f-BIHARMONIC SUBMANIFOLDS OF GENERALIZED SPACE FORMS

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ABSTRACT. We study f-biharmonic submanifolds in both generalized complex and Sasakian space forms. We prove necessary and sufficient conditions for f-biharmonicity in the general case and many particular cases. Some geometric estimates as well as non-existence results are also obtained.

1. Introduction

Harmonic maps between two Riemannian manifolds (M^m, g) and (N^n, h) are critical points of the energy functional

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dv_g,$$

where ψ is a map from M to N and dv_g denotes the volume element of g. The Euler-Lagrange equation associated with $E(\psi)$ is given by $\tau(\psi) = 0$, where $\tau(\psi) = Trace\nabla d\psi$ is the tension field of ψ, which vanishes precisely for harmonic maps.

In 1983, Eells and Lemaire [11] suggested to consider biharmonic maps which are a natural generalization of harmonic maps. A map ψ is called *biharmonic* if it is a critical point of the bi-energy functional

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 dv_g,$$

on the space of smooth maps between two Riemannian manifolds. In [16], Jiang studied the first and second variation formulas of E_2 for which critical points are biharmonic maps. The Euler-Lagrange equation associated with this bi-energy functional is $\tau_2(\psi) = 0$, where $\tau_2(\psi)$ is the so-called bi-tension field given by

$$\tau_2(\psi) = \Delta\tau(\psi) - \mathrm{tr}\big(R^N(d\psi, \tau(\psi))d\psi\big). \tag{1}$$

Here, Δ is the rough Laplacian acting on the sections of $\psi^{-1}(TN)$ given by $\Delta V = \mathrm{tr}(\nabla^2 V)$ for any $V \in \Gamma(\psi^{-1}(TN))$ and R^N is the curvature tensor of the target manifold N defined as $R^N(X,Y) = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$ for any $X, Y \in \Gamma(TN)$.

Over the past years, many geometers studied biharmonic submanifolds and obtained a great variety of results in this domain (see [5, 12, 13, 14, 24, 28, 29], for instance). If the map $\psi : (M, g) \to (N, h)$ is an isometric immersion from a manifold (M, g) into an ambient manifold (N, h) then M is called *biharmonic submanifold* of N. Since it is obvious that any harmonic map

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is a biharmonic map, we will call *proper biharmonic submanifolds* the biharmonic submanifolds which are not harmonic, that is, minimal.

The main problem concerning biharmonic submanifold is the Chen Conjecture [7]:

"the only biharmonic submanifolds of Euclidean spaces are the minimal ones."

The Chen biharmonic conjecture is still an open problem, but lots of results on submanifolds of Euclidean spaces provide affirmative partial solutions to the conjecture (see [6, 8] and references therein for an overview). On the other hand, the generalized Chen's conjecture replacing Euclidean spaces by Riemannian manifolds of non-positive sectional curvature turns out to be false (see [19, 23] for counter-examples). Nevertheless, this generalized conjecture is true in various situations and obtaining non-existence results in non-positive sectional curvature is still an interesting question. In [29], the authors gave two new contexts where such results hold.

In [20], Lu gave a natural generalization of biharmonic maps and introduced f-biharmonic maps. He studied the first variation and calculated the f-biharmonic map equation as well as the equation for the f-biharmonic conformal maps between the same dimensional manfolds. Ou also studied f-biharmonic map and f-biharmonic submanifolds in [25], where he proved that an f-biharmonic map from a compact Riemannian manifold into a non-positively curved manifold with constant f-bienergy density is a harmonic map; any f-biharmonic function on a compact manifold is constant, and that the inversion about S^m for $m \geq 3$ are proper f-biharmonic conformal diffeomorphisms. He also derived f-biharmonic submanifolds equation and proved that a surface in a manifold (N^n, h) is an f-biharmonic surface if and only if it can be biharmonically conformally immersed into (N^n, h). Further in [26], the author characterizes harmonic maps and minimal submanifolds by using the concept of f-biharmonic maps and obtained an improved equation for f-biharmonic hypersurfaces.

