) is the solution of problem (12).

After another extraction, we get \(u_n(t) \xrightarrow{a.e.} u(t)\) and consequently \(de^{u_n} \xrightarrow{a.e.} de^u\). But \(de^{u_n}\) is relatively compact in \(L^1_{loc}\) so \(de^u \in L^1\) and \(de^{u_n} \xrightarrow{L^1} de^u\). We also get \(v_n \xrightarrow{L^1_{loc}} \rho(t) - de^{u(t)}\), \(v(t) = \rho(t) - de^{u(t)}\) and \(u(t) = e \ast v(t) = e \ast (\rho(t) - de^{u(t)})\).

Now we use the Fourier transform in \(\mathbb{R}^3\). Knowing that \(v_n \xrightarrow{L^1_{loc}} v(t)\) we deduce that

\[
v_n(t) \xrightarrow{L^1_{loc}} v(t) \quad \text{pointwise},
\]

and by Fatou's lemma

\[
\int_{\mathbb{R}^3} \left| \hat{v}(t)(\xi) \right|^2 \frac{d\xi}{|\xi|^2} \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} \left| \hat{v_n}(t)(\xi) \right|^2 (2\pi)^{\frac{3}{2}} \frac{1}{|\xi|^2} |\sigma_n(\xi)| \frac{d\xi}{|\xi|^2}.
\]

But the right integral is \(\sigma_n(v_n(t))\), which is bounded by lemma 3.4. Thus we deduce the bound

\[
\int_{\mathbb{R}^3} \left| \hat{v}(t)(\xi) \right|^2 \frac{d\xi}{|\xi|^2} < \infty.
\]

Finally, we set \(\omega(t) = e \ast v(t) \in L^1_{loc} \cap S' \in L^1 + C_0\). We have

\[
\omega(t) = \frac{ie^i\xi}{|\xi|^2} \hat{v}(t) \in L^1_{\nu}
\]

and

\[
\int_{\mathbb{R}^3} \left| \omega(t)(\xi) \right|^2 \frac{d\xi}{|\xi|^2} = \int_{\mathbb{R}^3} \left| v(t)(\xi) \right|^2 \frac{d\xi}{|\xi|^2} < \infty
\]

so \(\omega(t) \in L^2\). But \(-\nabla u(t) = \omega(t)\) so \(u(t) \in L^6 + \mathbb{R}\). \(u(t) = e \ast v(t) \in L^1 + C_0\) so \(u(t) \in L^6\) and \(u(t)\) is effectively the solution of (12).

Now, since the solution of (12) is unique, we do not need any longer to extract subsequences to get the announced convergence. \(\square\)
Lemma 3.6.

For every $t \geq 0$ we have $E_n(t) \xrightarrow{L^1_{loc}} E(t) \equiv -\nabla u(t)$ and $u, E$ satisfy

$$u \in L^\infty(R_+, L^6(R^3)), \text{ ess sup } u < \infty,$$

$$E \in L^\infty(R_+, L^2(R^3)).$$

Proof: The convergence of $E_n$ is proved in the same way as the convergence of $u_n$ in the previous lemma. For the regularity of $u$ and $E$, the only non trivial assertion is that $u$ and $E$ are measurable in the $(t, x)$ variables. For example, it suffices to prove that $u(t) \ast \zeta_\epsilon$ is $(t, x)$ measurable. But it is the limit a.e. of $u_n(t) \ast \zeta_\epsilon$. \[ ]

Lemma 3.7.

The equation (4) for $f$ and the energy estimate (11) are satisfied.

Proof: by Lebesgue’s theorem in the $t$ variable we get

$$E_n f_n \xrightarrow{n \to \infty} E f \quad \text{in } D'(0; \infty[ \times R^6)$$

and (4) follows.

