Renormalized solutions to the Vlasov equation with coefficients of bounded variation

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Abstract
We prove that weak bounded solutions to the Vlasov equation with BV coefficient have the renormalization property, and we obtain that when the renormalization property holds for a general transport equation, it also holds for only Lipschitz nonlinearities.


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1 Introduction

It is well-known that the classical Vlasov equation

$$\partial_t f + \xi \cdot \nabla_x f + \text{div}_x [E(t, x)f] = 0$$  \(1.1\)

describes the evolution in phase space of the density $f$ of particles satisfying the fundamental law of mechanics $\dot{x} = \xi, \dot{\xi} = E(t, x)$. The optimal smoothness of the force field $E(t, x)$ that is necessary in order that the Cauchy problem associated to (1.1) is well-posed is an important problem for applications. According to [10], a way to obtain this is to ask that any weak $L^\infty$ solution $f$ to (1.1) should be a renormalized solution, in the sense that $g(f)$ is also a weak solution to (1.1) for any sufficiently smooth function $g$. Thus we are led to the problem of justifying some kind of chain rule for functions with low regularity.

The classical chain rule

$$\frac{\partial}{\partial x_i} [g(u)] = g'(u) \frac{\partial u}{\partial x_i}$$  \(1.2\)

for a scalar function $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ was obtained for $u \in W^{1,1}_{loc}(\Omega)$ and $g$ Lipschitz continuous $\mathbb{R} \to \mathbb{R}$ by Stampacchia. The proof can be found in [3] and [12]. It is also known that in this context $u \mapsto g(u)$ is continuous [13]. Functions $u$ of bounded variation are also considered in [1], [7]. The relation (1.2) is indeed strongly related to the following property which can be interpreted as an inverse Sard lemma: if $u \in W^{1,1}_{loc}(\Omega)$ and $Z$ is a Borel set in $\mathbb{R}$ with $|Z| = 0$, then

$$\{|x \in \Omega ; u(x) \in Z \text{ and } \nabla u(x) \neq 0\}| = 0.$$  \(1.3\)

This was proved in [3].

In the first part of this paper, we discuss some extensions of (1.2) and (1.3). The formulation (1.2) is indeed adapted to elliptic equations when we a priori know that $\nabla u \in L^{1}_{loc}$. However, if only $\partial u/\partial x_i \in L^{1}_{loc}$ for some $i_0$, it still holds for $i = i_0$. More generally, if $u$ solves an hyperbolic problem, one may only know that

$$a(x) \cdot \nabla u \in L^{1}_{loc}$$  \(1.4\)

for some coefficient $a(x)$ which may be not smooth. Then the chain rule becomes the so called "renormalization property"

$$a(x) \cdot \nabla [g(u)] = g'(u)[a(x) \cdot \nabla u],$$  \(1.5\)

and the inverse Sard lemma becomes

$$\{|x \in \Omega ; u(x) \in Z \text{ and } a(x) \cdot \nabla u(x) \neq 0\}| = 0.$$  \(1.6\)

A general context where (1.5) is true is provided in [10] for $a \in W^{1,1}_{loc}$, with some extensions in [6], [8], [11]. With different methods, coefficients $a$ with a one-sided Lipschitz condition can be treated [6], [5], [17], [16]. A special case with only continuous coefficient $a$ is also treated in [4].

The second part of this paper is concerned with properties (1.5) and (1.6). Our main result is that they hold for a Vlasov type equation with coefficients of bounded variation. More precisely, we have the following theorem.
Theorem 1.1 Let $\Omega$ be an open subset of $\mathbb{R}_x^N \times \mathbb{R}_\xi^M$, 

$$v \in L^1([0,T], (W^{1,1}_{loc}(\Omega))^N),$$

and $E \in L^1([0,T], (L^1_{loc}(\Omega))^M)$ such that for a.e. $t$, $E(t,\cdot) \in BV_{loc}(\Omega)$,

$$\int_0^T TV_\omega(E(t,\cdot)) \, dt < \infty, \quad \omega \subset \subset \Omega,$$

and

$$\nabla_\xi E \in L^1([0,T], L^1_{loc}(\Omega)).$$

Assume that $f \in L^\infty([0,T], L^\infty_{loc}(\Omega))$ and

$$\partial_t f + \text{div}_x[v(t,x,\xi)f] + \text{div}_\xi[E(t,x,\xi)f] \in L^1([0,T], L^1_{loc}(\Omega)).$$

Then

(i) for any Borel set $Z \subset \mathbb{R}$ with $|Z| = 0$,

$$\left| \{(t,x,\xi) \in [0,T] \times \Omega \mid f(t,x,\xi) \in Z \text{ and } \partial_t f + \text{div}_x(vf) + \text{div}_\xi(Ef) - f(\text{div}_x v + \text{div}_\xi E)(t,x,\xi) \neq 0 \} \right| = 0,$$

(ii) for any $g : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz continuous

$$\partial_t[g(f)] + \text{div}_x[v(g(f)) + \text{div}_\xi(Eg(f))] - g(f)(\text{div}_x v + \text{div}_\xi E)$$

$$= g'(f)[\partial_t f + \text{div}_x(vf) + \text{div}_\xi(Ef) - f(\text{div}_x v + \text{div}_\xi E)].$$

We have to notice that assumptions (1.7), (1.8) and (1.9) roughly mean that locally

$$\nabla_x \xi(v(t,x,\xi), E(t,x,\xi)) \in \begin{pmatrix} L^1 & L^1 \\ \mathcal{M} & L^1 \end{pmatrix}.$$ 

The paper is organized as follows. In Section 2 we give some extensions to (1.3) in a vector context with elliptic estimate, and in Section 3 we prove property (1.6) for the hyperbolic case if $a \in W^{1,1}_{loc}$, and for $a \in BV$ in the case of a Vlasov type equation (Theorem 1.1). We also give a strong time continuity that remains true in general for coefficients $a$ with only bounded deformation in evolution problems.