By definition, for a positive, well defined and C^∞ differentiable function $f : M \to R$, *f-biharmonic* maps are critical points of the f-bienergy functional for maps $\psi : (M, g) \to (N, h)$, between Riemannian manifolds, i.e.,

$$E_{2,f}(\psi) = \frac{1}{2}\int_M f|\tau(\psi)|^2 dv_g.$$

Lu also obtained the corresponding Euler-Lagrange equation for f-biharmonic maps, i.e.,

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + (\Delta f)\tau(\psi) + 2\nabla^{\psi}_{\mathrm{grad} f}\tau(\psi) = 0. \tag{2}$$

An f-biharmonic map is called a *proper f-biharmonic map* if it is neither a harmonic nor a biharmonic map. Also, we will call *proper f-biharmonic submanifolds* a f-biharmonic submanifols which is neither minimal nor biharmonic.

Very recently, Karaca and Özgür [17] studied f-biharmonic submanifolds in products sapce and extended the results obtained by the first author [28] for biharmonic submanifolds. In the present paper, we continue to explore f-biharmonic submanifolds. Precisely, we focus here on f-biharmonic submanifolds of both (generalized) complex space forms and generalized Sasakian space forms. After a section of basics about generalized complex and Sasakian space forms as well as their submanifolds. we study of f-biharmonic submanifolds. For both classes of ambient spaces, we first give the general necessary and sufficient condition for submanifolds to be f-biharmonic. Then, we focus of many particular cases and obtain some non-existence results for spaces with holomorphic (or ϕ-holomorphic) sectional curvature bounded from above. Finally, the last section is devoted to the study of Legendre curves in generalized Sasakian space forms.

2. **Preliminaries**

2.1. **Generalized complex space forms and their submanifolds.** A Hermitian manifold (N, g, J) with constant sectional holomorphic curvature $4c$ is called a complex space form. We denote by $M^n_{\mathbb{C}}(4c)$ be the simply connected complex n-dimensional complex space form of constant holomorphic sectional curvature $4c$. The curvature tensor R of $M^n_{\mathbb{C}}(4c)$ is given by

$$R^{\mathbb{C}}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX - g(Z,JX)JY + 2g(X,JY)JZ\},$$

for $X, Y, Z \in \Gamma(TM^n_{\mathbb{C}}(4c))$, where $< \cdot, \cdot >$ is the Riemannian metric on $M^n_{\mathbb{C}}(4c)$ and J is the canonical almost complex structure of $M^n_{\mathbb{C}}(4c)$. The complex space from $M^n_{\mathbb{C}}(4c)$ is the complex projective space $\mathbb{C}P^n(4c)$, the complex Euclidean space $\mathbb{C}^n$ or the complex hyperbolic space $\mathbb{C}H^n(4c)$ according to $c > 0$, $c = 0$ or $c < 0$.

Now, we consider a natural generalization of complex space forms, namely the generalized complex space forms. After defining them, we will give some basic information about generalized complex space forms and their submanifolds. Generalized complex space forms form a particular class of Hermitian manifolds which has not been intensively studied. In 1981, Tricerri and Vanhecke [30] introduced the following generalization of the complex space forms ($\mathbb{C}^n$, $\mathbb{C}P^n$ and $\mathbb{C}H^n$). Let (N^{2n}, g, J) be an almost Hermitian manifold. We denote the generalized curvature tensors by R_1 and R_2 which is defined as

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g(JY,Z)JX - g(JX,Z)JY + 2g(JY,X)JZ, \quad \forall \;\; X,Y,Z \in \Gamma(TN).$$

The manifold (N, g, J) is called *generalized complex space form* if its curvature tensor R has the following form

$$R = \alpha R_1 + \beta R_2,$$

where α and β are smooth functions on N. The terminology comes obviously from the fact that complex space forms satisfy this property with constants $\alpha = \beta$.
In the same paper [30], Tricerri and Vanhecke showed that if N is of (real) dimension $2n \geq 6$, then (N, g, J) is a complex space form. They also showed that $\alpha + \beta$ is necessarily constant. This implies that $\alpha = \beta$ are constants in dimension $2n \geq 6$, but this is not the case in dimension 4. Hence, the notion of generalized complex space form is of interest only in dimension 4. Further, Olszak [22] constructed examples in dimension 4 with α and β non-constant. These examples are obtained by conformal deformation of Bochner flat Kählerian manifolds of non constant scalar curvature. Examples of Bochner flat Kählerian manifolds can be found in [10]. From now on, we will denote by $N(\alpha, \beta)$ a (4-dimensional) generalized complex space form with curvature given by $R = \alpha R_1 + \beta R_2$. Note that these spaces are Einstein, with constant scalar curvature equal to $12(\alpha + \beta)$. Of course, they are not Kählerian because if they were, they would be complex space forms.

