A Non-linear Stochastic Differential Equation Involving the Hilbert Transform

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We consider a non-linear stochastic differential equation which involves the Hilbert transform,

\[ X_t = \sigma B_t + 2 \int_0^t H(u(s, X_s)) \, ds. \]

In the previous equation, \( u(t, \cdot) \) is the density of \( +t \), the lax of \( X_t \), and \( H \) represents the Hilbert transform in the space variable. In order to define correctly the solutions, we first study the associated non-linear second-order integro-partial differential equation which can be reduced to the holomorphic Burgers equation. The real analyticity of solutions allows us to prove existence and uniqueness of the non-linear diffusion process. This stochastic differential equation has been introduced when studying the limit of systems of Brownian particles with electrostatic repulsion when the number of particles increases to infinity. More precisely, it has been shown that the empirical measure process tends to the unique solution \( \mu = (\mu_t)_{t \geq 0} \) of the non-linear second-order integro-partial differential, equation studied here.

1. INTRODUCTION

In their paper [1], E. Cépa and D. Lépingle have considered systems of \( N \) interacting Brownian particles on the real line which are submitted to an electrostatic repulsion and an affine drift. More precisely, they have studied the following system of stochastic differential equations

\[
\begin{align*}
    dX^{(i)}_t &= \sigma dW^{(i)}_t + \frac{2t_i}{N} \sum_{1 \leq j \neq i \leq N} \frac{dt}{X^{(i)}_t - X^{(j)}_t} + (\theta X^{(i)}_t + \rho) \, dt, \\
    i &= 1, 2, \ldots, N
\end{align*}
\]

with the further assumptions

\[
\begin{align*}
    X^{(1)}_t &\leq X^{(2)}_t \leq \cdots \leq X^{(N)}_t, \quad 0 \leq t < \infty, \\
    X^{(i)}_0 &= \xi_i, \quad i = 1, 2, \ldots, N.
\end{align*}
\]
The constants $\lambda$ and $\sigma$ are positive and the constants $\theta$, $\rho$ are real. The initial condition, $\xi$, is a real random variable. Strong existence and uniqueness have been proved in [1]. Moreover, E. Cépa and D. Lépingle considered the behavior of the unique solution when $N$ tends to infinity. They proved weak convergence of the empirical distributions

$$\mu_i^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^{(i)}}.$$  

Here, for real $x$, $d_x$ denotes the Dirac probability. They proved that the limiting measure-valued process $\mu$ is the unique continuous probability measure-valued function which satisfies

$$\int f(x) \mu_i(dx) = f(x) + \int_0^t \left( \int (\theta x + \rho) \cdot f'(x) \mu_i(dx) \right) dx$$

$$+ \frac{\sigma^2}{2} \int_0^t \left( \int f''(x) \mu_i(dx) \right) ds + \lambda \int_0^t \left( \int \frac{f'(x) - f'(y)}{x - y} \mu_i(dx) \mu_j(dy) \right) ds$$

for all functions $f$ which have bounded continuous derivatives up to order 2, and such that $\sigma f'(x)$ is bounded.

Systems of interacting diffusing particles governed by

$$dX_i^{(i)} = b_N(X_i^{(i)}) \, dt + \sigma_N(X_i^{(i)}) \, dW_i^{(i)} - D_i \Psi_N(X_i^{(i)}, X_i^{(2)}, ..., X_i^{(N)}) \, dt,$$

$i = 1, 2, ..., N,$

where $\Psi_N$ is an interaction potential of the form

$$\Psi_N(x^{(1)}, x^{(2)}, ..., x^{(N)}) = \gamma_N \sum_{i \neq j} V(x^{(i)} - x^{(j)}),$$

have been studied by many authors, going back at least to H. P. McKean [3]. The reader may consult A. S. Sznitman [8] as well. In our situations, the potential $V$ has a logarithmic singularity at 0 and usual results and tools are no more available.

Let us notice that the eigenvalues of a randomly-diffusing symmetric matrix diffusion obey an equation like (1.1) with $\rho = 0$, $\sigma = 1/\sqrt{N}$, $\lambda = 1/4$ and $\theta < 0$. The case $\theta \neq 0$ corresponds to a matrix diffusion of Ornstein-Uhlenbeck processes rather than Brownian motions. It is the reason for which the system (1.1) has been studied by T. Chan [2] and L. C. G. Rogers and Z. Shi [5]. In this situation, there are no collisions between particles and, since the diffusion coefficient tends to 0, the limit
process has no diffusive part. In contrast with [2, 5], collisions between particles are possible in [1] and the limiting equation is a second-order limiting integro-partial differential equation (which will be referred as I-PDE from now).

