Spinorial Representation of Surfaces into 4-dimensional Space Forms

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Abstract

In this paper we give a geometrically invariant spinorial representation of surfaces in four-dimensional space forms. In the Euclidean space, we obtain a representation formula which generalizes the Weierstrass representation formula of minimal surfaces. We also obtain as particular cases the spinorial characterizations of surfaces in $\mathbb{R}^3$ and in S^3 given by T. Friedrich and by B. Morel.

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1 Introduction

The Weierstrass representation describes a conformal minimal immersion of a Riemann surface M into the three-dimensional Euclidean space $\mathbb{R}^3$. Precisely, the immersion is expressed using two holomorphic functions $f, g : M \longrightarrow \mathbb{C}$ by the following integral formula

$$(x_1, x_2, x_3) = \Re e \left(\int f(1-g^2)dz, \int if(1+g^2)dz, \int 2fgdz \right) : M \longrightarrow \mathbb{R}^3.$$

On the other hand, the spinor bundle ΣM over M is a two-dimensional complex vector bundle splitting into

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M = \Lambda^0 M \oplus \Lambda^{0,1} M.$$

Hence, a pair of holomorphic functions (g, f) can be considered as a spinor field $\varphi = (g, fdz)$. Moreover, the Cauchy-Riemann equations satisfied by f and g are equivalent to the Dirac equation

$$D\varphi = 0.$$

This representation is still valid for arbitrary surfaces. In the general case, the functions f and g are not holomorphic and the Dirac equation becomes

$$D\varphi = H\varphi,$$

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where H is the mean curvature of the immersion. This fact is well-known and has been studied in the last years by many authors (see [8, 9, 15, 16]).

In [5], T. Friedrich gave a geometrically invariant spinorial representation of surfaces in $\mathbb{R}^3$. This approach was generalized to surfaces of other three-dimensional spaces [13, 14] and also in the pseudo-Riemannian case [10, 11].

The aim of the present paper is to extend this approach to the case of codimension 2 and then provide a geometrically invariant representation of surfaces in the 4-dimensional space form $\mathbb{M}^4(c)$ of sectional curvature c by spinors solutions of a Dirac equation.

2 Preliminaries

2.1 The fundamental theorem of surfaces in $\mathbb{M}^4(c)$

Let (M^2, g) be an oriented surface isometrically immersed into the four-dimensional space form $\mathbb{M}^4(c)$. Let us denote by E its normal bundle and by $B : TM \times TM \longrightarrow E$ its second fundamental form defined by

$$B(X,Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

where ∇ and $\overline{\nabla}$ are the Levi-Civita connections of M and $\mathbb{M}^4(c)$ respectively. For $\xi \in \Gamma(E)$, the shape operator associated to ξ is defined by

$$S_\xi(X) = -\left(\overline{\nabla}_X \xi\right)^T,$$

for all $X \in \Gamma(TM)$, where the upper index T means that we take the component of the vector tangent to M. Then, the following equations hold:

1. $K = \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)|^2 + c$, (Gauss equation)
2. $K_N = -\langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2\rangle$, (Ricci equation)
3. $(\nabla'_X B)(Y,Z) - (\nabla'_Y B)(X,Z) = 0$, (Codazzi equation)

where K and K_N are the curvatures of (M,g) and E, (e_1,e_2) and (e_3,e_4) are orthonormal and positively oriented bases of TM and E respectively, and where ∇' is the natural connection induced on the bundle $T^*M^{\otimes 2} \otimes E$. Reciprocally, there is the following theorem:

Theorem (Tenenblat [17]). *Let (M^2, g) be a Riemannian surface and E a vector bundle of rank 2 on M, equipped with a metric $\langle .,. \rangle$ and a compatible connection. We suppose that M and E are oriented. Let $B : TM \times TM \longrightarrow E$ be a bilinear map satisfying the Gauss, Ricci and Codazzi equations above, where, if $\xi \in E$, the shape operator $S_\xi : TM \to TM$ is the symmetric operator such that*

$$g\left(S_\xi(X), Y\right) = \langle B(X,Y), \xi\rangle$$

for all $X, Y \in TM$. Then, there exists a local isometric immersion $V \subset M \longrightarrow \mathbb{M}^4(c)$ so that E is identified with the normal bundle of M into $\mathbb{M}^4(c)$ and with B as second fundamental form.

2.2 Twisted spinor bundle

Let (M^2, g) be an oriented Riemannian surface, with a given spin structure, and E an oriented and spin vector bundle of rank 2 on M. We consider the spinor bundle Σ over M twisted by E and defined by

$$\Sigma = \Sigma M \otimes \Sigma E.$$

We endow Σ with the spinorial connection ∇ defined by

$$\nabla = \nabla^{\Sigma M} \otimes \mathrm{Id}_{\Sigma E} + \mathrm{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

We also define the Clifford product $\cdot$ by

$$\begin{cases} X \cdot \varphi = (X \cdot_{_M} \alpha) \otimes \overline{\sigma} & \text{if } X \in \Gamma(TM) \\ \\ X \cdot \varphi = \alpha \otimes (X \cdot_{_E} \sigma) & \text{if } X \in \Gamma(E) \end{cases}$$

for all $\varphi = \alpha \otimes \sigma \in \Sigma M \otimes \Sigma E$, where $\cdot_{_M}$ and $\cdot_{_E}$ denote the Clifford products on ΣM and on ΣE respectively and where $\overline{\sigma} = \sigma^+ - \sigma^-$. We finally define the Dirac operator D on $\Gamma(\Sigma)$ by

$$D\varphi = e_1 \cdot \nabla_{e_1}\varphi + e_2 \cdot \nabla_{e_2}\varphi,$$

where (e_1, e_2) is an orthonormal basis of TM.

We note that Σ is also naturally equipped with a hermitian scalar product $\langle .,. \rangle$ which is compatible to the connection ∇, since so are ΣM and ΣE, and thus also with a compatible real scalar product $\Re e\langle .,. \rangle$. We also note that the Clifford product $\cdot$ of vectors belonging to $TM \oplus E$ is antihermitian with respect to this hermitian product. Finally, we stress that the four subbundles $\Sigma^{\pm\pm} := \Sigma^{\pm} M \otimes \Sigma^{\pm} E$ are orthogonal with respect to the hermitian product. Throughout the paper we will assume that the hermitian product is $\mathbb{C}-$linear w.r.t. the first entry, and $\mathbb{C}-$antilinear w.r.t. the second entry.

2.3 Spin geometry of surfaces in $\mathbb{M}^4(c)$

It is a well-known fact (see [1, 6, 7]) that there is an identification between the spinor bundle $\Sigma\mathbb{M}^4(c)_{|M}$ of $\mathbb{M}^4(c)$ over M, and the spinor bundle of M twisted by the normal bundle $\Sigma := \Sigma M \otimes \Sigma E$. Moreover, we have the spinorial Gauss formula: for any $\varphi \in \Gamma(\Sigma)$ and any $X \in TM$,

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi,$$

where $\widetilde{\nabla}$ is the spinorial connection of $\Sigma\mathbb{M}^4(c)$ and ∇ is the spinoral connection of Σ defined by

$$\nabla \ = \ \nabla^{\Sigma M} \otimes \mathrm{Id}_{\Sigma E} \ + \ \mathrm{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

Here $\cdot$ is the Clifford product on $\mathbb{M}^4(c)$. Therefore, if φ is a Killing spinor of $\mathbb{M}^4(c)$, that is satisfying

$$\widetilde{\nabla}_X \varphi = \lambda X \cdot \varphi,$$

where the Killing constant λ is 0 for the Euclidean space, $\pm\frac{1}{2}$ for the sphere and $\pm\frac{i}{2}$ for the hyperbolic space, that is, $4\lambda^2 = c$, then its restriction over M satisfies

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi. \tag{1}$$

Taking the trace in (1), we obtain the following Dirac equation

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi, \tag{2}$$

where we have again $D\varphi = \sum_{j=1}^{2} e_j \cdot \nabla_{e_j}\varphi$ and $\vec{H} = \frac{1}{2}\sum_{j=1}^{2} B(e_j, e_j)$ is the mean curvature vector of M in $\mathbb{M}^4(c)$.