Now, let M be a submanifold of the (generalized) complex space form $M^n_{\mathbb{C}}(4c)$ or $N(\alpha, \beta)$. The almost complex structure J on $M^n_{\mathbb{C}}(4c)$ (or $N(\alpha, \beta)$) induces the existence of four operators on

M, namely

$$j : TM \longrightarrow TM,\ k : TM \longrightarrow NM, l : NM \longrightarrow TM \text{ and } m : NM \longrightarrow NM,$$

defined for all $X \in TM$ and all $\xi \in NM$ by

$$JX = jX + kX \quad \text{and} \quad J\xi = l\xi + m\xi. \tag{3}$$

Since J is an almost complex structure, it satisfies $J^2 = -Id$ and for X, Y tangent to $M^n_{\mathbb{C}}(4c)$ (or $N(\alpha, \beta)$), we have $g(JX, Y) = -g(X, JY)$. Then, we deduce that the operators j, k, l, m satisfy the following relations

$$j^2X + lkX = -X, \tag{4}$$

$$m^2\xi + kl\xi = -\xi, \tag{5}$$

$$jl\xi + lm\xi = 0, \tag{6}$$

$$kjX + mkX = 0, \tag{7}$$

$$g(kX, \xi) = -g(X, l\xi), \tag{8}$$

for all $X \in \Gamma(TM)$ and all $\xi \in \Gamma(NM)$. Moreover j and m are skew-symmetric.

2.2. **Generalized Sasakian space forms and their submanifolds.** Now, we give some recalls about almost contact metric manifolds and generalized Sasakian space forms. For more details, one can refer to ([1, 4, 31]) for instance. A Riemannian manifold $\widetilde{M}$ of odd dimension is said almost contact if there exists globally over $\widetilde{M}$, a vector field ξ, a 1-form η and a field of $(1, 1)$-tensor ϕ satisfying the following conditions:

$$\eta(\xi) = 1 \quad \text{and} \quad \phi^2 = -Id + \eta \otimes \xi. \tag{9}$$

Remark that this implies $\phi\xi = 0$ and $\eta \circ \phi = 0$. The manifold $\widetilde{M}$ can be endowed with a Riemannian metric $\widetilde{g}$ satisfying

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \eta(X)\eta(Y) \quad \text{and} \quad \eta(X) = \widetilde{g}(X, \xi), \tag{10}$$

for any vector fields X, Y tangent to $\widetilde{M}$. Then, we say that $(\widetilde{M}, \widetilde{g}, \xi, \eta, \phi)$ is an almost contact metric manifold. Three class of this family are of particular interest, namely, the Sasakian, Kenmotsu and cosymplectic manifolds. We will give some recalls about them.

First, we introduce the fundamental 2-form (also called Sasaki 2-form) Ω defined for $X, Y \in \Gamma(TM)$ by

$$\Omega(X, Y) = \widetilde{g}(X, \phi Y).$$

We consider also N_ϕ, the Nijenhuis tensor defined by

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y],$$

for any vector fields X, Y. An almost contact metric manifold is said normal if and only if the Nijenhuis tensor N_ϕ satisfies

$$N_\phi + 2d\eta \otimes \xi = 0.$$

An almost contact metric manifold is said *Sasakian manifold* if and only if it is normal and $d\eta = \Omega$. This is equivalent to

$$(\nabla_X \phi)Y = \widetilde{g}(X, Y)\xi - \eta(Y)X,\ \ \forall\ \ X, Y \in \Gamma(\widetilde{M}). \tag{11}$$

It also implies that

$$\nabla_X \xi = -\phi(X). \tag{12}$$

An almost contact metric manifold is said *Kenmotsu manifold* if and only if $d\eta = 0$ and $d\Omega = 2\eta \wedge \Omega$. Equivalently, this means

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{13}$$

for any X and Y. Hence, we also have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{14}$$