In the present work we consider the case $\theta = \rho = 0$ and $\xi = 0$. According to (1.5), $\mu = (\mu_t)$, is the weak solution of the following non-linear second-order I-PDE

$$\begin{cases}
\frac{\partial \mu_t}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \mu_t}{\partial x^2} - 2\lambda \frac{\partial (\mu_t H(\mu_t))}{\partial x} \\
\mu = \delta_0,
\end{cases} \quad (1.6)$$

the so-called McKean–Vlasov equation, where $H(\mu_t)$ stands for the Hilbert transform of $\mu_t$ given by $H(v) = pv(1/x) * v$. We prove, using complex variables methods, that, for each $t > 0$, the probability $\mu_t$ has a density $u(t, \cdot)$, and that $u_t$ and its Hilbert transform $H(u_t)$ are real analytic. Thus, $u = (u_t)_t$ is a classical solution of

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - 2\lambda \frac{\partial (u H(u))}{\partial x} \\
\mu(t, x) \, dx \to \delta_0 \quad \text{as} \quad t \to 0.
\end{cases} \quad (1.7)$$

Moreover, $H(u_t)$ can be written in terms of an explicit bounded analytic function $\psi$ as

$$H(u_t)(x) = q(t, x) = \frac{1}{2\lambda} \varphi\left(\frac{x}{\sqrt{t}}\right). \quad (1.8)$$

The non-linear SDE (in the sense of McKean) associated with (1.6) is given by

$$\begin{cases}
dX_t = \sigma \, dB_t + 2\lambda q(t, X_t) \, dt \\
X_0 = 0.
\end{cases} \quad (1.9)$$

Indeed, when applying Itô's formula, it is not difficult to prove that the law of a solution $X$ has density $u_t$ at time $t$, so that we are linked to the non-linear stochastic differential equation

$$\begin{cases}
X_t = \sigma \cdot B_t + 2\lambda \int_0^t H(u(s, X_s)) \, ds \\
X_0 = 0; \quad X_t \sim u(t, x) \, dx, \quad t > 0,
\end{cases} \quad (1.10)$$

where the notation “$Y \sim \nu$” means that the random variable $Y$ has the law $\nu$. 
The main probabilistic result of this paper is existence and uniqueness for the strong solution of (1.10). As B. Roynette and P. Vallois in their study of Burgers equation for example (see [6]), we can separate the unknowns $X$ and $u$. We first find $u$ and replace this function in the above equation, which becomes an ordinary stochastic differential equation in $X$ (non-homogeneous, with a singularity at $t = 0$).

As a conclusion, we are able to adapt to singular drifts the scheme used in [8] to link systems of particles and non-linear processes. Let us notice however that we did not yet obtain any result of chaos propagation in our situation.

2. A NON-LINEAR I-PDE WITH A HILBERT TRANSFORM

The aim of this section is to provide some existence/uniqueness and, especially, regularity results for the non-linear I-PDE

$$\frac{\partial \mu_j}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \mu_j}{\partial x^2} - \frac{2\lambda}{\partial x} \frac{\partial (\mu_j \mathcal{H}(\mu_j))}{\partial x}.$$  \(2.11\)

The notion of weak solution for (2.11) is understood as usual in the sense of Schwartz distributions, after integration by parts against suitable test functions, and existence of a weak solution has been proved in [1] with a probabilistic method. More precisely, there is an unique continuous probability measure-valued function $\mu$ which satisfies for all functions $f$ which have bounded continuous derivatives up to order 2 and such that $\int f(x) \mu_t(dx) = \int f(x) \mu_0(dx) + \frac{\sigma^2}{2} \int_0^t ds \left( \int f''(x) \mu_s(dx) \right)$

$$\begin{align*}
\int & f(x) \mu_t(dx) = \int f(x) \mu_0(dx) + \frac{\sigma^2}{2} \int_0^t ds \left( \int f''(x) \mu_s(dx) \right) \\
& + \lambda \int_0^t ds \left( \int \frac{f'(x) - f'(y)}{x-y} \mu_s(dx) \mu_s(dy) \right). \quad 2.12
\end{align*}
$$

The following result gives us the existence of a classical solution as well as its regularity. We use the notation $\alpha = 2\lambda/\sigma^2$ in the following.

**Theorem 2.1.** Let $\mu_0$ be a probability measure on $\mathbb{R}$. Then, there is exactly one (classical) solution $u$ for

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{2\lambda}{\partial x} \frac{\partial (u \mathcal{H}(u))}{\partial x} \\
u(t, x) \to \mu_0 \quad \text{as} \quad t \to 0.
\end{cases}$$  \(2.13\)
Moreover, $u$ and its Hilbert transform $\mathcal{H}(u)$ are real analytic functions in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Let us now take $\mu_0 = \delta_0$. We put $q(t, x) = \mathcal{H}(u(t, \cdot))(x)$.

**Corollary 2.2.** We have

$$q(t, x) = \frac{1}{2\sqrt{i}} \varphi \left( \frac{x}{\sqrt{t}} \right),$$

where $\varphi$ is the bounded real analytic function given by

$$\frac{1}{\pi} \varphi(\alpha x) = \lim_{y \to 0} \Re \left\{ \frac{(x - v) e^{-(x - v)^2/2} e^{-i\alpha v}}{\sqrt{2\pi} e^{-(v + iy)^2/2}} \right\},$$

with the principal branch for the power function.