2 Chain rule with elliptic estimate

2.1 Scalar functions

Before coming to the vector case, let us first give a simplified proof for the scalar case. The spirit is the same as that of [3].
Theorem 2.1 Let $\Omega$ be an open subset of $\mathbb{R}^N$, and $u \in L^1_{\text{loc}}(\Omega)$ such that $\nabla u \in L^1_{\text{loc}}(\Omega)$. Then for any Borel set $Z \subset \mathbb{R}$ such that $|Z| = 0$,
\[ |\{ x \in \Omega ; u(x) \in Z \text{ and } \nabla u(x) \neq 0 \}| = 0. \tag{2.1} \]

Proof. We need to prove that
\[ a.e. \ x \in \Omega \quad \mathbf{1}_{u^{-1}(Z)}(x) \nabla u(x) = 0. \tag{2.2} \]

First step. Assume that $Z = K$ is a compact subset of $\mathbb{R}$ with zero Lebesgue measure. Then we can write $K = \bigcap_{n=1}^{\infty} V_n$, $V_n$ open, $V_{n+1} \subset V_n$, and there exist $\theta_n \in C^\infty_c(V_n)$, $0 \leq \theta_n \leq 1$, $\theta_n = 1$ on $\overline{K}$. We define
\[ g_n(x) = \int_0^x \theta_n(y) \, dy, \quad x \in \mathbb{R}. \tag{2.3} \]
Then $g_n \in C^\infty(\mathbb{R})$, $|g_n(x)| \leq |x|$ and
\[ g'_n(x) \to \mathbf{1}_K(x), \quad x \in \mathbb{R}. \tag{2.4} \]
Therefore, by Lebesgue’s theorem,
\[ g_n(x) = \int_0^\infty \mathbf{1}_K(y) \, dy = 0, \quad x \in \mathbb{R}, \tag{2.5} \]
because $|K| = 0$. But since $g_n$ is smooth, $g_n(u) \in W^{1,1}_{\text{loc}}$, and
\[ \frac{\partial}{\partial x_i} g_n(u) = g'_n(u) \frac{\partial u}{\partial x_i}, \tag{2.6} \]
\[ - \int_{\Omega} g_n(u) \frac{\partial \varphi}{\partial x_i} = \int_{\Omega} g'_n(u) \frac{\partial u}{\partial x_i} \varphi, \quad \varphi \in C^\infty_c(\Omega). \tag{2.7} \]
Then, by letting $n \to \infty$ in (2.7), and according to (2.4), (2.5) and Lebesgue’s theorem, we get
\[ \int_{\Omega} \mathbf{1}_K(u) \frac{\partial \varphi}{\partial x_i} = 0, \quad \varphi \in C^\infty_c(\Omega). \tag{2.8} \]
This proves (2.2).

Second step. Reduction. Fix a compact set $Q \subset \Omega$ and define for any Borel set $E \subset \mathbb{R}$
\[ \mu(E) = \int_{u^{-1}(E)} \left| \frac{\partial u}{\partial x_i} \right| \mathbf{1}_Q. \tag{2.9} \]
Then $\mu$ is a bounded nonnegative Borel measure on $\mathbb{R}$, and we want to prove that it is absolutely continuous with respect to the Lebesgue measure. Since any finite Borel measure on $\mathbb{R}$ is regular, it holds that
\[ \mu(E) = \sup_{K \text{compact} \subset E} \mu(K). \tag{2.10} \]
Therefore, for any Borel set $E$ such that $|E| = 0$, (2.10) gives that $\mu(E) = 0$ because for any compact subset $K$ of $E$ we have $|K| \leq |E| = 0$, thus $\mu(K) = 0$ by the first step above. Finally, since this holds for any compact $Q \subset \Omega$, the theorem is proved. \[\Box\]

The chain rule can be deduced easily from Theorem 2.1.

**Theorem 2.2** Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then for any $u \in L^1_{\text{loc}}(\Omega)$ such that $\nabla u \in L^1_{\text{loc}}(\Omega)$, we have

$$\nabla [g(u)] = g'(u)\nabla u \in L^1_{\text{loc}}(\Omega). \quad (2.11)$$

**Proof.** Approximate $g$ by a sequence of smooth functions $g_n$, such that $|g_n| \leq C$ and $g'_n \to g'$ a.e. We have $\nabla [g_n(u)] = g'_n(u)\nabla u$, and by Theorem 2.1 $g'_n(u)\nabla u \to g'(u)\nabla u$ a.e. Thus we conclude by Lebesgue’s theorem. \[\Box\]

**Remark 2.1** The function $g'$ is only defined almost everywhere, but thanks to Theorem 2.1, the product $g'(u)\nabla u$ in (2.11) is well defined.

### 2.2 An extension to vector functions

It was obtained in [14] that if $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^d)$, then $g(u) \in W^{1,1}_{\text{loc}}(\Omega)$. However, no formula such as (1.2) is available in general (see [12]). The problem comes from the lack of a property like (1.3) that would enable to give a sense to the products in $\sum_k \partial g/\partial u_k \partial u_k/\partial x_i$. Nevertheless, it holds for piecewise smooth $g$ [15].

Let us recall Sard’s lemma for smooth functions (see for example [9]).

**Lemma 2.3** (Sard) If $u : \Omega \to \mathbb{R}^d$ is of class $C^k$, $k = 1 + (N - d)\plus$, and $E = \{x \in \Omega : Du(x) : \mathbb{R}^N \to \mathbb{R}^d \text{ is not surjective}\}$, then $u(E)$ has Lebesgue measure 0 in $\mathbb{R}^d$. In the scalar case $d = 1$, Theorem 2.1 states that somehow the converse is true. Here we provide an extension to a vector case.