Now we have $\rho_n(0) \xrightarrow{L^1 \cap L^{\frac{6}{5}}} \rho(0)$ because

$$\rho_n(0) - \rho(0) = \int_{R^3} (f_{n0} - f_0) \, dv$$

and

$$\|\rho_n(0) - \rho(0)\|_{L^{\frac{6}{5}}} \leq C \left( \int_{R^3} |v|^2 |f_{n0} - f_0| \right)^{\frac{1}{2}} \|f_{n0} - f_0\|_{L^2}^{\frac{1}{2}}.$$

After an extraction (see lemma 3.5) we have $u_n \rightharpoonup u$ so

$$\iint |v|^2 f \leq \liminf \iint |v|^2 f_n \leq \limsup \iint |v|^2 f_n \leq \iint |v|^2 f_n,$$

$$|E|^2 \leq \lim q_n''(v_n''),$$

$$\int d((u - 1) e^{u''} + 1) \leq \lim \int d((u_n'' - 1) e^{u''} + 1), \quad \text{(Fatou's lemma)}$$

so

$$\int d(u - 1) e^u \leq \lim \int d(u_n'' - 1) e^{u_n''}$$.  

and then

$$\mathcal{E}(t) \leq \lim \iint |v|^2 f_n'' + \lim q_n''(v_n'') + 2 \lim \int d(u_n'' - 1) e^{u_n''}$$

$$\leq \lim \mathcal{E}_n''(t) + \lim E_n''(0).$$

It suffices to prove that $\mathcal{E}_n''(0) \xrightarrow{n \to \infty} \mathcal{E}(0)$, which is easy. \[ ]

Theorem 2 is now proved.
III.4. Modifications for the proof of theorem 1.

The proof is similar. Let us just mention the main modifications. $f_n$ is now the solution of

$$
\begin{cases}
\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n + \text{div}_v (E_n f_n) = 0, \\
f_n(0, x, v) = f_{n0}(x, v), \\
E_n = -\nabla u_n, \\
\frac{d}{dt} u_n = u_n + \left( \frac{\rho_n - \frac{de_u^*}{dt}}{\int de_u^*} \right), \\
\rho_n = \int_{\mathbb{R}_+^3} f_n dv,
\end{cases}
$$

and we set $\phi_n = u_n - \ln \int_{\mathbb{R}_+^3} de_u^*$ $\in C^1(\mathbb{R}_+, C^1(\mathbb{R}^3))$, $v_n = \rho_n - \frac{de_u^*}{\int de_u^*}$.

we obtain the energy bound as before; $q_n(v_n) \in C^1(\mathbb{R}_+)$ and

$$
\frac{d}{dt} q_n(v_n) = 2 \left[ \rho_n - \frac{de_u^*}{\int de_u^*} \frac{\partial \rho_n}{\partial t} - \frac{\partial}{\partial t} \left( \frac{de_u^*}{\int de_u^*} \right) \right] u_n
$$

$$
\begin{align*}
&= 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial \rho_n}{\partial t} - 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial}{\partial t} \frac{de_u^*}{\int de_u^*} \\
&= 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial \rho_n}{\partial t} - 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial}{\partial t} \phi_n \\
&= 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial \rho_n}{\partial t} - 2 \frac{d}{dt} \int_{\mathbb{R}_+^3} u_n \phi_n + 2 \int_{\mathbb{R}_+^3} \frac{\partial u_n}{\partial t} \phi_n \\
&= 2 \int_{\mathbb{R}_+^3} u_n \frac{\partial \rho_n}{\partial t} - 2 \frac{d}{dt} \int_{\mathbb{R}_+^3} u_n \phi_n + 2 \frac{d}{dt} \ln \int_{\mathbb{R}_+^3} de_u^*.
\end{align*}
$$

Thus the total energy defined by

$$
E_n(t) = \iint |v|^2 f_n \, dx \, dv + q_n(v_n) + 2 \int \phi_n \, de_u^*
$$

still satisfies $E_n \in C^1(\mathbb{R}_+)$ and $\frac{dE_n}{dt} = 0$.

