One-dimensional transport equations with discontinuous coefficients

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Abstract

We consider one-dimensional linear transport equations with bounded but possibly discontinuous coefficient $a$. The Cauchy problem is studied from two different points of view. In the first case we assume that $a$ is piecewise continuous. We give an existence result and a precise description of the solutions on the lines of discontinuity. In the second case, we assume that $a$ satisfies a one-sided Lipschitz condition. We give existence, uniqueness and general stability results for backward Lipschitz solutions and forward measure solutions, by using a duality method. We prove that the flux associated to these measure solutions is a product by some canonical representative $\hat{a}$ of $a$.

Key-words. Linear transport equations, discontinuous coefficients, weak stability, duality, product of a measure by a discontinuous function, nonnegative solutions.

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Equations de transport unidimensionnelles à coefficients discontinus

Résumé
Nous considérons des équations de transport linéaires en une dimension avec coefficient $a$ borné mais éventuellement discontinu. Le problème de Cauchy est abordé par deux points de vue différents. Dans le premier cas nous supposons que $a$ est continu par morceaux. Nous donnons un résultat d’existence et une description précise des solutions sur les lignes de discontinuité. Dans le deuxième cas, nous supposons que $a$ vérifie une condition de Lipschitz d’un seul côté. Nous donnons des résultats d’existence, d’unicité et de stabilité faible générale pour les solutions rétrogrades lipschitziennes et les solutions directes mesurées par une méthode de dualité. Nous montrons que le flux associé à ces solutions mesurées est un produit par un représentant canonique $\tilde{a}$ de $a$.

Mots-clés. Equations de transport linéaires, coefficients discontinus, stabilité faible, dualité, produit d’une mesure par une fonction discontinue, solutions positives.
1 Introduction

This paper is devoted to one-dimensional homogeneous linear transport equations

$$\partial_t u + a(t, x) \partial_x u = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R},$$

(1.1)

with $T > 0$ and $a$ a given bounded coefficient. This equation will be referred as the nonconservative problem. By differentiating (1.1) with respect to $x$, we obtain the conservative problem

$$\partial_t \mu + \partial_x (a(t, x) \mu) = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R},$$

(1.2)

with $\mu = \partial_x u$. This type of equations appears naturally in the study of some systems of conservation laws where solutions belong to some measure space, as in H.C. Kranzer and B.L. Keyfitz [17], P. Le Floch [18], D. Tan, T. Zhang and Y. Zheng [25], Y. Zheng and A. Majda [26]. It is especially the case for the system of pressureless gases, see F. Bouchut [2], E. Grenier [14], Y. Brenier and E. Grenier [4], W. E, Y.G. Rykov and Y.G. Sinai [11]. Therefore, we are interested in solutions $\mu$ to (1.2) that are measures in $x$ (for a fixed time $t$), or solutions $u$ to (1.1) that are functions of bounded variation in $x$ (for a fixed time $t$). Notice that we have an $a$ priori estimate on the total mass of $\mu$ since by multiplying (1.2) by $\text{sgn} \mu$ we get

$$\partial_t |\mu| + \partial_x (a|\mu|) = 0,$$

(1.3)

and thus

$$\frac{d}{dt} \int_{\mathbb{R}} |\mu(t, dx)| = 0.$$

(1.4)

This computation is only formal, but in the cases of interest we can prove that (1.3) is actually true with an inequality (Theorem 4.3.6).

Both equations (1.1) and (1.2) also appear in problems of identification or control of the non-linearity of a scalar conservation law

$$\partial_t v + \partial_x f(v) = 0,$$

(1.5)

where the coefficient is given by $a = f'(v)$, see F. James and M. Sepúlveda [16]. Eq. (1.1) can actually be seen as a linearized version of (1.5). It should be noted that there is a very extended literature on non-linear problems such as (1.5), although very few papers consider linear equations. It is due to the very specific difficulty of (1.2) which lies in the product of the measure $\mu$ by the possibly discontinuous coefficient $a$. In this situation, the theory of R.J. DiPerna and P.-L. Lions [10] (see also I. Capuzzo Dolcetta and B. Perthame [5]) does not apply. In [9], G. Dal Maso, P. Le Floch and F. Murat have introduced a definition for the product $a \partial_x u$ when $a$ is a function of $u$ (and for vector valued functions). This condition happens to be well-suited to treat certain non-linear systems.
In this paper, we are interested in defining the product $a \mu$ when no \textit{a priori} relation is assumed between $a$ and $\mu$. We actually only use the implicit relation induced by the fact that $\mu$ solves (1.2). In order to handle this product, we use two different approaches.

In the first case, we use the classical product of a measure by a bounded Borel function, which is well-defined provided that $a$ is everywhere defined. We assume that $a$ is piecewise continuous, and we define \textit{admissibility conditions} ((AC1)-(AC2) in Section 3) which limit the possible behaviors of $a$ on discontinuity lines. In this context, we have an existence result for $BV_{loc}$ solutions to (1.1) with initial data $u^0 \in BV_{loc}(\mathbb{R})$. Moreover, we can give a very precise description of the solutions along discontinuity lines (Proposition 3.2). There is no uniqueness for a general piecewise smooth $a$. However, we prove that under the one-sided Lipschitz condition

$$\partial_x a \leq a(t) \quad \text{in} \quad ]0, T[ \times \mathbb{R}, \quad a \in L^1([0, T[),$$

(1.6)

uniqueness holds. We have similar results for (1.2) with initial data $\mu^0 \in \mathcal{M}_{loc}(\mathbb{R})$.

In the second case, we assume that

$$a \in L^\infty([0, T[ \times \mathbb{R})$$

(1.7)

only satisfies the one-sided Lipschitz condition (1.6). In the literature, this condition is often written under the equivalent form

$$a(x, t, y) \in [0, T[ \times \mathbb{R} \times \mathbb{R} \quad (a(t, x) - a(t, y))(x - y) \leq a(t)(x - y)^2,$$

(1.8)

with the same $a \in L^1([0, T[)$. We prove that the problems (1.1) and (1.2) are well-posed for Cauchy data $u^0 \in BV_{loc}(\mathbb{R})$ and $\mu^0 \in \mathcal{M}_{loc}(\mathbb{R})$ respectively. These solutions are understood in the \textit{duality} sense. This means that we consider locally Lipschitz solutions $p$ to

$$\partial_t p + a \partial_x p = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R},$$

(1.9)

with final data $p_T \in \text{Lip}_{loc}(\mathbb{R})$. Then, a formal computation shows that

$$\partial_t (p \mu) + \partial_x (a p \mu) = 0,$$

(1.10)

and hence

$$\frac{d}{dt}(\mu, p) = 0.$$  

(1.11)

This last formula makes up the definition of \textit{duality} solutions $\mu$ (see Definition 4.2.4). It is well-known that (1.6) ensures the existence of Lipschitz solutions to (1.9), see O.A. Oleinik [21], E.D. Conway [7], D. Hoff [15], E. Tadmor [23], P. Le Floch and Z. Xin [19]. However, there is no uniqueness. The corner stone of this paper is the introduction of the notion of \textit{reversible} solutions $p$ to (1.9), a class for which there is existence and uniqueness for the backward Cauchy problem. These \textit{reversible} solutions can be
characterized by various properties: support properties (Definition 4.1.4), monotonicity properties (Theorem 4.1.9), total variation properties (Proposition 4.1.7), entropy inequality or equality (Theorem 4.3.13). Then, only reversible solutions are taken into account in (1.11) for the definition of duality solutions (see Definition 4.2.4).

Finally, we prove that duality solutions $\mu$ actually satisfy in the distribution sense

$$\partial_t \mu + \partial_x (\hat{a} \mu) = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R}$$

(1.12)

for some canonical representative $\hat{a}$ of $a$ (Theorem 4.3.4), and this result answers the question raised by the product $a \times \mu$. In the piecewise continuous case, $\hat{a}$ can be computed explicitly.

Then, we have a general stability result (Theorem 4.3.2) which states that a bounded sequence of duality solutions $\mu_n$ to (1.2) with coefficient $a_n$ satisfying the one-sided Lipschitz condition (1.6) with $a_n$ bounded in $L^1([0, T[)$ will converge weakly to the duality solution $\mu$ associated to the $L^\infty - w^*$ limit $a$ of $(a_n)$. We also obtain that $\hat{a}_n \mu_n \to \hat{a} \mu$.

Notice that the above cited authors only used existence for the backward problem in order to get uniqueness for some non-linear problems. In [14], E. Grenier proves that the one-sided Lipschitz condition holds for the system of pressureless gases, with $a(t) = 1/t$. Therefore, our stability results apply to that system.

In the classical theory (i.e. for a smooth $a$), transport equations such as (1.1), (1.2) or (1.9) are closely related to characteristics. The flow $X(s,t,x)$ is defined to be the solution to the ODE

$$\begin{cases}
\frac{dX}{ds} = a(s, X), \\
X(t) = x.
\end{cases}$$

(1.13)

In this formulation, $t$ and $x$ are parameters (the initial time and initial position), and $s$ is the only variable. Then, the solution $p$ to (1.9) with data $p(s,.) = p^*$ (for a fixed $s$) is given by

$$p(t, x) = p^*(X(s, t, x)).$$

(1.14)

For discontinuous $a$, this equivalence between characteristics and transport equations is no longer valid. However, the theory of A.F. Filippov [12] ensures that (1.13) has got a solution in a generalized sense for bounded $a$. Moreover, if $a$ satisfies the one-sided Lipschitz condition (1.6), this solution is unique on the right (i.e. for $s > t$). This theory is used to study differential equations and inclusions, see J.-P. Aubin and A. Cellina [1], A.F. Filippov [13], and also in non-linear hyperbolic problems, see C.M. Dafermos [8], E. Tadmor and T. Tassa [24]. As noticed by E.D. Conway [7], this situation leads to the following paradox: when $a$ satisfies (1.6), uniqueness holds for Filippov solutions to the forward problem (1.13) (for $s > t$), although the corresponding backward problem (1.9) with final data $p^*$ (solved for $t < s$) can have many solutions. Similarly, the backward problem (1.13) (for $s < t$) can have many solutions, although duality can only provide uniqueness for the corresponding forward problem (1.1) with initial data $u^*$ (solved for $t > s$). In that case existence was an open problem.
In this paper, we give the following answer: the formula (1.14) with $X$ the Filippov flow (for $s > t$) actually gives the unique reversible backward solution to (1.9). As stated above, it is the only one that is stable by approximation of $a$. Notice that partial results in that direction were already obtained by Conway in [7]. By duality, we then obtain existence for the forward problem (1.1). Incidentally, we also obtain a stability result for forward Filippov solutions to (1.13) with the assumption that the sequence of coefficients $a_n$ satisfies (1.6) with $a_n$ bounded in $L^1$.

The paper is organized as follows. In Section 2, we introduce some notations and prove our main lemma which states that conservative and nonconservative equations are equivalent. We also prove two sharp results of uniqueness. Section 3 is concerned with the case of a piecewise continuous $a$. Finally, Section 4 is devoted to the case where $a$ satisfies the one-sided Lipschitz condition. In 4.1 we study the backward problem and Lipschitz solutions, in 4.2 we define duality solutions for the forward problem, 4.3 contains more sophisticated results and the relation with the generalized Filippov flow, and 4.4 is devoted to some comments about viscous problems.

The main results of this paper were announced in [3].

2 Notations and preliminaries

This section is devoted to notations that are used throughout the paper and to general results concerning equations (1.1) and (1.2).

2.1 Some notations

The following sets will often be used.

- $\mathcal{B}(Y, Z)$: set of bounded functions from $Y$ to $Z$,
- $C(Y, Z)$: set of continuous functions from $Y$ to $Z$,
- $C_c(Y, Z)$: set of continuous functions with compact support from $Y$ to $Z$,
- $\text{Lip}(Y, Z)$: set of Lipschitz continuous functions from $Y$ to $Z$,
- $\mathcal{M}(\mathbb{R})$: space of Borel measures on $\mathbb{R}$.

An index $c$ as for $C_c$ will always refer to functions of compact support, and an index $\text{loc}$ will refer to the corresponding local spaces. For example, $\mathcal{M}_{\text{loc}}(\mathbb{R})$ denotes the space of local Borel measures on $\mathbb{R}$, i.e. the space of distributions of order 0. In the above sets, a missing $Z$ (as in $C(Y)$) means that $Z = \mathbb{R}$. 
In the study of (1.1) and (1.2) we fix $T > 0$, and set
\[ \Omega = [0, T] \times \mathbb{R}. \] (2.1.1)

We introduce the four following spaces
\[ \mathcal{S}_{BV} = C([0, T], L^1_{loc}(\mathbb{R})) \bigcap \mathcal{B}([0, T], BV_{loc}(\mathbb{R})), \]
\[ \mathcal{S}_{M} = C([0, T], \mathcal{M}_{loc}(\mathbb{R})) - \sigma(\mathcal{M}_{loc}(\mathbb{R}), C_t(\mathbb{R})), \]
\[ \mathcal{S}_{Lip} = \text{Lip}_{loc}([0, T] \times \mathbb{R}), \]
\[ \mathcal{S}_{L^\infty} = C([0, T], L^\infty_{loc}(\mathbb{R})) - \sigma(L^\infty_{loc}(\mathbb{R}), L^1(\mathbb{R})). \] (2.1.2)

In (2.1.2), we denote by $\sigma(Y, Z)$ the weak topology on $Y$ generated by the natural bilinear form on $Y \times Z$.

We will sometimes use the space $L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$, which is defined as follows.

1. A function $f : [0, T] \rightarrow \mathcal{M}(\mathbb{R}^N)$ is said to be measurable if for any $\varphi \in C_0(\mathbb{R}^N)$, $t \mapsto \langle f(t), \varphi \rangle$ is a Borel function, where $C_0(\mathbb{R}^N)$ denotes the space of functions in $C(\mathbb{R}^N)$ that tend to 0 at $\infty$. Then, $\mathcal{M}([0, T], \mathcal{M}(\mathbb{R}^N))$ is defined to be the quotient of the space of measurable functions $[0, T] \rightarrow \mathcal{M}(\mathbb{R}^N)$ by the subspace of functions that vanish a.e. in $[0, T]$. Finally,

\[ L^\infty([0, T], \mathcal{M}(\mathbb{R}^N)) = \left\{ \mu \in \mathcal{M}([0, T], \mathcal{M}(\mathbb{R}^N)) : \text{ess sup}_{t \in [0, T]} \| \mu(t) \|_{\mathcal{M}(\mathbb{R}^N)} < \infty \right\}. \] (2.1.3)

It is a Banach space with the obvious norm. Let us now state without proof a few properties related to this space.

1. If $f : [0, T] \rightarrow \mathcal{M}(\mathbb{R}^N)$ is measurable, and $\varphi$ is a bounded Borel function $[0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$, then $t \mapsto \int_{\mathbb{R}^N} \varphi(t, x) f(t, dx)$ is a Borel function.

2. If $f : [0, T] \rightarrow \mathcal{M}(\mathbb{R}^N)$ is measurable, then $t \mapsto \| f(t, \cdot) \|$ is also measurable. Therefore, for any $\mu \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$, $|\mu| \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$.

3. If $\mu \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$, we define the measure $dt \mu(t, dx) \in \mathcal{M}_{loc}([0, T] \times \mathbb{R}^N)$ by

\[ \int_A dt \mu(t, dx) = \int_0^T dt \int_{\mathbb{R}^N} 1_A(t, x) \mu(t, dx). \]

Then for any bounded Borel function $\varphi : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$

\[ \int_{[0, T] \times \mathbb{R}^N} \varphi(t, x) dt \mu(t, dx) = \int_0^T dt \int_{\mathbb{R}^N} \varphi(t, x) \mu(t, dx). \]

Moreover, we have $|dt \mu(t, dx)| = dt |\mu(t, dx)|$.

4. If $\mu_n$ is a bounded sequence in $L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$, then there exists a measure $\mu \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$ and a subsequence $\mu_{n_k}$ such that $\mu_{n_k} \rightarrow \mu$ in the distribution sense in $[0, T] \times \mathbb{R}^N$.

5. Let $u \in C([0, T], L^1(\mathbb{R}^N))$. Then $u \in \text{Lip}_{loc}([0, T], L^1(\mathbb{R}^N))$ if and only if $\partial_t u \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))$. Moreover, we have $\| u \|_{\text{Lip}_{loc}([0, T], L^1(\mathbb{R}^N))} = \| \partial_t u \|_{L^\infty([0, T], \mathcal{M}(\mathbb{R}^N))}$. 

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2.2 From conservative to nonconservative by integration

Let $u \in \mathcal{S}_{BV}$ solve (1.1). Then it is obvious that $\mu = \partial_x u \in \mathcal{S}_M$ and solves (1.2). We here prove that conversely, it is possible to go from (1.2) to (1.1).

**Lemma 2.2.1** Let $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ a bounded Borel function, and assume that $\mu \in \mathcal{S}_M$ solves (1.2). Then there exists $u \in \mathcal{S}_{BV}$ solving (1.1) such that $\mu = \partial_x u$. Moreover, $u$ is unique up to an additive constant.

Here, (1.1) and (1.2) are understood in the usual distribution sense, and the product $a \times \mu$ is the usual product of a measure by a bounded Borel function.

**Remark 2.2.1** If $\mu$ has more regularity, we automatically get more regularity for $u$. For example, if $\mu \in \mathcal{S}_{L^\infty}$, then $u \in \mathcal{S}_{L^p}$.

**Proof of Lemma 2.2.1.** Uniqueness up to a constant is obvious. For existence, denote by $v(t, \cdot)$ the primitive of $\mu(t, \cdot)$ such that $v(t, 0^+) = 0$. Then $v \in \mathcal{S}_{BV}$ and $\partial_x v = \mu$,

$$\partial_x (\partial_t v + a \partial_x v) = 0 \quad \text{in} \quad \Omega.$$

Therefore, there exists $\psi \in \mathcal{D}'([0, T])$ such that

$$\partial_t v + a \partial_x v = \psi \quad \text{in} \quad \Omega.$$

Let $\varphi \in C^\infty_c (\mathbb{R})$. Then for any $\chi \in C^\infty_c ([0, T])$,

$$\left( \int \varphi \right) \langle \psi, \chi \rangle = \langle \partial_t v + a \partial_x v, \chi \otimes \varphi \rangle = -\iint v(t, x)\chi'(t)\varphi(x) \, dt \, dx + \int \chi(t) \, dt \int a(t, x)\varphi(x) \partial_x v(t, dx),$$

and thus

$$\left( \int \varphi \right) \psi(t) = \partial_t \int v(t, x)\varphi(x) \, dx + \int a(t, x)\varphi(x) \partial_x v(t, dx),$$

so that $\psi = \partial_t \eta$ with $\eta \in C([0, T])$. We finally define $u = v - \eta(t)$ which solves the problem. $\square$

2.3 Uniqueness when the coefficient is continuous with respect to $x$

**Lemma 2.3.1** Let $a \in L^\infty ([0, T] \times \mathbb{R})$ such that

$$a.e. \ t \in [0, T] \quad a(t, \cdot) \in C(\mathbb{R}). \quad (2.3.1)$$

Assume that $v \in \mathcal{S}_{L^\infty}$ solves

$$\begin{cases} 
\partial_t v + \partial_x (av) = 0 & \text{in} \quad [0, T] \times \mathbb{R}, \\
v(0, \cdot) = 0.
\end{cases} \quad (2.3.2)$$

Then $v = 0$. 