**Theorem 2.4** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$, $d \leq N$, such that $\partial_{x_1} u, \ldots, \partial_{x_N} u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$. Then for any Borel set $Z \subset \mathbb{R}^d$ such that $|Z| = 0$ and any $\sigma : N_d \to N_N$ injective,

$$\left| \left\{ x \in \Omega ; u(x) \in Z \text{ and } \det \left( \frac{\partial u_i}{\partial x_{\sigma(j)}} \right)_{1 \leq i,j \leq d} \neq 0 \right\} \right| = 0. \quad (2.12)$$

In other words, $\text{rank}(Du(x)) < d$ almost everywhere on $u^{-1}(Z)$.

Before proving Theorem 2.4, let us write the associated "divergence chain rule" in the smooth case $g \in C^1$. 


Lemma 2.5 Let \( g \in C^1(\mathbb{R}^d, \mathbb{R}^d) \) such that
\[
\text{div } g \in L^\infty, \quad |g(y)| \leq C(1 + |y|). \tag{2.13}
\]
Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), and \( u \in L^1(\Omega, \mathbb{R}^d) \), \( d \leq N \), such that \( \partial_x u, \cdots, \partial_{x,N} u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d) \). Then for any \( \sigma : \mathbb{N}_d \to \mathbb{N}_N \) injective,
\[
\sum_{k=1}^d \frac{\partial}{\partial x_{\sigma(k)}} \left[ I \left( \begin{bmatrix} \partial u_i \\ \partial x_{\sigma(j)} \end{bmatrix}_{1 \leq i,j \leq d} \right) g(u) \right] = (\text{div } g) \circ u \det \left[ \frac{\partial u_i}{\partial x_{\sigma(j)}} \right]_{1 \leq i,j \leq d}. \tag{2.14}
\]

**Proof.** Here and below, \( I(A) \) denotes the pseudo inverse of \( A \in M_d(\mathbb{R}) \), \( I(A) = (\text{com} A)^t \). We have \( |I(A)| \leq C_d |A|^{d-1} \). At first, by the Sobolev imbedding, we have \( u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d) \), thus by (2.13) \( g(u) \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d) \). Denote \( A(x) = [\partial u_i / \partial x_{\sigma(j)}]_{1 \leq i,j \leq d} \). Then \( A \in L^1_{\text{loc}}(\Omega, M_d(\mathbb{R})) \), and \( I(A) \in L^1_{\text{loc}}(\Omega, M_d(\mathbb{R})) \), thus \( I(A)g(u) \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d) \). Also \( (\text{div } g) \circ u \in L^\infty(\Omega) \) and \( \det A \in L^1_{\text{loc}}(\Omega) \), hence (2.14) makes sense. In order to prove that this identity holds, observe that if \( u \in C^2 \) this is a classical formula. Then, for \( u \in L^1_{\text{loc}} \) such that \( Du \in L^1_{\text{loc}} \), just approximate \( u \) by convolution. \( \square \)

**Proof of Theorem 2.4.** We argue as in Theorem 2.1. We need to prove that
\[
\mathbb{I}_{u^{-1}(Z)} \det A(x) = 0 \quad a.e. \ x \in \Omega, \tag{2.15}
\]
where \( A(x) = [\partial u_i / \partial x_{\sigma(j)}]_{1 \leq i,j \leq d} \). As in Theorem 2.1, it is enough to consider the case when \( Z = K \) is a compact subset of \( \mathbb{R}^d \) with \( |K| = 0 \). Then we can write \( K = \cap_{n=1}^\infty V_n \), \( V_n \) open, \( V_{n+1} \subset V_n \subset \cdots \subset V_1 \) bounded, and there exist \( \theta_n \in C^\infty_0(V_n), 0 \leq \theta_n \leq 1, \theta_n = 1 \) on \( K \). We define
\[
g_n(y) = \frac{1}{|\mathbb{S}^{d-1}| ||y||^d} * \theta_n. \tag{2.16}
\]
Then \( g_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap L^\infty, \) \( \text{div } g_n = \theta_n. \) We get by Lemma 2.5
\[
\sum_{k=1}^d \frac{\partial}{\partial x_{\sigma(k)}} [I(A)g_n(u)]_k = \theta_n(u) \det A. \tag{2.17}
\]
But for any \( y \in \mathbb{R}^d, \theta_n(y) \to \mathbb{I}_K(y) \) when \( n \to \infty \), thus \( \theta_n \to 0 \) in \( L^p(\mathbb{R}^d) \) for any \( 1 \leq p < \infty \), and \( \|g_n\|_{L^\infty} \to 0 \). Since \( \theta_n(u) \to \mathbb{I}_K(u) \) a.e., we get by letting \( n \to \infty \) in (2.17)
\[
0 = \mathbb{I}_K(u) \det A \quad \text{in } \mathcal{D}', \tag{2.18}
\]
which yields (2.15). \( \square \)

3 Chain rule with hyperbolic estimate

3.1 Coefficient in \( W^{1,1} \)

The aim of this section is to prove the inverse Sard lemma in the context of the theory provided in [10] on transport equations.
**Theorem 3.1** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $a \in (W^{1,p'}_{{\text{loc}}} (\Omega))^N$ for some $1 \leq p \leq \infty$. Assume that $u \in L^p_{\text{loc}}(\Omega)$ and

$$
\text{div}(au) \in L^1_{\text{loc}}(\Omega).
$$

Then
(i) for any Borel set $Z \subset \mathbb{R}$ with $|Z| = 0$,

$$
\left| \{ x \in \Omega : u(x) \in Z \text{ and } (\text{div}(au) - u\text{div }a)(x) \neq 0 \} \right| = 0,
$$

(ii) for any $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous,

$$
\text{div}[ag(u)] - g(u) \text{div } a = g'(u) [\text{div}(au) - u\text{div }a].
$$

A similar result is valid in a time dependent context.