Finally, an almost contact metric manifold is said *cosymplectic manifold* if and only if $d\eta = 0$ and $d\Omega = 0$, or equivalently

$$\nabla \phi = 0, \tag{15}$$

and in this case, we have

$$\nabla \xi = 0. \tag{16}$$

The ϕ-sectional curvature of an almost contact metric manifold is defined as the sectional curvature on the 2-planes $\{X, \phi X\}$. When the ϕ-sectional curvature is constant, we say that the manifold is a space form (Sasakian, Kenmotsu or cosymplectic in each of the three cases above). It is well known that the ϕ-sectional curvature determines entirely the curvature of the manifold. When the ϕ-sectional curvature is constant, the curvature tensor is expressed explicitely. Let $R_1^\star$, $R_2^\star$ and $R_3^\star$ be the generalized curvature tensors defined by

$$R_1^\star(X, Y)Z = \widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y, \tag{17}$$

$$R_2^\star(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X, Z)\eta(Y)\xi - \widetilde{g}(Y, Z)\eta(X)\xi \tag{18}$$

and

$$R_3^\star(X, Y)Z = \Omega(Z, Y)\phi X - \Omega(Z, X)\phi Y + 2\Omega(X, Y)\phi Z. \tag{19}$$

For the three cases we are interested in, the curvature of a space form of constant ϕ-sectional curvature c is given by

- Sasaki: $R^\star = \frac{c+3}{4}R_1^\star + \frac{c-1}{4}R_2^\star + \frac{c-1}{4}R_3^\star$.
- Kenmotsu: $R^\star = \frac{c-3}{4}R_1^\star + \frac{c+1}{4}R_2^\star + \frac{c+1}{4}R_3^\star$.
- Cosymplectic: $R^\star = \frac{c}{4}R_1^\star + \frac{c}{4}R_2^\star + \frac{c}{4}R_3^\star$.

In the sequel, for more clarity, we will denote the Sasakian (*resp.* Kenmotsu, cosymplectic) space form of constant ϕ-sectional curvature c by $\widetilde{M_S}(c)$ (*resp.* $\widetilde{M_K}(c)$, $\widetilde{M_C}(c)$). These space forms appear as particular cases of the so-called generalized Sasakian space forms, introduced by Alegre, Blair and Carriazo in [1]. A generalized Sasakian space form, denoted by $\widetilde{M}(f_1, f_2, f_3)$, is a contact metric manifold with curvature tensor of the form

$$f_1 R_1^\star + f_2 R_2^\star + f_3 R_3^\star, \tag{20}$$

where f_1, f_2 and f_3 are real functions on the manifold. The most simple examples of generalized Sasakian space forms are the warped products of the real line by a complex space form or a generalized complex space forms. Their conformal deformations as well as their so-called $\mathcal{D}$-homothetic deformations are also generalized Sasakian space forms (see [1]). Other examples can be found in [2].

Now, let (M,g) be a submanifold of an almost contact metric manifold $(\widetilde{M},\widetilde{g},\xi,\eta,\phi)$. The field of tensors ϕ induces on M, the existence of the following four operators:

$$P:TM\longrightarrow TM,\ N:TM\longrightarrow NM,\ t:NM\longrightarrow TM \text{ and } s:NM\longrightarrow NM,$$

defined for any $X\in TM$ and $\nu\in NM$. Now, we have

$$\phi X=PX+NX \quad \text{and} \quad \phi\nu=s\nu+t\nu, \tag{21}$$

where PX and NX are tangential and normal components of ϕX, respectively, whereas $t\nu$ and $s\nu$ are the tangential and normal components of $\phi\nu$, respectively. A submanifold M is said invariant (*resp.* anti-invariant) if N (*resp.* P) vanishes identically. In [18], Lotta shows that if the vector field ξ is normal to M, then M is anti-invariant.

3. f-Biharmonic submanifolds of generalized complex space forms

At first, we will calculate necessary and sufficient condition of f-biharmonic submanifold of generalized complex space forms and then we make a exposition about the results which could characterize these type of submanifolds.