The bounds on $\iint |v|^2 f_n$, $\|\rho_n\|_{L^1}$, $\|\rho_n\|_{L^\infty}$, $q_n(v_n)$, $\|u_n\|_{L^1}$, $\text{ess sup} \, u_n$, and $\|(1 + |\phi_n|) e_u^*\|_{L^\infty}$ are still true and the above compactness results hold.

Finally, $\phi(t) = u(t) - \ln \int de_u(t)$ $\in L^1(\mathbb{R}^3) \oplus \mathbb{R}$, where $u(t)$ is solution of the problem (15) in the second section, and we obtain here

$$
\begin{align*}
&u_n(t) \rightarrow u(t), \\
&de_u^*(t) \rightarrow de_u^*(t), \\
&de_u^*(t) \rightarrow de_u^*(t).
\end{align*}
$$

Also the integral $\int de_u^*$ is bounded, as stated in proposition A.2.

Appendix.

We give here several results concerning the approximated elliptic equation.

$\rho \in L^1(\mathbb{R}^3)$ is given and we consider the equation

$$
\begin{cases}
\frac{\partial u}{\partial t} = a * \left( \frac{de_u}{\int de_u} \right), \\
u \in C_0(\mathbb{R}^3).
\end{cases}
$$

(All results would be true in the simplified case of theorem 2.)

a is defined in section III.1.

Proposition A.1.

For every $\rho \in L^1(\mathbb{R}^3)$ there exists a unique solution $u$ to (26).
We will only indicate that a possible proof uses Hadamard's theorem in order to prove that the function $u \mapsto u + a \ast \frac{de^u}{deu}$ is bijective.

The essential point is how we get bounds on $u$. This is explained in the

**Proposition A.2.**

If $\rho \in L^1 \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ and $\|\rho\|_{L^1} + \|\rho\|_{L^{\frac{3}{2}}} \leq M$ then the associated solution $u$ to (26) satisfies

$$\text{ess sup } u \leq C,$$

$$\frac{1}{C} \leq \int_{\mathbb{R}^3} de^u \leq C,$$

$$q \left( \rho - \frac{de^u}{deu} \right) \leq C,$$

$$\|v\|_{L^3} \leq C,$$

$$\|(1 + |\phi|)e^\phi\|_{L^1} \leq C;$$

with $\phi = u - \ln \int_{\mathbb{R}^3} de^u$ and $C$ depends only on $M$ and $d$.

**Proof:** $u = a \ast \left( \rho - \frac{de^u}{deu} \right) \leq a \ast \rho = \zeta \ast (e \ast \rho)$,

$e \ast \rho \in C_0$ and $\|e \ast \rho\|_{C_0} \leq C \|\rho\|_{L^\frac{3}{2}} \|\rho\|_{L^\frac{3}{2}}$, so $a \ast \rho \in C_0$ with $\|a \ast \rho\|_{C_0} \leq \|e \ast \rho\|_{C_0} \leq CM$ and thus

$\text{ess sup } u \leq CM$, therefore we also get

$$\int_{\mathbb{R}^3} de^u \leq \left( \int d \right) e^{CM}.$$

We set $v = \rho - \frac{de^u}{deu}$; $v$ is bounded in $L^1$ by $M + 1$ and thus $u = e \ast [\zeta \ast v]$ is bounded in $M^3$ and proposition 2.7. gives the lower bound on $\int de^v$.

Finally $u$ satisfies $u = a \ast v$ so

$$q(v) = \int uv$$

$$= \int \rho u - \int \frac{du e^u}{deu}$$

$$\leq \|\rho\|_{L^\frac{3}{2}} \|u\|_{L^3} + e^{-1} \int \frac{d}{deu}$$

$$\leq M \sqrt{C_1 \int q(v) \frac{1}{2} + C}$$

(because of (23)) so $q(v)$ is bounded, and also $\|u\|_{L^3}$. The bound concerning $\phi$ is a consequence of

$$e^\phi = \frac{e^u}{\int de^u} \quad \text{and} \quad \phi e^\phi = \frac{ue^u}{\int de^u} - \frac{\ln \int de^u}{\int de^u} e^u.$$

**References:**