**Theorem 3.2** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $a \in L^1([0,T[,(W^{1,p'}_{{\text{loc}}} (\Omega))^N)$ for some $1 \leq p \leq \infty$ and $T > 0$. Assume that $u \in L^\infty([0,T[, L^p_{\text{loc}}(\Omega))$ and

$$
\partial_t u + \text{div}(au) \in L^1([0,T[, L^1_{\text{loc}}(\Omega)).
$$

Then
(i) for any Borel set $Z \subset \mathbb{R}$ with $|Z| = 0$,

$$
\left| \{ (t,x) \in [0,T[ \times \Omega : u(t,x) \in Z \text{ and } (\partial_t u + \text{div}(au) - u\text{div }a)(t,x) \neq 0 \} \right| = 0,
$$

(ii) for any $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous,

$$
\partial_t [g(u)] + \text{div}[ag(u)] - g(u) \text{div } a = g'(u) [\partial_t u + \text{div}(au) - u\text{div }a].
$$

**Remark 3.1** The function $g'$ is only defined almost everywhere but as in Remark 2.1, the products in (3.3) and (3.6) are well-defined.

Before proving Theorems 3.1 and 3.2, let us recall the main approximation lemma from [10].

**Lemma 3.3 (DiPerna, Lions)** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $1 \leq p, q \leq \infty$ such that $p' \leq q$. We let

$$
1/r = 1/q + 1/p \quad \text{if } p < \infty \text{ or } q < \infty,
$$

$$
1 \leq r < \infty \text{ is arbitrary } \quad \text{if } p = q = \infty.
$$

Then,
(i) if $a \in (W^{1,p}_{{\text{loc}}} (\Omega))^N$ and $u \in L^p_{\text{loc}}(\Omega)$ then

$$
\rho_{\varepsilon} * \text{div}(au) - \text{div} [a(\rho_{\varepsilon} * u)] \xrightarrow{\varepsilon \to 0} 0 \quad \text{in } L^r_{\text{loc}}(\Omega),
$$

(ii) if $a \in L^1([0,T[, (W^{1,q}_{{\text{loc}}} (\Omega))^N)$ and $u \in L^\infty([0,T[, L^p_{\text{loc}}(\Omega))$ then

$$
\rho_{\varepsilon} * \text{div}(au) - \text{div} [a(\rho_{\varepsilon} * u)] \xrightarrow{\varepsilon \to 0} 0 \quad \text{in } L^1([0,T[, L^r_{\text{loc}}(\Omega)).
$$

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Here $\rho_\varepsilon$ is a smoothing kernel in $\mathbb{R}^N$, $\rho_\varepsilon(x) = \varepsilon^{-N} \rho_1(x/\varepsilon)$, $\varepsilon > 0$, $x \in \mathbb{R}^N$, with $\rho_1 \in C_0^\infty(B(0,1))$, $\rho_1(x) \geq 0$, $\int \rho_1 = 1$, and the convolution in (3.8), (3.9) is in $x$. Since we are going to consider variants of this lemma, let us rewrite the proof of [10].

**Proof of Lemma 3.3.** The result (ii) can easily be obtained from (i) by applying Lebesgue's theorem in the variable $t$. Therefore, we only prove (i). At first, we observe that

$$
\rho_\varepsilon \ast \text{div} (au) - \text{div} [a(\rho_\varepsilon \ast u)] = I + u[\rho_\varepsilon \ast (\text{div} a)] - (\rho_\varepsilon \ast u)(\text{div} a),
$$

with

$$
I(x) = \int [u(x-y) - u(x)] [a(x-y) - a(x)] \cdot \nabla \rho_\varepsilon(y) \, dy.
$$

Thus we have to prove that $I \to 0$ in $L^\infty_\text{loc}(\Omega)$. Define $1/\tilde{p} = 1/q + 1/p$, and consider an open subset $\omega \subset \subset \Omega$. We take $\varepsilon$ small enough so that $\mathcal{W}+\text{supp} \rho_\varepsilon \subset \omega_1 \subset \subset \Omega$. We have

$$
\begin{align*}
&\|I\|_{L^\infty_\omega(\omega)} \\
&\leq \int \|u(x-y) - u(x)\| \|a(x-y) - a(x)\| \cdot \|\nabla \rho_\varepsilon(y)\| \, dy \\
&\leq \|u(x-y) - u(x)\|_{L^\infty_\omega(\omega)} \|a(x-y) - a(x)\|_{L^\infty_\omega(\omega)} \|\nabla \rho_\varepsilon(y)\| \, dy \\
&\leq \|\nabla a\|_{L^\infty(\omega_1)} \int \|u(x-y) - u(x)\|_{L^\infty_\omega(\omega)} \|\nabla \rho_\varepsilon(y)\| \, dy,
\end{align*}
$$

thus we obtain

$$
\|I\|_{L^\infty_\omega(\omega)} \leq \left( \int \|\nabla \rho_\varepsilon\| \, dy \right) \|\nabla a\|_{L^\infty(\omega_1)} \sup_{|b| \leq \varepsilon} \|u(x-y) - u(x)\|_{L^\infty_\omega(\omega)}.
$$

Since $\int \|\nabla \rho_\varepsilon\| \leq C$, we conclude obviously that $I \to 0$ in $L^\infty(\omega)$ in the case $p < \infty$. If $p = q = \infty$, we can apply the previous case with some $\tilde{p} < \infty$, thus $I \to 0$ in $L^\infty_\text{loc}(\Omega)$ for any $r < \infty$.