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\textbf{Proof.} By Lemma 2.2.1 (and Remark 2.2.1), there exists \( u \in \mathcal{S}_{\text{Lip}} \) solving (1.1) such that \( \partial_x u = v \). Since \( u \) is defined up to a constant and \( \partial_x u(0,.) = v(0,.) = 0 \), we can assume that \( u(0,.) = 0 \). Let \( \rho_n \) be a smoothing sequence in \( \mathbb{R}^2, \rho_n(t,x) = \rho_n'(t)\rho_n(x) \), with \( \text{supp} \rho_n \subset ]-\infty,0[ \). Let us define \( u_n = \rho_n \circ u \in \text{Lip}_{\text{loc}}([0,T-\varepsilon] \times \mathbb{R}) \) (for a fixed \( \varepsilon \in ]0,T[ \) such that \( 1/n \leq \varepsilon \)), and \( a_n = \rho_n \ast a \in L^\infty([0,T-\varepsilon[ \times \mathbb{R}) \). We have

\[
\partial_t u_n + \rho_n \ast (a \partial_x u) = 0 \quad \text{in} \quad ]0,T-\varepsilon[ \times \mathbb{R},
\]

so that

\[
\partial_t u_n + a_n \partial_x u_n = f_n \quad \text{in} \quad ]0,T-\varepsilon[ \times \mathbb{R},
\]

with

\[
f_n = a_n \partial_x u_n - \rho_n \ast (a \partial_x u).
\]

Since \( a_n, u_n, f_n \in C^\infty([0,T-\varepsilon] \times \mathbb{R}) \) we can consider the flow \( X_n(s,t,x) \) of \( a_n \), and for \( 0 \leq t \leq T-\varepsilon \) we have

\[
\begin{align*}
\partial_t u_n + a_n \partial_x u_n & = f_n(s,X_n(s,t,x)) + \int_0^s f_n(s,X_n(s,t,x)) \, ds, \\
\end{align*}
\]

Therefore for any \( 0 < t \leq T-\varepsilon \) and \( x_1 < x_2 \)

\[
\begin{align*}
\| u_n(t,\cdot) \|_{L^\infty([x_1,x_2])} & \leq \| u_n(0,\cdot) \|_{L^\infty([x_1-\varepsilon,x_2+\varepsilon])} + \int_0^t \| f_n(s,\cdot) \|_{L^\infty([x_1-\varepsilon,x_2+\varepsilon])} \, ds,
\end{align*}
\]

Now since \( u_n \to u \) locally uniformly in \( [0,T-\varepsilon] \times \mathbb{R} \), it is enough to prove that for any \( x_1 < x_2 \)

\[
\begin{align*}
\int_0^{T-\varepsilon} \| f_n(s,\cdot) \|_{L^\infty([x_1,x_2])} \, ds & \to 0.
\end{align*}
\]

We have

\[
f_n = (a_n - a) \partial_x u_n + \left[ a(\rho_n \ast \partial_x u) - \rho_n \ast (a \partial_x u) \right] = f^n_1 + f^n_2.
\]

In order to treat the first term \( f^n_1 \), we bound \( \partial_x u_n \) by a constant, and \( a_n \to a \to 0 \) in \( L^1([0,T-\varepsilon[, L^\infty([x_1,x_2]]) \) because by (2.3.1), \( a \) can be approached in \( L^1([0,T[, L^\infty_{\text{loc}}(\mathbb{R})) \) by \( C^\infty \) functions. For the second term,

\[
\begin{align*}
f^n_2 & = \int [a(t,x) - a(t-\tau,x-y)] \partial_x u(t-\tau,x-y) \rho_n(\tau,y) \, d\tau dy,
\end{align*}
\]

\[
\begin{align*}
\| f^n_2 \|_{L^1([0,T-\varepsilon[, L^\infty([x_1,x_2]))} & \leq \| \partial_x u \|_{L^\infty([0,T-\varepsilon[, L^\infty([x_1-\varepsilon,x_2+\varepsilon]))} \times \sup_{-1/n \leq \tau \leq 0} \| a(t,x) - a(t-\tau,x-y) \|_{L^1([0,T-\varepsilon[, L^\infty([x_1,x_2]))}
\end{align*}
\]

which tends to 0 by the same approximation argument. Therefore, (2.3.6) is proved. \( \square \)
2.4 Uniqueness for bounded nonnegative solutions

We prove here a uniqueness result for a bounded coefficient, by a slight modification of the method of T.-P. Liu and M. Pierre [20].

**Lemma 2.4.1** Let $a \in L^\infty ([0, T] \times \mathbb{R})$ and $v \in \mathcal{S}_{L^\infty}$ solve

$$
\partial_t v + \partial_x (av) = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R},
$$

with $v(0, .) = 0$. Assume moreover that there exists $\tilde{v} \in \mathcal{S}_{L^\infty}$ some nonnegative solution to (2.4.1) such that

$$|v| \leq \tilde{v}. \quad (2.4.2)$$

Then $v = 0$.

In particular, notice that condition (2.4.2) is fulfilled if $v = v_1 - v_2$ with $v_1, v_2 \in \mathcal{S}_{L^\infty}$ some nonnegative solutions to (2.4.1) (take $\tilde{v} = v_1 + v_2$). Therefore, if $v_1, v_2 \in \mathcal{S}_{L^\infty}$ are two nonnegative solutions to (2.4.1) and have the same initial data, then they are equal.

**Proof of Lemma 2.4.1.** By Lemma 2.2.1 (and Remark 2.2.1), there exists $u \in \mathcal{S}_{Lip}$ solving (1.1) such that $\partial_x u = v$. Since $u$ is defined up to a constant and $\partial_x u(0, .) = v(0, .) = 0$, we can assume that $u(0, .) = 0$. Let us define $\tilde{v}_n = \rho_n * \tilde{v}$ and $b_n = \rho_n * (av)$ (with $\rho_n(t, x)$ a smoothing sequence in $\mathbb{R}^2$), so that

$$
\partial_t \tilde{v}_n + \partial_x b_n = 0 \quad \text{in} \quad ]\varepsilon, T - \varepsilon[ \times \mathbb{R} \quad (2.4.3)
$$
as soon as $1/n < \varepsilon$. Since $u$ is locally Lipschitz continuous, we can multiply (1.1) by $\text{sgn} u$ to get

$$
\partial_t |u| + a \partial_x |u| = 0. \quad (2.4.4)
$$

Now (2.4.3) and (2.4.4) yield

$$
\int_{x_1}^{x_2} \tilde{v}_n(t, x)|u(t, x)| \, dx \leq \int_{x_1 - \|a\|_{L^\infty}(t_2 - t_1)}^{x_2 + \|a\|_{L^\infty}(t_2 - t_1)} \tilde{v}_n(t_1, x)|u(t_1, x)| \, dx \quad + \quad \int_{t_1 \leq t < t_2} (b_n - a \tilde{v}_n) \partial_x |u| \, dt \, dx.
$$

By letting $n \to \infty$ we get

$$
\int_{x_1}^{x_2} \tilde{v}(t, x)|u(t, x)| \, dx \leq \int_{x_1 - \|a\|_{L^\infty}(t_2 - t_1)}^{x_2 + \|a\|_{L^\infty}(t_2 - t_1)} \tilde{v}(t_1, x)|u(t_1, x)| \, dx,
$$

\[10\]
for any $x_1 < x_2$ and $0 < t_1 \leq t_2 < T$. Then we let $t_1 \to 0$ and obtain for $0 < t_2 < T$

$$\int_{x_1}^{x_2} \tilde{v}(t_2, x) |u(t_2, x)| \, dx \leq 0,$$

so that by the nonnegativity of $\tilde{v}$

$$\forall \, 0 < t < T, \quad a.e. \ x \quad \tilde{v}(t, x) |u(t, x)| = 0.$$ 

Now since $|\partial_x u| = |v| \leq \tilde{v}$, we get $u \partial_x u = 0$ a.e., and (1.1) gives also $u \partial_t u = 0$ a.e. Therefore, $\partial_x u^2 = 0$ and $\partial_t u^2 = 0$ so that $u^2 = \text{cst} = 0$, and hence $u = 0$, $v = \partial_x u = 0$. \Box

3 Transport equations with piecewise continuous coefficient

In this section $a: \Omega = ]0, T[ \times \mathbb{R} \to \mathbb{R}$ is a bounded Borel function (everywhere defined), and we consider solutions $u \in S_{BV}$ to (1.1), where the product is understood in the usual sense since $\partial_x u \in L^\infty(]0, T[; \mathcal{M}_{loc}(\mathbb{R})) \subset \mathcal{M}_{loc}(\Omega)$. We assume in all Section 3 that $a$ satisfies the following piecewise continuity condition: $\Omega = \mathcal{C} \cup \mathcal{D} \cup \mathcal{S}$ with $\mathcal{C}$, $\mathcal{D}$, $\mathcal{S}$ some disjoint sets and

\begin{enumerate}
    \item[(PC1)] $\mathcal{C}$ is open, $a$ is continuous in $\mathcal{C}$,
    \item[(PC2)] $\mathcal{S}$ is locally finite in $\Omega$,
    \item[(PC3)] $\mathcal{D}$ is a one-dimensional submanifold of class $C^1$ of $\Omega$, such that $a$ has got limit values at $(t_0, x_0) \in \mathcal{D}$ on both sides of $\mathcal{D}$. These limits are denoted by $a_-(t_0, x_0)$ and $a_+(t_0, x_0)$, and it is assumed that they can be locally chosen continuous with respect to $(t_0, x_0) \in \mathcal{D}$.
\end{enumerate}

Since the values of $a$ on $\mathcal{D}$ cannot be arbitrary, we have to prescribe them where the measure $dt \partial_x u(t, dx)$ can concentrate. We thus assume that the following admissibility conditions are fulfilled for any $(t_0, x_0) \in \mathcal{D}$.

\begin{enumerate}
    \item[(AC1)] $a(t_0, x_0) \in [a_-(t_0, x_0), a_+(t_0, x_0)]$,
    \item[(AC2)] if $(dt) \partial_x u(t_0, x_0) \neq 0$ and $\left(\frac{dx}{dt}\right)_\mathcal{D}(t_0, x_0) \in [a_-(t_0, x_0), a_+(t_0, x_0)]$ then $a(t_0, x_0) = \left(\frac{dx}{dt}\right)_\mathcal{D}(t_0, x_0)$.
\end{enumerate}

Remark 3.1 Notice that $\mathcal{C} \cup \mathcal{D}$ is an open set.

Remark 3.2 The values of $a$ on the set $\mathcal{S}$ do not matter, neither in the assumptions nor in Eq. (1.1).

Remark 3.3 Any function $a \in C_b(\Omega)$ (continuous and bounded) verifies all the above hypotheses.
**Remark 3.4** For a given function $a$, the sets $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{S}$ are not uniquely determined. For example, it is possible to add artificial lines of discontinuity. Then since Eq. (1.1) does not depend on this choice it is a way to get informations on the behavior of $u$ along any curve in $\mathcal{C}$ (by Lemma 3.1 and Proposition 3.2 below).

With (PC1)-(PC3) and (AC1)-(AC2), we are able to characterize the solutions to (1.1) (Proposition 3.2) and to prove an existence result (Theorem 3.4). Let us begin by a lemma.

**Lemma 3.1** If $u \in \mathcal{S}_{BV}$ solves (1.1), then the measure $dt \partial_x u(t, dx)$ does not concentrate in the set of inconsistency

$$\mathcal{D}_{uc} = \left\{ (t_0, x_0) \in \mathcal{D} : (dt)_{\mathcal{D}}(t_0, x_0) \neq 0 \text{ and } a(t_0, x_0) \neq \left( \frac{dx}{dt} \right)_D(t_0, x_0) \right\}. \quad (3.1)$$

More precisely, it means that $\int_{\mathcal{D}_{uc}} |dt \partial_x u(t, dx)| = 0$.

**Proof of Lemma 3.1.** Let $(t_0, x_0) \in \mathcal{D}$ such that $(dt)_{\mathcal{D}}(t_0, x_0) \neq 0$. Then $\mathcal{D}$ is defined by a curve $x = \xi(t)$ in a neighborhood $V$ of $(t_0, x_0)$, with $\xi \in C^1$. Let us denote by $\zeta_\varepsilon(x)$ a cutoff function, $\zeta_\varepsilon(x) = \zeta_1(x/\varepsilon)$, $\zeta_1 \in C^\infty(I - 1, 1]$, $0 \leq \zeta_1 \leq 1$, $\zeta_1(x) = 1$ if $|x| \leq 1/2$. Then we have

$$\zeta_\varepsilon(x - \xi(t)) (\partial_t u + a \partial_x u) = 0 \quad \text{in} \quad V. \quad (3.2)$$

Since by Lebesgue’s theorem $\zeta_\varepsilon(x - \xi(t)) a(t, x) \partial_x u \to \mathbb{D} a \partial_x u$ when $\varepsilon \to 0$, and similarly

$$\zeta_\varepsilon(x - \xi(t)) \partial_t u = \partial_t (\zeta_\varepsilon(x - \xi(t)) u) + \zeta_\varepsilon'(x - \xi(t)) \xi'(t) u$$

$$= \partial_t (\zeta_\varepsilon(x - \xi(t)) u) + \partial_x (\zeta_\varepsilon(x - \xi(t)) \xi'(t) u) - \zeta_\varepsilon(x - \xi(t)) \xi'(t) \partial_x u$$

$$\to \partial_t (\mathbb{D} u) + \partial_x (\mathbb{D} \zeta'(t) u) - \mathbb{D} \zeta'(t) \partial_x u, \quad (3.3)$$

we get by letting $\varepsilon \to 0$ in (3.2) and by using that $\text{meas}(\mathcal{D}) = 0$

$$\mathbb{D} \left( a(t, x) - \left( \frac{dx}{dt} \right)_D(t, x) \right) \partial_x u = 0 \quad \text{in} \quad V. \quad (3.4)$$

The result follows easily. \qed

Notice that the above proof does not use the admissibility conditions.

**Proposition 3.2 (Characterization of solutions)** Let $u \in \mathcal{S}_{BV}$. Then $u$ solves (1.1) if and only if the three following conditions are satisfied

(i) $u \in \text{Lip}([0, T], L^1_{loc}(\mathbb{R}))$, \hspace{1cm} (3.5)

(ii) $\partial_t u + a \partial_x u = 0$ in $\mathcal{C}$, \hspace{1cm} (3.6)

(iii) the measure $dt \partial_x u(t, dx)$ does not concentrate in the set of transversality

$$\mathcal{D}_{tr} = \left\{ (t_0, x_0) \in \mathcal{D} : (dt)_{\mathcal{D}}(t_0, x_0) \neq 0 \text{ and } \left( \frac{dx}{dt} \right)_D(t_0, x_0) \notin [a_-(t_0, x_0), a_+(t_0, x_0)] \right\}. \quad (3.7)$$

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Proof. First, if \( u \) solves (1.1), then (ii) is obvious, (i) is deduced from the equation (1.1) and Property 5 in §2.1, and (iii) results from Lemma 3.1 and (AC1).

Let us now assume that (i), (ii), (iii) are verified. By (i) and Property 5 in §2.1, we have \( \partial_t u \in L^\infty ([0, T[. M_{loc}(\mathbb{R})) \) and thus \( \partial_t u \) and \( \partial_x u \) do not concentrate on points. Then since \( S \) is locally finite we just have to prove that \( \partial_t u + a \partial_x u = 0 \) in the open set \( \mathcal{C} \cup \mathcal{D} \). But (ii) indicates that it holds in \( \mathcal{C} \). Therefore, we take \((t_0, x_0) \in \mathcal{D} \) and we just have to prove that \( \partial_t u + a \partial_x u = 0 \) in a neighborhood \( V \) of \((t_0, x_0) \). We distinguish two cases.

1\textsuperscript{st} case: \( (dt)_D (t_0, x_0) \neq 0 \).

As in the proof of Lemma 3.1, we have

\[ \zeta (x - \xi(t)) (\partial_t u + a \partial_x u) \rightarrow \mathbb{I}_D \left( a(t, x) - \frac{d x}{d t}_D (t, x) \right) \partial_x u \quad \text{in} \quad V. \]  \hfill (3.8)

Since by (ii) \( [1 - \zeta (x - \xi(t))] (\partial_t u + a \partial_x u) = 0 \) in \( V \), we obtain by addition

\[ \partial_t u + a \partial_x u = \mathbb{I}_D \left( a(t, x) - \frac{d x}{d t}_D (t, x) \right) \partial_x u \quad \text{in} \quad V. \]  \hfill (3.9)

Then by (iii) and (AC2) the right-hand side vanishes and we get the result.

2\textsuperscript{nd} case: \( (dt)_D (t_0, x_0) = 0 \).

Let \( V \) be a small open set containing \((t_0, x_0) \), and define

\[ N = \{(t, x) \in V \cap \mathcal{D} \; ; \; (dt)_D (t, x) = 0 \}. \]  \hfill (3.10)

By the first case, we know that the measure \( \partial_t u + a \partial_x u \) vanishes in the open set \( V \setminus N \). It remains to prove that \( \lambda = \partial_t u + a \partial_x u \) does not concentrate in \( N \). Since \( \lambda \in L^\infty ([0, T[. M_{loc}(\mathbb{R})) \) we have

\[ \iint_N |dt \lambda (t, dx)| = \int dt \int \mathbb{I}_N (t, x) |\lambda (t, dx)|, \]  \hfill (3.11)

and by Sard’s lemma \( \{t \in \mathbb{R} \; ; \; \exists x \in \mathbb{R} \; (t, x) \in N \} \) = 0. This proves that the right-hand side of (3.11) vanishes, and the proof is complete. \( \square \)

**Proposition 3.3 (Renormalization)** If \( u \in S_{BV} \) solves (1.1) then for any \( S : \mathbb{R} \rightarrow \mathbb{R} \) Lipschitz continuous, \( S(u) \in S_{BV} \) solves (1.1).

Before proving Proposition 3.3, let us state a simple estimate, which proof is easy: if \( v \in BV_{loc}(\mathbb{R}) \) and \( S : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous, then \( S(v) \in BV_{loc}(\mathbb{R}) \) and \( |\partial_x [S(v)]| \leq \text{Lip}(S) |\partial_x v| \) in the sense of measures.