Theorem 3.1. *Let M^p, $p<4$ be a submanifold of the generalized complex space form $N(\alpha,\beta)$ with second fundamental form B, shape operator A, mean curvature H and a positive C^∞-differentiable function f on M. Then M is f-biharmonic submanifold of $N(\alpha,\beta)$ if and only if the following two equations are satisfied*

(1)
$$-\Delta^\perp H+\mathrm{tr}\left(B(\cdot,A_H\cdot)\right)-p\alpha H+3\beta klH+\frac{\Delta f}{f}H+2\nabla^\perp_{\mathrm{grad}(\ln f)}H=0,$$

(2)
$$\frac{p}{2}\mathrm{grad}|H|^2-2A_H\mathrm{grad}(\ln f)+2\mathrm{tr}\left(A_{\nabla^\perp H}(\cdot)\right)+6\beta jlH=0.$$

Proof: It is a classic fact that the tension field of the isometric immersion ψ is given by

$$\tau(\psi)=\mathrm{tr}\nabla d\psi=\mathrm{tr}B=pH. \tag{22}$$

Using equation (22) in equation (1), we have

$$\tau_2(\psi)=p\Delta H-\mathrm{tr}\big(R^N(d\psi,pH)d\psi\big). \tag{23}$$

Moreover, we recall that, by some classical and straightforward computations, we have

$$\Delta H=\frac{p}{2}\mathrm{grad}|H|^2+\mathrm{tr}\left(B(\cdot,A_H\cdot)\right)+2\mathrm{tr}\left(A_{\nabla^\perp H}(\cdot)\right)+\Delta^\perp H.$$

Reporting this into (23), we get

$$\tau_2(\psi)=-\Delta^\perp H+\mathrm{tr}\left(B(\cdot,A_H\cdot)\right)+\frac{p}{2}\mathrm{grad}|H|^2+2\mathrm{tr}\left(A_{\nabla^\perp H}(\cdot)\right)+2\mathrm{tr}\left(R^N(\cdot,H)\cdot\right). \tag{24}$$

Now, the curvature tensor of generalized complex space form, $N(\alpha,\beta)$, is given by

$$\mathrm{tr}\left(R(\cdot,H)\cdot\right) \;=\; \alpha\mathrm{tr}\left(R_1(\cdot,H)\cdot\right)+\beta\mathrm{tr}\left(R_2(\cdot,H)\cdot\right).$$

Let $\{e_1,\cdots,e_p\}$ be a local orthonormal frame of TM. Then, we have

$$\mathrm{tr}\left(R(\cdot,H)\cdot\right)=\alpha\sum_{i=1}^p R_1(e_i,H)e_i+\beta\sum_{i=1}^p R_2(e_i,H)e_i$$

or,

$$\operatorname{tr}(R(\cdot,H)\cdot)=\alpha\sum_{i=1}^{p}\left[g(H,e_i)e_i-g(e_i,e_i)H\right]$$
$$+\beta\sum_{i=1}^{p}\left[g(JH,e_i)Je_i-g(Je_i,e_i)JH+2g(JH,e_i)Je_i\right].$$

or,

$$\operatorname{tr}(R(\cdot,H)\cdot)=\alpha(-pH)+\beta(3jlH+3klH). \tag{25}$$

From equation (2), M is f-biharmonic if and only if

$$f\tau_2(\psi)+\Delta f\tau(\psi)+2\nabla^{\psi}_{\operatorname{grad}f}\tau(\psi)=0,$$

which is equivalent to

$$\tau_2(\psi)+p\frac{\Delta f}{f}H+2p(-A_H\operatorname{grad}(\ln f)+\nabla^{\perp}_{\operatorname{grad}(\ln f)}H)=0. \tag{26}$$

Now, using equations (24) and (25) in equation (26) and considering that jlH is tangent and klH is normal, we get the statement of the theorem by identification of tangent and normal parts. □

We can easily obtain by the same computations an analogous result for f-biharmonic submanifolds of complex space forms $M^n_{\mathbb{C}}(4c)$. Namely, we have

Corollary 3.2. *Let M^p, $p\leqslant 2n$, be a submanifold of the complex space form $M^n_{\mathbb{C}}(4c)$ of complex dimension n and constant holomorphic sectional curvature $4c$, with second fundamental form B, shape operator A, mean curvature H and a positive C^∞-differentiable function f on M. Then M is f-biharmonic submanifold of $M^n_{\mathbb{C}}(4c)$ if and only if the following two equations are satisfied*