**Remark.** There is a similar way to write $u$ as $u(t, x) = (1 - t) \psi(x/\sqrt{t})$.

**Proof.** We start with the proof of the theorem. Now, for $z = x + iy \in \mathbb{C}$, $y > 0$, we consider the function

$$M_t(z) = \int \frac{\mu_t(du)}{z - u},$$

which is holomorphic in the upper half-plane. Simple calculations, with (2.12) written for the function $f(u) = 1/(z - u)$, show that $M$ satisfies the holomorphic Burgers equation

$$
\begin{align*}
\frac{\partial M_t(z)}{\partial t} &= \frac{1}{2} \frac{\partial^2 M_t(z)}{\partial z^2} - 2\lambda M_t(z) \cdot \frac{\partial M_t(z)}{\partial z}, \\
M_0(z) &= \int \frac{\mu_0(du)}{z - u}.
\end{align*}
$$

The previous equation has already been introduced in [2, 5] and used in [1]. We may write

$$M_t(z) = \int \frac{(x - u) \mu_t(du)}{(x - u)^2 + y^2} - iy \int \frac{\mu_t(du)}{(x - u)^2 + y^2}.$$  

From (2.18), it is easy to show the well-known fact that $M_t(\cdot + iy)$ converges in the sense of distributions in the $x$ variable to

$$pv \left( \frac{1}{\chi_x} \right) * \mu_t - i\mu_t.$$
as \( y \) tends to 0. Here \( \text{pv}(1/x) \) is the Schwartz distribution \( f \rightarrow \lim_{\epsilon \rightarrow 0} \int_{|x| > \varepsilon} f(x)/x \, dx \), and by the very definition of the Hilbert transform \( \mathcal{H} \),
\[
\mathcal{H}(\mu_1) = \text{pv} \left( \frac{1}{x} \right) * \mu_1.
\]

Let us point out that the Hilbert transform is usually defined as \( \mathcal{H}/\pi \) but we omit the constant here.

Let us consider, for \( z = x + iy \in \mathbb{C}, \ y > 0 \), the contour integral in the upper half-plane
\[
N_I(z) = \int_i^{\infty} M_I(v) \, dv,
\]
which defines a holomorphic function such that \( (\partial N_I/\partial z)(z) = M_I(z) \).

Moreover, from integrating along the path \((i; \ x + i; \ x + iy)\), we obtain
\[
N_I(z) = \int \mu_1(du) \int_0^{\infty} \frac{(v-u) \, dv}{(v-u)^2 + 1} - i \int \mu_1(du) \int_0^{\infty} \frac{dv}{(v-u)^2 + 1}
\]
\[
+ \int \mu_1(du) \int_1^{y} \frac{v \, dv}{(x-u)^2 + v^2} + i \int \mu_1(du) \int_1^{y} \frac{(x-u) \, dv}{(x-u)^2 + v^2}.
\]

Therefore, we have
\[
\mathcal{R}e(N_I(z)) = \int \mu_1(du) \ln \left( \frac{y^2 + (x-u)^2}{1 + (x-u)^2} \right)
\]
\[
+ \int \mu_1(du) \ln \left( \frac{1 + (x-u)^2}{1 + u^2} \right)
\]
\[
\geq \min(0; \ \ln y) + \ln \left( \frac{1}{|x|+1} \right).
\]

Following the Hopf–Cole transformation, we define the new function
\[
H_I(z) = \exp(-\pi N_I(z)),
\]
which is holomorphic in the upper half-plane and satisfies
\[
\frac{\partial H_I}{\partial z} (z) = -\pi M_I(z) \, H_I(z).
\]
Using the estimate for $\Re(N_t(z))$, we get
\[
|H_t(z)| = \exp(-x\Re(N_t(z))) \leq \max(1, y^{-\gamma})(|x| + 1)^\gamma. \tag{2.24}
\]

Starting from (2.17), (2.21), it is easy to obtain
\[
\frac{\partial}{\partial z} \left( \frac{\partial N_t}{\partial t}(z) - \frac{\sigma^2}{2} \frac{\partial^2 N_t}{\partial z^2}(z) + \lambda M_t^2 \right) = 0, \tag{2.25}
\]
so that
\[
\frac{\partial N_t}{\partial t}(z) - \frac{\sigma^2}{2} \frac{\partial^2 N_t}{\partial z^2}(z) + \lambda M_t^2 = f(t), \tag{2.26}
\]
for a given continuous function $f: \mathbb{R}_+ \to \mathbb{C}$. We now set $K_t = g(t) H_t$, where the function $g: \mathbb{R}_+ \to \mathbb{C}$ will be chosen later. Using (2.22) and (2.26), we obtain
\[
\frac{\partial K_t}{\partial t}(z) - \frac{\sigma^2}{2} \frac{\partial^2 K_t}{\partial z^2}(z) = H_t(g(t) - zg(t) f(t)), \tag{2.27}
\]
with $g'$ the derivative of $g$. If we choose $g$ such that $g' = \sigma g$ and $g(0) = 1$, the function $g$ does not vanish and $K$ is a solution of
\[
\frac{\partial K_t}{\partial t}(z) - \frac{\sigma^2}{2} \frac{\partial^2 K_t}{\partial z^2}(z) = 0. \tag{2.28}
\]
Moreover $K$ is the unique solution with polynomial growth in $x$. Thus, if we set $G_t(z) = (1/\sigma \sqrt{2\pi t}) \exp(-z^2/2\sigma^2 t)$, $K_t$ is given by the formula
\[
K_t(x + iy) = \int_{\mathbb{R}} H_0(u + iy) G_t(x - u) \, du, \tag{2.29}
\]
which becomes, using Cauchy Formula
\[
K_t(x + iy) = \int_{\mathbb{R}} H_0(u + iy) G_t(x - u) \, du \\
= \int_{x = u + iy} H_0(z) G_t(x + iy - z) \, dz \\
= \int_{x = u + i} H_0(z) G_t(x + iy - z) \, dz \\
= \int_{x = u + i} H_0(u + i) G_t(x + iy - i - u) \, du.
We have used the fact that $G_t$ decreases exponentially in horizontal bands, while $H_0$ has polynomial growth. As a conclusion, we find that $K_t$ is the restriction to the upper half-plane of the holomorphic function over the whole complex plane defined by