It only remains to treat the case where $p = \infty$ and $q < \infty$. In order to do so, for any $\delta > 0$ there exists $a_\delta \in C^\infty$ such that $\|a_\delta - a\|_{W^{1,q}(\omega)} \leq \delta$. By (3.13) we can write $\|I_\delta - I\|_{L^\infty(\omega)} \leq C\delta$, where $I_\delta$ is defined by (3.11) with $a$ replaced by $a_\delta$. Since $a_\delta$ is smooth, we can apply the case $p = q = \infty$ which has been treated above, and we deduce that $I_\delta \to 0$ when $\varepsilon \to 0$ in $L^\infty(\omega)$ for any $s < \infty$, at fixed $\delta$. Then, for $\varepsilon$ small enough, $\|I\|_{L^\infty(\omega)} \leq \|I - I_\delta\|_{L^\infty(\omega)} + \|I_\delta\|_{L^\infty(\omega)} \leq (C+1)\delta$. This proves that $I \to 0$ in $L^\infty_\text{loc}(\Omega)$ when $\varepsilon \to 0$. \qed

**Remark 3.2** The critical case in the previous lemma is when $p = \infty$ and $q = r = 1$. The fact that smooth functions are dense in $W^{1,1}_{\text{loc}}$ is crucial in the proof. If only $a \in BV_{\text{loc}}$ (and $\text{div} a \in L^\infty_{\text{loc}}$ so that all terms make sense), we have $\|I\|_{L^1_{\text{loc}}} \leq C$ because (3.13) is valid with $\|\nabla a\|_{L^1(\omega_1)}$ replaced by $TV_{\omega_1}(a)$, and $I \to 0$ but a priori $\|I\|_{L^1_{\text{loc}}}$ does not tend to zero. In this general setting it is not known whether Theorem 3.1 holds. Only particular cases are solved, the case of piecewise $W^{1,1}$ coefficient $a$ in [11], and the case of Vlasov type equation in Section 3.2.
Proof of Theorem 3.1. It is obtained in three steps.

1st step. Validity of (ii) if \( g \in C^1 \). In this case there is no ambiguity in the product in the right-hand side of (3.3). We define \( u_\varepsilon = \rho_\varepsilon * u \in C^\infty \), and write
\[
\div[a g(u_\varepsilon)] - g(u_\varepsilon) \div a = g'(u_\varepsilon) [\div(a u_\varepsilon) - u_\varepsilon \div a] = g'(u_\varepsilon) [\rho_\varepsilon \ast \div(a u) - u_\varepsilon \div a + \{\div(a u_\varepsilon) - \rho_\varepsilon \ast \div(a u)\}].
\]
By Lemma (3.3)(i) applied with \( q = p' \), the term between braces tends to 0 in \( L^1_{\text{loc}} \) when \( \varepsilon \to 0 \). Since by assumption \( \div(a u) \in L^1_{\text{loc}} \), we can let \( \varepsilon \to 0 \) in (3.14) and this yields (3.3).

2nd step. Proof of (i). We need to prove that
\[
\mathbf{1}_Z(u) [\div(a u) - u \div a] = 0 \quad \text{a.e. in } \Omega.
\]
We proceed as in Theorem 2.1. We assume that \( Z = K \) is a compact subset of \( \mathbb{R} \) with \( \|K\| = 0 \), and write \( K = \cap_{n=1}^\infty V_n \), \( V_n \) open, \( V_{n+1} \subset V_n \). There exist \( \theta_n \in C^\infty_c (V_n) \), \( 0 \leq \theta_n \leq 1 \), \( \theta_n = 1 \) on \( K \), and we define
\[
g_n(x) = \int_0^x \theta_n(y)dy, \quad x \in \mathbb{R}.
\]
By the first step above,
\[
\div[a g_n(u)] - g_n(u) \div a = g'_n(u) [\div(a u) - u \div a].
\]
Since \( g'_n \to g' \) and \( g_n \to 0 \) everywhere, we get by using Lebesgue's theorem
\[
0 = \mathbf{1}_K(u) [\div(a u) - u \div a]
\]
in the sense of distributions, which gives (3.15). The general case when \( Z \) is not compact is obtained exactly as in Theorem 2.1.

3rd step. Proof of (ii) for \( g \) only Lipschitz continuous. We define \( g_n = \rho_n * g \). Since \( g_n \in C^1 \), we know by the first step that
\[
\div[a g_n(u)] - g_n(u) \div a = g'_n(u) [\div(a u) - u \div a].
\]
Since \( g'_n \to g' \) a.e., we deduce with the second step that
\[
g'_n(u) [\div(a u) - u \div a] \to g'(u) [\div(a u) - u \div a] \quad \text{a.e.}
\]
Therefore, we can apply Lebesgue's theorem and let \( n \to \infty \) in (3.19). □

Proof of Theorem 3.2. It is similar to the one of Theorem 3.1. Only the first step needs further smoothing in time. We define \( u_{\tau,\varepsilon} = (\rho_\tau \ast \rho_\varepsilon)_t \ast u \) for \( \varepsilon > 0 \), \( \tau > 0 \), where \( \rho_\tau \) is a smoothing kernel in \( \mathbb{R} \). Then for \( g \in C^1 \) globally Lipschitz continuous,
\[
\partial_t [g(u_{\tau,\varepsilon})] + \div[a g(u_{\tau,\varepsilon})] - g(u_{\tau,\varepsilon}) \div a = g'(u_{\tau,\varepsilon}) [\partial_t u_{\tau,\varepsilon} + a \cdot \nabla u_{\tau,\varepsilon}].
\]
But
\[
\frac{\partial}{\partial \tau} u_{\tau,\varepsilon} + a \cdot \nabla u_{\tau,\varepsilon} = (\rho_\varepsilon \otimes \rho_\varepsilon) \ast (\partial_t u + \text{div}(au)) + a \cdot \nabla u_{\tau,\varepsilon} - (\rho_\varepsilon \otimes \rho_\varepsilon) \ast \text{div}(au) \quad (3.22)
\]
in $L^1_{\text{loc}}$. Therefore, letting $\tau \to 0$ in (3.21) we get
\[
\frac{\partial}{\partial t} [g(u_\varepsilon)] + \text{div}[a g(u_\varepsilon)] - g(u_\varepsilon) \text{div } a = g'(u_\varepsilon) [\rho_\varepsilon \ast (\partial_t u + \text{div}(au)) - u_\varepsilon \text{div } a + \text{div}(au_\varepsilon) - \rho_\varepsilon \ast \text{div}(au)] , \quad (3.23)
\]
and by Lemma 3.3(ii), we can let $\varepsilon \to 0$ and obtain (3.6). \(\square\)

### 3.2 Vlasov equation with BV field

We here extend Theorems 3.1 and 3.2 to the case of Theorem 1.1. Only the time dependent case is stated in Theorem 1.1, but of course a stationary analogue also holds. We are not going to give all the details of the proof since it is very close to the proofs of Section 3.1, with $a(t, x, \xi) = (v(t, x, \xi), E(t, x, \xi))$ and $p = \infty$. The difficulty is that now $\nabla_\xi E \notin L^1_{\text{loc}}$. As in Section 3.1, it is enough to prove (3.8), (3.9). Thus the result follows from the following lemmas.