Let us consider $\omega_4 = -e_1 \cdot e_2 \cdot e_3 \cdot e_4$. We recall that $\omega_4^2 = 1$ and ω_4 has two eigenspaces for eigenvalues 1 and -1 of same dimension. We denote by Σ^+ and Σ^- these subbundles. They decompose as follows:

$$\begin{cases} \Sigma^+ = (\Sigma^+ M \otimes \Sigma^+ E) \oplus (\Sigma^- M \otimes \Sigma^- E) \\ \\ \Sigma^- = (\Sigma^+ M \otimes \Sigma^- E) \oplus (\Sigma^- M \otimes \Sigma^+ E), \end{cases}$$

where $\Sigma^\pm M$ and $\Sigma^\pm E$ are the spaces of half-spinors for M and E respectively. In the sequel, for $\varphi \in \Sigma$, we will use the following convention:

$$\varphi = \varphi^{++} + \varphi^{--} + \varphi^{+-} + \varphi^{-+},$$

with

$$\begin{cases} \varphi^{++} \in \Sigma^{++} := \Sigma^+ M \otimes \Sigma^+ E, \\ \varphi^{--} \in \Sigma^{--} := \Sigma^- M \otimes \Sigma^- E, \\ \varphi^{+-} \in \Sigma^{+-} := \Sigma^+ M \otimes \Sigma^- E, \\ \varphi^{-+} \in \Sigma^{-+} := \Sigma^- M \otimes \Sigma^+ E. \end{cases}$$

Finally, we set

$$\varphi^+ = \varphi^{++} + \varphi^{--} \quad \text{and} \quad \varphi^- = \varphi^{+-} + \varphi^{-+}.$$

If φ is a Killing spinor of $\mathbb{M}^4(c)$, an easy computation yields

$$X|\varphi^+|^2 = 2\Re e\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \qquad \text{and} \qquad X|\varphi^-|^2 = 2\Re e\langle \lambda X \cdot \varphi^+, \varphi^- \rangle.$$

3 Main result

Theorem 1. *Let (M^2, g) be an oriented Riemannian surface, with a given spin structure, and E an oriented and spin vector bundle of rank 2 on M. Let $\Sigma = \Sigma M \otimes \Sigma E$ be the twisted spinor bundle and D its Dirac operator. We assume that λ is a constant belonging to $\mathbb{R} \cup i\mathbb{R}$ and $\vec{H}$ is a section of E. Then the three following statements are equivalent:*

1. *There exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of the Dirac equation*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \tag{3}$$

such that φ^+ and φ^- do not vanish and satisfy

$$X|\varphi^+|^2 = 2\Re e\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \quad \text{and} \quad X|\varphi^-|^2 = 2\Re e\langle \lambda X \cdot \varphi^+, \varphi^- \rangle. \tag{4}$$

2. *There exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of*

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi,$$

where $B : TM \times TM \longrightarrow E$ is bilinear and $\frac{1}{2}\mathrm{tr}\,(B) = \vec{H}$ and such that φ^+ and φ^- do not vanish.

3. *There exists a local isometric immersion of (M, g) into $\mathbb{M}^4(c)$ with normal bundle E, second fundamental form B and mean curvature $\vec{H}$.*

The form B and the spinor field φ are linked by (6).

In order to prove Theorem 1 we consider the following equivalent technical

Proposition 3.1. *Let M, E and Σ as in Theorem 1 and assume that there exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \tag{5}$$

with φ^+ and φ^- nowhere vanishing spinors satisfying (4). Then the symmetric bilinear map

$$B : TM \times TM \longrightarrow E$$

defined by

$$\begin{aligned} \langle B(X,Y), \xi \rangle \; = \; & \frac{1}{2|\varphi^+|^2} \Re e \left\langle X \cdot \nabla_Y \varphi^+ + Y \cdot \nabla_X \varphi^+ + 2\lambda \langle X, Y \rangle \varphi^-, \xi \cdot \varphi^+ \right\rangle \\ & + \frac{1}{2|\varphi^-|^2} \Re e \left\langle X \cdot \nabla_Y \varphi^- + Y \cdot \nabla_X \varphi^- + 2\lambda \langle X, Y \rangle \varphi^+, \xi \cdot \varphi^- \right\rangle \end{aligned} \tag{6}$$

for all $X, Y \in \Gamma(TM)$ and all $\xi \in \Gamma(E)$ satisfies the Gauss, Codazzi and Ricci equations and is such that

$$\vec{H} = \frac{1}{2} \mathrm{tr}\, B.$$

Remark 1. *If $\lambda = 0$, and if $\varphi \in \Gamma(\Sigma)$ is a solution of*

$$D\varphi = \vec{H} \cdot \varphi \qquad \text{with} \qquad |\varphi^+| = |\varphi^-| = 1,$$

formula (6) simplifies to

$$\begin{aligned} \langle B(X,Y), \xi \rangle &= \frac{1}{2} \Re e \left\langle X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \xi \cdot \varphi \right\rangle \\ &= \Re e \left\langle X \cdot \nabla_Y \varphi, \xi \cdot \varphi \right\rangle, \end{aligned} \tag{7}$$

since this last expression is in fact symmetric in X and Y.

To prove proposition 3.1 we first state the following lemma.

Lemma 3.2. *Assume that φ is a solution of the Dirac equation (5) with φ^+ and φ^- non-vanishing spinors satisfying (4). Then, for all $X \in \Gamma(TM)$,*

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi, \tag{8}$$

with

$$\eta(X) = -\frac{1}{2} \sum_{j=1}^{2} e_j \cdot B(e_j, X), \tag{9}$$

where the bilinear map B is defined by (6).

The proof of this lemma will be given in Section 4.

Proof of Proposition 3.1: The equations of Gauss, Codazzi and Ricci appear to be the integrability conditions of (8). Indeed computing the spinorial curvature $\mathcal{R}$ for φ, we first observe that (9) implies

$$X\cdot\eta(Y)-\eta(Y)\cdot X=B(X,Y)=Y\cdot\eta(X)-\eta(X)\cdot Y$$

for all $X,Y\in TM$. Then, a direct computation yields

$$\begin{aligned}\mathcal{R}(X,Y)\varphi &= d^{\nabla}\eta(X,Y)\cdot\varphi+\big(\eta(Y)\cdot\eta(X)-\eta(X)\cdot\eta(Y)\big)\cdot\varphi \\ &\quad+\lambda^2(Y\cdot X-X\cdot Y)\cdot\varphi,\end{aligned} \tag{10}$$

where

$$d^{\nabla}\eta(X,Y)=\nabla_X(\eta(Y))-\nabla_Y(\eta(X))-\eta([X,Y]).$$

Here we also denote by ∇ the natural connection on $Cl(TM\oplus E)\simeq Cl(M)\hat{\otimes}Cl(E)$, the graded tensor product of the Clifford algebras $Cl(M)$ and $Cl(E)$.

Lemma 3.3. *We have:*

1. *The left-hand side of* (10) *satisfies*

$$\mathcal{R}(e_1,e_2)\varphi=-\frac{1}{2}Ke_1\cdot e_2\cdot\varphi-\frac{1}{2}K_Ne_3\cdot e_4\cdot\varphi.$$

2. *The first term of the right-hand side of* (10) *satisfies*

$$d^{\nabla}\eta(X,Y)=-\frac{1}{2}\sum_{j=1}^{2}e_j\cdot\Big((\nabla'_XB)(Y,e_j)-(\nabla'_YB)(X,e_j)\Big)$$

where ∇' stands for the natural connection on $T^*M\otimes T^*M\otimes E$.