**Proof of Proposition 3.3.** By the above mentioned estimate, we have \( S(u) \in S_{BV} \), \( |\partial_x [S(u)]| \leq \text{Lip}(S) |\partial_x u| \). By Proposition 3.2, we have to prove that \( S(u) \) verifies conditions (i), (ii), (iii). Since \( u \) verifies these properties, it is obvious with the above
estimate that (i) and (iii) hold. It remains to prove (ii), and since \( a \) is continuous in \( C \), it is easy to reduce the problem to the case where \( S \in C^1 \).

Therefore, let us prove that \( \partial_t [S(u)] + a \partial_x [S(u)] = 0 \) in \( C \) with \( S \in C^1 \). Consider an open subset \( \omega \) of \( C \) with compact closure in \( C \), and \( a_\varepsilon(t, x) \) a sequence of bounded \( C^1 \) functions such that \( a_\varepsilon \to a \) locally uniformly in \( C \). Since \( u \) solves (1.1) we have

\[
\partial_t u + a_\varepsilon \partial_x u = (a_\varepsilon - a) \partial_x u = \mu_\varepsilon
\]

(3.12)

with \( \mu_\varepsilon \) a locally bounded measure, and \( \mu_\varepsilon \to 0 \) strongly in \( \omega \). Let us define \( u_n = \rho_n(x) \ast u \), with \( \rho_n \) a standard smoothing sequence in \( \mathbb{R} \). By using an estimate from the paper of R.J. DiPerna and P.-L. Lions [10], we have

\[
\rho_n \ast (a_\varepsilon \partial_x u) - a_\varepsilon \partial_x u_n = r_{\varepsilon,n} \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\Omega),
\]

(3.13)

and since

\[
\partial_t u_n + a_\varepsilon \partial_x u_n = \rho_n \ast \mu_\varepsilon - r_{\varepsilon,n},
\]

we have

\[
\partial_t [S(u_n)] + a_\varepsilon \partial_x [S(u_n)] = S'(u_n) \rho_n \ast \mu_\varepsilon - S'(u_n) r_{\varepsilon,n}.
\]

Then we let \( n \to \infty \) and obtain by (3.13)

\[
\partial_t [S(u)] + a_\varepsilon \partial_x [S(u)] = \lambda_{S,\varepsilon},
\]

with \( \lambda_{S,\varepsilon} \) a locally bounded measure, such that \( \lambda_{S,\varepsilon} \to 0 \) in \( \omega \) strongly when \( \varepsilon \to 0 \). We finally let \( \varepsilon \to 0 \), and using that \( a_\varepsilon \to a \) uniformly in \( \omega \) we obtain

\[
\partial_t [S(u)] + a \partial_x [S(u)] = 0 \quad \text{in} \quad \omega,
\]

which ends the proof. \( \Box \)

**Theorem 3.4 (Existence)** For any \( u^0 \in BV_{\text{loc}}(\mathbb{R}) \) there exists \( u \in S_{BV} \) solving (1.1) with \( u(0,.) = u^0 \), and such that for any \( x_1 < x_2 \) and \( t \in [0,T] \)

\[
TV_I(u(t,\cdot)) \leq TV_I(u^0),
\]

(3.14)

\[
\|u(t,\cdot)\|_{L^\infty(I)} \leq \|u^0\|_{L^\infty(I)}
\]

(3.15)

with \( I = ]x_1, x_2[ \) and \( J = ]x_1 - \|a\|_\infty t, x_2 + \|a\|_\infty t[ \).

**Proof.** Let \( a_\varepsilon(t, x) \) a sequence of bounded \( C^1 \) functions such that \( a_\varepsilon \to a \) locally uniformly in \( C \) (for example a convolution of \( a \) by a smoothing kernel in \( (t, x) \)), and \( u^0_\varepsilon \in C^1(\mathbb{R}) \) a bounded sequence in \( BV_{\text{loc}}(\mathbb{R}) \) such that \( u^0_\varepsilon \to u^0 \). Let us define \( u_\varepsilon \) as the (classical) solution to

\[
\partial_t u_\varepsilon + a_\varepsilon \partial_x u_\varepsilon = 0 \quad \text{in} \quad \Omega, \quad u_\varepsilon(0,\cdot) = u^0_\varepsilon.
\]

(3.16)
We are going to prove that for suitable \( a_\epsilon \) and \( u_0^0 \) (such as convolutions of \( a \) and \( u^0 \)), and after extraction of a subsequence, \( u_\epsilon \) tends to a solution. We have \( u_\epsilon(t, x) = u_0^0(X_\epsilon(0, t, x)) \) with \( X_\epsilon(s, t, x) \) the solution to

\[
\partial_s X_\epsilon = a_\epsilon(s, X_\epsilon), \quad X_\epsilon(t, t, x) = x, \tag{3.17}
\]

and thus \( \partial_s u_\epsilon = \partial_s u_0^0(X_\epsilon(0, t, x)) \partial_s X_\epsilon(0, t, x) \). We have

\[
\begin{align*}
\partial_s \partial_x X_\epsilon &= \partial_s a_\epsilon(s, X_\epsilon) \partial_x X_\epsilon, \\
\partial_x X_\epsilon &= \exp \left[ \int_t^s \partial_x a_\epsilon(\tau, X_\epsilon(\tau, t, x)) \, d\tau \right] > 0, \\
X_\epsilon &= x + \int_t^s a_\epsilon(\tau, X_\epsilon(\tau, t, x)) \, d\tau, \\
|X_\epsilon - x| &\leq \| a \| \infty |s - t|.
\end{align*}
\]

Therefore, \( |\partial_s u_\epsilon| = |\partial_s u_0^0|(X_\epsilon(0, t, x)) |\partial_s X_\epsilon(0, t, x) | \), and

\[
\int_{x_1}^{x_2} |\partial_x u_\epsilon| \, dx = \int_{X_\epsilon(0, t, x_2)}^{X_\epsilon(0, t, x_1)} |\partial_x u_0^0|(x) \, dx \leq \int_{x_1}^{x_2 + \| a \| \infty t} |\partial_x u_0^0|(x) \, dx. \tag{3.19}
\]

Hence, we conclude that \( u_\epsilon \) is bounded in \( S_{BV} \). Then, since \( \partial_t u_\epsilon = -a_\epsilon \partial_x u_\epsilon \) is bounded in \( C([0, T], L^1_{loc}(\mathbb{R})) \), we get by Property 5 in §2.1 that \( u_\epsilon \) is bounded in \( \text{Lip}([0, T], L^1_{loc}(\mathbb{R})) \). Therefore after extraction of a subsequence

\[
u_\epsilon \rightarrow u \quad \text{in} \quad C([0, T], L^1_{loc}(\mathbb{R})), \tag{3.20}
\]

with \( u \in S_{BV} \cap \text{Lip}([0, T], L^1_{loc}(\mathbb{R})) \). It is easy to check that \( u(0, \cdot) = u^0 \) and that \( u \) verifies (3.14)-(3.15) (if the variation of \( u_\epsilon^0 \) is not too large, for example \( u_\epsilon^0 = \rho_\epsilon * u^0 \)). Since \( a_\epsilon \to a \) locally uniformly in \( C \), we obtain

\[
\partial_t u + a \partial_x u = 0 \quad \text{in} \quad C. \tag{3.21}
\]

By Proposition 3.2, it now remains to prove (iii). We are going to prove that property by a dispersion estimate.

Let us consider an open set \( V \ni (t_0, x_0) \) with \( (t_0, x_0) \in D \), \( V \) as small as we want, such that for any \( (t, x) \in V \cap D \)

\[
(dt)_{\mathcal{D}}(t, x) \neq 0 \quad \text{and} \quad \left( \frac{dx}{dt} \right)_{\mathcal{D}}(t, x) \notin [a_-(t, x), a_+(t, x)]. \tag{3.22}
\]

Then \( D \) is defined in \( V \) by a curve \( x = \xi(t) \), \( \xi \in C^1 \). Let \( s_1 < t_0 < s_2 \) such that \( (t, \xi(t)) \in V \) for \( s_1 \leq t \leq s_2 \), \( \chi \in C_c([s_1, s_2]) \) a given test function, and define

\[
I = \int \chi(t) \Pi_{(t, x) \in V \cap D} dt \partial_x u(t, x). \tag{3.23}
\]

We have to prove that \( I = 0 \). We have \( I = \lim_{\eta \to 0} I_\eta \), \( I_\eta = \lim_{\epsilon \to 0} I_{\eta \epsilon} \), with

\[
I_\eta = \int \chi(t) \, dt \int \zeta_\eta(x - \xi(t)) \partial_x u(t, x) \, dx = \left\langle \partial_x u, \chi(t) \zeta_\eta(x - \xi(t)) \right\rangle,
\]

\[
I_{\eta \epsilon} = \left\langle \partial_x u_\epsilon, \chi(t) \zeta_\eta(x - \xi(t)) \right\rangle = \int \chi(t) \, dt \int \zeta_\eta(x - \xi(t)) \partial_x u_\epsilon(t, x) \, dx. \tag{3.23}
\]

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Here $\zeta_\eta$ is the same cutoff function as in Lemma 3.1. In order to use a duality formula, let us define the test function $\varphi_{\eta z} = \zeta_\eta(X_z(s, t, x) - \xi(s))$, which is compactly supported in the $x$ variable and solves
\[
\partial_t \varphi_{\eta z} + a_z(t, x)\partial_x \varphi_{\eta z} = 0, \quad \varphi_{\eta z}(s, s, x) = \zeta_\eta(x - \xi(s)). \tag{3.24}
\]
Since $\partial_x u_z(t, x) = \partial_x u_z(s, X_z(s, t, x)) \partial_x X_z(s, t, x)$, we have
\[
\begin{align*}
I_{\eta z} &= \int \varphi_{\eta z}(s, t, x) \partial_x u_z(t, x) \, dx \\
&= \int \zeta_\eta(x - \xi(s)) \varphi_{\eta z}(s, t, x) \partial_x u_z(t, x) \, dx \\
&= \int \zeta_\eta(x - \xi(s)) \partial_x u_z(s, X_z(s, t, x)) \partial_x X_z(s, t, x) \, dx \\
&= \int \zeta_\eta(x - \xi(s)) \partial_x u_z(s, x) \, dx,
\end{align*}
\]
which is independent of $t$. Then
\[
I_{\eta z} = \int \chi(s) \, ds \int \zeta_\eta(x - \xi(s)) \partial_x u_z(s, x) \, dx
\]
\[
= \int \chi(s) \, ds \int \varphi_{\eta z}(s, t, x) \partial_x u_z(t, x) \, dx
\]
\[
= \int \partial_x u_z(t, x) \psi_{\eta z}(x) \, dx
\]
\[
= \int \partial_x u_z(t, x) \psi_{\eta z}(x) \, dx,
\]
with
\[
\psi_{\eta z}(x) = \int \chi(s) \varphi_{\eta z}(s, t, x) \, ds
\]
\[
= \int \chi(s) \zeta_\eta(X_z(s, t, x) - \xi(s)) \, ds. \tag{3.26}
\]

Since $\text{supp} \psi_{\eta z}$ remains in a compact set independent of $\eta$ and $\varepsilon$, and because of (3.25), it is enough to prove that $\|\psi_{\eta z}\|_{L^\infty} \leq C\eta$.

By (3.21) and by reducing $V$ if necessary, we can assume that for some $\sigma \in \{-1, 1\}$ and $\alpha > 0$
\[
\forall (t, x) \in V_-, \quad \sigma(a_z(t, x) - \xi'(t)) \geq \alpha,
\]
\[
\forall (t, x) \in V_+, \quad \sigma(a_z(t, x) - \xi'(t)) \geq \alpha. \tag{3.27}
\]

Then by reducing $V$ and $\alpha$ again, for small enough $\varepsilon$ and for a well chosen $a_z$ (such as a convolution in $t, x$ of $\zeta$), we get
\[
\forall (t, x) \in V \quad \sigma(a_z(t, x) - \xi'(t)) \geq \alpha. \tag{3.28}
\]
Let us denote $\delta_{\varepsilon, \eta}(s) = X_z(s, t_0, x) - \xi(s)$. The set $J_{\eta z}(x) = \{s \in ]s_1, s_2[; \ |\delta_{\varepsilon, \eta}(s)| < \eta\}$ is open, and for small $\eta$ we have
\[
\forall s \in J_{\eta z}(x) \quad (s, X_z(s, t_0, x)) \in V,
\]
and thus by (3.28)
\[
\forall s \in J_{\eta z}(x) \quad \sigma \delta''_{\varepsilon, \eta}(s) = \sigma (a_z(s, X_z(s, t_0, x)) - \xi'(s)) \geq \alpha.
\]
This implies that $J_{te}(x)$ is an interval, and $\delta_{t,\bar{x}}$ is a diffeomorphism from $J_{te}(x)$ onto its image. We have

$$\psi_{te}(x) = \int_{J_{te}(x)} \chi(s) \zeta_\eta(\delta_{t,\bar{x}}(s)) \, ds = \int_{\delta_{t,\bar{x}}(J_{te}(x))} \chi(s) \zeta_\eta(y) \frac{dy}{|\delta'_{t,\bar{x}}(s)|},$$

with $\delta_{t,\bar{x}}(s) = y$, and thus

$$|\psi_{te}(x)| \leq \frac{1}{\alpha} \|\chi\|_\infty \int \zeta_\eta(y) \, dy \leq \frac{2}{\alpha} \|\chi\|_\infty \eta.$$

Therefore we get $I_\eta \to 0 = I$ when $\eta \to 0$ as announced and the proof is complete. \(\blacksquare\)

**Remark 3.5** The above proof actually provides a stability result for a sequence of solutions associated to a smooth approximate coefficient $a_\varepsilon$. The limit $u$ is a solution associated to $a$, provided that $a_\varepsilon$ satisfies a convexity assumption ensuring (3.28). This result has to be compared with the stability result of A.F. Filippov [12] for differential equations. Notice that in both cases, the convexity assumption compensates for non-unicities.

**Remark 3.6** In the above proof, we obtain condition (iii) of Proposition 3.2 by a dispersion estimate in a neighborhood of $(t_0, x_0) \in D_t$. This estimate means that $u$ is "approximately continuous" through $D_t$, which comes from the transversality expressed in (3.7). That transversality property has been used by N. Caroff [6] and P. Serfati [22] to prove some continuity estimates for $u$ when only allowing discontinuity lines $x = cst$ for $a$, and assuming $\text{ess inf } a > 0$.

In general, solutions to (1.1) are not unique. Some examples of non-unicities are shown in §3.1 and at the beginning of Section 4 (Example 4.1.1). However, when $a$ satisfies the one-sided Lipschitz condition, we are able to prove uniqueness by the method of P. Le Floch [18].

**Theorem 3.5 (Uniqueness)** Let us assume that $a$ satisfies the one-sided Lipschitz condition (1.6). Then any solution $u \in S_{BV}$ to (1.1) verifies

$$\partial_t |u| + \partial_x (a|u|) \leq \alpha(t) |u| \quad \text{in} \quad [0, T[ \times \mathbb{R},$$

and for any $x_0 \in \mathbb{R}$, $R > 0$, $0 \leq t \leq T$ we have

$$\int_{|x-x_0|<R} |u(t,x)| \, dx \leq e^{\int_0^t \alpha} \int_{|x-x_0|<R+\|u\|_{C^t}} |u(0,x)| \, dx.$$  \(\text{(3.30)}\)

Therefore, under condition (1.6), there exists at most one solution $u \in S_{BV}$ to (1.1) with Cauchy data $u^0 \in BV_{loc}(\mathbb{R})$. 

\[17\]
**Proof.** The estimate (3.30) and the uniqueness property can be deduced from (3.29) by standard integration. Therefore, we only prove that (3.29) holds in distribution sense.

First, Proposition 3.3 ensures that
\[ \partial_t |u| + a \partial_x |u| = 0 \quad \text{in} \quad \Omega. \tag{3.31} \]

Then let us define \( a_n = \rho_n * a \) (with \( \rho_n \) a smoothing sequence in \( \mathbb{R} \)), which is a bounded Borel function. We have by (3.31)
\[ \partial_t |u| + \partial_x (a_n |u|) = (\partial_t a_n) |u| + (a_n - a) \partial_x |u| \leq a(t) |u| + (a_n - a) \partial_x |u|, \tag{3.32} \]

because \( \partial_x a_n \leq a(t) \) by (1.6). Therefore, since \( a_n \to a \) locally uniformly in \( C \),
\[ \partial_t |u| + \partial_x (a |u|) \leq a(t) |u| \quad \text{in} \quad C. \tag{3.33} \]

It now remains to study what happens around \( S \) and \( D \). Let us consider the nonnegative measure
\[ \mu_n = (a(t) - \partial_x a_n) |u| = a(t) |u| + a_n \partial_x |u| - \partial_x (a_n |u|). \tag{3.34} \]

After extraction of a subsequence, \( \mu_n \to \mu \geq 0 \). Then for any \( \chi \in C_c^\infty([0, T[) \) and \( \psi \in C_c^\infty(\mathbb{R}) \), \( \psi \geq 0 \),
\[ \langle \mu, \chi \otimes \psi \rangle = \lim \langle \mu_n, \chi \otimes \psi \rangle = \lim \left[ \iint a(t) |u| \chi(t) \psi(x) \, dt \, dx + \langle a_n \partial_x |u|, \chi \otimes \psi \rangle + \iint a_n |u| \chi(t) \psi(x) \, dt \, dx \right]. \]

Therefore for a fixed \( \psi \) we get
\[ |\langle \mu, \chi \otimes \psi \rangle| \leq C \int_0^T (1 + |a(t)|) |\chi(t)| \, dt, \tag{3.35} \]

and hence \( \langle \mu, \mathbb{I}_{t=0} \psi(x) \rangle = 0 \). Since \( \partial_x (a_n |u|) \to \partial_x (a |u|) \), we conclude that \( \partial_x (a |u|) \) is a local measure in \( (t, x) \) in \( \Omega \), and does not concentrate in points. Then since \( S \) is locally finite, it remains to prove that (3.29) holds in a small neighborhood \( V \) of \((t_0, x_0) \in D \). As in the proof of Proposition 3.2 we distinguish two cases.