**Lemma 3.4** Let $\Omega$ be an open subset of $\mathbb{R}^N \times \mathbb{R}^M$,
\[
v \in (W^{1,1}_{\text{loc}}(\Omega))^N , \quad (3.24)
\]
\[
E \in \text{BV}_{\text{loc}}(\Omega)^M \quad \text{with} \quad \nabla_\xi E \in L^1_{\text{loc}}(\Omega) , \quad (3.25)
\]
and
\[
f \in L^\infty_{\text{loc}}(\Omega) . \quad (3.26)
\]
Then there exist two sequences $\varepsilon_n > 0$, $\mu_n > 0$, $\varepsilon_n \to 0$, $\mu_n \to 0$ such that
\[
(\rho_{\varepsilon_n} \otimes \rho_{\mu_n}) \ast \text{div}_x (v f) + \text{div}_\xi (E f) - \text{div}_x [v (\rho_{\varepsilon_n} \otimes \rho_{\mu_n}) \ast f] - \text{div}_\xi [E ((\rho_{\varepsilon_n} \otimes \rho_{\mu_n}) \ast f)] \longrightarrow 0 \quad \text{in } L^1_{\text{loc}}(\Omega) , \quad (3.27)
\]
where $(\rho_{\varepsilon_n} \otimes \rho_{\mu_n})(x, \xi) = \rho_{\varepsilon_n}(x) \rho_{\mu_n}(\xi)$ and $\rho_{\varepsilon}$, $\rho_{\mu}$ denote smoothing kernels in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively.

**Lemma 3.5** Let $v$ and $E$ be as in Theorem 1.1. Then for any function $f \in L^\infty([0, T], L^\infty_{\text{loc}}(\Omega))$, there exist two sequences $\varepsilon_n > 0$, $\mu_n > 0$, $\varepsilon_n \to 0$, $\mu_n \to 0$ such that the convergence (3.27) holds in $L^1([0, T], L^1_{\text{loc}}(\Omega))$.

**Proof of Lemma 3.4.** As in the proof of Lemma 3.3, we have to prove that $I \to 0$ in $L^1_{\text{loc}}(\Omega)$, with
\[
I = I^v + I^E , \quad (3.28)
\]
\[
I^v(x, \xi) = \int \int f(x-y, \xi-\eta) - f(x, \xi) [v(x-y, \xi-\eta) - v(x, \xi)] \nabla \rho_{\varepsilon}(y) \rho_{\mu}(\eta) dy d\eta , \quad (3.29)
\]
\[ I^E(x, \xi) = \iint \left[ f(x-y, \xi-\eta) - f(x, \xi) \right] \left[ E(x-y, \xi-\eta) - E(x, \xi) \right] \mu_0(y) \nabla \rho_\mu(\eta) dy d\eta. \] (3.30)

We can further decompose
\[ I^E = I^{E,x} + I^{E,\xi}, \] (3.31)
where \( I^{E,x} \) involves \( E(x-y, \xi) - E(x, \xi) \), and \( I^{E,\xi} \) involves \( E(x-y, \xi-\eta) - E(x-y, \xi) \). We consider an open set \( \omega \subset \subset \Omega \), and assume that \( \varpi + \sup \rho_\varepsilon \otimes \rho_\mu \subset \omega \subset \subset \Omega \). Then, since
\[ \| E(x-y, \xi) - E(x, \xi) \|_{L^1_{\varepsilon\xi}(\omega)} \leq \| \nabla_x E \|_{\mathcal{M}(\omega)} |y|, \quad |y| < \varepsilon, \] (3.32)
we get easily that
\[ \| I^{E,x} \|_{L^1_{\varepsilon\xi}(\omega)} \leq \frac{\varepsilon}{\mu} \left( \int |\mu \nabla \rho_\mu| \right) \| \nabla_x E \|_{\mathcal{M}(\omega)} \times \sup_{|b_\varepsilon| < \varepsilon, |b_\mu| < \mu} \| f(x-y, \xi-\eta) - f(x, \xi) \|_{L^\infty_{\varepsilon\xi}(\omega)}. \] (3.33)

Similarly, we get
\[ \| I^{E,\xi} \|_{L^1_{\varepsilon\xi}(\omega)} \leq \left( \int |\mu \nabla \rho_\mu| \right) \| \nabla_x \xi \|_{L^1(\omega_\varepsilon)} \times \sup_{|b_\varepsilon| < \varepsilon, |b_\mu| < \mu} \| f(x-y, \xi-\eta) - f(x, \xi) \|_{L^\infty_{\varepsilon\xi}(\omega)}, \] (3.34)
and
\[ \| I^v \|_{L^1_{\varepsilon\xi}(\omega)} \leq (1 + \mu/\varepsilon) \left( \int |\mu \nabla \rho_\mu| \right) \| \nabla_x \xi \|_{L^1(\omega_\varepsilon)} \times \sup_{|b_\varepsilon| < \varepsilon, |b_\mu| < \mu} \| f(x-y, \xi-\eta) - f(x, \xi) \|_{L^\infty_{\varepsilon\xi}(\omega)}. \] (3.35)