3. *The second term of the right-hand side of* (10) *satisfies*

$$\begin{aligned}\eta(e_2)\cdot\eta(e_1)-\eta(e_1)\cdot\eta(e_2) &= \frac{1}{2}\big(|B(e_1,e_2)|^2-\langle B(e_1,e_1),B(e_2,e_2)\rangle\big)e_1\cdot e_2 \\ &\quad+\frac{1}{2}\langle(S_{e_3}\circ S_{e_4}-S_{e_4}\circ S_{e_3})(e_1),e_2\rangle\, e_3\cdot e_4.\end{aligned}$$

Proof: First, we compute $\mathcal{R}(e_1,e_2)\varphi$. We recall that $\Sigma=\Sigma M\otimes\Sigma E$ and suppose that $\varphi=\alpha\otimes\sigma$ with $\alpha\in\Sigma M$ and $\sigma\in\Sigma E$. Thus,

$$\mathcal{R}(e_1,e_2)\varphi=\mathcal{R}^M(e_1,e_2)\alpha\otimes\sigma+\alpha\otimes\mathcal{R}^E(e_1,e_2)\sigma,$$

where $\mathcal{R}^M$ and $\mathcal{R}^E$ are the spinorial curvatures on M and E respectively. Moreover, by the Ricci identity on M, we have

$$\mathcal{R}^M(e_1,e_2)\alpha=-\frac{1}{2}Ke_1\cdot e_2\cdot\alpha,$$

where K is the Gauss curvature of (M,g). Similarly, we have

$$\mathcal{R}^E(e_1,e_2)\sigma=-\frac{1}{2}K_Ne_3\cdot e_4\cdot\sigma,$$

where K_N is the curvature of the connection on E. These last two relations give the first point of the lemma.
For the second point of the lemma, we choose e_j so that at $p \in M$, $\nabla e_{j|p} = 0$. Then, we have

$$\begin{aligned}
d^\nabla \eta(X,Y) &= \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X,Y]) \\
&= \sum_{j=1,2} -\frac{1}{2}\nabla_X\big(e_j \cdot B(Y,e_j)\big) + \frac{1}{2}\nabla_Y\big(e_j \cdot B(X,e_j)\big) + \frac{1}{2}e_j \cdot B([X,Y],e_j) \\
&= \sum_{j=1,2} -\frac{1}{2}e_j \cdot \nabla^E_X(B(Y,e_j)) + \frac{1}{2}e_j \cdot \nabla^E_Y(B(X,e_j)) + \frac{1}{2}e_j \cdot B(\nabla_X Y, e_j) \\
&\qquad -\frac{1}{2}e_j \cdot B(\nabla_Y X, e_j) \\
&= -\frac{1}{2}\sum_{j=1,2} e_j \cdot \Big((\nabla'_X B)(Y,e_j) - (\nabla'_Y B)(X,e_j)\Big)
\end{aligned}$$

since $[X,Y] = \nabla_X Y - \nabla_Y X$ and $(\nabla'_X B)(Y,e_j) = \nabla^E_X(B(Y,e_j)) - B(\nabla_X Y, e_j)$. Here ∇^E stands for the given connection on E.
We finally prove the third assertion of the lemma. In order to simplify the notation, we set $B(e_i,e_j) = B_{ij}$. We have

$$\eta(e_2)\cdot\eta(e_1) - \eta(e_1)\cdot\eta(e_2) = -\frac{1}{4}\sum_{j,k=1}^{2} e_j \cdot B_{1j} \cdot e_k \cdot B_{2k} + \frac{1}{4}\sum_{j,k=1}^{2} e_j \cdot B_{2j} \cdot e_k \cdot B_{1k}$$

$$\begin{aligned}
= \frac{1}{4}\Big[&- e_1 \cdot B_{11} \cdot e_1 \cdot B_{21} - e_1 \cdot B_{11} \cdot e_2 \cdot B_{22} - e_2 \cdot B_{12} \cdot e_1 \cdot B_{21} - e_2 \cdot B_{12} \cdot e_2 \cdot B_{22} \\
&+ e_1 \cdot B_{21} \cdot e_1 \cdot B_{11} + e_1 \cdot B_{21} \cdot e_2 \cdot B_{12} + e_2 \cdot B_{22} \cdot e_1 \cdot B_{11} + e_2 \cdot B_{22} \cdot e_2 \cdot B_{12}\Big]
\end{aligned}$$

$$= \frac{1}{2}\Big[|B_{12}|^2 - \langle B_{11}, B_{22}\rangle\Big]e_1 \cdot e_2 + \frac{1}{4}\Big[- B_{11} \cdot B_{21} + B_{21} \cdot B_{11} - B_{12} \cdot B_{22} + B_{22} \cdot B_{12}\Big].$$

Now, if we write $B_{ij} = B^3_{ij}e_3 + B^4_{ij}e_4$, we have

$$-B_{11} \cdot B_{21} + B_{21} \cdot B_{11} = 2(-B^3_{11}B^4_{21} + B^3_{21}B^4_{11})e_3 \cdot e_4$$

and

$$-B_{12} \cdot B_{22} + B_{22} \cdot B_{12} = 2(-B^3_{12}B^4_{22} + B^4_{12}B^3_{22})e_3 \cdot e_4.$$

Moreover

$$-B^3_{11}B^4_{21} + B^3_{21}B^4_{11} - B^3_{12}B^4_{22} + B^4_{12}B^3_{22} = \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2\rangle$$

since for $j \in \{1,2\}$ and $k \in \{3,4\}$, we have $S_{e_k}e_j = B^k_{j1}e_1 + B^k_{j2}e_2$. The formula follows. □

Now, we give this final lemma

Lemma 3.4. *If T is an element of $\mathcal{C}l(M)\hat{\otimes}\mathcal{C}l(E)$ of order 2, that is of*

$$\Lambda^2 M \otimes 1 \;\oplus\; TM \otimes E \;\oplus\; 1 \otimes \Lambda^2 E,$$

so that

$$T\cdot\varphi=0,$$

where φ is a spinor field of Σ such that φ^+ and φ^- do not vanish, then $T=0$.

Proof: We have

$$\mathcal{C}l_2\hat{\otimes}\mathcal{C}l_2\simeq\mathcal{C}l_4\simeq\mathbb{H}(2),$$

where $\mathbb{H}(2)$ is the set of 2×2 matrices with quaternionic coefficients. The spinor bundle Σ and the Clifford product come from the representation

$$\mathbb{H}(2)\longrightarrow \mathrm{End}_{\mathbb{H}}(\mathbb{H}\oplus\mathbb{H}).$$

The first factor of $\mathbb{H}\oplus\mathbb{H}$ correspond to Σ^+ and the second to Σ^-. Moreover, elements of order 2 of $\mathcal{C}l_4$ are matrices

$$\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix},$$

where p,q are purely imaginary quaternions. Hence $T\cdot\varphi=0$ is equivalent to

$$\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix}\begin{pmatrix}\alpha\\ \sigma\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

with α,σ non zero quaternions. Thus $p=q=0$, and so T vanishes identically. $\square$

We deduce from (10) and Lemma 3.4 and comparing terms, that

$$\begin{cases} K=\langle B(e_1,e_1),B(e_2,e_2)\rangle-|B(e_1,e_2)|^2+4\lambda^2,\\[2ex] K_N=-\left\langle(S_{e_3}\circ S_{e_4}-S_{e_4}\circ S_{e_3})(e_1),e_2\right\rangle,\\[2ex] (\nabla'_XB)(Y,e_j)-(\nabla'_YB)(X,e_j)=0,\quad\forall j=1,2,\end{cases}$$

which are respectively the Gauss, Ricci and Codazzi equations. $\square$

From Proposition 3.1 and the fundamental theorem of submanifolds, we deduce that a spinor field solution of (5) such that (4) holds defines a local isometric immersion of M into $\mathbb{M}^4(c)$ with normal bundle E and second fundamental form B. This implies the equivalence between assertions 1 and 3 in Theorem 1. The equivalence between assertions 1 and 2 is given by Lemma 3.2 and will be proven in the next section.

