Phénomènes de localisation et d’universalité pour des polymères aléatoires

Niccolò Torri

Université Claude Bernard - Lyon 1
&
Università degli Studi di Milano-Bicocca

Lyon, September 18, 2015
1 Polymers

2 Pinning Model

3 Universality of the Pinning Model

4 Pinning Model with Heavy Tailed disorder

5 Perspectives
Polymers

Interactions with
- Itself
- External environment

Depend on some parameters

Aim
- Spatial configuration
- Phase transition? Critical Points?
Intermezzo: Some Basic Probabilistic Processes

Random Walk on $\mathbb{Z}^d$

$$S_n = \sum_{i=1}^{n} X_i, \quad X_i = \text{increments.}$$

Self Avoiding Random Walk (SAW) on $\mathbb{Z}^d$

Conditioned to visit at most ones each state.
Abstract Monomers and Abstract Polymers

Increment $X_i \leftrightarrow$ a monomer

Use SAW:

Abstract Polymer ($N$ monomers):
$N$-increments of a SAW
Abstract Monomers and Abstract Polymers

Increment $X_i \Leftrightarrow$ a monomer

Use SAW:

SAW: challenging object!

Subclass: Directed Random Walks (Directed Polymers)

Abstract Polymer ($N$ monomers):

$N$-increments of a SAW
1 Polymers
2 Pinning Model
3 Universality of the Pinning Model
4 Pinning Model with Heavy Tailed disorder
5 Perspectives
Pinning Model

Pinning Model: interactions polymer ⇔ environment.

region of interaction with the random walk

Henceforth S one-dimensional.

N. Torri (Lyon 1 & Milano-Bicocca)
Pinning Model

Pinning Model: interactions polymer ⇔ environment.

Interaction with a membrane ⇔ region = straight line

Henceforth S one-dimensional.
Pinning Model

Modify Random Walk $S$:
reward/penalty whenever $S$ visits 0

$$P_{N,\beta,h}^\omega(s_1, \ldots, s_N) = \frac{1}{Z_{N,\beta,h}^\omega} \exp \left\{ \sum_{i=1}^{N} g_i^\omega(\beta, h) \mathbb{1}_{\{s_i=0\}} \right\} \cdot P(s_1, \ldots, s_N).$$

$N$: polymer length

$$g_i^\omega(\beta, h) = \begin{cases} 
h, & \text{Homogeneous model,} \\
\beta \omega_i + h, & \text{Disordered model.} \end{cases}$$

$Z_{N,\beta,h}^\omega$ Partition Function
Pinning Model

Modify Random Walk $S$: reward/penalty whenever $S$ visits 0

$$P_{N, \beta, h}^{\omega}(s_1, \ldots, s_N) = \frac{1}{Z_{N, \beta, h}^{\omega}} \exp \left\{ \sum_{i=1}^{N} g_i^{\omega}(\beta, h) 1_{\{s_i=0\}} \right\} \cdot P(s_1, \ldots, s_N).$$

$N$: polymer length

$$g_i^{\omega}(\beta, h) = \begin{cases} h, & \text{Homogeneous model,} \\ \beta \omega_i + h, & \text{Disordered model.} \quad \omega = (\omega_i)_{i \in \mathbb{N}} \text{ disorder} \end{cases}$$

$Z_{N, \beta, h}^{\omega}$ Partition Function

Remark:

Need only $\{n : S_n = 0\}$ to define the model.
Random Walk choice

**symmetric simple random walk**

\[ S_n = X_1 + \ldots + X_n \]

\[ p(x, x+1) = \frac{1}{2}, \quad p(x, x-1) = \frac{1}{2} \]

\[ P(S_n = 0 \text{ for the first time}) \sim \frac{c}{n^2} \quad n \to \infty \]

Explicit examples: \( \alpha \in (0, 1) \) Bessel-like Random walk.
Random Walk choice

**symmetric simple random walk**

\[ S_n = X_1 + \ldots + X_n \]

\[ p(x, x+1) = \frac{1}{2} \]

\[ p(x, x-1) = \frac{1}{2} \]

\[ P(S_n = 0 \text{ for the first time}) \sim \frac{c}{n^{3/2}} \quad n \to \infty \]

Generalize

\[ P(S_n = 0 \text{ for the first time}) \sim \frac{c_S}{n^{1+\alpha}}, \quad \alpha > 0. \]

Explicit examples: \( \alpha \in (0, 1) \) Bessel-like Random walk.
Disorder Assumptions

Disorder $\omega$: *quenched* realization of an *i.i.d.* random sequence.

$$\mathbb{E}(\omega_1) = 0, \quad \text{Var}(\omega_1) = 1, \quad \Lambda(\beta) = \log \mathbb{E}(e^{\beta \omega_1}) < \infty, \forall \beta \text{ small.}$$

Examples:

- $\omega_1$ bounded,
- $\omega_1$ Gaussian.
Disorder Assumptions

Disorder $\omega$: *quenched* realization of an *i.i.d.* random sequence.

\[ E(\omega_1) = 0, \quad \text{Var}(\omega_1) = 1, \quad \Lambda(\beta) = \log E(e^{\beta \omega_1}) < \infty, \forall \beta \text{ small}. \]

Examples:
- $\omega_1$ bounded,
- $\omega_1$ Gaussian.