Let us choose \( \varepsilon_n/\mu_n = 1/n \). Since \( f \in L^\infty_{\text{loc}}(\Omega) \), (3.33) gives obviously that \( \| I^{E,x} \| \to 0 \) when \( n \to \infty \). Then, there exist \( E_n, v_n \) some smooth approximations to \( E, v \) such that
\[ \| \nabla_x \xi (v - v_n) \|_{L^1(\omega_\varepsilon)} \leq 1/n^2, \quad \| \nabla \xi (E - E_n) \|_{L^1(\omega_\varepsilon)} \leq 1/n. \] (3.36)

This enables to write \( \| I^{E,\xi} \|_{L^1(\omega)} \leq \| I^{E-E_n,\xi} \|_{L^1(\omega)} + \| I^{E,\xi} \|_{L^1(\omega)} \), \( \| I^v \|_{L^1(\omega)} \leq \| I^{v-v_n} \|_{L^1(\omega)} + \| I^{v_n} \|_{L^1(\omega)} \), and by estimates (3.34), (3.35) applied to \( E - E_n \) and \( v-v_n \), \( \| I^{E-E_n,\xi} \|_{L^1(\omega)} \to 0 \), \( \| I^{v-v_n} \|_{L^1(\omega)} \to 0 \). It remains to estimate \( \| I^{E_n,\xi} \|_{L^1(\omega)} \) and \( \| I^{v_n} \|_{L^1(\omega)} \). We introduce an exponent \( 1 \leq p < \infty \), and by estimates similar to those of Lemma 3.3 we obtain
\[ \| I^{E_n,\xi} \|_{L^1_{\varepsilon\xi}(\omega)} \leq \left( \int |\mu \nabla \rho_\mu| \right) \| \nabla \xi E_n \|_{L^p(\omega_\varepsilon)} \times \sup_{|b_\varepsilon| < \varepsilon, |b_\mu| < \mu} \| f(x-y, \xi-\eta) - f(x, \xi) \|_{L^p_{\varepsilon\xi}(\omega)}, \] (3.37)
\[ \|P^\nu\|_{L^1_{t,x}(\omega)} \leq (1 + \mu/\varepsilon) \left( \int |\varepsilon \nabla \rho_k| \right) \|\nabla_x \xi v_n\|_{L^1_{t,x}(\omega)} \times \sup_{|y|<\varepsilon,|z|<\mu} \|f(t, x - y, \xi - \eta) - f(t, x, \xi)\|_{L^1_{t,x}(\omega)}. \]  

(3.38)

But since

\[ \sup_{|y|<\varepsilon,|z|<\mu} \|f(t, x - y, \xi - \eta) - f(t, x, \xi)\|_{L^1_{t,x}(\omega)} \rightarrow 0, \quad \varepsilon, \mu \rightarrow 0, \]  

(3.39)

we can choose \( \varepsilon_n, \mu_n \) so small, with \( \varepsilon_n/\mu_n = 1/n \), that the right-hand sides of (3.37), (3.38) are less than \( 1/n \), and this completes the proof. \( \square \)

**Proof of Lemma 3.5.** Here we cannot apply Lebesgue’s theorem in time, because the sequences \( \varepsilon_n, \mu_n \) must be independent of \( t \). However, one can easily adapt the above proof of Lemma 3.4. Now, \( E_n, v_n \) are smooth functions of \( (t, x, \xi) \) and the estimates (3.36) hold in \( L^1_{t,x,\xi} \). The estimate (3.38) is replaced by

\[ \|P^\nu\|_{L^1_{t,x,\xi}} \leq (1 + \mu/\varepsilon) \left( \int |\varepsilon \nabla \rho_k| \right) \|\nabla_x \xi v_n\|_{L^1_{t,x,\xi}} \times \sup_{|y|<\varepsilon,|z|<\mu} \|f(t, x - y, \xi - \eta) - f(t, x, \xi)\|_{L^1_{t,x,\xi}}, \]  

(3.40)

and we conclude similarly. \( \square \)

**Remark 3.3** The idea of choosing \( \varepsilon_n, \mu_n \rightarrow 0 \), but at different rates is already present in the classical case of Section 3.1 in the time-dependent case. Indeed in (3.22) the time smoothing parameter \( \tau \) tends to 0 before that \( \varepsilon \) tends to 0. This can be interpreted as "\( \tau \rightarrow 0 \) infinitely faster than \( \varepsilon \rightarrow 0 \)" and this enables to avoid the time regularity of \( a \). In the same spirit, if in Theorem 1.1 we assume \( \nabla_x v = 0 \), we can even allow fields \( E \) such that only \( \nabla_x E \in L^1_{t,\xi} \), because the term in \( \mu/\varepsilon \) disappears in (3.35), and instead of (3.32) we have

\[ \|E(t, x - y, \xi) - E(t, x, \xi)\|_{L^1_{t,\xi}} \leq \iota(\varepsilon), \quad |y| < \varepsilon, \]  

(3.41)

with \( \iota(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). Therefore instead of choosing \( \varepsilon_n/\mu_n = 1/n \), we take \( \iota(\varepsilon_n)/\mu_n \leq 1/n \). In particular, by exchanging the role of \( x \) and \( \xi \), we deduce the same result for \( \partial_t + v(t, \xi) \cdot \nabla_x \) with \( v(t, \xi) \in L^1_{t,\xi} \). A similar technique is also used in [11] for piecewise smooth \( a \in BV \).

### 3.3 Time continuity with values in strong \( L^1 \) for BD coefficients

One of the consequences of the theory of [10] is that a function \( u(t, x) \) satisfying a transport equation with coefficient \( a \in L^1([0, T], W^{1,1}_{loc}) \) is necessarily strongly continuous with respect to time. This result is in fact not necessarily related to uniqueness for the Cauchy problem. According to the following theorem, it holds also for BD coefficients (see Lemma 3.7 for the definition), and in this situation uniqueness does not hold, even in one dimension.
Then, by (3.47) we deduce that $\nabla a(x) \cdot h \cdot h \in \mathcal{M}(\omega)$, $\|\nabla a(x) \cdot h \cdot h\|_{\mathcal{M}(\omega)} \leq C|h|^2$. Since this holds for any $\omega \subset \subset \Omega$, we conclude that $\nabla a(x) \cdot h \cdot h \in \mathcal{M}(\Omega)$, and that $\|\nabla a(x) \cdot h \cdot h\|_{\mathcal{M}(\Omega)} \leq C|h|^2$.