Remark 2. *If in Theorem 1 we assume moreover that the manifold is simply connected, the spinor field solution of (5) defines a global isometric immersion of M into $\mathbb{M}^4(c)$.*

4 Proof of Lemma 3.2

In order to prove Lemma 3.2, we need some preliminary results. First, we remark that

$$\begin{cases} D\varphi^{--}=\vec{H}\cdot\varphi^{++}-2\lambda\varphi^{+-},\\ D\varphi^{++}=\vec{H}\cdot\varphi^{--}-2\lambda\varphi^{-+},\\ D\varphi^{+-}=\vec{H}\cdot\varphi^{-+}-2\lambda\varphi^{--},\\ D\varphi^{-+}=\vec{H}\cdot\varphi^{+-}-2\lambda\varphi^{++}.\end{cases}$$

We fix a point $p \in M$, and consider e_3 a unit vector in E_p so that $\vec{H} = |\vec{H}|e_3$ at p. We complete e_3 by e_4 to get a positively oriented and orthonormal frame of E_p. We first assume that φ^{--}, φ^{++}, φ^{+-} and φ^{-+} do not vanish at p. We see easily that

$$\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} \right\}$$

is an orthonormal frame of Σ^{++} for the real scalar product $\Re e \langle \cdot, \cdot \rangle$. Indeed, we have

$$\begin{aligned} \Re e \left\langle e_1 \cdot e_3 \cdot \varphi^{--}, e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle &= \Re e \left\langle \varphi^{--}, e_3 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle \\ &= \Re e \left(i|\varphi^{--}|^2 \right) = 0. \end{aligned}$$

Analogously,

$$\begin{aligned} &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|} \right\} \end{aligned}$$

are orthonormal frames of Σ^{--}, Σ^{+-} and Σ^{-+} respectively. We define the following bilinear forms

$$\begin{aligned} F_{++}(X,Y) &= \Re e \left\langle \nabla_X \varphi^{++}, Y \cdot e_3 \cdot \varphi^{--} \right\rangle, \\ F_{--}(X,Y) &= \Re e \left\langle \nabla_X \varphi^{--}, Y \cdot e_3 \cdot \varphi^{++} \right\rangle, \\ F_{+-}(X,Y) &= \Re e \left\langle \nabla_X \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{-+} \right\rangle, \\ F_{-+}(X,Y) &= \Re e \left\langle \nabla_X \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{+-} \right\rangle, \end{aligned}$$

and

$$\begin{aligned} B_{++}(X,Y) &= -\Re e \left\langle \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \right\rangle, \\ B_{--}(X,Y) &= -\Re e \left\langle \lambda X \cdot \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{++} \right\rangle, \\ B_{+-}(X,Y) &= -\Re e \left\langle \lambda X \cdot \varphi^{--}, Y \cdot e_3 \cdot \varphi^{-+} \right\rangle, \\ B_{-+}(X,Y) &= -\Re e \left\langle \lambda X \cdot \varphi^{++}, Y \cdot e_3 \cdot \varphi^{+-} \right\rangle. \end{aligned}$$

We have this first lemma:

Lemma 4.1. *We have*

1. $\mathrm{tr}(F_{++}) = -|\vec{H}||\varphi^{--}|^2 + 2\Re e \langle \lambda \varphi^{-+}, e_3 \cdot \varphi^{--} \rangle$,
2. $\mathrm{tr}(F_{--}) = -|\vec{H}||\varphi^{++}|^2 + 2\Re e \langle \lambda \varphi^{+-}, e_3 \cdot \varphi^{++} \rangle$,
3. $\mathrm{tr}(F_{+-}) = -|\vec{H}||\varphi^{-+}|^2 + 2\Re e \langle \lambda \varphi^{--}, e_3 \cdot \varphi^{-+} \rangle$,
4. $\mathrm{tr}(F_{-+}) = -|\vec{H}||\varphi^{+-}|^2 + 2\Re e \langle \lambda \varphi^{++}, e_3 \cdot \varphi^{+-} \rangle$.

This second lemma gives the defect of symmetry:

Lemma 4.2. *We have*

1. $F_{++}(e_1,e_2) = F_{++}(e_2,e_1) - 2\Re e\left\langle \lambda\varphi^{-+}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle,$
2. $F_{--}(e_1,e_2) = F_{--}(e_2,e_1) - 2\Re e\left\langle \lambda\varphi^{+-}, e_1\cdot e_2\cdot e_3\cdot\varphi^{++}\right\rangle,$
3. $F_{+-}(e_1,e_2) = F_{+-}(e_2,e_1) - 2\Re e\left\langle \lambda\varphi^{--}, e_1\cdot e_2\cdot e_3\cdot\varphi^{-+}\right\rangle,$
4. $F_{-+}(e_1,e_2) = F_{-+}(e_2,e_1) - 2\Re e\left\langle \lambda\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{+-}\right\rangle.$

For sake of brevity, we only prove Lemma 4.2. The proof of Lemma 4.1 is very similar.

Proof: We have

$$\begin{aligned} F_{++}(e_1,e_2) &= \Re e\left\langle \nabla_{e_1}\varphi^{++}, e_2\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= \Re e\left\langle e_1\cdot\nabla_{e_1}\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= \Re e\left\langle \vec{H}\cdot\varphi^{--} - 2\lambda\varphi^{-+} - e_2\cdot\nabla_{e_2}\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle. \end{aligned}$$

The first term is

$$\begin{aligned} \Re e\left\langle \vec{H}\cdot\varphi^{--}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle &= -\Re e\left\langle \varphi^{--}, i\vec{H}\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= -\Re e\left(i|\vec{H}||\varphi^{--}|^2\right) = 0. \end{aligned}$$

Hence, we get

$$\begin{aligned} F_{++}(e_1,e_2) + 2\Re e\left\langle \lambda\varphi^{-+}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle &= -\Re e\left\langle e_2\cdot\nabla_{e_2}\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= \Re e\left\langle \nabla_{e_2}\varphi^{++}, e_2\cdot e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= \Re e\left\langle \nabla_{e_2}\varphi^{++}, e_1\cdot e_3\cdot\varphi^{--}\right\rangle \\ &= F_{++}(e_2,e_1). \end{aligned}$$

The proof is similar for the three other forms. □

By analogous computations, we also get the following lemmas:

Lemma 4.3. *We have*

1. $tr(B_{++}) = -2\Re e\left\langle \lambda\varphi^{-+}, e_3\cdot\varphi^{--}\right\rangle,$
2. $tr(B_{--}) = -2\Re e\left\langle \lambda\varphi^{+-}, e_3\cdot\varphi^{++}\right\rangle,$
3. $tr(B_{+-}) = -2\Re e\left\langle \lambda\varphi^{--}, e_3\cdot\varphi^{-+}\right\rangle,$
4. $tr(B_{-+}) = -2\Re e\left\langle \lambda\varphi^{++}, e_3\cdot\varphi^{+-}\right\rangle.$

Lemma 4.4. *We have*

1. $B_{++}(e_1,e_2) = B_{++}(e_2,e_1) + 2\Re e\left\langle \lambda\varphi^{-+}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle,$
2. $B_{--}(e_1,e_2) = B_{--}(e_2,e_1) + 2\Re e\left\langle \lambda\varphi^{+-}, e_1\cdot e_2\cdot e_3\cdot\varphi^{++}\right\rangle,$
3. $B_{+-}(e_1,e_2) = B_{+-}(e_2,e_1) + 2\Re e\left\langle \lambda\varphi^{--}, e_1\cdot e_2\cdot e_3\cdot\varphi^{-+}\right\rangle,$
4. $B_{-+}(e_1,e_2) = B_{-+}(e_2,e_1) + 2\Re e\left\langle \lambda\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{+-}\right\rangle.$

Now, we set

$$\begin{cases} A_{++} := F_{++} + B_{++}, \\ A_{--} := F_{--} + B_{--}, \\ A_{+-} := F_{+-} + B_{+-}, \\ A_{-+} := F_{-+} + B_{-+}, \end{cases}$$

and

$$F_+ = \frac{A_{++}}{|\varphi^{--}|^2} - \frac{A_{--}}{|\varphi^{++}|^2} \qquad \text{and} \qquad F_- = \frac{A_{+-}}{|\varphi^{-+}|^2} - \frac{A_{-+}}{|\varphi^{+-}|^2}.$$

From the last four lemmas we deduce immediately that F_+ and F_- are symmetric and trace-free. Moreover, by a direct computation using the conditions (4) on the norms of φ^+ and φ^-, we get the following lemma:

Lemma 4.5. *The symmetric operators F^+ and F^- of TM associated to the bilinear forms F_+ and F_-, defined by*

$$F^+(X) = F_+(X, e_1)e_1 + F_+(X, e_2)e_2 \quad \textit{and} \quad F^-(X) = F_-(X, e_1)e_1 + F_-(X, e_2)e_2$$

for all $X \in TM$, satisfy

1. $\Re e \, \langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle = 0$,
2. $\Re e \, \langle F^-(X) \cdot e_3 \cdot \varphi^{-+}, \varphi^{+-} \rangle = 0$.