(1)
$$-\Delta^{\perp}H+\operatorname{tr}(B(\cdot,A_H\cdot))-pcH+3cklH+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\operatorname{grad}(\ln f)}H=0,$$

(2)
$$\frac{p}{2}\operatorname{grad}|H|^2-2A_H\operatorname{grad}(\ln f)+2\operatorname{tr}(A_{\nabla^{\perp}H}(\cdot))+6cjlH=0.$$

Proof: For complex space forms the computations are essentially the same as for the generalized complex space forms with the only differences that $\alpha=\beta=c$ and dimension is not necessarily equal to 4. □

In the sequel, we will state many results for biharmonic subamnifolds of the generalized complex space forms $N(\alpha,\beta)$. They have of course analogue for the complex space forms but for a sake of briefness, we do not write then since the results are the same with $\alpha=\beta=c$. Assuming particular cases such as hypersurfaces, Lagrangian or complex surfaces and curves of generalized complex space form $N(\alpha,\beta)$, we have the following conclusion.

Corollary 3.3. *Let M^p, $p<4$ be a submanifold of the generalized complex space form $N(\alpha,\beta)$ with second fundamental form B, shape operator A, mean curvature H and a positive C^∞-differentiable function f on M.*

(1) *If M is a hypersurface then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H+\mathrm{tr}\,(B(\cdot,A_H\cdot))-3(\alpha+\beta)H=0,$$

and

$$\frac{3}{2}\mathrm{grad}|H|^2-2A_H grad(\ln f)+2\mathrm{tr}\,(A_{\nabla^{\perp}H}(\cdot))=0.$$

(2) *If M is a complex surface then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H+\mathrm{tr}\,(B(\cdot,A_H\cdot))-2\alpha H=0,$$

and

$$\mathrm{grad}|H|^2-2A_H\mathrm{grad}(\ln f)+2\mathrm{tr}\,(A_{\nabla^{\perp}H}(\cdot))=0.$$

(3) *If M is a Lagrangian surface then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H+\mathrm{tr}\,(B(\cdot,A_H\cdot))-2\alpha H-3\beta H=0,$$

and

$$\mathrm{grad}|H|^2-2A_H\mathrm{grad}(\ln f)+2\mathrm{tr}\,(A_{\nabla^{\perp}H}(\cdot))=0.$$

(4) *If M is a curve then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H+\mathrm{tr}\,(B(\cdot,A_H\cdot))-\alpha H-3\beta(H+m^2H)=0,$$

and

$$\frac{1}{2}\mathrm{grad}|H|^2-2A_H\mathrm{grad}(\ln f)+2\mathrm{tr}\,(A_{\nabla^{\perp}H}(\cdot))=0.$$

Proof: The proof is a consequence of Theorem 3.1 using the facts that

(1) if M is a hypersurface, then $m=0$ and so $jlH=0$, $kjH=0$ and $klH=-H$,
(2) if M is a complex surface then $k=0$ and $l=0$,
(3) if M is a Lagrangian surface, then $j=0$, $m=0$,
(4) if M is a curve, then $j=0$.

□

Remark 3.4. *It is a well known fact that any complex submanifold of a Kähler manifold is necessarily minimal. But as mentioned above, the generalized space forms $N(\alpha,\beta)$ are not Kählerian unless there are the complex projective plane or the complex hyperbolic plane. Hence, considering f-biharmonic surfaces into $N(\alpha,\beta)$ is of real interest, since they are not necessarily minimal.*

Similarly, if we assume mean curvature vector H as parallel vector then for curves and complex or Lagranian surfaces, we obtain the following corollaries.