Let us now prove the converse, (ii) implies (iii). Let $\omega \subset \subset \Omega$, and define $a_n = \rho_n * a$, which is defined on $\omega$ for sufficiently large $n$. From (3.45) we
deduce \( \| \nabla a_n \cdot h \cdot h \|_{L^1(\omega)} \leq C|h|^2 \). If \( h \) is fixed and \( n \) large enough, one has 
\[ \nabla + B(0, |h| + 1/n) \subset \Omega, \]
thus
\[ \left\| [a_n(x+h) - a_n(x)] \cdot h \right\|_{L^1(\omega)} = \left\| \int_0^1 \nabla a_n(x + \theta h) \cdot h \cdot h d\theta \right\|_{L^1(\omega)} \leq C|h|^2. \quad (3.48) \]

By letting \( n \to \infty \), we obtain \( \left\| [a(x+h) - a(x)] \cdot h \right\|_{L^1(\omega)} \leq C|h|^2 \). \( \square \)

**Proof of Theorem 3.6.** It follows the lines of [10] and [8], with the trick of [6] that enables to only involve the symmetric part of \( \nabla a \). We define \( u_n = \rho_n \ast u \in C([0, T], C_x) \), and \( r = \partial_t u + \text{div}(au) \in L^1([0, T], L^1_{loc}) \). Thus

\[ \partial_t u_n + \text{div}[\rho_n \ast (au)] = \rho_n \ast r \quad \in L^1([0, T], L^1_{loc}). \quad (3.49) \]

We introduce \( a_k = \rho_k \ast a \), and write

\[ \partial_t u_n + \text{div}(a_k u_n) = \text{div}[a_k u_n - \rho_n \ast (au)] + \rho_n \ast r \quad \in L^1([0, T], L^1_{loc}). \quad (3.50) \]

Let us now consider a nonlinearity \( g : \mathbb{R} \to \mathbb{R} \) bounded, \( C^1 \) and Lipschitz continuous. We deduce from (3.50) that

\[ \partial_t [g(u_n)] + \text{div}[a_k g(u_n)] - g(u_n) \text{div} a_k = g'(u_n)[a_k \cdot \nabla u_n - \nabla \rho_n \ast (au) + \rho_n \ast r]. \quad (3.51) \]

Next, we integrate (3.51) against a test function \( \varphi(x) \in C_c^\infty \), which gives

\[ \frac{d}{dt} \int g(u_n) \varphi dx - \int a_k g(u_n) \cdot \nabla \varphi dx = \int g(u_n)(\text{div} a_k) \varphi dx + \int g'(u_n) \left[ a_k \cdot \nabla u_n - \nabla \rho_n \ast (au) + \rho_n \ast r \right] \varphi dx. \quad (3.52) \]

Therefore, \( d/dt(\int g(u_n) \varphi dx) \in L^1([0, T]) \), and by defining

\[ \alpha(t) = \sum_{1 \leq i, j \leq N} \int_{\omega} |\partial_j a_i + \partial_i a_j| \in L^1([0, T]), \quad (3.53) \]

we obtain

\[ \left| \frac{d}{dt} \int g(u_n) \varphi dx \right| \leq C\|a(t, .)\|_{L^1} + C \alpha(t) + C\|a_k \cdot \nabla u_n - \nabla \rho_n \ast (au) + \rho_n \ast r\|_{L^1(t, .)}. \quad (3.54) \]

Here and below \( L^1 \) norms are taken over \( x \in \omega \subset \subset \Omega \), a sufficiently large set containing \( \text{supp} \varphi \), that may change line to line. Now we let \( k \to \infty \), which yields that (3.54) holds with \( a \) instead of \( a_k \), and we write

\[ a \cdot \nabla u_n - \nabla \rho_n \ast (au) = \int u(t, x - h)[a(t, x) - a(t, x - h)] \cdot \nabla \rho_n(h) dh. \quad (3.55) \]
By choosing $\rho_n(x)$ depending only on $|x|$, we can use the characterization (iii) of Lemma 3.7, and get
\[ \|a \cdot \nabla u_n - \nabla \rho_n \ast (au)\|_{L^1} \leq C \alpha(t). \tag{3.56} \]

Thus (3.54) gives
\[ \left| \frac{d}{dt} \int g(u_n) \varphi \, dx \right| \leq C \left( \|a(t, \cdot)\|_{L^1} + \alpha(t) + \|r(t, \cdot)\|_{L^1} \right). \tag{3.57} \]

Therefore, for any $0 \leq t_1 \leq t_2 \leq T$,
\[ \left[ \int_{t_1}^{t_2} \int g(u_n) \varphi \, dx \right]_{t_1}^{t_2} \leq C \int_{t_1}^{t_2} \left( \|a(t, \cdot)\|_{L^1} + \alpha(t) + \|r(t, \cdot)\|_{L^1} \right) \, dt. \tag{3.58} \]

We finally let $n \to \infty$. Since for any $t \in [0, T]$, $u_n(t, \cdot) = \rho_n \ast u(t, \cdot) \to u(t, \cdot)$ a.e., we get
\[ \left[ \int_{t_1}^{t_2} \int g(u(t, x)) \varphi \, dx \right]_{t_1}^{t_2} \leq C \int_{t_1}^{t_2} \left( \|a(t, \cdot)\|_{L^1} + \alpha(t) + \|r(t, \cdot)\|_{L^1} \right) \, dt, \tag{3.59} \]

and this yields that $t \mapsto \int g(u(t, x)) \varphi(x) \, dx$ is continuous in $[0, T]$. This is true in particular for $g(u) = u^2$ since $u$ is bounded, thus $u \in C([0, T], L^2_{loc})$. \hfill \Box

References


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