Proof. Since

$$A_{++}(X, Y) = \Re e \, \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \rangle,$$

and since $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$ is an orthonormal frame of Σ^{++}, we have

$$\begin{aligned} & \Re e \, \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, \varphi^{++} \rangle \\ = \quad & \frac{A_{++}}{|\varphi^{--}|^2}(X, e_1) \, \Re e \, \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle + \frac{A_{++}}{|\varphi^{--}|^2}(X, e_2) \, \Re e \, \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} & \Re e \, \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle \\ & = \frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \, \Re e \, \langle e_1 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle + \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \, \Re e \, \langle e_2 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle \\ & = -\frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \, \Re e \, \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle - \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \, \Re e \, \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

These two formulas imply that

$$\Re e \left\langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \right\rangle = \Re e \, \langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, \varphi^+ \rangle;$$

by the first condition in (4), this last expression is zero. □

Hence, the operators F^+ and F^- are of rank at most ≤ 1. Since they are symmetric and trace-free, they vanish identically.

Using again that $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$ is an orthonormal frame of Σ^{++}, we have

$$\nabla_X \varphi^{++} = F_{++}(X, e_1) e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} + F_{++}(X, e_2) e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}.$$

Since $F_{++} = A_{++} - B_{++}$ and denoting by A^{++} and B^{++} the symmetric operators of TM associated to A_{++} and B_{++} and defined by

$$A^{++}(X) = A_{++}(X, e_1) e_1 + A_{++}(X, e_2) e_2, \ B^{++}(X) = B_{++}(X, e_1) e_1 + B_{++}(X, e_2) e_2,$$

we get

$$\nabla_X \varphi^{++} = \frac{1}{|\varphi^{--}|^2} \left[A^{++}(X) \cdot e_3 \cdot \varphi^{--} - B^{++}(X) \cdot e_3 \cdot \varphi^{--} \right]. \tag{11}$$

Similarly, if A^{--} and B^{--} denote the symmetric operators of TM associated to A_{--} and B_{--}, we have

$$\nabla_X \varphi^{--} = \frac{1}{|\varphi^{++}|^2} \left[A^{--}(X) \cdot e_3 \cdot \varphi^{++} - B^{--}(X) \cdot e_3 \cdot \varphi^{++} \right]. \tag{12}$$

Moreover, we easily get

$$B^{++}(X) \cdot e_3 \cdot \varphi^{--} = -|\varphi^{--}|^2 \lambda X \cdot \varphi^{-+} \ \text{ and } \ B^{--}(X) \cdot e_3 \cdot \varphi^{++} = -|\varphi^{++}|^2 \lambda X \cdot \varphi^{+-}.$$

Thus

$$\begin{aligned} \nabla_X \varphi^{+} &= \frac{1}{|\varphi^{--}|^2} A^{++}(X) \cdot e_3 \cdot \varphi^{--} + \lambda X \cdot \varphi^{-+} \\ &\quad + \frac{1}{|\varphi^{++}|^2} A^{--}(X) \cdot e_3 \cdot \varphi^{++} + \lambda X \cdot \varphi^{+-}. \end{aligned}$$

Setting $A^+ = A^{++} + A^{--}$ we get from the definition of A^{++} and A^{--} and from $F^+ = 0$ that $\frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}$. Bearing in mind that $|\varphi^+|^2 = |\varphi^{++}|^2 + |\varphi^{--}|^2$, we get finally

$$\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}. \tag{13}$$

Thus

$$\nabla_X \varphi^+ = \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \cdot \varphi^+ + \lambda X \cdot \varphi^-. \tag{14}$$

Similarly, denoting by A^{+-} and A^{-+} the symmetric operators of TM associated to A_{+-} and A_{-+}, setting $A^- = A^{+-} + A^{-+}$ and using $F^- = 0$ we get

$$\begin{aligned} \nabla_X \varphi^{-} &= \frac{1}{|\varphi^{+-}|^2} A^{-+}(X) \cdot e_3 \cdot \varphi^{+-} + \lambda X \cdot \varphi^{++} \\ &\quad + \frac{1}{|\varphi^{-+}|^2} A^{+-}(X) \cdot e_3 \cdot \varphi^{-+} + \lambda X \cdot \varphi^{--} \\ &= \frac{1}{|\varphi^{-}|^2} A^{-}(X) \cdot e_3 \cdot \varphi^{-} + \lambda X \cdot \varphi^{+}. \end{aligned} \tag{15}$$

We now observe that formulas (14) and (15) also hold if φ^{++} or φ^{--}, (resp. φ^{+-} or φ^{-+}) vanishes at p : indeed, assuming for instance that $\varphi^{++}(p) = 0$, and thus that $\varphi^{--}(p) \neq 0$ since $\varphi^{+}(p) \neq 0$, equation (11) holds, and, from the first condition in (4),

$$\Re e\ \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle = 0.$$

Since $\left(\frac{\varphi^{--}}{|\varphi^{--}|}, i\frac{\varphi^{--}}{|\varphi^{--}|}\right)$ is an orthonormal basis of Σ^{--}, we deduce that

$$\nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-} = i\alpha(X)\frac{\varphi^{--}}{|\varphi^{--}|}$$

for some real 1-form α. Since $D\varphi^{--} + 2\lambda\varphi^{+-} = 0$ ($\varphi^{++} = 0$ at p), this implies that

$$(\alpha(e_1)e_1 + \alpha(e_2)e_2) \cdot \frac{\varphi^{--}}{|\varphi^{--}|} = 0,$$

and thus that $\alpha = 0$. We thus get $\nabla_X \varphi^{--} = \lambda X \cdot \varphi^{+-}$ instead of (12), which, together with (11), easily implies (14).

Now, we set

$$\eta^+(X) = \left(\frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3\right)^+ \qquad \text{and} \qquad \eta^-(X) = \left(\frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3\right)^-$$

where, if σ belongs to $\mathcal{C}l^0(TM \oplus E)$, we denote by $\sigma^+ := \frac{1+\omega_4}{2} \cdot \sigma$ and by $\sigma^- := \frac{1-\omega_4}{2} \cdot \sigma$ the parts of σ acting on Σ^+ and on Σ^- only, i.e., such that

$$\sigma^+ \cdot \varphi = \sigma \cdot \varphi^+ \ \in\ \Sigma^+ \qquad \text{and} \qquad \sigma^- \cdot \varphi = \sigma \cdot \varphi^- \ \in\ \Sigma^-.$$

Setting $\eta = \eta^+ + \eta^-$ we thus get

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi,$$

as claimed in Lemma 3.2.

Explicitely, setting $A_+(X,Y) := \langle A^+(X), Y\rangle$ and $A_-(X,Y) := \langle A^-(X), Y\rangle$, the form η is given by

$$\begin{aligned}\eta(X) \quad &= \quad \frac{1}{2|\varphi^+|^2}\left[A_+(X,e_1)(e_1 \cdot e_3 - e_2 \cdot e_4) + A_+(X,e_2)(e_2 \cdot e_3 + e_1 \cdot e_4)\right] \\ &\quad + \frac{1}{2|\varphi^-|^2}\left[A_-(X,e_1)(e_1 \cdot e_3 + e_2 \cdot e_4) + A_-(X,e_2)(e_2 \cdot e_3 - e_1 \cdot e_4)\right]\end{aligned}$$

with

$$A_+(X,Y) = \Re e \left\langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, Y \cdot e_3 \cdot \varphi^+ \right\rangle$$

and

$$A_-(X,Y) = \Re e \left\langle \nabla_X \varphi^- - \lambda X \cdot \varphi^+, Y \cdot e_3 \cdot \varphi^- \right\rangle.$$

By direct computations we get that

$$B(X,Y) := X \cdot \eta(Y) - \eta(Y) \cdot X$$

is a vector belonging to E which is such that

$$\begin{aligned}\langle B(X,Y), \xi\rangle \quad &= \quad \frac{1}{|\varphi^+|^2}\Re e \left\langle X \cdot \nabla_Y \varphi^+ - \lambda X \cdot Y \cdot \varphi^-, \xi \cdot \varphi^+ \right\rangle \\ &\quad + \frac{1}{|\varphi^-|^2}\Re e \left\langle X \cdot \nabla_Y \varphi^- - \lambda X \cdot Y \cdot \varphi^+, \xi \cdot \varphi^- \right\rangle\end{aligned}$$

for all $\xi \in E$. This last expression appears to be symmetric in X, Y (the proof is analogous to the proof of the symmetry of $A_{++} = F_{++} + B_{++}$ above). Computing

$$\langle B(X,Y),\xi\rangle = \frac{1}{2}\left(\langle B(X,Y),\xi\rangle + \langle B(Y,X),\xi\rangle\right)$$

we finally obtain that B is given by formula (6).