Different interesting choice:
heavy-tail case $\mathbb{P}(\omega_1 > t) \sim_{t \to \infty} c t^{-\xi}$
Our Goal

Goal: behavior of $S$ under $P_{\beta,h,N}^\omega$ when $N$ gets large.

- $N$ "abstract polymer" length,
- $h$ Homogeneous parameter,
- $\beta$ Tunes the presence of the disorder $\omega$

- localized? de-localized?
- phase transition on $\beta$, $h$? Critical point?
Localization/Delocalization

**Free Energy**

\[
F^{(\alpha)}(\beta, h) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \log Z_{\beta, h, N}^\omega \right]
\]

\[
\frac{\partial}{\partial h} F^{(\alpha)}(\beta, h) = \lim_{N \to \infty} \mathbb{E} E_{\beta, h, N}^\omega \left( \frac{\# \{ n \leq N : S_n = 0 \}}{N} \right) \quad (\ast)
\]

\forall \beta \geq 0 there exists a critical point \( h_c(\beta) \)

- \( h > h_c(\beta) \) localization, \( (\ast) > 0 \),
- \( h < h_c(\beta) \) de-localization, \( (\ast) = 0 \).
Analysis of the model

Goal: understand $h_c(\beta)$.

- **Homogeneous model** $h_c(0)$ explicit ($= 0$ if $S$ is recurrent).

It provides Lower/Upper bounds

\[ h_c(0) - \Lambda(\beta) \leq h_c(\beta) < h_c(0), \]

- $h_c(0) - \Lambda(\beta)$ *annealed critical point*. 

N. Torri (Lyon 1 & Milano-Bicocca)
Relevance/Irrelevance of the disorder

\( \alpha > 1/2 \)
relevant disorder

\[ h_c(\beta) > h_c(0) - \Lambda(\beta) \quad \forall \beta > 0 \]

\( 0 < \alpha < 1/2 \)
irrelevant disorder

\[ h_c(\beta) = h_c(0) - \Lambda(\beta) \quad \text{if } \beta \text{ small} \]

Aim: When \( \alpha > 1/2 \), asymptotics of \( h_c(\beta) \) as \( \beta \to 0 \).
An overview of the literature

**case \( \alpha > 1 \)**

**Theorem (Q. Berger, F. Caravenna, J. Poisat, R. Sun, N. Zygouras, 2014)**

Let \( \alpha > 1 \), then

\[
h_c(\beta) \sim \tilde{c}_\beta^2, \quad \beta \to 0
\]

where \( \tilde{c} \) is **explicit** depending on \( \alpha \) and \( c_S \).

**case \( \alpha \in (1/2, 1) \):** several authors K. S. Alexander, B. Derrida, G. Giacomin, H. Lacoin, F. L. Toninelli and N. Zygouras (2008 – 2011).

**Theorem**

Let \( \alpha \in (1/2, 1) \), then there exist \( 0 < c < C < \infty \) such that

\[
c \beta^{\frac{2\alpha}{2\alpha - 1}} \leq h_c(\beta) \leq C \beta^{\frac{2\alpha}{2\alpha - 1}},
\]

for \( \beta \) small.
1. Polymers

2. Pinning Model

3. Universality of the Pinning Model

4. Pinning Model with Heavy Tailed disorder

5. Perspectives
Results

(Add some Technical assumptions on Disorder and Random Walk)

Theorem (Caravenna, Toninelli, T., 2015)

Universal feature of the Free Energy

\[ \mathcal{F}^{(\alpha)}(\hat{\beta}, \hat{h}) = \lim_{\varepsilon \downarrow 0} \frac{F^{(\alpha)}(\hat{\beta} \varepsilon^{\alpha - \frac{1}{2}}, \hat{h} \varepsilon^{\alpha})}{\varepsilon} \]

\[ \mathcal{F}^{(\alpha)}(\hat{\beta}, \hat{h}) = \beta^{\frac{2}{2\alpha - 1}} \mathcal{F}^{(\alpha)}(1, \hat{h} \beta^{-\frac{2\alpha}{2\alpha - 1}}) \quad \Rightarrow \quad \mathcal{H}^{(\alpha)}(\hat{\beta}) = \hat{c} \hat{\beta}^{\frac{2\alpha}{2\alpha - 1}} \]

\( \hat{c} \) non-trivial constant depending on \( \alpha \) and \( c_S \).