$$K_t(z) = \int_{-\Re} H_0(u+i) G_t(z-i-u) \, du.$$  \hfill (2.30)

Let us remark that $K_t$ satisfies (2.23) as well. From (2.22), we already know that $K_t$ does not vanish in the upper half-plane. Let us suppose that $K_t(x_0) = 0$ for $x_0$ on the real axis. Since $K_t$ is holomorphic, there exist $k \in \mathbb{N}^*$ and a holomorphic function $R$ such that $R$ does not vanish in the ball $B(x_0; \delta)$ and

$$K_t(z) = (z-x_0)^k R(z).$$  \hfill (2.31)

Thus, for $z \in B(x_0; \delta) \setminus \{x_0\}$,

$$\frac{K_t'(z)}{K_t(z)} = \frac{k}{z-x_0} + \frac{R'(z)}{R(z)}.$$  \hfill (2.32)

In particular, for $\varepsilon < \delta$

$$\frac{K_t'(x_0+\varepsilon i)}{K_t(x_0+\varepsilon i)} = \frac{k}{\varepsilon} + O(1),$$  \hfill (2.33)

when $\varepsilon$ tends to 0. On the other hand, we have

$$\frac{K_t'(z)}{K_t(z)} = -\pi M_t(z),$$  \hfill (2.34)

so that its imaginary part is positive. We get a contradiction. Finally, $K_t$ never vanishes on $\Re$. Furthermore (2.19) and (2.34) give

$$\mu_t(dx) = \frac{1}{2\pi} \text{Re} \left( \frac{K_t'(x)}{K_t(x)} \right) dx,$$  \hfill (2.35)

$$\mathcal{H}(\mu_t) = -\frac{1}{\pi} \text{Im} \left( \frac{K_t'(x)}{K_t(x)} \right).$$  \hfill (2.36)

Since $K_t'(z)/K_t(z)$ is analytic in a neighborhood of the closed upper half-plane, both $\mu_t$ and its Hilbert transform $\mathcal{H}(\mu_t)$ possess a real analytic density. This ends the proof of Theorem 2.1. \hfill $\blacksquare$
Moreover, we get an explicit expression for both $u$ and $H(u)$, which will be studied in the special case covered by Corollary 2.2, that is, $\mu_0 = \delta_0$. Let us now prove this corollary.

Formula (2.14) is a direct consequence of (2.36), and we already know that $\varphi$ is real analytic. To prove boundedness it is sufficient to consider the behavior of $\varphi$ at infinity. Let us remark, using an integration by parts, that the numerator is, up to the constant $\pi$, the same function as the denominator, with $\pi$ replaced by $\pi + 1$. So the fact that $\varphi(x)$ is equivalent to $\pi/x$ follows from the next lemma.

**Lemma 2.3.** For $x \to +\infty$,

$$
\lim_{y \to 0^+} \int_{-\infty}^{+\infty} e^{-\nu^2/2} \frac{dv}{(x-v+iy)^2} = \frac{\sqrt{2\pi}}{x^2} + o\left(\frac{1}{|x|^3}\right).
$$

For $x \to -\infty$,

$$
\lim_{y \to 0^+} \int_{-\infty}^{+\infty} e^{-\nu^2/2} \frac{dv}{(x-v+iy)^2} = \frac{\sqrt{2\pi}}{|x|^2} + o\left(\frac{1}{|x|^3}\right).
$$

**Proof.** We shall prove (2.37), a similar proof giving (2.38). Let us write for $x > 1$,

$$
\lim_{y \to 0^+} \int_{-\infty}^{+\infty} e^{-\nu^2/2} \frac{dv}{(x-v+iy)^2}
= \int_{-\infty}^{-\sqrt{x}} e^{-\nu^2/2} \frac{dv}{(x-v)^2} + \int_{-\sqrt{x}}^{+\sqrt{x}} e^{-\nu^2/2} \frac{dv}{(x-v)^2}
+ \lim_{y \to 0^+} \int_{-\sqrt{x}}^{+\infty} e^{-\nu^2/2} \frac{dv}{(x-v+iy)^2}
= I + II + III.
$$

We can directly estimate

$$
I \leq \frac{1}{x^2} \int_{-\sqrt{x}}^{-\sqrt{x}} e^{-\nu^2/2} dv = o\left(\frac{1}{x^3}\right).
$$