Since $B(e_j, X) = e_j \cdot \eta(X) - \eta(X) \cdot e_j$, we obtain

$$\sum_{j=1,2} e_j \cdot B(e_j, X) = -2\eta(X) - \sum_{j=1,2} e_j \cdot \eta(X) \cdot e_j.$$

Writing $\eta(X)$ in the form $\sum_k e_k \cdot n_k$ for some vectors n_k belonging to E, we easily get that $\sum_j e_j \cdot \eta(X) \cdot e_j = 0$. Thus

$$\eta(X) = -\frac{1}{2}\sum_{j=1,2} e_j \cdot B(e_j, X).$$

The last claim in Lemma 3.2 is proved. □

5 Weierstrass representation of surfaces in $\mathbb{R}^4$

We are interested here in isometric immersions in euclidean space $\mathbb{R}^4$ (thus $c = \lambda = 0$); we obtain that the immersions are given by a formula which generalizes the representation formula given by T. Friedrich in [5]. Such a formula was also found in [4] using a different method involving twistor theory.

We consider the scalar product $\langle\langle .,. \rangle\rangle$ defined on Σ^+ by

$$\begin{array}{rccc} \langle\langle .,. \rangle\rangle : & \Sigma^+ \times \Sigma^+ & \to & \mathbb{H} \\ & (\varphi^+, \psi^+) & \mapsto & \overline{[\psi^+]}[\varphi^+], \end{array}$$

where $[\varphi^+]$ and $[\psi^+] \in \mathbb{H}$ represent the spinors φ^+ and ψ^+ in some frame, and where, if $q = q_1 1 + q_2 I + q_3 J + q_4 K$ belongs to $\mathbb{H}$,

$$\overline{q} = q_1 1 - q_2 I - q_3 J - q_4 K.$$

We also define the product $\langle\langle .,. \rangle\rangle$ on Σ^- by an analogous formula:

$$\begin{array}{rccc} \langle\langle .,. \rangle\rangle : & \Sigma^- \times \Sigma^- & \to & \mathbb{H} \\ & (\varphi^-, \psi^-) & \mapsto & \overline{[\psi^-]}[\varphi^-]. \end{array}$$

The following properties hold: for all $\varphi, \psi \in \Sigma$ and all $X \in TM \oplus E$,

$$\langle\langle \varphi^+, \psi^+ \rangle\rangle = \overline{\langle\langle \psi^+, \varphi^+ \rangle\rangle}, \qquad \langle\langle \varphi^-, \psi^- \rangle\rangle = \overline{\langle\langle \psi^-, \varphi^- \rangle\rangle} \tag{16}$$

and

$$\langle\langle X \cdot \varphi^+, \psi^- \rangle\rangle = -\langle\langle \varphi^+, X \cdot \psi^- \rangle\rangle. \tag{17}$$

Assume that we have a spinor φ solution of the Dirac equation $D\varphi = \vec{H} \cdot \varphi$ so that $|\varphi^+| = |\varphi^-| = 1$, and define the $\mathbb{H}$-valued 1-form ξ by

$$\xi(X) = \langle\langle X \cdot \varphi^-, \varphi^+ \rangle\rangle \qquad \in \qquad \mathbb{H}.$$

Proposition 5.1. *The form $\xi \in \Omega^1(M,\mathbb{H})$ is closed.*

Proof: By a straightforward computation, we get

$$d\xi(e_1,e_2) = \langle\langle e_2\cdot\nabla_{e_1}\varphi^-,\varphi^+\rangle\rangle - \langle\langle e_1\cdot\nabla_{e_2}\varphi^-,\varphi^+\rangle\rangle + \langle\langle e_2\cdot\varphi^-,\nabla_{e_1}\varphi^+\rangle\rangle - \langle\langle e_1\cdot\varphi^-,\nabla_{e_2}\varphi^+\rangle\rangle.$$

First observe that

$$\begin{aligned}\langle\langle e_2\cdot\nabla_{e_1}\varphi^-,\varphi^+\rangle\rangle - \langle\langle e_1\cdot\nabla_{e_2}\varphi^-,\varphi^+\rangle\rangle &= -\langle\langle e_1\cdot\nabla_{e_1}\varphi^-,e_1\cdot e_2\cdot\varphi^+\rangle\rangle + \langle\langle e_2\cdot\nabla_{e_2}\varphi^-,e_2\cdot e_1\cdot\varphi^+\rangle\rangle\\ &= -\langle\langle D\varphi^-,e_1\cdot e_2\cdot\varphi^+\rangle\rangle\end{aligned}$$

and similarly that

$$\begin{aligned}\langle\langle e_2\cdot\varphi^-,\nabla_{e_1}\varphi^+\rangle\rangle - \langle\langle e_1\cdot\varphi^-,\nabla_{e_2}\varphi^+\rangle\rangle &= \langle\langle e_1\cdot e_2\cdot\varphi^-,e_1\cdot\nabla_{e_1}\varphi^+\rangle\rangle - \langle\langle e_2\cdot e_1\cdot\varphi^-,e_2\cdot\nabla_{e_2}\varphi^+\rangle\rangle\\ &= \langle\langle e_1\cdot e_2\cdot\varphi^-,D\varphi^+\rangle\rangle.\end{aligned}$$

Thus

$$d\xi(e_1,e_2) = \langle\langle e_1\cdot e_2\cdot D\varphi^-,\varphi^+\rangle\rangle + \langle\langle e_1\cdot e_2\cdot\varphi^-,D\varphi^+\rangle\rangle.$$

Since $D\varphi = \vec{H}\cdot\varphi$, then $D\varphi^+ = \vec{H}\cdot\varphi^+$ and $D\varphi^- = \vec{H}\cdot\varphi^-$, which implies

$$d\xi(e_1,e_2) = \langle\langle (e_1\cdot e_2\cdot\vec{H} - \vec{H}\cdot e_1\cdot e_2)\cdot\varphi^-,\varphi^+\rangle\rangle = 0.$$

□

Assuming that M is simply connected, there exists a function $F : M \longrightarrow \mathbb{H}$ so that $dF = \xi$. We now identify $\mathbb{H}$ to $\mathbb{R}^4$ in the natural way.

Theorem 2. *1. The map $F = (F_1,F_2,F_3,F_4) : M \longrightarrow \mathbb{R}^4$ is an isometry.*

2. The map

$$\begin{array}{rccl}\Phi_E : & E & \longrightarrow & M\times\mathbb{R}^4\\ & X\in E_m & \longmapsto & (F(m),\xi_1(X),\xi_2(X),\xi_3(X),\xi_4(X))\end{array}$$

is an isometry between E and the normal bundle $N(F(M))$ of $F(M)$ into $\mathbb{R}^4$, preserving connections and second fundamental forms.