1st case: \((dt)_D(t_0, x_0) \neq 0 \).
We use the same approach as in Lemma 3.1. By (3.33) we have
\[ \left[ 1 - \zeta_\varepsilon(x - \xi(t)) \right] \left[ \partial_t |u| + \partial_x (a |u|) - a(t) |u| \right] \leq 0 \quad \text{in} \quad V, \]

and since
\[ \zeta_\varepsilon(x - \xi(t)) \partial_t |u| \xrightarrow{\varepsilon \to 0} \mathbb{I}_D \left( \frac{dx}{dt} \right)_D(t, x) \partial_x |u|, \]

we get
\[ \zeta_\varepsilon(x - \xi(t)) \partial_t |u| \xrightarrow{\varepsilon \to 0} \mathbb{I}_D \left( \frac{dx}{dt} \right)_D(t, x) \partial_x |u|, \]

for any \( \zeta_\varepsilon \) such that
\[ \int \zeta_\varepsilon \, d\mu = 1 \quad \text{and} \quad \int \zeta_\varepsilon \, d\mu = 0. \]
we get
\[
\partial_t |u| + \partial_x (a|u|) + \mathbb{D} \left( \frac{dx}{dt} \right)_\mathbb{D} \partial_x |u| - \mathbb{D} \partial_x (a|u|) \leq \alpha(t) |u| \quad \text{in} \quad V.
\]

Therefore, it is enough to prove that
\[
\mathbb{D} \left( \frac{dx}{dt} \right)_\mathbb{D} \partial_x |u| - \mathbb{D} \partial_x (a|u|) \geq 0 \quad \text{in} \quad V.
\]

We have
\[
\mathbb{D} \partial_x (a|u|) = \lim_{\varepsilon \to 0} \zeta_\varepsilon (x - \xi(t)) \partial_x (a|u|) = \lim_{\varepsilon \to 0} \frac{\partial_x (\zeta_\varepsilon (x - \xi(t)) a|u|) - \zeta_\varepsilon (x - \xi(t)) a|u|}{\varepsilon} \leq \lim_{\varepsilon \to 0} -\zeta_\varepsilon (x - \xi(t)) a|u|,
\]
and thus for any \( \varphi \in C^\infty_c (V) \)
\[
\langle \mathbb{D} \partial_x (a|u|), \varphi(t, x) \rangle = \lim_{\varepsilon \to 0} -\int dt \int dx \varphi(t, x) a(t, x) |u(t, x)| \zeta_\varepsilon (x - \xi(t))
= \int dt \varphi(t, \xi(t)) \left[ a(t, \xi(t)_+) \left| u(t, \xi(t)_+) \right| - a(t, \xi(t)_-) \left| u(t, \xi(t)_-) \right| \right].
\]

Similarly
\[
\langle \mathbb{D} \left( \frac{dx}{dt} \right)_\mathbb{D} \partial_x |u|, \varphi(t, x) \rangle = \lim_{\varepsilon \to 0} -\int dt \int dx \varphi(t, x) \zeta_\varepsilon (x - \xi(t)) \xi'(t) |u|, \varphi \rangle
= \int dt \varphi(t, \xi(t)) \xi'(t) \left[ \left| u(t, \xi(t)_+) \right| - \left| u(t, \xi(t)_-) \right| \right].
\]

Therefore, the inequality to be proved (3.36) can be written
\[
a.e. \ t \quad a(t, \xi(t)_+) |u(t, \xi(t)_+) - a(t, \xi(t)_-) | u(t, \xi(t)_-) \leq \xi'(t) \left[ \left| u(t, \xi(t)_+) \right| - \left| u(t, \xi(t)_-) \right| \right].
\]

Let us notice that (1.6) is equivalent to a.e. \( t \partial_x [a(t, .)] \leq \alpha(t) \) in \( \mathbb{R} \), and thus
\[
a.e. \ t \quad \forall x \in \mathbb{R} \quad a(t, x_+) \leq a(t, x_-).
\]

Now, by property (iii) of Proposition 3.2 (applied to \( |u| \)), for any \( t \) out of a set of Lebesgue measure zero, one of the two following cases occurs.

a. \( \xi'(t) \not\in [a_-(t, \xi(t)), a_+(t, \xi(t))] \) and \( u(t, \xi(t)_+) = u(t, \xi(t)_-) \). Then (3.38) gives (3.37).

b. \( \xi'(t) \in [a_-(t, \xi(t)), a_+(t, \xi(t))] \). Choosing \( a_+(t, x) = a(t, x_+) \) and \( a_-(t, x) = a(t, x_-) \), we get
\[
a(t, \xi(t)_+) |u(t, \xi(t)_+) - a(t, \xi(t)_-) | u(t, \xi(t)_-) \leq -\xi'(t) \left[ \left| u(t, \xi(t)_+) \right| - \left| u(t, \xi(t)_-) \right| \right],
\]
\[
a(t, \xi(t)_-) |u(t, \xi(t)_-) - a(t, \xi(t)_+) | u(t, \xi(t)_+) \leq -\xi'(t) \left[ \left| u(t, \xi(t)_-) \right| - \left| u(t, \xi(t)_+) \right| \right].
\]
and (3.37) follows by addition.

2nd case: $(dt)_{D}(t_0, x_0) = 0$.

Let us define

$$N = \{(t, x) \in V \cap D; (dt)_{D}(t, x) = 0\}.$$  

By the first case, we know that the measure $\partial |u| + \partial_x (a|u| - \alpha(t)|u|$ is nonpositive in the open set $V \setminus N$. In order to prove that it is nonpositive in $V$, it is enough to prove that it does not concentrate in $N$. Since $\alpha(t)|u|$ and $\partial_t |u|$ do not, it remains to check that $\partial_x (a|u|)$, or equivalently $\mu$, does not concentrate in $N$.

Let $\psi \in C_c^\infty(\mathbb{R})$, $\psi \geq 0$. We define the measure $\lambda$ on $[0, T]$ by

$$\lambda(E) = \int_E \psi(x) \mu(dt, dx).$$

Then for any $\chi \in C_c^\infty([0, T])$

$$\int \chi(t) \lambda(dt) = \int \chi(t) \psi(x) \mu(dt, dx),$$

$$\left| \int \chi(t) \lambda(dt) \right| \leq C \int_0^T (1 + |\alpha(t)|) |\chi(t)| dt$$

because of (3.35). Therefore, $\lambda$ is absolutely continuous with respect to the Lebesgue measure, and $\lambda(\{t \in [0, T]; \exists y \in \mathbb{R} (t, y) \in N\}) = 0$. Thus, $\mu(\{(t, x); \exists y \in \mathbb{R} (t, y) \in N\}) = 0$ and $\mu(N) = 0$. 

By using Lemma 2.2.1, we can also give existence and uniqueness results for (1.2). The following result is a straightforward consequence of Theorems 3.4 and 3.5.

**Theorem 3.6 (Conservative Cauchy problem)** For any $\mu^0 \in \mathcal{M}_{loc}(\mathbb{R})$, there exists $\mu \in S_\mathcal{M}$ solution to (1.2) with initial datum $\mu^0$, and such that for any $x_1 < x_2$ and $t \in [0, T]$

$$\int_I |\mu(t, dx)| \leq \int_J |\mu^0(dx)|,$$  

(3.39)

with $I = [x_1, x_2]$ and $J = [x_1 - \|a\|_{\infty} t, x_2 + \|a\|_{\infty} t]$. Moreover, if $a$ satisfies the one-sided Lipschitz condition (1.6), then this solution is unique.

Notice that by Proposition 3.2, any solution $\mu$ satisfies

$$\int_{D_{tr}} dt |\mu(t, dx)| = 0.$$  

(3.40)

3.1 Two examples of non-uniqueness

In this paragraph we show how non-uniqueness can occur. We also refer to Section 4 for the sgn function example.
1. Non-uniqueness when (1.6) is not satisfied

Let $a_l < a_r$ two real numbers, and define

$$a(t, x) = \begin{cases} a_l & \text{if } \frac{x}{t} \leq a_l, \\ \frac{x}{t} & \text{if } a_l \leq \frac{x}{t} \leq a_r, \\ a_r & \text{if } \frac{x}{t} \geq a_r. \end{cases}$$

Then $a \in C_b(]0, \infty[ \times \mathbb{R})$. Given $u_l, u_r \in \mathbb{R}$ we consider the initial datum

$$u^0(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases}$$

Then $u \in S_{BV}$ (for any $T > 0$) and solves (1.1) in $]0, \infty[ \times \mathbb{R}$. To see this, just add artificial discontinuity lines and apply Proposition 3.2. Therefore, there is no uniqueness, although $\partial_x a \leq 1/t$.

2. Non-uniqueness for a nonlinear problem

Let $u_l, u_r \in \mathbb{R}$ and consider the initial datum

$$u^0(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases}$$

Given $v \in [u_l, u_r]$ we define

$$u(t, x) = \begin{cases} u_l & \text{if } x < vt, \\ v & \text{if } x = vt, \\ u_r & \text{if } x > vt. \end{cases}$$

Then $u$ is a piecewise continuous admissible coefficient in the sense of (PC1)-(PC3) and (AC1)-(AC2), and $u \in S_{BV}$ (for any $T > 0$) at the same time. By using Proposition 3.2 we easily obtain that

$$\partial_t u + u \partial_x u = 0 \quad \text{in } ]0, \infty[ \times \mathbb{R}.$$
4 Transport equations with coefficient satisfying a one-sided Lipschitz condition

In all Section 4 we consider a coefficient $a \in L^\infty(\Omega)$, $\Omega = [0, T] \times \mathbb{R}$, satisfying the one-sided Lipschitz condition

$$\partial_x a \leq \alpha(t) \quad \text{in} \quad \Omega, \quad \alpha \in L^1([0, T]).$$

Notice that in contrast with Section 3, $a$ is now defined almost everywhere.

Condition (4.1) implies some regularity for $a$. In fact, it can be seen by appropriate smoothing that (4.1) is equivalent to

$$\alpha \text{ a.e. } t \in [0, T[ \quad \partial_x[a(t, ..)] \leq \alpha(t) \quad \text{in} \quad \mathbb{R}. \quad (4.2)$$

Therefore, it is quite easy to obtain that for almost every $t \in [0, T]$, $a(t, ..) \in BV_{loc}(\mathbb{R})$ and for any $x_1 < x_2$

$$TV_{[x_1, x_2]}(a(t, ..)) \leq 2 \left( |\alpha(t)| (x_2 - x_1) + \|a\|_{L^\infty} \right). \quad (4.3)$$

Notice also that since $a$ is bounded, (4.2) implies that $\alpha \geq 0$.

In Section 4.1 we consider the backward problem, in Section 4.2 the forward problem by the duality method, in Section 4.3 we prove some sharp properties for both problems by using the generalized flow, and in Section 4.4 we make some comments about viscous problems.

4.1 Lipschitz solutions, backward problem

We are here interested in solutions $p \in \mathcal{L}_{lip}$ to

$$\partial_t p + a \partial_x p = 0 \quad \text{in} \quad \Omega. \quad (4.1.1)$$

The space of these solutions will be denoted by $\mathcal{L}$. We study solutions to the backward problem, consisting of all $p \in \mathcal{L}$ such that

$$p(T, \cdot) = p^T, \quad (4.1.2)$$

for a given $p^T \in \operatorname{Lip}_{loc}(\mathbb{R})$.

Remark 4.1.1 One can of course transform (4.1.1)-(4.1.2) in a direct problem, by setting $\tilde{a}(t, x) = -a(T - t, x)$, and $u(t, x) = p(T - t, x)$. Condition (4.1) then becomes $\partial_x \tilde{a} \geq -\alpha(T - t)$.

In general, the problem (4.1.1)-(4.1.2) does not have a unique solution, even in the class of Lipschitz continuous functions. Let us consider the sign example, already studied by E.D. Conway [7].
Example 4.1.1 Take

\[
a(t, x) = -\text{sgn}\ x = \begin{cases} 
-1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
1 & \text{if } x < 0.
\end{cases}
\]

For any \( h \in \text{Lip}([0, T]) \) such that \( h(0) = p^T(0) \), we set for \((t, x) \in [0, T] \times \mathbb{R} \),

\[
p(t, x) = \begin{cases} 
p^T(x - (T - t)\text{sgn}\ x) & \text{if } T - t \leq |x|, \\
h(T - t - |x|) & \text{if } T - t \geq |x|.
\end{cases}
\]  

(4.1.3)

Then \( p \in \mathcal{S}_{\text{Lip}} \) solves (4.1.1)-(4.1.2), while \( p(t, 0) = h(T - t) \) is arbitrary. Actually any Lipschitz continuous solution to (4.1.1)-(4.1.2) is of the form (4.1.3) for some \( h \).

Notice that there is a ”canonical choice” for the above \( h \), namely \( h \equiv p^T(0) \). The aim of this section is to give a criterion which selects this solution.

In all the following, \((\rho_n)_n\) will stand for a sequence of standard \( C^\infty_c \) mollifiers on \( \mathbb{R} \).

4.1.1 General properties

We first give some properties of Lipschitz solutions to (4.1.1).

Lemma 4.1.1 For any \( p \in \mathcal{L} \) we have the following properties.

(i) Forward stability. Let \( a_n \in C^1([0, T] \times \mathbb{R}) \) be a bounded sequence in \( L^\infty([0, T] \times \mathbb{R}) \), such that \( a_n \rightarrow \alpha \) in \( L^1_{\text{loc}}([0, T[ \times \mathbb{R}) \), and \( \partial_x a_n \leq \alpha_n(t) \), \( \alpha_n \rightarrow \alpha \) in \( L^1([0, T[) \). Let \( p^0_n \in C^1(\mathbb{R}) \), \( p^0_n \rightarrow p(0, .) \) in \( L^1_{\text{loc}}(\mathbb{R}) \). Then, denoting by \( p_n \) the (classical) solution to

\[
\partial_t p_n + a_n \partial_x p_n = 0 \quad \text{in} \quad \Omega, \quad p_n(0, .) = p^0_n,
\]

we have \( p_n \rightarrow p \) in \( C([0, T[, L^1_{\text{loc}}(\mathbb{R}) \) ).

(ii) Decrease of the total variation. The application

\[
t \mapsto \int_\mathbb{R} |\partial_x p(t, x)| \, dx \in [0, \infty]
\]

is nonincreasing in \([0, T]\).

(iii) Let \( x_0 \in \mathbb{R} \). Then, \( \forall t \in [0, T] \), \( \exists x \in [x_0 - \|a\|_\infty(T - t), x_0 + \|a\|_\infty(T - t)] \) such that \( p(t, x) = p(T, x_0) \).

Proof. (i) We write

\[
\partial_t(p - p_n) + a_n \partial_x(p - p_n) = (a_n - a) \partial_x p,
\]

so that by the Lipschitz functions calculus

\[
\partial_t |p - p_n| + \partial_x (a_n |p - p_n|) = (a_n - a) \text{sgn}(p - p_n) \partial_x p + |p - p_n| \partial_x a_n \\
\leq |a_n - a| \partial_x p + a_n |p - p_n|.
\]

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Then, denoting \( q_n = |p - p_n| \exp \left( -\int_0^t \alpha_n \right) \), we get

\[
\partial_t q_n + \partial_x (a_n q_n) \leq |a_n - a| \partial_x p \xi^{-\int_0^t \alpha_n},
\]

and by standard integration on a cone

\[
\int_{|x_0| < R} q_n(t, x) \, dx \leq \int_{|x_0| < R + \|a_n\|_\infty} q_n(0, x) \, dx + \int_0^t \int_{|x_0| < R + \|a_n\|_\infty} (|a_n - a| \partial_x p \xi^{-\int_0^t \alpha_n} ) \, d\tau \, dx.
\]

But \( \partial_x p \xi^{-\int_0^t \alpha_n} \) is bounded uniformly, \( a_n \to a \) in \( L^1_{loc} \). Thus \( \int q_n(t, x) \, dx \to 0 \) uniformly for \( t \in [0, T] \).

(ii) Take \( a_n \) as in (i), and set \( p_n = \rho_n \ast p(0, \cdot) \), so that for \( t \in [0, T] \)

\[
\int_{\mathbb{R}} |\partial_x p_n(t, x)| \, dx = \int_{\mathbb{R}} |\partial_x p_0(0, x)| \, dx \leq \int_{\mathbb{R}} |\partial_x p(0, x)| \, dx.
\]

Then by (i)

\[
\int |\partial_x p(t, x)| \, dx \leq \lim_{n \to \infty} \int |\partial_x p_n(t, x)| \, dx \leq \int |\partial_x p(0, x)| \, dx.
\]

We get the result by a similar proof on any subinterval of \([0, T]\).

(iii) Assume on the contrary that for some \( t \in [0, T] \), for any \( x \in [x_0 - \|a\|_\infty (T-t), x_0 + \|a\|_\infty (T-t)] \), \( p(t, x) \neq p(T, x) \equiv y \). Then, since \( p \) is continuous, we have for some \( \varepsilon > 0 \) and \( \eta > 0 \)

\[
\forall x \in [x_0 - \|a\|_\infty (T-t) - \eta, x_0 + \|a\|_\infty (T-t) + \eta], \quad |p(t, x) - y| \geq \varepsilon/2.
\]

Now we approximate \( a \) by \( a_n \) as in (i) with \( \|a_n\|_\infty \leq \|a\|_\infty \), we set \( p_n = \rho_n \ast p(0, \cdot) \), and we call \( p_n \) the solution to \( \partial_t p_n + a_n \partial_x p_n = 0 \), \( p_n(t, \cdot) = p_n^0 \). By (i), \( p_n \to p \) in \( C([t, T], L^1_{loc}(\mathbb{R})) \). Now since \( p_n \to p(\cdot, \cdot) \) locally uniformly, we have for \( n \) large enough

\[
\forall x \in [x_0 - \|a\|_\infty (T-t) - \eta, x_0 + \|a\|_\infty (T-t) + \eta], \quad |p_n(t, x) - y| \geq \varepsilon/2.
\]

Going forward up to \( T \) by characteristics, we obtain \( |p_n(T, x) - y| \geq \varepsilon/2 \) for any \( x \in [x_0 - \eta, x_0 + \eta] \). Since \( p_n(T, \cdot) \to p(T, \cdot) \) in \( L^1_{loc}(\mathbb{R}) \), we get \( |p(T, x) - y| \geq \varepsilon/2 \), for a.e. \( x \in [x_0 - \eta, x_0 + \eta] \). But this contradicts \( p(T, x_0) = y \), since \( p \) is continuous.

**Remark 4.1.2** Properties (i) and (ii) reflect some kind of *irreversibility* of the solution, in the sense that the inequality \( \partial_t a \leq a \) gives some control on \( p(t, \cdot) \) and \( \partial_x p(t, \cdot) \) in terms of \( p(0, \cdot) \) and \( \partial_x p(0, \cdot) \), and not the converse. It remains true when replacing 0 by any \( t_1 \), for \( t > t_1 \).

**Remark 4.1.3** Property (iii) asserts that the final data is somewhat transported by the backward equation. Going back to Example 4.1.1, this means that for any \( t < T \), \exists \exists x \in [T - T, T - t] \) such that \( p(t, x) = p^0(0) \). A "natural" solution is then to transport the constant \( p^0(0) \) in the whole fan \( \{(t, x); \quad |x| \leq T - t, \quad t \in [0, T]\} \).
The remainder of this section is devoted to the introduction of some criteria selecting the "natural" solutions. As we shall see, these solutions are "reversible" in a sense to be precised, and we have existence and uniqueness of a reversible solution to (4.1.1)-(4.1.2). However, before that, we have to build up such solutions, by a completely different technique.