Let us write

$$
II = \int_{-\sqrt{x}}^{+\sqrt{x}} e^{-\nu^2/2} \frac{dv}{x^2} + \int_{-\sqrt{x}}^{+\sqrt{x}} e^{-\nu^2/2} \left[\frac{1}{(x-v)^2} - \frac{1}{x^2}\right] dv.
$$
The first term is clearly equal to $\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi}$. For the second term, we use the mean value inequality to get

$$\left| \frac{1}{(x-v)^2} - \frac{1}{x^2} \right| \leq C \frac{\sqrt{x}}{x^{3/2}}$$

for $v \in [-\sqrt{x}, \sqrt{x}]$. It remains to consider III. We use integrations by parts $l$ times, with $l > \alpha$, in order to deal with an antiderivative of $(x-v+iy)^{-\alpha}$ which has no singularity anymore. If we denote by $P^{(j)}$ the Hermite polynomials defined by $P^{(j)}(x) = (d/dx)^j e^{-x^2/2}$, we may write for $\alpha$ non-integer,

$$\int_{\sqrt{x}}^{+\infty} e^{-v^2/2} \frac{dv}{(x-v+iy)^\alpha} = \sum_{j=0}^{l-1} c_j \int_{\sqrt{x}}^{+\infty} P^{(j)}(v) e^{-x^2/2} (x-v+iy)^{\alpha-j-1} \, dv + c_l \int_{\sqrt{x}}^{+\infty} v e^{-v^2/2} (x-v+iy)^{\alpha-1} \, dv,$$

which may be bounded by $Ce^{-\sqrt{x}}$ for $0 < y < 1$. For $\alpha$ an integer we have to add logarithms in the above formula, but the estimate is still valid.

**Remark.** In their paper [1], E. Cépa and D. Lépingle have added an affine drift in the equation satisfied by the particles. This drift has been omitted up to now to simplify expressions, but a change of variables allows also to conclude in this situation. Let $u$ be the solution of

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - 2\lambda \frac{\partial (u \Psi(u))}{\partial x} \\
u(t, x) \, dx \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\end{cases} \tag{2.41}$$

Then, for $\theta > 0$ and $\rho \in \mathbb{R}$,

$$v(t, x) = e^{\theta t} \left( e^{2\theta t} - 1, xe^{\theta t} + \frac{\rho}{\theta} (e^{\theta t} - 1) \right) \tag{2.42}$$

is the solution of

$$\begin{cases}
\frac{\partial v}{\partial t} = \theta \sigma^2 \frac{\partial^2 v}{\partial x^2} - 4\lambda \theta \frac{\partial (v \Psi(v))}{\partial x} + \frac{\partial [(\theta x + \rho) v]}{\partial x} \\
v(t, x) \, dx \rightarrow \delta_0 \quad \text{as} \quad t \rightarrow 0.
\end{cases} \tag{2.43}$$

Moreover $v(t, x)$ converges as $t \rightarrow \infty$ to $\psi(x+\rho/\theta)$, where $\psi$ has been introduced above.
3. NON-LINEAR PROCESS

We have obtained the weak solution \((\mu_t)\) of the equation

\[
\frac{\partial \mu_t}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \mu_t}{\partial x^2} - 2i \frac{\partial (\mu_t \mathcal{H}(\mu))}{\partial x},
\]  
(3.44)

where \(\mathcal{H}(\mu)\) is the Hilbert transform of \(\mu\). In fact, we have proved (see Theorem 2.1) that \(\mu_t(dx) = u(t, x) dx\) with \(u, \mathcal{H}(u)\) analytic (for the \(x\) variable) and \(u\) satisfies

\[
\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - 2i \frac{\partial (u \mathcal{H}(u))}{\partial x}.
\]  
(3.45)

Let us suppose now that \(\mu_0 = \delta_0\); we recall that, from Corollary 2.2, \(q(t, x) = \mathcal{H}(u)(t, x) = (1/2\sqrt{t}) \phi(x/\sqrt{t})\) with \(\phi\) a bounded real analytic function.

The non-linear stochastic equation associated with (3.45) is

\[
\begin{aligned}
X_t &= \sigma \cdot B_t + 2\int_0^t \mathcal{H}(v)(x, X_s) \, ds \\
X_0 &= 0; \quad X_t \sim v(t, x) \, dx \quad \text{if } t > 0.
\end{aligned}
\]  
(3.46)

Indeed, for \(f \in C^2_0(\mathbb{R}; \mathbb{R})\), Itô’s formula allows us to write

\[
\begin{aligned}
f(X_t) &= f(0) + \int_0^t f'(X_s) \sigma dB_s + \int_0^t f'(X_s) \, 2\lambda \mathcal{H}(v)(s, X_s) \, ds \\
&\quad + \frac{\sigma^2}{2} \int_0^t f''(X_s) \, ds.
\end{aligned}
\]  
(3.47)

Taking expectation in both members and using the fact that \(v(t, \cdot)\) is the density of \(X_t\), we obtain

\[
\begin{aligned}
\int f(x) v(t, x) \, dx &= f(0) + \int_0^t f'(x) 2\lambda \mathcal{H}(v)(s, x) \, v(x, s) \, ds \, dx \\
&\quad + \frac{\sigma^2}{2} \int_0^t f''(x) \, v(x, s) \, ds \, dx.
\end{aligned}
\]  
(3.48)

After derivation in \(t\) and integration by parts, we get the desired equation

\[
\begin{aligned}
\frac{\partial v_t}{\partial t} &= \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2} - 2\lambda \frac{\partial (v \mathcal{H}(v))}{\partial x} \\
v(t, x) \, dx \rightarrow \delta_0 \quad \text{as } t \rightarrow 0,
\end{aligned}
\]  
(3.49)

and uniqueness for this I-PDE implies that \(v\) is the solution \(u\) given above.
**Theorem 3.1.** The non-linear stochastic equation (3.46) has a unique strong solution.