Corollary 3.5. *Let M^p, $p<4$ be a submanifold of the generalized complex space form $N(\alpha,\beta)$ with second fundamental form B, shape operator A, mean curvature H and a positive C^{∞}-differentiable function f on M.*

(1) *If M be a Lagrangian surface of $N(\alpha,\beta)$ with parallel mean curvature then M is f-biharmonic if and only if*

$$\mathrm{tr}\,(B(\cdot,A_H\cdot))=2\alpha H+3\beta H-\tfrac{\Delta f}{f}H,\ \ \textit{and}\ \ A_H\mathrm{grad}f=0.$$

(2) *If M be a complex surface of $N(\alpha,\beta)$ with parallel mean curvature then M is f-biharmonic if and only if*

$$\operatorname{tr}(B(\cdot, A_H\cdot)) = 2\alpha H - \tfrac{\Delta f}{f}H \quad \text{and} \quad A_H \operatorname{grad} f = 0.$$

(3) *If M is a curve in $N(\alpha,\beta)$ with parallel mean curvature then M is f-biharmonic if and only if*

$$\operatorname{tr}(B(\cdot, A_H\cdot)) = \alpha H + 3\beta(H + m^2 H) - \tfrac{\Delta f}{f}H, \quad \text{and} \quad A_H \operatorname{grad} f = 0.$$

Proof: Since M has parallel mean curvature so that the terms $\Delta^{\perp}H$, $\nabla^{\perp}_{\operatorname{grad} f}H$, $\operatorname{grad}|H|^2$ and $\operatorname{tr}(A_{\nabla^{\perp}_{\cdot}H}\cdot)$ vanish and we obtain immediately the result from the previous Corollary. □

Further, for constant mean curvature hypersurfaces in $N(\alpha,\beta)$, we have the following result.

Proposition 3.6. (1) *Let M^3 be a hypersurface of the generalized complex space form $N(\alpha,\beta)$ with second fundamental form B, non zero constant mean curvature H and f a positive C^{∞}-differentiable function on M. Then M is f biharmonic if and only if*

$$|B|^2 = 3(\alpha+\beta) - \frac{\Delta f}{f} \quad \text{and} \quad A \operatorname{grad} f = 0$$

or equivalently, M is proper f-biharmonic if and only if the scalar curvature of M satisfies

$$\operatorname{Scal}_M = 3(\alpha+\beta) + 9H^2 + \frac{\Delta f}{f} \quad \text{and} \quad A \operatorname{grad} f = 0.$$

(2) *There exists no proper f-biharmonic hypersurfaces with constant mean curvature and constant scalar curvature.*

Proof: For the first point, since M is a hypersurface, by Corollary 3.2, M is f-biharmonic if and only if

$$\begin{cases} -\Delta^{\perp}H + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\operatorname{grad}(\ln f)}H + \operatorname{tr}(B(\cdot, A_H\cdot)) - 3(\alpha+\beta)H = 0, \\ \frac{3}{2}\operatorname{grad}|H|^2 - 2A_H \operatorname{grad}(\ln f) + 2\operatorname{tr}(A_{\nabla^{\perp}H}(\cdot)) = 0. \end{cases}$$

Since M has constant mean curvature, the above equation reduces to

$$\begin{cases} \operatorname{tr}(B(\cdot, A_H\cdot)) = 3(\alpha+\beta)H - \frac{\Delta f}{f}H, \\ A_H \operatorname{grad}(\ln f) = 0. \end{cases}$$

Using condition $A_H = HA$ for hypersurfaces, we get

$$\operatorname{tr}\Big(B(\cdot, A_H(\cdot))\Big) = H\operatorname{tr}\Big(B(\cdot, A(\cdot))\Big) = H|B|^2.$$

Reporting this result in first equation of the above condition and from the assumption that H is a non-zero constant, we get the desired identity $|B|^2 = 3(\alpha+\beta) - \frac{\Delta f}{f}$.

For the second equivalence, by the Gauss equation, we have

$$\operatorname{Scal}_M = \sum_{i,j=1}^{3} g\left(R^N(e_i,e_j)e_j, e_i\right) - |B|^2 + 9H^2,$$

where $\{e_1,e_2,e_3\}$ is a local orthonormal frame of M. From the expression of the curvature tensor of $N(\alpha,\beta)$, we get

$$\operatorname{Scal}_M = 6(\alpha+\beta) - ||B||^2 + 9H^2.$$

Moreover, since $\mathrm{grad}(\ln f) = \frac{1}{f}\mathrm{grad}f$ and $A_H = HA$ with H is a non-zero constant, then $A_H\mathrm{grad}(\ln f) = 0$ reduces to $A\,\mathrm{grad}f = 0$.