### 4.1.2 Limit solutions

The limit solutions to (4.1.1) are obtained by testing against some "pseudo-solutions" to the direct problem (1.1) with initial data \( u^0 \in BV_{loc}(\mathbb{R}) \). This looks very much as a duality argument, and will appear finally to be actually one. Let us first build the "pseudo-solutions".

Let \( a_n \to a \) as in Lemma 4.1.1 with \( \|a_n\|_\infty \leq \|a\|_\infty \), \( u^0 \in BV_{loc}(\mathbb{R}) \) and set \( u_n^0 = \rho_n * u^0 \). Some classical estimates along the characteristics prove that the solution \( u_n \) to

\[
\partial_t u_n + a_n \partial_x u_n = 0, \quad u_n(0, \cdot) = u_n^0
\]

is bounded in \( S_{BV} \). Since \( \partial_t u_n = -a_n \partial_x u_n \), \( u_n \) is also bounded in \( \text{Lip}([0, T], L^1_{loc}(\mathbb{R})) \), and therefore compact in \( C([0, T], L^1_{loc}(\mathbb{R})) \). Up to a subsequence, we get that \( u_n \to u \) in \( C([0, T], L^1_{loc}(\mathbb{R})) \) with \( u \in S_{BV} \cap \text{Lip}([0, T], L^1_{loc}(\mathbb{R})) \), and \( u(0, \cdot) = u^0 \). We wish now to extract a subsequence for which convergence holds for any \( u^0 \in BV_{loc}(\mathbb{R}) \).

By a diagonal process, we can choose a subsequence for which convergence holds for a countable subset of initial data \( \mathcal{A} \subset C^1(\mathbb{R}) \), dense in \( L^1_{loc}(\mathbb{R}) \). Then we deduce from the estimate

\[
\int_{|x - x_0| < R} |u_n(t, x) - v_n(t, x)| \, dx \leq e^{\int_0^t a_n} \int_{|x - x_0| < R + \|a_n\|_\infty} |u_n^0(x) - v_n^0(x)| \, dx
\]

that the Cauchy criterion is verified in \( C([0, T], L^1_{loc}(\mathbb{R})) \) for any \( u^0 \in BV_{loc}(\mathbb{R}) \) (and actually for any \( u^0 \in L^1_{loc}(\mathbb{R}) \)). One can also deduce that for another approximation \( u_n^0 \in C^1(\mathbb{R}) \), bounded in \( BV_{loc}(\mathbb{R}) \), such that \( u_n^0 \to u^0 \) in \( L^1_{loc}(\mathbb{R}) \), the corresponding solution also converges, to the same limit.

Now this subsequence is fixed once for all, and we denote by \( U(u^0) \) the limit \( u \). It is quite easy to check that \( U(u^0) \geq 0 \) whenever \( u^0 \geq 0 \); \( U(u^0) \) is compactly supported in \( x \) if \( u^0 \) is; and \( U(u^0 u^0) = U(u^0)U(u^0) \).

**Definition 4.1.2** Let \( p \in \mathcal{L} \). We say that \( p \) is a limit solution to (1.1) if

\[
\forall u^0 \in BV_{loc}(\mathbb{R}), \quad \partial_t \left( U(u^0) \partial_x p \right) + \partial_x \left( a U(u^0) \partial_x p \right) = 0 \quad \text{in} \quad \Omega.
\]

Notice that the operator \( U \) and the notion of limit solution depend on the sequence of approximate coefficients \( (a_n) \).
Remark 4.1.4 As a consequence of the whole construction we shall achieve, we shall see that
1. They actually not depend on the choice of the sequence \((a_n)\), and it is not necessary to extract any subsequence,
2. \(U(w^0)\) is actually solution to (1.1), in the sense of duality which will be defined later.

Proposition 4.1.3 Let \(p^T \in \text{Lip}_{loc}(\mathbb{R})\). Then there exists a unique \(p \in \mathcal{L}\) limit solution to (4.1.1) with final data \(p(T,.) = p^T\).

Proof. Uniqueness. It follows from a duality argument. Integrate (4.1.6) with respect to \(x\), for a compactly supported \(u^0\). We obtain that \(t \mapsto \int U(u^0) \partial_x p dx\) is constant in \([0,T]\). Assuming that \(p(T,.) = 0\), it follows that
\[
0 = \int_{\mathbb{R}} U(u^0)(T, x) \partial_x p(T, x) \, dx = \int_{\mathbb{R}} u^0(x) \partial_x p(0, x) \, dx.
\]
Since this holds for any \(u^0\), we have \(\partial_x p(0,.) = 0\). But by Lemma 4.1.1(ii) the total variation is nonincreasing, so that \(\partial_x p(t,.) = 0\) for any \(t\). Using the equation, we find that \(\partial_t p = 0\) and \(p = 0\).

Existence. Let \(p_n^T \to p^T\), \(p_n^T\) bounded in \(\text{Lip}_{loc}(\mathbb{R})\), with \(p_n^T \in C^1(\mathbb{R})\), and \(p_n\) solve
\[
\partial_t p_n + a_n \partial_x p_n = 0, \quad p_n(T,.) = p_n^T.
\]
If \(X_n\) is the flow corresponding to \(a_n\), one has
\[
0 \leq \partial_x X_n(T, t, x) \leq e^{\int_0^T a_n}.
\]
Since \(p_n(t, x) = p_n^T(X_n(T, t, x))\), we easily deduce from this inequality a \(L^\infty_{loc}\) bound on \(\partial_x p_n\) and, using the equation, on \(\partial_t p_n\). Thus \((p_n)\) is bounded in \(\text{Lip}_{loc}([0,T] \times \mathbb{R})\), and therefore has a subsequence converging in \(C([0,T] \times [-R,R])\) to some \(p \in \text{Lip}_{loc}([0,T] \times \mathbb{R})\), which satisfies (4.1.1)-(4.1.2). Now for any \(n\), we multiply (4.1.4) and \(\partial_x (4.1.7)\) respectively by \(\partial_x p_n\) and \(u_n\) to obtain
\[
\partial_t (u_n \partial_x p_n) + \partial_x (a_n u_n \partial_x p_n) = 0.
\]
When \(n \to \infty\), \(\partial_x p_n \to \partial_x p\) weakly, but \(a_n \to a\) and \(u_n \to U(w^0)\) strongly so we recover (4.1.6).

4.1.3 Reversible solutions
We wish to prove now that the limit solution is indeed the "natural" solution mentioned in Remark 4.1.3.

Definition 4.1.4 (Nonconservative reversible solutions) (i) We call exceptional solution any function \(p_e \in \mathcal{L}\) such that \(p_e(T,.) = 0\). We denote by \(\mathcal{E}\) the vector space of exceptional solutions.
(ii) We call domain of support of exceptional solutions the open set
\[ \mathcal{V}_c = \{(t, x) \in [0, T] \times \mathbb{R} ; \exists p_c \in \mathcal{E} \; p_c(t, x) \neq 0\} \]

(iii) Any \( p \in \mathcal{L} \) is called reversible if \( p \) is locally constant in \( \mathcal{V}_c \).

In the case of Example 4.1.1 \( a(t, x) = - \text{sgn } x \), the exceptional solutions are given by \( p_c(t, x) = h((T - t - |x|)_+) \) with \( h \in \text{Lip}([0, T]) \), \( h(0) = 0 \), and we have \( \mathcal{V}_c = \{(t, x) \in \Omega ; |x| < T - t\} \).

Remark 4.1.5 1. The vector space of reversible solutions to (4.1.1) will be denoted by \( \mathcal{R} \). It is a subspace of \( \mathcal{L} \), containing constants.
2. If \( p \) is a reversible solution, then \( S(p) \) is also a reversible solution, for any \( S : \mathbb{R} \rightarrow \mathbb{R} \) Lipschitz continuous. Also, the product of two reversible solutions is a reversible solution.
3. It is easy to see that \( \mathcal{R} \cap \mathcal{E} = \{0\} \). For a proof see Corollary 4.1.6.
4. We have the equivalence
\[ \mathcal{E} = \{0\} \Leftrightarrow \mathcal{V}_c = \emptyset \Leftrightarrow \text{any } p \in \mathcal{L} \text{ is reversible}. \]

For example, it is the case if a.e. \( t \) \( a(t, \cdot) \in C(\mathbb{R}) \), see Lemma 2.3.1. Another example is \( a(t, x) = 1_{x < 0} \).

Theorem 4.1.5 (Nonconservative backward Cauchy problem) Let be given a final datum \( p^T \in \text{Lip}_{\text{loc}}(\mathbb{R}) \). Then there exists a unique \( p \in \mathcal{L} \) reversible solution to (4.1.1) such that \( p(T, \cdot) = p^T \). Moreover we have for any \( x_1 < x_2 \) and \( t \in [0, T] \)
\[
\| p(t, \cdot) \|_{L^\infty(I)} \leq \| p^T \|_{L^\infty(J)},
\| \partial_x p(t, \cdot) \|_{L^\infty(I)} \leq e^{\int_t^T a} \| \partial_x p^T \|_{L^\infty(J)},
\]
with \( I = [x_1, x_2] \) and \( J = [x_1 - \| a \|_\infty (T - t), x_2 + \| a \|_\infty (T - t)] \).

Proof. Uniqueness follows from Remark 4.1.5(3). For existence, we actually prove that the limit solution \( p \) obtained in Proposition 4.1.3 is reversible.

Let \( p_c \in \mathcal{E} \). By Lemma 4.1.1(i), we have \( p_c = U(p_c(0, \cdot)) \). Then since \( p \) is a limit solution,
\[ \partial_t (p_c \partial_x p) + \partial_x (ap_c \partial_x p) = 0. \]

Now by Lemma 2.2.1 (and Remark 2.2.1), there exists \( q \in \mathcal{L} \) such that \( \partial_x q = p_c \partial_x p \). Then, by the formula \( U(u^0 v^0) = U(u^0)U(v^0) \), we easily see that \( q \) is a limit solution.

Now by uniqueness of limit solutions (Proposition 4.1.3) and since \( \partial_x q(T, \cdot) = 0 \), we find that \( q \) is a constant and \( p_c \partial_x p = \partial_x q = 0 \). Since this holds for any \( p_c \in \mathcal{E} \), we get \( \partial_x p = 0 \) in \( \mathcal{V}_c \), and (4.1.1) gives also \( \partial_t p = 0 \) in \( \mathcal{V}_c \). The estimates are left to the reader. \( \square \)
Remark 4.1.6 The above proof shows that the notions of limit and reversible solutions actually coincide.

Corollary 4.1.6 Recall that \( \mathcal{L}, \mathcal{R}, \mathcal{E} \) are the vector spaces of Lipschitz, reversible and exceptional solutions respectively. Then we have

\[
\mathcal{L} = \mathcal{R} \oplus \mathcal{E}.
\]

Moreover, for any \( p_r \in \mathcal{R}, \ p_e \in \mathcal{E} \) and \( t \in [0,T] \), \( \partial_x p_r(t, \cdot) \) and \( \partial_x p_e(t, \cdot) \) are supported by disjoint sets.

**Proof.** The decomposition of \( \mathcal{L} \) follows from Theorem 4.1.5. Then, let \( p_r \in \mathcal{R}, \ p_e \in \mathcal{E} \) and \( t \in [0,T] \), and define \( A = \{ x \in \mathbb{R}; (t,x) \in \mathcal{V}_e \} \), which is open. Since \( p_r \) is reversible, \( p_r(t, \cdot) \) is locally constant in \( A \). Therefore, a.e. \( x \in A \partial_x p_r(t, x) = 0 \). Now, by the definition of \( \mathcal{V}_e \), we have \( \forall x \in A^c \ p_r(t, x) = 0 \), and thus a.e. \( x \in A^c \partial_x p_e(t, x) = 0 \). This proves the support property if \( 0 < t < T \). If \( t = T \), we just take \( A = \emptyset \). If \( t = 0 \), take \( A = \{ x \in \mathbb{R}; \exists q_e \in \mathcal{E} \ q_e(0, x) \neq 0 \} \). The proof is then similar to the above one. \( \Box \)

We now come to more handable characterizations of reversible solutions.

**Proposition 4.1.7.** Let \( p \in \mathcal{L} \). Then

1. If \( p \) is reversible then \( t \mapsto \int_{\mathbb{R}} |\partial_x p(t,x)| \, dx \) is constant in \([0,T]\).
2. If the above function is constant and finite, then \( p \) is reversible.

We recall that for any \( p \in \mathcal{L} \), the total variation is in general a nonincreasing function of \( t \) (Lemma 4.1.1(ii)).

**Proof of Proposition 4.1.7.** In order to prove (i), let us assume that \( p \) is reversible. Then by Theorem 4.1.5 and by construction, \( p \) is a limit solution, obtained as the limit of \( p_n \) solution to \( \partial_t p_n + a_n \partial_x p_n = 0 \), \( p_n(T, \cdot) = \rho_n * p^T \), \( p^T = p(T, \cdot) \). Therefore

\[
\int_{\mathbb{R}} |\partial_x p_n(0,x)| \, dx = \int_{\mathbb{R}} |\partial_x p_n^T(x)| \, dx \leq \int_{\mathbb{R}} |\partial_x p^T(x)| \, dx.
\]

Thus, by lower semicontinuity we get

\[
\int_{\mathbb{R}} |\partial_x p(0,x)| \, dx \leq \int_{\mathbb{R}} |\partial_x p(T,x)| \, dx,
\]

and we conclude since we already know that the total variation is nonincreasing. The converse (ii) is easily obtained by (i) and the decomposition of Corollary 4.1.6. \( \Box \)

The most useful criterion for reversibility (Theorem 4.1.9) is closely related to monotone solutions. We first state a lemma.
Lemma 4.1.8 Let \( p \in \mathcal{L} \). We have

(i) if \( p \in \mathcal{E} \) and \( \partial_x p \geq 0 \) then \( p = 0 \),
(ii) if \( \partial_x p \geq 0 \) then \( p \) is reversible,
(iii) if \( p \) is reversible and \( \partial_x p(T,.,:) \geq 0 \) then \( \partial_x p \geq 0 \).

Proof. (i) Since \( p(T,.) = 0 \), by Lemma 4.1.1(iii), for any \( t \in [0,T] \) and any \( R > 0 \), there exists \( x_1 < -R \) and \( x_2 > R \) such that \( p(t, x_1) = p(t, x_2) = 0 \). Thus \( \int_{-R}^R \partial_x p(t, x) \, dx = 0 \), and by nonnegativity \( \partial_x p(t, x) = 0 \) a.e. \( x \in [x_1, x_2] \), and therefore for a.e. \( x \in \mathbb{R} \) since \( R \) is arbitrary. The result easily follows.

(ii) By Corollary 4.1.6 we have \( p = p_r + p_e \) with \( p_r \) reversible and \( p_e \) exceptional, and such that \( \partial_x p_r(t,.) \) and \( \partial_x p_e(t,.) \) have disjoint supports. Therefore, the condition \( \partial_x p \geq 0 \) is equivalent to \( \partial_x p_r \geq 0 \) and \( \partial_x p_e \geq 0 \). Then by (i) \( p_e = 0 \), and thus \( p = p_r \) is reversible.

(iii) This property easily follows from the construction of reversible solutions by an approximate problem.  

Theorem 4.1.9 (Main criterion for reversibility) Let \( p \in \mathcal{L} \). Then \( p \) is reversible if and only if there exists \( p_1, p_2 \in \mathcal{L} \) such that \( \partial_x p_1 \geq 0, \partial_x p_2 \geq 0 \) and \( p = p_1 - p_2 \).

Proof. If \( p \) is reversible, we write \( p(T,.) = p_1^T - p_2^T \) with \( p_k^T \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) and \( \partial_x p_k^T \geq 0 \). Then \( p = p_1 - p_2 \) with \( p_1, p_2 \) the reversible solutions with final data \( p_1^T, p_2^T \), and Lemma 4.1.8(iii) ensures that \( \partial_x p_k \geq 0 \). The converse follows from Lemma 4.1.8(ii) and from the fact that \( \mathcal{R} \) is a vector space.

Remark 4.1.7 1. The above decomposition was already used by D. Hoff [15].

2. From the characterization in Theorem 4.1.9 we deduce that the time restriction of a reversible solution is also reversible. This property was not obvious in Definition 4.1.4.

3. If we choose the above property as a definition for reversible solutions, then there is a direct proof of Theorem 4.1.5. Existence is easily obtained by any reasonable approximation, and uniqueness follows from Lemma 2.4.1. However, the basic renormalization property of Remark 4.1.5(2) is not obvious with this definition.

We next give an important feature of reversible solutions, namely the stability with respect to coefficient and final data.

Theorem 4.1.10 (Stability) Let \( (a_n) \) be a bounded sequence in \( L^\infty([0,T] \times \mathbb{R}) \), with \( a_n \to a \) in \( L^\infty([0,T] \times \mathbb{R}) \) w*. Assume \( \partial_x a_n \leq a_n(t) \), where \( (a_n) \) is bounded in \( L^1([0,T]) \), \( \partial_t a \leq a \in L^1([0,T]) \). Let \( (p_n^T) \) be a bounded sequence in \( \text{Lip}_{\text{loc}}(\mathbb{R}) \), \( p_n^T \to p^T \), and denote by \( p_n \) the reversible solution to

\[
\begin{align*}
\partial_t p_n + a_n(\partial_x p_n = 0 & \quad \text{in} \quad [0,T] \times \mathbb{R}, \\
p_n(T,.) = p_n^T.
\end{align*}
\]

Then \( p_n \to p \) in \( C([0,T] \times [-R,R]) \) for any \( R > 0 \), where \( p \) is the reversible solution to

\[
\begin{align*}
\partial_t p + a(\partial_x p = 0 & \quad \text{in} \quad [0,T] \times \mathbb{R}, \\
p(T,.) = p^T.
\end{align*}
\]
**Proof.** By the bounds of Theorem 4.1.5 and the equation on \( p_n \), we easily get that \( p_n \) is bounded in \( S_{L_{\text{Lip}}} \). Therefore, we can extract a subsequence \( p_{n'} \), such that

\[
p_{n'} \to p \quad \text{in} \quad C([0, T] \times [-R, R]), \quad p \in \text{Lip}_{\text{loc}}([0, T] \times \mathbb{R}), \quad p(T, \cdot) = p^T.
\]

Let us first prove that \( p \) satisfies the equation. Let \( \mu_n = a_n(t) - \partial_x a_n \geq 0 \). Since \( \mu_n \) is a measure, so is \( \partial_x a_n \). We have \( \partial_x a_n \to \partial_x a \) in \( D'(\Omega) \), and, up to a subsequence, \( \alpha_n \to \lambda \) in \( M([0, T]) - w^* \), so that \( \mu_n \to \mu = \lambda - \partial_x a \) in \( D'(\Omega) \), and \( \mu \geq 0 \). The convergence holds therefore weakly in the space of measures. Going back to \( \partial_x a_n \), we have now \( \partial_x a_n \to \partial_x a \) locally weakly in the space of measures. Thus we can pass to the limit in the product \( p_n \partial_x a_n \), so that \( a_n \partial_x p_n = \partial_x(a_n p_n) - p_n \partial_x a_n \to \partial_x(ap) - p \partial_x a = a \partial_x p. \) This yields \( \partial_x p + a \partial_x p = 0. \)

It remains to prove that \( p \) is reversible. By Theorem 4.1.9 we have \( p_n = p_{n,1} - p_{n,2} \), \( \partial_x p_{n,i} \geq 0 \), where \( p_{n,i} \) are solutions corresponding to \( a_n \) and \( \partial_x a_n \) chosen bounded in \( \text{Lip}_{\text{loc}}(\mathbb{R}) \). Up to a subsequence, \( p_{n,i}^T \to p_i^T \in \text{Lip}_{\text{loc}}(\mathbb{R}) \), and we can pass to the limit for \( p_{n,i} \), as for \( p_r \). Thus \( p_{n,i} \to p_i \) (again up to a subsequence), \( p_i \) is solution to (4.1.1), \( \partial_x p_i \geq 0, \) and \( p = p_1 - p_2. \) Thus \( p \) is reversible. Finally, uniqueness ensures that it is not necessary to extract any subsequence. \( \square \)

### 4.1.4 The conservative case

Let us now consider the conservative problem. Let \( \pi \in S_{L_{\text{loc}}} \) solve

\[
\partial_t \pi + \partial_x (a \pi) = 0 \quad \text{in} \quad \Omega.
\]

Then by the integration lemma 2.2.1 (and Remark 2.2.1), there exists a unique (up to an additive constant) \( p \in S_{L_{\text{Lip}}} \) which satisfies

\[
\partial_x p = \pi, \quad \partial_t p + a \partial_x p = 0 \quad \text{in} \quad \Omega.
\]

**Definition 4.1.11** We say that \( \pi \in S_{L_{\text{loc}}} \) solving (4.1.8) is a reversible solution to (4.1.8) if \( p \) is reversible.