**Proof of Uniqueness.** If $X^{(1)}$ and $X^{(2)}$ are solutions, then the previous computation shows that the corresponding densities $u^{(1)}$ and $u^{(2)}$ are solutions of the same Eq. (3.45), for which uniqueness is known. So $u^{(1)} = u^{(2)} = u$, which means that the solutions have the same martingals. Moreover, since

$$ X^{(1)}_t = \sigma B_t + \int_0^t \frac{1}{\sqrt{s}} \varphi \left( \frac{X^{(1)}_s}{\sqrt{s}} \right) \, ds, $$

$$ X^{(2)}_t = \sigma B_t + \int_0^t \frac{1}{\sqrt{s}} \varphi \left( \frac{X^{(2)}_s}{\sqrt{s}} \right) \, ds, $$

the process $X^{(1)} - X^{(2)}$ has bounded variation. Therefore its local time at 0 identically vanishes, and both $\inf(X^{(1)}, X^{(2)})$ and $\sup(X^{(1)}, X^{(2)})$ are solutions (see [4]). Since they have same martingals, we finally obtain $X^{(1)}_t = X^{(2)}_t$.

**Proof of Existence.** It suffices now to prove that there is a weak solution since pathwise uniqueness holds. We consider for $r > 0$ the equation

$$ Y^r_t = Y + \sigma B_t + \int_0^t \frac{1}{\sqrt{r+s}} \varphi \left( \frac{Y^r_s}{\sqrt{r+s}} \right) \, ds, $$

where $Y$ is a real random variable with density $u(r, y)$ given by the solution of the I-PDE, and independent of the Brownian motion $B$. The function $b$ defined by $b(s, x) = (1/\sqrt{r+s}) \varphi(x/\sqrt{r+s})$ is locally Lipschitz and bounded, so there is an unique strong solution to (3.51). Let us set $X^*_t = (t/r) Y$ for $t \leq r$ and $X^*_t = Y^r_{t-r}$ for $t > r$. If $\mu^*_t$ is the law of $Y^r_t$, then

$$ \frac{\partial \mu^*_t}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \mu^*_t}{\partial x^2} - 2 \frac{\partial (\mu^*_t \mathcal{H}(\mu^*_t + \lambda))}{\partial x}. $$

Using uniqueness for the linear PDE, we get $\mu^*_t = \mu^*_{t+r}$. Let us consider the application $\tau'$ defined on $C([0; +\infty[; \mathbb{R})$ by $\tau'(\omega)(t) = (t/r) \omega(0)$ if $t \leq r$ and $\omega(t-r)$ if $t > r$. We note $Q'$ the lax of $Y^r$ and $P'$ the image of $Q'$ by $\tau'$. In fact, $P'$ is the law of $X^r$. If we note $K'(\omega)(t) = \omega(t+r)$, the $P^{1, \theta}$ and $P^{2, \theta}$ have same image under $K'$ when $r \geq \max(r_1, r_2)$. We will now use Aldous criterion to prove that the sequence $(P^{1, \theta})$ is tight. For $N > 0, \theta > 0$,
we consider stopping times $S$ and $T$ bounded by $N$ and such that
$S \leq T \leq S + \theta$. We can write for each $n \geq 1$

$$P_{1/n}(|w(T) - w(S)| > \varepsilon)$$

\[= P_{1/n}(|w(T) - w(S)| > \varepsilon; T \leq 1/n) + P_{1/n}(|w(T) - w(S)| > \varepsilon; S \leq 1/n < T) + P_{1/n}(|w(T) - w(S)| > \varepsilon; S > 1/n) \leq P_{1/n}(T - S) |w(1/n)| > \varepsilon, T \leq 1/n + P_{1/n}(|w(T) - w(S)| > \varepsilon; S > 1/n) + P_{1/n}(|w(T) - nS(1/n)| > \varepsilon; S < 1/n < T)\]

\[\leq \int_{|\cdot| > \varepsilon} u(1/n, x) \, dx + P_{1/n}\left( \left| w(1/n)(1 - nS) + \sigma(T - 1/n) \right| > \varepsilon; S \leq 1/n < T \right) + P_{1/n}\left( \left| \sigma(T - 1/n) - \sigma(S - 1/n) \right| > \varepsilon; S > 1/n \right)
\times P_{1/n}(|w(T) - w(S)| > \varepsilon)\]