Theorem (Caravenna, Toninelli, T., 2015)

Universal Critical Behavior

\[ h_c(\beta) \sim \hat{c} \beta^{\frac{2\alpha}{2\alpha - 1}} , \]

\( \beta \rightarrow 0 \)
Proof - Continuum Model

- $\tau := \{ n : S_n = 0 \} \Rightarrow \varepsilon \tau \xrightarrow{\varepsilon \downarrow 0} \hat{\tau}$
- Thm (C.S.Z., 15)
  (cond.) Pinning Model converges:
  
  $\beta = \hat{\beta} \varepsilon^{\alpha - \frac{1}{2}}$, $h = \hat{h} \varepsilon^{\alpha}$:

Continuum ingredients:
- regenerative set ($\hat{\tau}$)
- White Noise (Cont. disorder)

Problem: No Gibbs representation

Wiener Chaos Expansion
Proof - Strategy: Coarse-Graining

\[ N = \frac{t}{\varepsilon}. \] Consider \[ \frac{1}{N} \mathbb{E} \log Z_N \]

Compare \( \lim_t \lim_{\varepsilon} \) & \( \lim_{\varepsilon} \lim_t \)

Coarse-Graining of \( \tau = \varepsilon \tau \) or \( \hat{\tau} \)

Partition function decomposition

\[ Z^c(\tau) = Z^c_s(\tau), Z^c_{\varepsilon}(\tau), Z^c_t(\tau), Z^c_{\varepsilon, t}(\tau) \]

Convergence on each block

\[ Z^c_{\varepsilon, \text{disc}}(\tau) = Z^c_{\varepsilon, \text{cont}}(\tau) \]

Technical difficulty

Couple together convergence of \( (s_i(\varepsilon), t_i(\varepsilon)) \) with \( Z^c_{t/\varepsilon}(a, b) \)

\[ \Rightarrow \text{Get: } \forall \eta > 0 \exists \varepsilon_0 : \forall \varepsilon < \varepsilon_0 \]

\[ F^{(\alpha)}(\beta, \hat{h} - \eta) \leq \varepsilon^{-1} F^{(\alpha)}(\hat{\beta}, \varepsilon^{\alpha - \frac{1}{2}}, \hat{h}, \varepsilon^\alpha) \leq F^{(\alpha)}(\hat{\beta}, \hat{h} + \eta) \]
Proof - Strategy: Coarse-Graining

\[ N = \frac{t}{\varepsilon} \]. Consider \( \frac{1}{N} \mathbb{E} \log Z_N \)

Compare \( \lim_t \) lim_\( \varepsilon \) & lim_\( \varepsilon \) lim_\( t \)

Coarse-Graining of \( \tau = \varepsilon \tau \) or \( \hat{\tau} \)

\[
\begin{align*}
J_1 &= 1 \\
J_2 &= 2 \\
J_3 &= 4 \\
J_4 &= 6
\end{align*}
\]

Partition function decomposition

\[
\begin{align*}
Z_c^c(\tau) &= Z_c^c(\tau) \\
Z_c^c(\tau) &= Z_c^c(\tau) \\
Z_c^c(\tau) &= Z_c^c(\tau)
\end{align*}
\]

\[
\begin{align*}
Z_{c, \text{disc}}^c(\tau/\varepsilon) &= Z_{c \epsilon}^c(s_i(\varepsilon), t_i(\varepsilon)) \\
Z_{c, \text{cont}}^c(\hat{\tau}) &= Z_{c \epsilon}^c(s_i, t_i)
\end{align*}
\]

\[ \Rightarrow \text{Get: } \forall \eta > 0 \exists \varepsilon_0 : \forall \varepsilon < \varepsilon_0 \]

\[
\begin{align*}
\mathcal{F}^{(\alpha)}(\hat{\beta}, \hat{h} - \eta) &\leq \\
\varepsilon^{-1} F^{(\alpha)}(\hat{\beta} \varepsilon^{\alpha - \frac{1}{2}}, \hat{h} \varepsilon^{\alpha}) &\leq \mathcal{F}^{(\alpha)}(\hat{\beta}, \hat{h} + \eta)
\end{align*}
\]

Similar to Copolymer Model

(den Hollander & Bolthausen, 1997 and Caravenna & Giacomin, 2010)
1 Polymers

2 Pinning Model

3 Universality of the Pinning Model

4 Pinning Model with Heavy Tailed disorder

5 Perspectives
Disorder with Heavy Tail

- **Disorder** $\omega$ quenched realization of an i.i.d. random sequence s.t.

  \[ P(\omega_1 > t) \sim ct^{-\xi}, \quad t \to \infty. \quad \xi \in (0, 1) \]

  and $\omega_1$ positive with a continuous distribution.
Disorder with Heavy Tail

- Disorder $\omega$ quenched realization of an i.i.d. random sequence s.t.

\[ \mathbb{P}(\omega_1 > t) \sim ct^{-\xi}, \quad t \to \infty. \quad \xi \in (0, 1) \]

and $\omega_1$ positive with a continuous distribution.