From Definition 4.1.4 and Theorem 4.1.9, we have the following characterization for reversible conservative solutions.

**Theorem 4.1.12 (Conservative reversible solutions)** The following three properties are equivalent for \( \pi \in S_{L_{\text{loc}}} \) solution to (4.1.8).

(i) \( \pi \) is reversible,

(ii) \( \pi = 0 \) in \( \mathcal{V}_r \),

(iii) \( \pi = \pi_1 - \pi_2 \), for some \( \pi_i \in S_{L_{\text{loc}}} \) solutions to (4.1.8), such that \( \pi_i \geq 0 \).

Obviously, the set of reversible conservative solutions is a vector space. Notice also that the time restriction of a reversible conservative solution is reversible as well. From the existence and uniqueness theorem 4.1.5 for the nonconservative Cauchy problem, we have immediately
Theorem 4.1.13 (Conservative backward Cauchy problem) Let \( \pi^T \in L^\infty_{loc}(\mathbb{R}) \). Then there exists a unique \( \pi \in \mathcal{S}_L \) reversible solution to (4.1.8) such that \( \pi(T,.) = \pi^T \). This solution satisfies for any \( x_1 < x_2 \) and \( t \in [0,T] \)

\[
\| \pi(t,\cdot) \|_{L^\infty(I)} \leq e^{\int_t^T a_1(s)ds} \| \pi^T \|_{L^\infty(J)},
\]

where \( I = [x_1, x_2] \) and \( J = [x_2 - \| a \|_\infty (T - t), x_2 + \| a \|_\infty (T - t)] \).

Moreover, \( \pi \geq 0 \) if \( \pi^T \geq 0 \).

4.1.5 The generalized backward flow

We can define in a natural way a backward flow associated to \( a \).

Definition 4.1.14 Let \( D_b = \{(s,t) \in \mathbb{R}^2; \ 0 \leq t \leq s \leq T \} \). We define the backward flow \( X \in \text{Lip}(D_b \times \mathbb{R}) \) in the following way.

If \( 0 < s \leq T \), \( X(s,\cdot,\cdot) \) is the unique reversible \( \mathcal{S}_{Lip} \) solution to

\[
\begin{align*}
\begin{cases}
\partial_t X + a(t,x) \partial_x X = 0 & \text{in } \ [0,s] \times \mathbb{R}, \\
X(s,s,x) = x.
\end{cases}
\end{align*}
\]

If \( s = 0 \), we simply set \( X(0,0,x) = x \).

Remark 4.1.8 If \( a \in C^1([0,T] \times \mathbb{R}) \), we recover the usual flow, solution to the ordinary differential equation \( \partial_t X = a(s,X), X(t,t,x) = x \).

The following Lipschitz estimates are easily proved by approximation:

\[
\| \partial_x X \|_{L^\infty(D_b \times \mathbb{R})} \leq \| a \|_\infty, \quad \| \partial_s X \|_{L^\infty(D_b \times \mathbb{R})} \leq \| a \|_\infty e^{\int_t^T a_1}, \quad \forall (s,t) \in D_b, \quad \| \partial_x X(s,t,\cdot) \|_{L^\infty(\mathbb{R})} \leq e^{\int_t^T a_1},
\]

(4.1.9)

We have also \( |X(s,t,x) - x| \leq \| a \|_\infty |s-t| \), and for any \( t \in [0,T] \), \( \partial_x \partial_a X \leq a(s) \partial_x X \) in \( [t,T] \times \mathbb{R} \). Notice also that since the final data is increasing, the reversibility reads as \( \partial_x X \geq 0 \) (by Lemma 4.1.8). From this we deduce that \( x \mapsto X(s,t,x) \) is surjective. We have also the usual composition formula: for \( 0 \leq s_1 \leq s_2 \leq s_3 \leq T \) and \( x \in \mathbb{R} \)

\[
X(s_3,s_2,X(s_1,\cdot,x)) = X(s_3,s_1,x).
\]

Example 4.1.2 (i) \( a(t,s) = -\text{sgn} \ x \). For \( s \geq t \geq 0 \), \( X(s,t,x) = (|x| - (s-t))_+ \text{sgn} \ x \) and \( \partial_x X(s,t,x) = \mathbb{I}_{x>0} \).

(ii) \( a(t,x) = 0 \). For \( s \geq t \geq 0 \), \( X(s,t,x) = x \) and \( \partial_x X(s,t,x) = \mathbb{I}_{x>0} \).

From the above Lipschitz estimates we easily deduce the following stability result.

Theorem 4.1.15 Let \( a_n \) and \( a \) be as in Theorem 4.1.10, and denote by \( X_n \) and \( X \) their respective backward flows. Then \( X_n \rightarrow X \) in \( C(D_b \times [-R,R]) \) for any \( R > 0 \).
Now, as a simple consequence of renormalization for nonconservative reversible solutions (Remark 4.1.5(2)), we can rewrite all reversible solutions in terms of the flow. Differentiating this formula, we obtain a similar representation for conservative solutions.

Proposition 4.1.16 Let $s \in ]0, T]$. We have
(i) If $p' \in \text{Lip}_{loc}(\mathbb{R})$, the reversible solution $p$ to
\[
\begin{cases}
\partial_t p + a \partial_x p = 0 \quad \text{in} \quad ]0, s[ \times \mathbb{R}, \\
p(s,.) = p',
\end{cases}
\]
is given by $p(t,x) = p^s(X(s,t,x))$ for any $0 \leq t \leq s$ and $x \in \mathbb{R}$.

(ii) If $\pi^s \in L_{loc}^\infty(\mathbb{R})$, the reversible solution $\pi$ to
\[
\begin{cases}
\partial_t \pi + \partial_x (a \pi) = 0 \quad \text{in} \quad ]0, s[ \times \mathbb{R}, \\
\pi(s,.) = \pi^s,
\end{cases}
\]
is given by $\pi(t,x) = \pi^t(X(s,t,x))\partial_x X(s,t,x)$, for any $0 \leq t \leq s$ and a.e. $x \in \mathbb{R}$.

4.2 Duality solutions, forward problem

4.2.1 The nonconservative case

We turn now to the study of the forward problem, in nonconservative form to begin with
\[
\partial_t u + a \partial_x u = 0,
\]
with $u \in \mathcal{S}_{BV}$. Recall that $a$ still satisfies the assumption (4.1), i.e. $\partial_x a$ is upper-bounded by some $L^1$ function $\alpha$. This corresponds to some compressive case, by analogy to fluid mechanics.

Definition 4.2.1 (Nonconservative duality solutions) We say that $u \in \mathcal{S}_{BV}$ is a duality solution to (4.2.1) if for any $0 < \tau \leq T$, and any reversible solution $\pi$ with compact support in $x$, to $\partial_t \pi + \partial_x (a \pi) = 0$ in $]0, \tau[ \times \mathbb{R}$, the function $t \mapsto \int_{\mathbb{R}} u(t,x)\pi(t,x)\,dx$ is constant on $[0, \tau]$.

Remark 4.2.1 1. A duality solution is not a solution in the sense of distributions. We do not intend yet to give a meaning to the product $a \partial_x u$.
2. The set of duality solutions is a vector space which contains the constants. Conversely, a duality solution $u$ satisfying $\partial_x u = 0$ is constant.
3. The time restriction of a duality solution is a duality solution also.

Theorem 4.2.2 (Nonconservative forward Cauchy problem) Let be given $u^0 \in BV_{loc}(\mathbb{R})$. Then there exists a unique $u \in \mathcal{S}_{BV}$ duality solution to (4.2.1), such that $u(0,.) = u^0$. This solution satisfies, for any $x_1 < x_2$ and any $t \in ]0, T]$,\[
TV_t(u(t, \cdot)) \leq TV_t(u^0),
\]
\[ \|u(t, \cdot)\|_{L^\infty(I)} \leq \|u^0\|_{L^\infty(J)}, \]  
(4.2.3)

with \( I = ]x_1, x_2[ \) and \( J = ]x_1 - \|a\|_\infty t, x_2 + \|a\|_\infty t[ \). Moreover, \( u \in \text{Lip}([0, T], L^1_{\text{loc}}(\mathbb{R})) \).

**Proof.** Uniqueness. Let \( u \) be a duality solution with \( u(0, \cdot) = 0 \). For any compactly supported reversible \( \pi \), we have

\[ \forall t \in [0, \tau], \quad \int u(t, x)\pi(t, x) \, dx = \int u(0, x)\pi(0, x) \, dx = 0. \]

Since the choice of \( \pi(\tau, \cdot) \) in \( L^\infty \) is arbitrary, we obtain \( u(\tau, \cdot) = 0 \), for any \( 0 < \tau \leq T \).

Existence. Consider the pseudo-solution \( U(u^0) \) we built up in § 4.1.2. Then it is actually a duality solution. Indeed, if \( \pi \) is a reversible conservative solution, by definition \( \pi = \partial_x p \), with \( p \) reversible nonconservative solution. Thus \( p \) is also a limit solution (by Remark 4.1.6). Therefore (since we can restrict the operator \( U \) on \([0, \tau])\),

\[ \partial_t \left(U(u^0)\partial_x p\right) + \partial_x \left(a U(u^0)\partial_x p\right) = 0 \quad \text{in} \quad ]0, \tau[ \times \mathbb{R}. \]

If \( \pi = \partial_x p \) is compactly supported, we can integrate this equality and obtain

\[ \frac{d}{dt} \int_{\mathbb{R}} U(u^0)(t, x)\pi(t, x) \, dx = 0 \quad \text{in} \quad ]0, \tau[. \]

Since the integral is continuous in \( t \), it is a constant and we can take \( u = U(u^0) \). The bounds on \( u \) are obtained easily. \( \square \)

**Remark 4.2.2** The uniqueness result proves that there is actually no need to extract any subsequence in the construction of \( U \) (see Remark 4.1.4).

**Remark 4.2.3** If \( u \in \mathcal{S}_{BV} \), \( u \) is a duality solution to (4.2.1) in \([0, T] \times \mathbb{R} \) if and only if for any \( T_1 \in ]0, T[ \), \( u \) is a duality solution to (4.2.1) in \([T_1, T] \times \mathbb{R} \). This last formulation can be taken as a definition of duality solutions when \( a \) satisfies the one-sided Lipschitz condition with only \( a \in L^1_{\text{loc}}([0, T]) \), such as \( a(t) = 1/t \).

One can easily check that, if \( S : \mathbb{R} \to \mathbb{R} \) is any Lipschitz continuous function, then \( S(U(u^0)) = U(S(u^0)) \). Since the duality solution is given by \( U(u(0, \cdot)) \), we can rewrite this as

**Proposition 4.2.3 (Renormalization)** Let \( u \in \mathcal{S}_{BV} \) be a duality solution to (4.2.1), and \( S : \mathbb{R} \to \mathbb{R} \) a Lipschitz continuous function. Then \( S(u) \) is also a duality solution.

**Remark 4.2.4** If \( p \in \mathcal{S}_{Lip} \) solves (4.2.1), then \( p \) is a duality solution to (4.2.1), even if \( p \) is not reversible. This results from Lemma 4.1.1(i).
4.2.2 The conservative case

**Definition 4.2.4 (Conservative duality solutions)** We say that \( \mu \in \mathcal{S}_M \) is a duality solution to
\[
\frac{\partial}{\partial t} \mu + \partial_x (a \mu) = 0 \quad \text{in} \quad ]0, T[ \times \mathbb{R}
\]
if for any \( 0 < \tau \leq T \), and any reversible solution \( p \) with compact support in \( x \), to \( \partial_t p + a \partial_x p = 0 \) in \([0, \tau] \times \mathbb{R} \), the function \( t \mapsto \int_{\mathbb{R}} p(t, x) \mu(t, dx) \) is constant on \([0, \tau]\).

**Remark 4.2.5** The set of conservative duality solutions is obviously a vector space, and the time restriction of a conservative duality solution is also a conservative duality solution.

**Theorem 4.2.5 (Conservative forward Cauchy problem)** Given \( \mu^0 \in \mathcal{M}_1(\mathbb{R}) \), there exists a unique \( \mu \in \mathcal{S}_M \) duality solution to (4.2.4), such that \( \mu(0, \cdot) = \mu^0 \). This solution satisfies for any \( x_1 < x_2 \) and \( t \in [0, T] \)
\[
\int_{[x_1, x_2]} |\mu(t, dx)| \leq \int_{[x_1, x_2]} |\mu^0(dx)|. \tag{4.2.5}
\]
Moreover, \( t \mapsto \int_{\mathbb{R}} |\mu(t, dx)| \) (which has values in \([0, \infty)\)) is nonincreasing on \([0, T]\).

**Remark 4.2.6** It is easy to prove that \( \mu^0 \geq 0 \) implies that \( \mu \geq 0 \).

**Proof of Theorem 4.2.5.** Uniqueness is obtained similarly as in Theorem 4.2.2. For existence, just define \( \mu = \partial_x \mu \), with \( \mu \) obtained by Theorem 4.2.2 with initial data \( u^0 \in BV_{loc}(\mathbb{R}) \) such that \( \partial_x u^0 = \mu^0 \). Then, regarding to Definition 4.1.11, it is easy to obtain that \( \mu \) is a duality solution, and (4.2.5) follows from (4.2.2). The final property is obtained by restriction on a subinterval and by letting \( x_1 \to -\infty, x_2 \to \infty \).

Two easy consequences of the above proofs are the following properties of derivation and integration, which generalize Lemma 2.2.1.

**Proposition 4.2.6** (i) Let \( u \in \mathcal{S}_{BV} \) be a duality solution to \( \partial_t u + a \partial_x u = 0 \). Then \( \mu = \partial_x u \in \mathcal{S}_M \) is a duality solution to \( \partial_t \mu + \partial_x (a \mu) = 0 \).

(ii) Let \( \mu \in \mathcal{S}_M \) be a duality solution to \( \partial_t \mu + \partial_x (a \mu) = 0 \). Then there exists \( u \in \mathcal{S}_{BV} \) duality solution to \( \partial_t u + a \partial_x u = 0 \), such that \( \mu = \partial_x u \). Moreover, \( u \) is unique up to an additive constant.

The following result justifies the terminology duality solutions.

**Proposition 4.2.7** (i) Let \( \pi \in \mathcal{S}_{L^\infty} \) be a reversible solution to \( \partial_t \pi + \partial_x (a \pi) = 0 \), and \( u \in \mathcal{S}_{BV} \) a duality solution to \( \partial_t u + a \partial_x u = 0 \). Then \( \pi u \in \mathcal{S}_{L^\infty} \) is a reversible solution to \( \partial_t (\pi u) + \partial_x (a \pi u) = 0 \).
(ii) Let \( p \in S_{\text{Lip}} \) be a reversible solution to \( \partial_t p + a \partial_x p = 0 \), and \( \mu \in S_{\mathcal{M}} \) a duality solution to \( \partial_t \mu + \partial_x (a \mu) = 0 \). Then \( p_\mu \in S_{\mathcal{M}} \) is a duality solution to
\[
\partial_t (p_\mu) + \partial_x (a p_\mu) = 0.
\]

**Proof.** For (i) the assertion solution is contained in the proof of Theorem 4.2.2. The reversibility follows from the cancellation in \( \mathcal{V} \), (Theorem 4.1.12).

For (ii), use the definition, and the fact that if \( p_1, p_2 \) are reversible, then so is \( p_1 p_2 \).

**Theorem 4.2.8** Let \( \mu^0 \in \mathcal{M}_{\text{loc}}(\mathbb{R}) \). Then the duality solution \( \mu \in S_{\mathcal{M}} \) obtained in Theorem 4.2.5 is given by
\[
\forall t \in [0, T], \quad \forall \varphi \in C_c(\mathbb{R}), \quad \int_{\mathbb{R}} \varphi(x) \mu(t, dx) = \int_{\mathbb{R}} \varphi(X(t, 0, x)) \mu^0(dx), \tag{4.2.6}
\]
where \( X \) is the backward flow (Definition 4.1.14).

**Proof.** Consider first \( \varphi \in \text{Lip}_c(\mathbb{R}) \), and let \( p \) be the reversible solution to
\[
\begin{cases}
\partial_t p + a \partial_x p = 0 \quad \text{in} \quad [0, \tau] \times \mathbb{R}, \\
p(\tau, \cdot) = \varphi,
\end{cases}
\]
for a given \( \tau \in [0, T] \). By Proposition 4.1.16, \( p(t, x) = \varphi(X(\tau, t, x)) \), so that
\[
\int \varphi(x) \mu(t, dx) = \int p(\tau, x) \mu(\tau, dx) = \int p(0, x) \mu(0, dx) = \int \varphi(X(\tau, 0, x)) \mu^0(dx),
\]
and the identity is proved for that \( \varphi \). The general case \( \varphi \in C_c(\mathbb{R}) \) follows by uniform approximation.