Proof. Note first that the euclidean norm of $\xi \in \mathbb{R}^4 \simeq \mathbb{H}$ is

$$|\xi|^2 = \langle\xi,\xi\rangle = \overline{\xi}\xi \qquad \in \qquad \mathbb{R},$$

and more generally that the real scalar product $\langle\xi,\xi'\rangle$ of $\xi,\xi' \in \mathbb{R}^4 \simeq \mathbb{H}$ is the component of *1* in $\langle\langle\xi,\xi'\rangle\rangle = \overline{\xi'}\xi \in \mathbb{H}$. We first compute, for all X,Y belonging to $E\cup TM$,

$$\begin{aligned}\overline{\xi(Y)}\xi(X) &= \overline{\langle\langle Y\cdot\varphi^-,\varphi^+\rangle\rangle}\langle\langle X\cdot\varphi^-,\varphi^+\rangle\rangle = \overline{\left(\overline{[\varphi^+]}[Y\cdot\varphi^-]\right)}\left(\overline{[\varphi^+]}[X\cdot\varphi^-]\right)\\ &= \overline{[Y\cdot\varphi^-]}[X\cdot\varphi^-]\end{aligned}$$

since $[\varphi^+]\overline{[\varphi^+]} = 1$ ($|\varphi^+| = 1$). Here and below the brackets $[.]$ stand for the components ($\in \mathbb{H}$) of the spinor fields in some local frame. Thus

$$\overline{\xi(Y)}\xi(X) = \langle\langle X\cdot\varphi^-,Y\cdot\varphi^-\rangle\rangle, \tag{18}$$

which in particular implies (considering the components of *1* of these quaternions)

$$\langle \xi(X), \xi(Y) \rangle = \Re e \ \langle X \cdot \varphi^-, Y \cdot \varphi^- \rangle. \tag{19}$$

This last identity easily gives

$$\langle \xi(X), \xi(Y) \rangle = 0 \qquad \text{and} \qquad |\xi(Z)|^2 = |Z|^2 \tag{20}$$

for all $X \in TM$, $Y \in E$ and $Z \in E \cup TM$. Thus $F = \int \xi$ is an isometry, and ξ maps isometrically the bundle E into the normal bundle of $F(M)$ in $\mathbb{R}^4$.

We now prove that ξ preserves the normal connection and the second fundamental form: let $X \in TM$ and $Y \in \Gamma(E) \cup \Gamma(TM)$; then $\xi(Y)$ is a vector field normal or tangent to $F(M)$. Considering $\xi(Y)$ as a map $M \to \mathbb{R}^4 \simeq \mathbb{H}$, we have

$$\begin{aligned} d(\xi(Y))(X) &= d\langle\langle Y \cdot \varphi^-, \varphi^+ \rangle\rangle(X) \qquad (21) \\ &= \langle\langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle\rangle + \langle\langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle\rangle + \langle\langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle\rangle \end{aligned}$$

where the connection $\nabla_X Y$ denotes the connection on E (if $Y \in \Gamma(E)$) or the Levi-Civita connection on TM (if $Y \in \Gamma(TM)$). We will need the following formulas:

Lemma 5.2. *We have*

$$\begin{aligned} \langle \ \langle\langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle\rangle \ , \ \xi(\nu) \ \rangle &= \Re e \ \langle \nabla_X Y \cdot \varphi^-, \nu \cdot \varphi^- \rangle \\ &= \Re e \ \langle \nabla_X Y \cdot \varphi^+, \nu \cdot \varphi^+ \rangle, \end{aligned}$$

$$\langle \ \langle\langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle\rangle \ , \ \xi(\nu) \ \rangle = \Re e \ \langle Y \cdot \nabla_X \varphi^-, \nu \cdot \varphi^- \rangle$$

and

$$\langle \ \langle\langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle\rangle \ , \ \xi(\nu) \ \rangle = \Re e \ \langle Y \cdot \nabla_X \varphi^+, \nu \cdot \varphi^+ \rangle.$$

In the expressions above, $\langle .,. \rangle$ defined on $\mathbb{H}$ for the left-hand side and $\Re e \ \langle .,. \rangle$ defined on Σ for the right-hand side of each identity, stand for the natural real scalar products.

Proof. The first identity is a consequence of (19) and the second identity may be obtained by a very similar computation observing that, by (16)-(17),

$$\langle \ \langle\langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle\rangle \ , \ \xi(\nu) \ \rangle = \langle \ \langle\langle \nabla_X Y \cdot \varphi^+, \varphi^- \rangle\rangle \ , \ \langle\langle \nu \cdot \varphi^+, \varphi^- \rangle\rangle \ \rangle.$$

The last two identities may be obtained by very similar computations. $\square$

From (21) and the lemma, we readily get the formula

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re e \ \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle + \Re e \ \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle. \tag{22}$$

We first suppose that $X, Y \in \Gamma(TM)$. The first term in the right-hand side of the equation above vanishes in that case since $\nabla_X Y \in \Gamma(TM)$, $\nu \in \Gamma(E)$. Recalling (7), we get then

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle \ = \ \Re e \ \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle = \langle B(X,Y), \nu \rangle = \langle \xi(B(X,Y)), \xi(\nu) \rangle.$$

Hence the component of $d(\xi(Y))(X)$ normal to $F(M)$ is given by

$$(d(\xi(Y))(X))^N = \xi(B(X,Y)). \tag{23}$$

We now suppose that $X \in \Gamma(TM)$ and $Y \in \Gamma(E)$. We first observe that the second term in the right-hand side of equation (22) vanishes. Indeed, if (e_3, e_4) stands for an orthonormal basis of E, for all $i, j \in \{3, 4\}$ we have

$$\Re e\ \langle e_i \cdot \nabla_X \varphi, e_j \cdot \varphi \rangle = -\Re e\ \langle \nabla_X \varphi, e_i \cdot e_j \cdot \varphi \rangle = -\Re e\ \langle \eta(X) \cdot \varphi, e_i \cdot e_j \cdot \varphi \rangle,$$

which is a sum of terms of the form $\Re e\ \langle e \cdot \varphi, e' \cdot \varphi \rangle$ with e and e' belonging to TM and E respectively; these terms are therefore all equal to zero. Thus, (22) simplifies to

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re e\ \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle = \langle \xi(\nabla_X Y), \xi(\nu) \rangle.$$

Hence

$$(d(\xi(Y))(X))^N = \xi(\nabla_X Y). \tag{24}$$

Equations (23) and (24) mean that $\Phi_E = \xi$ preserves the second fundamental form and the normal connection respectively. □

Remark 3. *The immersion $F : M \to \mathbb{R}^4$ given by the fundamental theorem is thus*

$$F = \int \xi = \left(\int \xi_1, \int \xi_2, \int \xi_3, \int \xi_4 \right).$$

This formula generalizes the classical Weierstrass representation: let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the $\mathbb{C}-$linear forms defined by

$$\alpha_k(X) = \xi_k(X) - i\xi_k(JX),$$

for $k = 1, 2, 3, 4$, where J is the natural complex structure of M. Let z be a conformal parameter of M, and let $\psi_1, \psi_2, \psi_3, \psi_4 : M \to \mathbb{C}$ be such that

$$\alpha_1 = \psi_1 dz,\ \ \alpha_2 = \psi_2 dz,\ \ \alpha_3 = \psi_3 dz,\ \ \alpha_4 = \psi_4 dz.$$

By an easy computation using $D\varphi = \vec{H} \cdot \varphi$, we see that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are holomorphic forms if and only if M is a minimal surface ($\vec{H} = \vec{0}$). Then if M is minimal,

$$\begin{aligned} F &= Re\left(\int \alpha_1, \int \alpha_2, \int \alpha_3, \int \alpha_4 \right) \\ &= Re\left(\int \psi_1 dz, \int \psi_2 dz, \int \psi_3 dz, \int \psi_4 dz \right) \end{aligned}$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are holomorphic functions. This is the Weierstrass representation of minimal surfaces.

Remark 4. *Theorem 2 also gives a spinorial proof of the fundamental theorem. We may integrate the Gauss, Ricci and Codazzi equations in two steps:*

1- first solving

$$\nabla_X \varphi = \eta(X) \cdot \varphi, \tag{25}$$

where

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(e_j, X)$$

(there is a unique solution in $\Gamma(\Sigma)$*, up to the natural right-action of* $Spin(4)$*);*

2- then solving

$$dF = \xi$$

where $\xi(X) = \langle\langle X \cdot \varphi^-, \varphi^+ \rangle\rangle$ *(the solution is unique, up to translations).*

Indeed, equation (25) is solvable, since its conditions of integrability are exactly the Gauss, Ricci and Codazzi equations; see the proof of Theorem 1. Moreover, the multiplication of φ *on the right by a constant belonging to* $Spin(4)$ *in the first step, and the addition to* F *of a constant belonging to* $\mathbb{R}^4$ *in the second step, correspond to a rigid motion in* $\mathbb{R}^4$.