\[\leq \int_{|\cdot| > \varepsilon} u(1/n, x) \, dx + \int_{|\cdot| > \varepsilon/3} u(1/n, x) \, dx + \frac{9\theta^2}{\varepsilon^2} + P_{1/n}\left( \left| c \int_0^{T-1/n} \frac{1}{\sqrt{s + 1/n}} \, ds > \varepsilon/2 \right) + \frac{4\theta^2}{\varepsilon^2} \right)
\]

\[+ P_{1/n}\left( c \int_{S-1/n}^{T-1/n} \frac{1}{\sqrt{s + 1/n}} \, ds > \varepsilon/2; S > 1/n \right)\]

\[\leq 2 \int_{|\cdot| > \varepsilon/3} u(1/n, x) \, dx + \frac{13\theta^2}{\varepsilon^2} + P_{1/n}(2\varepsilon(\sqrt{T} - \sqrt{S}) > \varepsilon/2) + P_{1/n}(2\varepsilon(\sqrt{\theta + 1/n} - \sqrt{1/n}) > \varepsilon/3) \leq 2 \int_{|\cdot| > \varepsilon/3} u(1/n, x) \, dx + \frac{13\theta^2}{\varepsilon^2} + 2P_{1/n}(2\varepsilon \sqrt{\theta} > \varepsilon/3).\]
From the last inequality, we deduce that

\[
\limsup \sup_{n,s,t} \mathbb{P}(|w(T) - w(S)| > \varepsilon) \leq \frac{130\sigma^2}{\epsilon^2} + 212\sigma \sqrt{3}\varepsilon
\]

and consequently

\[
\lim_{\theta \to 0} \limsup \sup_{n,s,t} \mathbb{P}(|w(T) - w(S)| > \varepsilon) = 0.
\]

So the sequence \((\mathbb{P}^n)\) is tight. Moreover, for \(0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty\), then the laws under \((\mathbb{P}^n)\) of \((w(t_0), w(t_1), \ldots, w(t_k))\) are identical as soon as \(1/n < t_1\). Finally the sequence \((\mathbb{P}^n)\) converges to a probability \(\mathbb{P}\) whose finite-dimensional distributions are given by \(\mathbb{P}^n\) if \(1/n < t_1\). We can assert that, under \(\mathbb{P}\),

\[
(1/\sigma)(w(t) - \int_0^t (1/\sqrt{s}) \varphi(w(s)/\sqrt{s}) \, ds)
\]

is a Brownian motion, which implies that \(\mathbb{P}\) is the weak solution of \(X_t = \sigma B_t + \int_0^t (1/\sqrt{s}) \varphi(X_s/\sqrt{s}) \, ds\). This ends the proof of the theorem.

Remark. If \(X_t\) is the solution of

\[
\begin{cases}
X_t = \sigma \cdot B_t + 2\lambda \int_0^t \mathcal{H}(u)(s, X_s) \, ds \\
X_0 = 0; \quad X_t \sim u(t, x) \, dx \quad \text{if} \quad t > 0,
\end{cases}
\]

then we can use Itô’s formula to prove that, for \(\theta > 0\),

\[
Y_t = e^{-\theta t} X_{e^{-\theta} t}
\]

is the solution

\[
\begin{cases}
Y_t = \sigma \sqrt{2\theta} \cdot B_t + 4\lambda \theta \int_0^t \mathcal{H}(v)(s, Y_s) \, ds - \theta \int_0^t Y_s \, ds \\
Y_0 = 0; \quad Y_t \sim v(t, x) \, dx \quad \text{if} \quad t > 0.
\end{cases}
\]

Moreover, \(Y_t\) converges in law as \(t \to \infty\) to \(Y_\infty \sim \psi(x) \, dx\). See the remark at the end of Section 2.

4. A DIRECT PROOF OF EXISTENCE OF WEAK SOLUTIONS

We deeply relied on the previous work [1] and on probabilistic methods to have existence of weak solutions for Eq. (2.11). Indeed, the fact that this equation comes from the limit law of systems of Brownian particles is a strong motivation for its study. Nevertheless it is natural to ask for a direct proof of the existence of weak solutions. We shall now sketch such a proof.
Let us first remark that what we really need is existence of holomorphic functions $M_t$, which are solutions of Eq. (2.17) and satisfy the inequality $\text{Im} M_t \leq 0$ in the upper half-plane. Once such a function $M_t$ has been obtained, the fact that it may be written as the Cauchy integral of a probability measure $\mu_t$, that is,

$$M_t(z) = \int_{-\infty}^{+\infty} \frac{\mu_t(d\zeta)}{z - \zeta},$$

follows from classical results on positive harmonic functions (see [7]) and from the conservation of the integral $\int_{-\infty}^{+\infty} \mu_t(d\zeta)$ in Eq. (2.11). Moreover, if $K_t$ is the solution of (2.28) with initial data $K_0$, then $M_t = -(1/\pi)(K'/K_t)$ is a solution of (2.17) under the condition that $K_t$ does not vanish so that $M_t$ is well defined. So existence of weak solutions for all times, which turn out to be strong solutions as we have seen, follows from the next proposition.

**Proposition 4.1.** Let $\mu_0$ be a probability measure, and let $K_0 = H_0$ be defined as before by (2.16), (2.21), (2.22). Then the solution $K_t$ of Eq. (2.28) with initial data $K_0$ does not vanish in the upper half-plane. Moreover the imaginary part of $M_t = -(1/\pi)(K'/K_t)$ is non-positive.