**Remark 4.2.7** The identity (4.2.6) means that \( \mu(t, dx) \) is the image of \( \mu^0(dx) \) by the application \( x \mapsto X(t, 0, x) \), which is well-known for a smooth \( a \). Therefore, (4.2.6) also holds for \( \varphi \) bounded, measurable with compact support. From this we deduce that for \( 0 \leq t \leq T \), \( |\mu(t, \cdot)| \leq \lambda(t, \cdot) \), where \( \lambda \) is the duality solution with initial data \( |\mu^0| \).

**Example 4.2.1** For \( \mu^0 = \delta(x - x^0) \), \( x^0 \in \mathbb{R} \), one has \( \mu(t, \cdot) = \delta(x - X(t, 0, x^0)) \). This gives the trajectory of a particle with initial position \( x^0 \).

Let us close this section by a lower estimate, which ensures that vacuum cannot appear if not present at time 0. An easy proof consists in smoothing \( a \) and \( \mu^0 \).

**Proposition 4.2.9** Let \( \mu \in S_{\mathcal{M}} \) be a duality solution to (4.2.4), and assume that \( \mu(0, dx) \geq \delta \, dx \) for some \( \delta > 0 \), in the sense of measures on \( \mathbb{R} \). Then for any \( t \in [0, T] \)
\[
\mu(t, dx) \geq \delta e^{-\int_0^t a} \, dx.
\]
4.3 Flux and universal representative

4.3.1 Flux of a conservative duality solution

The Definitions 4.2.1 and 4.2.4 of duality solutions are a priori not written in the distribution sense. We give here a simple way to define the product of the coefficient $a$ by the measure $\mu$, thus recovering a distribution equation. This product is in a way stable with respect to perturbations of $a$ and initial data. Notice that it is not defined for an arbitrary measure $\mu$, but only for duality solutions. That is the main reason why very general weak stability can hold. It cannot be true for more general definitions such as in G. Dal Maso, P. Le Floch, F. Murat [9] where the equation is not taken into account.

Definition 4.3.1 (Generalized flux) Let $\mu \in \mathcal{S}_M$ be a duality solution to (4.2.4). We define the flux corresponding to $\mu$ by

$$a \triangleleft \mu = -\partial_t u,$$

(4.3.1)

where $\mu = \partial_x u$ and $u \in \mathcal{S}_{BV}$ is a duality solution to the nonconservative problem (cf Proposition 4.2.6(ii)). We have therefore

$$\partial_t \mu + \partial_x (a \triangleleft \mu) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

(4.3.2)

The application $\mu \mapsto a \triangleleft \mu$ is of course linear, and since $u \in \text{Lip}([0, T], L^\infty_{\text{loc}}(\mathbb{R}))$, we have by Property 5 in §2.1 $a \triangleleft \mu \in L^\infty([0, T], \mathcal{M}_{1,\text{loc}}(\mathbb{R}))$, and for any $x_1 < x_2$,

$$\|a \triangleleft \mu\|_{L^\infty([0, T], \mathcal{M}(I_{x_1, x_2}))} \leq \|a\|_\infty \int_{x_1}^{x_2} |\mu(0, dx)|.$$

Remark 4.3.1 The flux corresponding to a time restriction is the time restriction of the flux.

Theorem 4.3.2 (Weak stability) Let $(a_n)$ be a bounded sequence in $L^\infty([0, T[ \times \mathbb{R})$, with $a_n \to a$ in $L^\infty([0, T[ \times \mathbb{R}) - w^*$. Assume $\partial_x a_n \leq \alpha_n(t)$, where $(\alpha_n)$ is bounded in $L^1([0, T[)$, $\partial_x a \leq \alpha \in L^1([0, T[)$. Consider a sequence $(\mu_n) \in \mathcal{S}_M$ of duality solutions to

$$\partial_t \mu_n + \partial_x (a_n \mu_n) = 0 \quad \text{in} \quad \Omega,$$

such that $\mu_n(0, )$ is bounded in $\mathcal{M}_{1,\text{loc}}(\mathbb{R})$, and $\mu_n(0, ) \rightharpoonup \mu^0 \in \mathcal{M}_{1,\text{loc}}(\mathbb{R})$. Then $\mu_n \rightharpoonup \mu$ in $\mathcal{S}_M$, where $\mu \in \mathcal{S}_M$ is the duality solution to

$$\partial_t \mu + \partial_x (a \mu) = 0 \quad \text{in} \quad \Omega, \quad \mu(0, ) = \mu^0.$$

Moreover, $a_n \triangleleft \mu_n \rightharpoonup a \triangleleft \mu$ weakly in $\mathcal{M}_{1,\text{loc}}(\Omega)$. 

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**Proof.** The sequence \((\mu_n)\) is bounded in \(\mathcal{S}_M\) by (4.2.5). By Proposition 4.2.6(ii), we can write \(\mu_n = \partial_x u_n\), with \(u_n \in \mathcal{S}_B\) duality solution to \(\partial_t u_n + a_n \partial_x u_n = 0\), such that \(u_n(0,\cdot)\) is bounded in \(BV_{loc}(\mathbb{R})\). Then \(u_n\) is bounded in \(\text{Lip}(0, T] , L^1_{loc}(\mathbb{R})\), and thus \((\mu_n)\) is equicontinuous in \(t\). Therefore, up to a subsequence, \(\mu_n \rightarrow \mu \in \mathcal{S}_M\).

Let us prove that \(\mu\) is a duality solution. Let \(p\) be a compactly supported reversible solution to \(\partial_t p + a \partial_x p = 0\) in \([0, \tau] \times \mathbb{R}\). From the stability theorem 4.1.10, the sequence \((p_n)\) of reversible solutions corresponding to \(a_n\) and \(p_n(\tau, \cdot) = p(\tau, \cdot)\) converges to \(p\). Since \(p_n\) is also compactly supported, \(t \mapsto \int p_n(t, x) \mu_n(t, dx)\) is constant in \([0, \tau]\), and so is \(t \mapsto \int p(t, x) \mu(t, dx)\) by passing to the limit. Therefore, \(\mu\) is a duality solution. Then uniqueness ensures that it is not necessary to extract any subsequence.

The convergence of the flux follows in a similar way from the convergence of \(u_n\). □

We now come to a more precise study of the flux.

**Lemma 4.3.3** Let \(\mu \in \mathcal{S}_M\) a conservative duality solution, and \(\varphi \in C_c(\Omega)\). Then

\[
\langle a \triangle \mu, \varphi \rangle = \int_{\mathbb{R}} \left( \int_0^T \varphi(t, X(t, 0, x)) \partial_x X(t, 0, x) \, dt \right) \mu^0(dx). \tag{4.3.3}
\]

Notice that the expression between brackets is continuous with respect to \(x\), because \(x \mapsto \partial_x X(\cdot, 0, x)\) is weakly continuous.

**Proof of Lemma 4.3.3.** We know that \(\mu = \partial_x u\), with \(u \in \mathcal{S}_B\) a nonconservative duality solution. Let us first assume that \(\varphi \in C_c^\infty(\Omega)\). We have by definition of the flux

\[\langle a \triangle \mu, \varphi \rangle = -\langle \partial_t \varphi, u \rangle = \int_{\mathbb{R}} \int_0^T u(t, x) \partial_t \varphi(t, x) \, dt \, dx.\]

By the definition of duality solutions and by Proposition 4.1.16(ii), we obtain

\[\int_{\mathbb{R}} u(t, x) \partial_t \varphi(t, x) \, dx = \int_{\mathbb{R}} u^0(x) \partial_t \varphi(t, X(t, 0, x)) \partial_x X(t, 0, x) \, dx,\]

so that \(\langle a \triangle \mu, \varphi \rangle = -\int_{-\infty}^0 u^0(x) I(x) \, dx\), where \(I\) is the following \(L^\infty(\mathbb{R})\) function:

\[I(x) = -\int_0^T \partial_t \varphi(t, X(t, 0, x)) \partial_x X(t, 0, x) \, dt.\]

Let us define

\[\Phi(t, x) = \int_{-\infty}^x \varphi(t, y) \, dy, \quad \psi(x) = -\int_0^T \partial_t \Phi(t, X(t, 0, x)) \, dt \in \text{Lip}_{loc}(\mathbb{R}),\]

so that \(\psi' = I\). We have

\[\psi(x) = \int_0^T \left\{ -\frac{d}{dt} [\Phi(t, X(t, 0, x))] + \partial_x \Phi(t, X(t, 0, x)) \partial_x X(t, 0, x) \right\} \, dt = \int_0^T \varphi(t, X(t, 0, x)) \partial_x X(t, 0, x) \, dt,\]

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thus $\psi \in \text{Lip}_c(\mathbb{R})$ and
\[
\langle a \ast \mu, \varphi \rangle = -\int_{\mathbb{R}} u^0(x) \psi'(x) \, dx = \int_{\mathbb{R}} \psi(x) \mu^0(dx),
\]
and we are done for a smooth $\varphi$. For $\varphi \in C_c(\Omega)$, the result follows by uniform approximation. \qed

We now state the main result of this section.

**Theorem 4.3.4 (Universal representative)** There exists a bounded Borel function $\hat{a} : [0,T[\times \mathbb{R} \to \mathbb{R}$ such that for any conservative duality solution $\mu$, one has
\[
a \ast \mu = \hat{a}(\mu).
\]

We call such a function a universal representative of $a$.

**Proof.** Define a Borel set $A$ by
\[
A = \left\{ (t,x) \in [0,T[\times \mathbb{R}; \; \frac{X(t+\varepsilon,t,x) - x}{\varepsilon} \text{ has a limit when } \varepsilon \to 0^+ \right\},
\]
and let
\[
\hat{a}(t,x) = \begin{cases} \lim_{\varepsilon \to 0^+} \frac{X(t+\varepsilon,t,x) - x}{\varepsilon} & \text{if } (t,x) \in A, \\ \text{anything} & \text{if } (t,x) \notin A. \end{cases}
\]

Let us first prove that $\hat{a}$ has the desired property for $\mu^0 = \delta(x-x^0)$, $x^0 \in \mathbb{R}$. We know in this case that $\mu(t) = \delta(x - X(t,0,x^0))$ (Example 4.2.1). On the one hand, by Lemma 4.3.3, we have for any $\varphi \in C_c(\Omega)$
\[
\langle a \ast \mu, \varphi \rangle = \int_0^T \varphi(t, X(t,0,x^0)) \partial_s X(t,0,x^0) \, dt,
\]
and on the other hand
\[
\langle \hat{a} \mu, \varphi \rangle = \int_0^T \varphi(t, X(t,0,x^0)) \hat{a}(t, X(t,0,x^0)) \, dt.
\]

We are thus led to prove
\[
\partial_s X(t,0,x^0) = \hat{a}(t, X(t,0,x^0)) \quad \text{a.e. in } t.
\]

But since $X$ is Lipschitz continuous, we have
\[
\partial_s X(t,0,x^0) = \lim_{\varepsilon \to 0^+} \frac{X(t+\varepsilon,t,x^0) - X(t,0,x^0)}{\varepsilon} \quad \text{a.e. in } t.
\]

Pick a $t$ for which this formula holds true, and set $y = X(t,0,x^0)$. Since $X(t+\varepsilon,t,x^0) = X(t+\varepsilon,t,y)$, we have $\partial_s X(t,0,x^0) = \lim_{\varepsilon \to 0^+} \frac{X(t+\varepsilon,t,y) - y}{\varepsilon}$, so that $(t,y) \in A$ and $\partial_s X(t,0,x^0) = \hat{a}(t,y) = \hat{a}(t, X(t,0,x^0))$. This concludes the proof for Dirac masses.
For a general duality solution \( \mu \), we have by Lemma 4.3.3 for any \( \varphi \in C_c(\Omega) \),

\[
\langle a \ast \mu, \varphi \rangle = \int \int_t^T \varphi(x(t), X(t, 0, x)) dt = \int_0^T \varphi(t, X(t, 0, x)) \hat{a}(t, X(t, 0, x)) dt
\]

by (4.3.7). Using Remark 4.2.7 we get

\[
\langle a \ast \mu, \varphi \rangle = \int_0^T dt \int_\mathbb{R} \varphi(t, X(t, 0, x)) \hat{a}(t, X(t, 0, x)) \mu^0(dx) \\
= \int_0^T dt \int_\mathbb{R} \varphi(t, x) \hat{a}(t, x) \mu(t, dx) \\
= \langle a, \mu, \varphi \rangle.
\]

Remark 4.3.2 By definition, if \( \hat{a}_1 \) and \( \hat{a}_2 \) are two universal representatives, then

\[
\hat{a}_1 = \hat{a}_2 \text{ a.e. } dt \mu(t, dx).
\]

Remark 4.3.3 In the above construction, since \( \hat{a} \) can take any value on \( A^c \), we have

\[
\int_{A^c} dt |\mu(t, dx)| = 0 \quad \text{for any duality solution } \mu.
\]

From this we deduce that for any duality solution \( \mu \)

\[
\frac{X(t + \varepsilon, t, x) - x}{\varepsilon} \to \hat{a}(t, x) \quad \text{dt} \mu(t, dx) \text{ a.e. in } \Omega.
\]

Proposition 4.3.5 (i) Let \( \hat{a}_1 \) be a universal representative of \( a \), and \( \hat{a}_2 : \Omega \to \mathbb{R} \) a bounded measurable function. Then \( \hat{a}_2 \) is a universal representative if and only if

\[
\forall x \in \mathbb{R} \quad a.e. \ s \in [0, T] \quad \hat{a}_2(s, X(s, 0, x)) = \hat{a}_1(s, X(s, 0, x)).
\]

(ii) One has for any \( (t, x) \in [0, T] \times \mathbb{R} \)

\[
a.e. \ s \in [t, T] \quad \partial_s X(s, t, x) = \hat{a}(s, X(s, t, x)).
\]

(iii) Let \( \hat{a}_1 \) and \( \hat{a}_2 \) be two universal representatives of \( a \). Then

\[
\hat{a}_1 = \hat{a}_2 \quad a.e. \ in \Omega.
\]

Proof. (i) Assume \( \hat{a}_2 \) is a universal representative, and take for \( \mu^0 \) a Dirac mass at \( x^0 \). Then by (4.3.7), (4.3.9) holds for \( x^0 \). Conversely, if (4.3.9) holds, then the definition property (4.3.4) is satisfied for Dirac masses. The general case follows as in the proof of Theorem 4.3.4.
(ii) The case \( t = 0 \) was treated in the proof of Theorem 4.3.4 (see (4.3.7)). In the general case, there exists \( x^0 \in \mathbb{R} \) such that \( x = X(t, 0, x^0) \). Thus, if \( s \in [t, T] \),
\[
X(s, t, x) = X(s, t, X(t, 0, x^0)) = X(s, 0, x^0).
\]
Differentiating in \( s \) and using (4.3.7) we obtain the result.

Remark 4.3.4 Formula (4.3.10) does not depend on the choice of \( \hat{a} \). Indeed, if \( \hat{a}_1 \) is another universal representative, we have by (4.3.9), for any \( t \in [0, T[ \) and \( x \in \mathbb{R} \),
\[
a.e. \ s \in [t, T[ \quad \hat{a}(s, X(s, t, X(t, 0, x))) = \hat{a}_1(s, X(s, t, X(t, 0, x))).
\]
But \( x \mapsto X(t, 0, x) \) is surjective, so that
\[
\forall t \in [0, T[, \forall y \in \mathbb{R}, \ a.e. \ s \in [t, T[, \quad \hat{a}(s, X(s, t, y)) = \hat{a}_1(s, X(s, t, y)).
\]

Remark 4.3.5 We give in Theorem 4.3.10 a more precise version of (iii), stating that actually \( \hat{a} = a \) a.e. in \( \Omega \).

Remark 4.3.6 Property (ii) shows that the "hat" operator and time restriction commute.

We are now able to prove an important feature for applications.

**Theorem 4.3.6 (Entropy inequality)** Let \( \mu \in \mathcal{S}_M \) be a conservative duality solution. Then
\[
\partial_t |\mu| + \partial_x (\hat{a}|\mu|) \leq 0 \quad in \quad \mathcal{D}'(\Omega).
\]

**Proof.** Let \( \varphi \in C_c^\infty(\mathbb{R}) \), \( \varphi \geq 0 \), \( t_0 \in [0, T[ \) and \( \varepsilon > 0 \) such that \( t_0 + \varepsilon < T \). Consider the conservative duality solution \( \lambda \) on \( [t_0, T] \), such that \( \lambda(t_0, dx) = |\mu(t_0, dx)| \). Making use of Remark 4.2.7 and of the representation formula (4.2.6), we obtain
\[
\int \varphi(x) |\mu(t_0 + \varepsilon, dx)| \leq \int \varphi(x) \lambda(t_0 + \varepsilon, dx) = \int \varphi(X(t_0 + \varepsilon, t_0, x)) |\mu(t_0, dx)|.
\]
Therefore
\[
\frac{1}{\varepsilon} \left[ \int \varphi(x) |\mu(t_0 + \varepsilon, dx)| - \int \varphi(x) |\mu(t_0, dx)| \right] \leq \int \frac{\varphi'(x) X(t_0 + \varepsilon, t_0, x) - \varphi(x)}{\varepsilon} |\mu(t_0, dx)|
\]
\[
+ \frac{1}{\varepsilon} \left[ \varphi(X(t_0 + \varepsilon, t_0, x)) - \varphi(x) \right](X(t_0 + \varepsilon, t_0, x) - x) |\mu(t_0, dx)|.
\]

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The second term in the right-hand side tends to 0 when \( \varepsilon \to 0 \), uniformly in \( t_0 \), and the first one, by Remark 4.3.3, satisfies for a.e. \( t_0 \)

\[
\int \varphi'(x) \frac{X(t_0 + \varepsilon, t_0, x) - x}{\varepsilon} \left| \mu(t_0, dx) \right| \xrightarrow{\varepsilon \to 0} \int \varphi'(x) \hat{a}(t_0, x) \left| \mu(t_0, dx) \right|.
\]

Thus we obtain in the sense of \( \mathcal{D}'(]0, T[) \), for any \( \varphi \in C^\infty_c(\mathbb{R}) \), \( \varphi \geq 0 \),

\[
\frac{d}{dt} \int \varphi(x) |\mu(t, dx)| \leq \int \varphi'(x) \hat{a}(t, x) |\mu(t, dx)|,
\]

or equivalently for any \( \varphi \in C^\infty_c(\mathbb{R}) \), \( \varphi \geq 0 \), and \( \chi \in C^\infty_c([0, T[) \), \( \chi \geq 0 \),

\[
\langle \partial_t |\mu| + \partial_x (\hat{a} |\mu|), \chi \otimes \varphi \rangle \leq 0.
\]

Now we take a mollifier \( \rho_n = \chi_n \otimes \varphi_n \) and obtain

\[
\rho_n * \left( \partial_t |\mu| + \partial_x (\hat{a} |\mu|) \right) \leq 0 \quad \text{pointwise.}
\]

Therefore, we get (4.3.12) by letting \( n \to \infty \). \( \square \)

We end this section by a geometric description in the piecewise continuous case.