6 Surfaces in $\mathbb{R}^3$ and S^3.

The aim of this section is to obtain as particular cases the spinor characterizations of T. Friedrich [5] and B. Morel [13] of surfaces in $\mathbb{R}^3$ and S^3. Assume that $M^2 \subset \mathcal{H}^3 \subset \mathbb{R}^4$, where $\mathcal{H}^3$ is a hyperplane, or a sphere of $\mathbb{R}^4$. Let N be a unit vector field such that

$$T\mathcal{H} = TM \oplus_{\perp} \mathbb{R}N.$$

The intrinsic spinors of M identify with the spinors of $\mathcal{H}$ restricted to M, which in turn identify with the positive spinors of $\mathbb{R}^4$ restricted to M :

Proposition 6.1. *There is an identification*

$$\begin{array}{ccc} \Sigma M & \overset{\sim}{\rightarrow} & \Sigma^+_{|M} \\ \psi & \mapsto & \psi^* \end{array}$$

such that

$$(\nabla\psi)^* = \nabla(\psi^*)$$

and such that the Clifford actions are linked by

$$(X \cdot_{{}_M} \psi)^* = N \cdot X \cdot \psi^*$$

for all $X \in TM$ *and all* $\psi \in \Sigma M$.

Using this identification, the intrinsic Dirac operator on M defined by

$$D_M\psi := e_1 \cdot_{{}_M} \nabla_{e_1}\psi + e_2 \cdot_{{}_M} \nabla_{e_2}\psi$$

is linked to D by

$$(D_M\psi)^* = N \cdot D\psi^*.$$

If $\varphi \in \Gamma(\Sigma)$ is a solution of

$$D\varphi = \vec{H} \cdot \varphi \qquad \text{and} \qquad |\varphi^+| = |\varphi^-| = 1$$

then $\varphi^+ \in \Sigma^+$ may be considered as belonging to ΣM; it satisfies

$$D_M\varphi^+ = N \cdot D\varphi^+ = N \cdot \vec{H} \cdot \varphi^+. \qquad (26)$$

We examine separately the case of a surface in a hyperplane, and in a 3-dimensional sphere:

1. If $\mathcal{H}$ is a hyperplane, then $\vec{H}$ is of the form HN, and (26) reads

$$D_M\varphi^+ = -H\varphi^+. \tag{27}$$

This is the equation considered by T. Friedrich in [5].

2. If $\mathcal{H} = S^3$, then $\vec{H}$ is of the form $HN - \nu$, where ν is the outer unit normal of S^3, and (26) reads

$$D_M\varphi^+ = -H\varphi^+ - i\overline{\varphi^+}. \tag{28}$$

This equation is obtained by B. Morel in [13].

Conversely, we now suppose that ψ is an intrinsic spinor field on M solution of (27) or (28). The aim is to construct a spinor field φ in dimension 4 which induces an immersion in a hyperplane, or in a 3-sphere. Define $E = M\times\mathbb{R}^2$, with its natural metric $\langle .,.\rangle$ and its trivial connection ∇', and consider $\nu, N \in \Gamma(E)$ such that

$$|\nu| = |N| = 1, \qquad \langle\nu, N\rangle = 0 \qquad \text{and} \qquad \nabla'\nu = \nabla'N = 0.$$

We first consider the case of an hyperplane:

Proposition 6.2. *Let $\psi \in \Gamma(\Sigma M)$ be a solution of*

$$D_M\psi = -H\psi$$

of constant length $|\psi| = 1$. There exists $\varphi \in \Gamma(\Sigma)$ solution of

$$D\varphi = \vec{H}\cdot\varphi \qquad \text{and} \qquad |\varphi^+| = |\varphi^-| = 1, \tag{29}$$

with $\vec{H} = HN$, such that

$$\varphi^+ = \psi$$

and the normal vector field

$$\xi(\nu) = \langle\langle \nu\cdot\varphi^-, \varphi^+\rangle\rangle$$

has a fixed direction in $\mathbb{H}$. In particular, the immersion given by φ belongs to the hyperplane $\xi(\nu)^\perp$ of $\mathbb{H}$. The spinor field φ is unique, up to the natural right-action of S^3 on φ^-.

Proof: define $\varphi = (\varphi^+, \varphi^-)$ by

$$\varphi^+ = \psi, \qquad \varphi^- = -\nu\cdot\psi.$$

We compute:

$$D\varphi^- = \nu\cdot D\varphi^+ = \nu\cdot\vec{H}\cdot\varphi^+ = \vec{H}\cdot\varphi^-,$$
$$\xi(\nu) = \langle\langle \nu\cdot\varphi^-, \varphi^+\rangle\rangle = 1,$$

and, for all $X \in TM$,

$$\xi(X) = \langle\langle X\cdot\varphi^-, \varphi^+\rangle\rangle = -\langle\langle X\cdot\nu\cdot\psi, \psi\rangle\rangle = \langle\langle \psi, X\cdot\nu\cdot\psi\rangle\rangle = \overline{\langle\langle X\cdot\nu\cdot\psi, \psi\rangle\rangle} = -\overline{\xi(X)},$$

that is $\xi(X) \in \Im m(\mathbb{H})$, the hyperplane of pure imaginary quaternions. Thus $F = \int \xi$ also belongs to the hyperplane $\Im m(\mathbb{H})$. Uniqueness is straightforward. $\Box$

We now consider the case of the 3-sphere:

Proposition 6.3. *Let $\psi \in \Gamma(\Sigma M)$ be a solution of*

$$D_M\psi \;=\; -H\psi - i\overline{\psi}$$

of constant length $|\psi| = 1$. There exists $\varphi \in \Gamma(\Sigma)$ solution of

$$D\varphi = \vec{H} \cdot \varphi \qquad \text{and} \qquad |\varphi^+| = |\varphi^-| = 1, \tag{30}$$

with $\vec{H} = HN - \nu$, such that

$$\varphi^+ = \psi$$

and the immersion F defined by φ is given by the unit normal vector field $\xi(\nu)$:

$$F = \xi(\nu) = \langle\langle \nu \cdot \varphi^-, \varphi^+ \rangle\rangle.$$

In particular $F(M)$ belongs to the sphere $S^3 \subset \mathbb{H}$. The spinor field φ is unique, up to the natural right-action of S^3 on φ^-.

Proof: The system

$$\begin{cases} F = \langle\langle \nu \cdot \varphi^-, \varphi^+ \rangle\rangle \\ dF(X) = \langle\langle X \cdot \varphi^-, \varphi^+ \rangle\rangle \end{cases}$$

is equivalent to

$$\varphi^- = -\nu \cdot \varphi^+ \cdot F$$

where $F : M \to \mathbb{H}$ solves the equation

$$dF(X) = \beta(X)F \tag{31}$$

in $\mathbb{H}$, with

$$\beta(X) = -\langle\langle X \cdot \nu \cdot \varphi^+, \varphi^+ \rangle\rangle.$$

By a direct computation, the compatibility equation

$$d\beta(X,Y) = \beta(X)\beta(Y) - \beta(Y)\beta(X)$$

of (31) is satisfied, and equation (31) is solvable. Uniqueness is straightforward. $\Box$

Remark 5. *Let M be a minimal surface in S^3 and N be such that*

$$TM \oplus_\perp \mathbb{R}N = TS^3.$$

For any $x \in S^3$, denote by $\vec{x} = \overrightarrow{0x}$ the position vector of x. At $x \in M$, $\vec{H} = -\vec{x}$. Thus, $M \subset S^3$ is represented by a solution $\varphi \in \Gamma(\Sigma)$ of

$$D\varphi = -\vec{x} \cdot \varphi.$$

The spinor field

$$\tilde{\varphi} := (\varphi^+, N \cdot \varphi^+)$$

defines a surface of constant mean curvature $H = -1$ in $\Im\mathrm{m}(\mathbb{H}) \simeq \mathbb{R}^3$. This is a classical transformation, described by B. Lawson in [12], and by T. Friedrich using spinors in dimension 3 in [5].

Remark 6. *In [13] B. Morel also obtained a spinor characterization of an isometric immersion of a surface in three-dimensional hyperbolic space $\mathbb{H}^3$. Very similarly to the case of a surface in $\mathbb{R}^3$ or in S^3 described above, such a characterization may be naturally obtained from a spinor characterization of a spacelike surface in four-dimensional Minkowski space; see [3].*

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