**Proof.** We shall first assume that $\mu_0$ has compact support. Under this assumption one can show that, for $|z|$ tending to $\infty$, $M_t$ behaves like $1/z$ and $K_t$ like $C/z^2$. Moreover $K_t$ has the same behavior at $\infty$, so that it does not vanish outside some compact set. The uniformity of estimates on compact sets in time allows to assume that for $0 \leq t \leq T$ the function $K_t(z)$ does not vanish for $|z| > R$. On this open set $M_t$ is well defined, and behaves like $1/z + O(1/|z|^2)$, which means in particular that $\text{Im} M_t$ is also $O(1/|z|^2)$ for $\text{Im} z$ bounded. Moreover derivatives of order $k$ of $M_t$ are $O(1/|z|^{k+1})$. We shall not give detailed proofs of these facts, which we have proved when $\mu_0$ is the Dirac mass.

Once given these preliminary estimates, let us prove that the conclusion of the proposition holds. We first remark that it is sufficient to prove it in the half-plane $\Pi = \{ z ; \text{Im} z > y_0 \}$ for every $y_0 > 0$. From now, we consider $y_0$ fixed. Let us consider the set of $s > 0$ for which the solution $K_s$ of Eq. (2.28) with initial data $K_0$ does not vanish in $\Pi$ for $t \leq s$. By continuity, using the behavior of $K_t$ at infinity, it is clearly an interval $[0, t_0]$ which is not reduced to 0. The conclusion is a consequence of the following lemma.

**Lemma 4.2.** Let $t_0 > 0$ and assume that $K_t$ does not vanish on $\Pi$ for $0 \leq t < t_0$. Then $\text{Im} M_t$ takes non-positive values on $\Pi$ for all $0 \leq t < t_0$. Moreover, $K_0$ does not vanish on $\Pi$. 
Our proof of the first fact relies on classical techniques used to get weak maximum principles for non-linear equations. Using the maximum principle, it is sufficient to prove that \( \text{Im } M_t \leq 0 \) on the boundary of \( \mathbb{I} \). Let us set \( u_t(x) = \text{Im } M_t(x + iy_0) \), and \( u^*_t = \max \{ 0, u_t \} \). We shall show that

\[
I(t) = \int_{-\infty}^{\infty} u^*_t(x) \, dx
\]

is a decreasing function of \( t \), from which we conclude directly. Since \( M_t \) is a real analytic function (both in \( z \) and \( t > 0 \)), the function \( u_t \) vanishes at isolated points, which depend continuously on the parameter \( t \). So \( I(t) \) is the sum of integrals \( \int_{a(t)}^{b(t)} \partial_z u_t(x) \, dx \), with \( u_t \) which is non-negative on the interval and vanishes at its boundary. It follows, using the behavior at infinity, that \( I(t) \) is the sum of integrals is non-positive. Using the fact that \( M_t \) is a solution of Eq. (2.17), we find that

\[
\int_{a(t)}^{b(t)} \partial_z u_t(x) \, dx = \frac{\sigma^2}{2} \left[ \partial_z u_t \right]_{a(t)}^{b(t)}.
\]

The derivative is non-positive at point \( b(t) \) if it is finite, tends to 0 at \( \pm \infty \) and is non-negative at point \( a(t) \) if it is finite, from which we conclude.

We have proved that \( u_t(z) \) is non-positive for \( t \in [0, t_0] \) and \( z \in \mathbb{I} \). It remains to prove that \( \mathcal{K}_q \) does not vanish on \( \mathbb{I} \). From the continuity of the zeroes as functions of \( t \), we already know that \( \mathcal{K}_q \) has no zeroes inside \( \mathbb{I} \). Let us proceed by contradiction, and assume that \( \mathcal{K}_q(z_0) = 0 \) for some \( z_0 = x_0 + iy_0 \). We use the same proof as in the proof of Theorem 2.1. Indeed, assume that \( \mathcal{K}_q(z) = (z - z_0)^k R(z) \), with \( R(z_0) \neq 0 \). Then, for \( z \in \mathbb{I} \) close to \( z_0 \), one has

\[
M_q(z) = -\frac{k}{\sigma} \frac{1}{z - z_0} + O(1)
\]

so that \( \text{Im } M_t \) is strictly positive for some \( z \in \mathbb{I} \) close to \( z_0 \), while we know, from the first part of the lemma, using continuity, that \( \text{Im } M_t \leq 0 \) on \( \mathbb{I} \). We have obtained the required contradiction.

This finishes the proof of the proposition when \( \mu_0 \) is compactly supported, so that the estimates at infinity are valid.

Now, let \( \mu_0 \) be an arbitrary probability measure. We can write it as the limit in norm of an increasing sequence of positive measures \( \mu_{0n} \) for which the conclusion of the proposition holds. One can check that the corresponding functions \( \mathcal{K}_q^{0n} \) tend to \( \mathcal{K}_q \) uniformly on compacts. Then the fact that \( \mathcal{K}_q \) has no zeroes is a classical result and the proof of the fact that \( \text{Im } M_t \) is negative is straightforward.
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REFERENCES