**Theorem 4.3.7 (Piecewise continuous coefficient)** Let us assume that in addition to the hypothesis (4.1) used in all Section 4, \( a \) is piecewise continuous in the sense of (PC1)-(PC3) in Section 3. Then there exists a function \( \hat{a} \) which coincides with \( a \) in \( C \), and which is admissible in the sense of (AC1)-(AC2).

With this \( \hat{a} \), \( \mu \in \mathcal{S}_M \) is a duality solution to (4.2.4) if and only if \( \partial_t \mu + \partial_x (\hat{a} |\mu|) = 0 \) in \( \mathcal{D}'(\Omega) \). Then \( a \ast \mu = \hat{a} \mu \).

In particular, \( \hat{a} \) is a universal representative of \( a \).

**Proof.** By Theorems 4.2.5 and 3.6, there exists unique solutions to the Cauchy problem for both kinds of solutions (duality solutions and distributional solutions). Since they are built by the same procedure (approximation of \( a \), they do coincide. Therefore, both notions are identical. The remaining assertions are obvious. \( \square \)

### 4.3.2 Characteristics in Filippov’s sense

**Lemma 4.3.8** Assume that a.e. \( t \in ]0, T[ \), \( a(t, \cdot) \in C(\mathbb{R}) \). Then for any conservative duality solution \( \mu \in \mathcal{S}_M \),

\[
a \ast \mu = a \mu.
\] (4.3.13)
\textbf{Proof.} Consider the same mollifier \( \rho_n \) as in Lemma 2.3.1, let \( a_n = \rho_n * a \) and \( \mu_n^0 = \rho_n^* \mu^0 \), and denote by \( \mu_n \) the (classical) solution to
\[
\partial_t \mu_n + \partial_x (a_n \mu_n) = 0 \quad \text{in} \quad ]0, T - \varepsilon[ \times \mathbb{R}, \quad \mu_n(0,.) = \mu_n^0.
\]
The stability result (Theorem 4.3.2) ensures that \( \mu_n \to \mu \) in \( C([0, T - \varepsilon], \mathcal{M}_{loc}(\mathbb{R}) - \sigma(\mathcal{M}_{loc}(\mathbb{R}), C_c(\mathbb{R}))) \) and \( a_n \mu_n \to a \Delta \mu \) in \( \mathcal{D}'([0, T - \varepsilon]\times \mathbb{R}) \). On the other hand, \( a_n \mu_n = a \mu_n + (a_n - a) \mu_n \), and since \( a \) is continuous in \( x \), \( a \mu_n \to a \mu \sigma(\mathcal{M}_{loc}(\mathbb{R}), C_c(\mathbb{R})) \) for a.e. \( t \in ]0, T - \varepsilon[ \). Integrating in \( t \) and using Lebesgue’s theorem leads therefore to \( a \mu_n \to a \mu \) in \( \mathcal{D}'([0, T - \varepsilon]\times \mathbb{R}) \). Next, as in the proof of Lemma 2.3.1, \( a_n \to a \) in \( L^1([0, T - \varepsilon], L_{loc}^\infty(\mathbb{R})) \) and thus \( (a_n - a) \mu_n \to 0 \) in \( L_{loc}^1([0, T - \varepsilon]\times \mathbb{R}) \). Therefore \( a_n \mu_n \to a \mu \) in \( \mathcal{D}'([0, T - \varepsilon]\times \mathbb{R}) \), which gives \( a \Delta \mu = a \mu \) in \( ]0, T - \varepsilon[ \times \mathbb{R} \), for any \( \varepsilon > 0 \).

We now prove that the backward flow actually coincides with the generalized forward solution introduced by A.F. Filippov [12]. Recall that \( g \in \text{Lip}([t, T]) \) solves \( g'(s) = a(s, g(s)) \) in the Filippov sense if
\[
a.e. \ s \in ]t, T[ \quad \lim_{\delta \to 0^+} \text{ess inf}_{|y-g(s)|<\delta} a(s, y) \leq g'(s) \leq \lim_{\delta \to 0^+} \text{ess sup}_{|y-g(s)|<\delta} a(s, y).
\]
In our case, since \( a \) satisfies the one-sided Lipschitz condition (4.1), \( a(t, \cdot) \in BV_{loc}(\mathbb{R}) \) for a.e. \( t \) so that \( a(t, x+) = \lim_{y \to x^+} a(t, y) \) and \( a(t, x-) = \lim_{y \to x^-} a(t, y) \) are well-defined for any \( x \in \mathbb{R} \). Moreover, again by (4.1), \( a(t, x+) \leq a(t, x-) \). Therefore Filippov’s condition reads as
\[
a.e. \ s \in ]t, T[ \quad g'(s) \in [a(s, g(s)+), a(s, g(s)-)]. \tag{4.3.14}
\]

\textbf{Proposition 4.3.9 (Filippov’s flow)} For any \( (t, x) \in ]0, T[ \times \mathbb{R} \), \( s \mapsto X(s, t, x) \) is the unique function \( g \in \text{Lip}([t, T]) \) which satisfies (4.3.14) and \( g(t) = x \).

\textbf{Proof.} Uniqueness is contained in Filippov’s paper [12]. In order to prove that \( g(s) \equiv X(s, t, x) \) is a solution, take \( \rho_n \) a smoothing sequence in \( \mathbb{R} \), and set \( a_n = \rho_n * a \), which satisfies \( \partial_t a_n \leq \alpha \). By Lemma 4.3.8, we can choose \( \tilde{a}_n = a_n \), and we have by \( (4.3.10) \)
\[
a.e. \ s \in ]t, T[ \quad \partial_x X_n(s, t, x) = a_n(s, X_n(s, t, x)),
\]
which means that \( X_n \) is solution in Filippov’s sense. By the stability theorem 4.1.15, \( X_n \to X \) uniformly on bounded subsets of \( D_t \times \mathbb{R} \). Therefore, by Filippov’s stability theorem [12], \( X \) verifies (4.3.14).

\textbf{Remark 4.3.7} Theorem 4.1.15 can be interpreted as a weak stability result for Filippov’s solutions.

\textbf{Theorem 4.3.10} One can choose a universal representative \( \tilde{a} \) of a such that
\[
a.e. \ t \in ]0, T[, \forall x \in \mathbb{R}, \tilde{a}(t, x) \in [a(t, x+), a(t, x-)]. \tag{4.3.15}
\]
In particular, we have
\[ \hat{a}(t, x) = a(t, x) = a(t, x+) = a(t, x-) \quad \text{a.e. in } [0, T] \times \mathbb{R}. \] (4.3.16)

Notice that by Proposition 4.3.5(iii), (4.3.16) therefore holds for any universal representative \( \hat{a} \) of \( a \).

**Proof of Theorem 4.3.10.** By changing \( a \) if necessary on a set of \( t \) of measure zero, we can assume that \( a(t, \cdot) \in BV_{loc}(\mathbb{R}) \) for any \( t \). Then the set
\[ B = \left\{ (t, x) \in \Omega: \lim_{\varepsilon \to 0^+} \frac{X(t + \varepsilon, t, x) - x}{\varepsilon} \right\} \]
is a Borel set, and we can define
\[ \hat{a}(t, x) = \begin{cases} \lim_{\varepsilon \to 0^+} \frac{X(t + \varepsilon, t, x) - x}{\varepsilon} & \text{if } (t, x) \in B, \\ \frac{a(t, x+) + a(t, x-)}{2} & \text{if } (t, x) \notin B. \end{cases} \]

By Proposition 4.3.9, we have for any \( x \in \mathbb{R} \)
\[ \text{a.e. } t \in ]0, T[, \quad \partial_x X(t, 0, x) \in [a(t, X(t, 0, x)+), a(t, X(t, 0, x)-)]. \]
Then, following the proof of Theorem 4.3.4, \( \hat{a} \) is a universal representative of \( a \).

### 4.3.3 Reversibility and renormalization

Let us now come back to the backward problem.

**Lemma 4.3.11 (Strong continuity)** Let \( \pi \in \mathcal{S}_{L^\infty} \) be a solution to \( \partial_t \pi + \partial_x (a \pi) = 0 \) in \( \Omega \). Then \( \pi \in C([0, T], L^1_{loc}(\mathbb{R})) \).

Notice that \( \pi \) is not assumed to be reversible.

**Proof of Lemma 4.3.11.** We use the same method as in R.J. DiPerna and P.-L. Lions [10]. Consider \( \rho_n \) a mollifier in \( \mathbb{R} \), and define \( \pi_n = \rho_n \ast \pi \in C([0, T] \times \mathbb{R}) \), and \( a_n = \rho_n \ast a \in L^\infty(\Omega) \). We notice that \( \pi_n \in \text{Lip}_{loc}(\mathbb{R}) \), since it satisfies
\[ \partial_t \pi_n + \partial_x (a_n \pi_n) = 0 \quad \text{in} \quad \Omega. \]
Writing
\[ \partial_t \pi_n + \partial_x (a_n \pi_n) = \partial_x (a_n \pi_n - \rho_n \ast (a \pi)) \in L^\infty_{loc}, \]
we multiply this equality by $\pi_n$, to obtain
\[
\frac{\partial}{\partial t} \left( \frac{\pi_n^2}{2} \right) + \partial_x \left( a_n \frac{\pi_n^2}{2} \right) = -\frac{\pi_n^2}{2} \partial_x a_n + \pi_n \partial_x (a_n \pi_n - \rho_n (a \pi)) \equiv f_n \quad \in L^\infty_{\text{loc}}.
\]

Let $\varphi \in C_c^\infty(\mathbb{R})$. The function $t \mapsto \int \frac{\pi_n^2}{2} \varphi \, dx$ is continuous on $[0, T]$, and satisfies
\[
\frac{d}{dt} \int a_n \frac{\pi_n^2}{2} \varphi \, dx = \int f_n \varphi \, dx \quad \text{in} \quad [0, T].
\]

In view of the regularity of $a (4.3)$, we write
\[
f_n = -\frac{\pi_n^2}{2} \rho_n \partial_x a_n + \pi_n \left[ \left( a_n - a \right) \rho_n \pi + \pi_n \rho_n \partial_x a + \alpha \rho_n \pi - \alpha' (a \pi) \right].
\]

Notice now that
\[
a_n - a = \int_{\mathbb{R}} \left[ a(t, x - y) - a(t, x) \right] \rho_n(y) \, dy,
\]
\[
\alpha' (a \pi) - \alpha' (a \pi) \pi = \int_{\mathbb{R}} \left[ a(t, x) - a(t, x - y) \right] \pi(t, x - y) \rho_n(y) \, dy.
\]

By using (4.3), we obtain for a.e. $t \in [0, T]$,
\[
\int_{x_2 - 1}^{x_2} |a(t, x - y) - a(t, x)| \, dx \leq 2 \|y\| (|a(t)| (x_2 - x_1 + 1/n) + \|a\|_{\infty}),
\]
and setting $M = \underset{x \in [-1/n, 1/n]}{\text{ess sup}} |\pi(t, x)|$,
\[
\int_{x_2}^{x_2} |\alpha (t) (x_2 - x_1 + 1/n) + \|a\|_{\infty} / 2 M^2 |\varphi|| \varphi'_1 |.
\]

Putting things together, and choosing $x_1 < x_2$ such that $\text{supp} \varphi \subset [x_1, x_2]$, we get that for a.e. $t \in [0, T]$,
\[
\frac{d}{dt} \int a_n \frac{\pi_n^2}{2} \varphi \, dx \leq C M^2 \left( |\alpha (t)| (x_2 - x_1 + 1/n) + \|a\|_{\infty} \right) \|\varphi\|_{\infty} + \frac{M^2}{2} \|a\|_{\infty} \|\varphi'_1 \|
\]

The family $t \mapsto \int \frac{\pi_n^2}{2} \varphi \, dx$ is therefore relatively compact in $C([0, T])$. Since it converges pointwise, the convergence is thus uniform, and the limit
\[
t \mapsto \int \frac{\pi (t, x)^2}{2} \varphi(x) \, dx
\]
is continuous in $[0, T]$. The result follows since it is true for any $\varphi \in C_c^\infty(\mathbb{R})$. $lacksquare$

In a similar way, one can prove

**Proposition 4.3.12** The backward flow $X$ satisfies $\partial_x X \in C(D_b, I_{loc}^1(\mathbb{R}))$, with $D_b = \{(s, t) \in \mathbb{R}^2; \ 0 \leq t \leq s \leq T\}$.

We can now prove the following characterization for reversibility by renormalization.

**Theorem 4.3.13 (Reversibility by renormalization)** Let $\pi \in S_{L^\infty}$ be a distributional solution to $\partial_t \pi + \partial_x (a \pi) = 0$ in $\Omega$. Then

$$\partial_t |\pi| + \partial_x (a |\pi|) \leq 0 \quad \text{in} \quad \Omega,$$

and equality holds if and only if $\pi$ is reversible.

**Proof.** By Remark 4.2.4 $\pi$ is a duality solution. Thus Theorem 4.3.6 gives the inequality since $\hat{a} = a$ a.e. by (4.3.16).

Concerning equality, first assume that $\pi$ is reversible. Then by Proposition 4.1.16(ii) $\pi(t, x) = \pi(T, X(T, t, x)) |\partial_x X(T, t, x)|$, and thus $|\pi(t, x)| = |\pi(T, X(T, t, x))| |\partial_x X(T, t, x)|$ is the reversible solution with final data $|\pi(T, .)|$. Conversely, if $|\pi|$ solves the equation with equality, we notice that by Lemma 4.3.11 $|\pi| \in S_{L^\infty}$. But $|\pi| \geq 0$, so $|\pi|$ is reversible by Theorem 4.1.12(iii). Therefore, $|\pi| = 0$ in $\mathcal{V}_e$ by Theorem 4.1.12(ii), and $\pi = 0$ in $\mathcal{V}_e$. The same criterion (ii) thus ensures that $\pi$ is reversible. $lacksquare$

## 4.4 On viscous problems

### 4.4.1 Duality for viscous problems

The duality method can be used to treat viscous problems. Let us show formally how it works for a constant viscosity $\varepsilon$. The forward conservative problem is written in this case

$$\partial_t \mu + \partial_x (a \mu) - \varepsilon \partial^2_{x,x} \mu = 0, \quad \mu(0, .) = \mu^0, \quad (4.4.1)$$

and the backward nonconservative problem is

$$\partial_t p + a \partial_x p + \varepsilon \partial^2_{x,x} p = 0, \quad p(T, .) = p_T. \quad (4.4.2)$$

For both problems, we have the same a priori estimates as for the non-viscous equations. Notice also that the decomposition of $p$ as a difference of monotone solutions is valid. The duality between (4.4.1) and (4.4.2) can be seen by multiplying (4.4.1) by $p$, (4.4.2) by $\mu$ and by adding the results. This yields

$$\partial_t (p \mu) + \partial_x (a \mu) + \varepsilon \partial_x (\mu \partial_x p - p \partial_x \mu) = 0. \quad (4.4.3)$$

Therefore,

$$\frac{d}{dt} \langle \mu, p \rangle = 0 \quad (4.4.4)$$

as in the non-viscous case. This shows that $p$ and $\mu$ tend respectively to the reversible and duality solutions to the non-viscous problems when $\varepsilon \to 0$. 

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4.4.2 Backward problem with discontinuous final data

Let us again consider the backward nonconservative problem

\[ \partial_t p + a \partial_x p = 0, \quad p(T, \cdot) = p^T. \tag{4.4.5} \]

To any \( p^T \in \text{Lip}_{loc}(\mathbb{R}) \) we can associate the reversible solution \( p \in \text{Lip}_{loc}([0, T] \times \mathbb{R}) \) to (4.4.5). Since this operator is linear and continuous for the topology of uniform convergence on compact sets (by Theorem 4.1.5), it extends in a unique linear continuous operator from \( C(\mathbb{R}) \) to \( C([0, T] \times \mathbb{R}) \). Actually this extension can be expressed in terms of the flow (see Proposition 4.1.16(i)).

We want here to show that there is no natural way to solve (4.4.5) if \( p^T \) is discontinuous, even if \( p^T \in BV_{loc}(\mathbb{R}) \).

Let us first choose a sequence \( p^T_n \in \text{Lip}_{loc}(\mathbb{R}) \) such that \( p^T_n \to p^T \) in \( L^1_{loc}(\mathbb{R}) \), with \( p^T_n \) bounded in \( BV_{loc}(\mathbb{R}) \). Then the corresponding reversible solution \( p_n \) is also a duality solution by Remark 4.2.4, which is bounded in \( S_{ BV} \) since \( p_n \) is reversible. Therefore, after extraction of a subsequence, \( p_n(0, \cdot) \to p^0 \in BV_{loc} \), and by Theorem 4.3.2 \( p_n \to p \) the duality solution with initial data \( p^0 \). Moreover, \( p \) is locally constant in \( V_c \) since \( p_n \) is. However, the values of \( p \) in \( V_c \) strongly depend on the sequence of approximations \( (p^T_n) \). For example, if \( a(t, x) = -\text{sgn} \, x \), this value is \( \lim p^T_n(0) \), which can be anything if \( p^T \) is discontinuous at 0.

However, it may be possible to determine what is \( p \) in \( V_c \) by selecting a specific approximation procedure, see F. James and M. Sepulveda [16]. For example, let us solve

\[ \partial_t p_\varepsilon + a \partial_x p_\varepsilon + \varepsilon \partial^2_{xx} p_\varepsilon = 0, \quad p_\varepsilon(T, \cdot) = p^T. \tag{4.4.6} \]

In the case where \( a = -\text{sgn} \, x \),

\[ p_\varepsilon \to p \quad \text{and} \quad p = (p^T(0^+) + p^T(0^-))/2 \quad \text{in} \quad V_c. \tag{4.4.7} \]

Actually by setting \( u_\varepsilon(t, x) = p_\varepsilon(T - t, x) \) we are led to

\[ \partial_t u_\varepsilon + (\text{sgn} \, x) \partial_x u_\varepsilon - \varepsilon \partial^2_{xx} u_\varepsilon = 0, \quad u_\varepsilon(0, \cdot) = u^0 = p^T. \tag{4.4.8} \]

The solution is given by

\[ u_\varepsilon(t, x) = \int_{\mathbb{R}} G_\varepsilon(t, x, y) u^0(y) \, dy, \tag{4.4.9} \]

\[ G_\varepsilon(t, x, y) = \frac{1}{\sqrt{4\pi t \varepsilon}} e^{-(x-\varepsilon \text{sgn}(x)-y)^2/4t\varepsilon - u_{yy} \text{sgn}(y) b_{bl}/\varepsilon} + \frac{e^{-b_{bl}/\varepsilon}}{2\varepsilon} \text{erf} \left( \frac{|x| + |y| - t}{\sqrt{2t\varepsilon}} \right), \tag{4.4.10} \]

\[ \text{erf}(y) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz, \tag{4.4.11} \]

and we easily get (4.4.7).
References