Inspired by

A. Auffinger and O. Louidor
Directed polymers in random environment with heavy tails

B. Hambly and J. B. Martin
Heavy tails in last-passage percolation

$\xi \in (0, 2)$. 
Pinning model:

\[
P_{N, \beta, h}^{\omega}(s_1, \ldots, s_N) = \frac{1}{Z_{N, \beta, h}^{\omega}} \exp \left\{ \sum_{i=1}^{N} (\beta \omega_i + h) \mathbb{1}_{\{s_i=0\}} \right\} \mathbb{1}_{\{s_N=0\}} \cdot P(s_1, \ldots, s_N).
\]

\[\mathbb{1}_{\{s_i=0\}} \iff \mathbb{1}_{\{i \in \tau\}}, \text{ where}\]

\[
\tau = \{\tau_0 = 0, \tau_1, \tau_2, \ldots\} = \{n : S_n = 0\}
\]

Renewal Process
Assumptions (Renewal Process)

\[ \tau = \{\tau_0 = 0, \tau_1, \tau_2, \cdots\} \quad \text{Renewal Process} \]

Assumptions

\[ K(n) := P(\tau_1 = n) \] probability on \( \mathbb{N}, \)

\[ K(n) \cong e^{-\gamma n} \quad \gamma \in (0, 1) \]

+ ”local regularity”.
Behavior of the Pinning Model \( \Rightarrow \)

Behavior of the Renewal Process \( \Rightarrow \) consider \( \tau/N \cap [0, 1] \).

- look at \( \tau/N \cap [0, 1] \) as r.v. in the space of all closed subsets of \([0, 1]\) (which contain 0, 1).
- Equip it with Hausdorff distance:

\[
d_{H}(A, B) < \varepsilon \quad \text{Def.} \quad \iff \quad \forall a \in A, \exists b \in B : |a - b| < \varepsilon
\]

and vice-versa, interchanging \( A \) and \( B \).
Concentration & Convergence

Take

- \( \beta = \beta_N = \hat{\beta} N^{\gamma - \frac{1}{\xi}} \rightarrow 0 \) as \( N \rightarrow \infty \).
- \( h < 0 \) fixed

**Theorem (T., 2014)**

For any \( \hat{\beta} > 0 \) there exists a universal closed subset \( \hat{I}^w_{\hat{\beta}, \infty} \subset [0, 1] \) such that

\[
\frac{\tau}{N} \cap [0, 1] \xrightarrow{(d)} \hat{I}^w_{\hat{\beta}, \infty}
\]

in the Hausdorff metric.
Random Critical Point

Theorem (T., 2014)

1. There exists $\hat{\beta}_c^w(h)$ random variable s.t.

$$\hat{I}_{\hat{\beta}, \infty}^w = \begin{cases} 
0 & \hat{\beta} < \hat{\beta}_c^w(h) \\
1 & \hat{\beta} > \hat{\beta}_c^w(h) 
\end{cases}$$

2. $\hat{\beta}_c^w(h) > 0$ for a.e.-w and for any choice of $\xi, \gamma \in (0, 1)$.

$\hat{\beta}$ small, ”weak disorder”

$\hat{\beta}$ big, ”strong disorder”
Directed polymers in random environment with heavy tails

\[ P_{N,\beta}(s) = \frac{e^{\beta \sum_i \omega_i s_i}}{Z_{N,\beta}} P(s) \]

- \( \beta = \hat{\beta} N^{1-\frac{2}{\xi}} \Rightarrow \exists \hat{\gamma}_{\beta,\infty} : s_N \xrightarrow{(d)} \hat{\gamma}_{\beta,\infty} \)

- \( \exists \hat{\beta}_c^w \) random critical point

- \( \beta_c^w = 0 \) if \( \xi \in [1/2, 2) \) and \( \beta_c^w > 0 \) if \( \xi \in [0, 1/3) \)

Improved: \( \beta_c^w > 0 \) also \( \xi \in [1/3, 1/2) \).
1 Polymers
2 Pinning Model
3 Universality of the Pinning Model
4 Pinning Model with Heavy Tailed disorder
5 Perspectives
Perspectives

- Universality for weak disorder: *free energy of directed polymer in random environment.*
- Structure of $\hat{l}_{\beta,\infty}$. Finite number of points?
- Renewal process with polynomial tails (and Heavy-Tailed disorder). Different limit structure?
Thanks for your attention!