Hence, we deduce that M is proper f-biharmonic if and only if $|B|^2 = 3(\alpha+\beta) - (\frac{\Delta f}{f})$ and $A\,\mathrm{grad}f = 0$, that is, if and only if $\mathrm{Scal}_M = 3(\alpha+\beta) + 9H^2 + \frac{\Delta f}{f}$ and $A\,\mathrm{grad}f = 0$.

Now, for the second point, if M is a hypersurface with constant mean curvature and constant scalar curvature, then by the first point, if M is f-biharmonic then

$$\mathrm{Scal}_M = 3(\alpha+\beta) + 9H^2 + \frac{\Delta f}{f}.$$

As we have already mentioned, $\alpha+\beta$ is constant, hence, since H and Scal_M are constant, then $\frac{\Delta f}{f}$ is constant, that is, f is an eigenvalue of the Laplacian. But f is a positive function, so the only possibility is that f is a positive constant and M is biharmonic. This concludes the proof of the second point. □

Now, we give this proposition which give an estimate of the mean curvature for a f-biharmonic Lagrangian surface.

Proposition 3.7. *Let M^2 be a Lagrangian surface of the generalized complex space form $N(\alpha,\beta)$ with second fundamental form B, shape operator A, non-zero constant mean curvature H and a positive C^∞-differentiable function f on M.*

(1) *If $\inf_M \left(2\alpha+3\beta-\frac{\Delta f}{f}\right)$ is non-positive then M is not f-biharmonic.*

(2) *If $\inf_M \left(2\alpha+3\beta-\frac{\Delta f}{f}\right)$ is positive and M is proper f-biharmonic then*

$$0 < |H|^2 \leqslant \inf_M \left(\frac{2\alpha+3\beta-\frac{\Delta f}{f}}{2}\right).$$

Proof: Assume that M is a f-biharmonic Lagrangian surface of $N(\alpha,\beta)$, considering third assertion of Corollary 3.2, we have

$$\begin{cases} -\Delta^\perp H + \frac{\Delta f}{f}H + \frac{2}{f}\nabla^\perp_{\mathrm{grad}f}H + \mathrm{tr}\left(B(\cdot, A_H\cdot)\right) - 2\alpha H - 3\beta H = 0, \\ \\ \mathrm{grad}|H|^2 - \frac{2}{f}A_H\mathrm{grad}f + 2\mathrm{tr}\left(A_{\nabla^\perp H}(\cdot)\right) = 0. \end{cases}$$

Hence, by taking the scalar product with H and taking the assumption that mean curvatutre $H \neq 0$, i.e., $|H|$ is constant, from the first part of the above equation, we have

$$- < \Delta^\perp H, H > + \frac{2}{f} < \nabla^\perp_{\mathrm{grad}f}H, H > + |A_H|^2 - \left(\frac{\Delta f}{f} - 2\alpha - 3\beta\right) < H, H >= 0.$$

This equation implies that

$$-\left\langle \Delta^\perp H, H\right\rangle = \left(2\alpha+3\beta-\frac{\Delta f}{f}\right)|H|^2 - |A_H|^2,$$

where we have used that $< \nabla^\perp_{\mathrm{grad}f}H, H >= 0$ since $|H|$ is constant. Now, with the help of the Bochner formula, we get

$$\left(2\alpha+3\beta-\frac{\Delta f}{f}\right)|H|^2 = |A_H|^2 + |\nabla^\perp H|^2.$$

Now, using Cauchy-Schwarz inequality, i.e., $|A_H|^2 \geqslant 2|H|^4$ in the above equation, we have

$$\left(2\alpha + 3\beta - \frac{\Delta f}{f}\right)|H|^2 \geqslant 2|H|^4 + |\nabla^\perp H|^2 \geqslant 2|H|^4. \tag{27}$$

So, we have $0 < |H|^2 \leqslant \inf_M \left(\frac{2\alpha+3\beta-\frac{\Delta f}{f}}{2}\right)$ because $|H|$ is a non-zero constant. This is only possible if the function $2\alpha + 3\beta - \frac{\Delta f}{f}$ has a positive infimum. This concludes the proof. □

Now, we have similar result for complex surfaces.

Proposition 3.8. *Let $\psi : M^2 \to N(\alpha, \beta)$ be a complex surface of generalized complex space form $N(\alpha, \beta)$ with second fundamental form B, shape operator A, mean curvature H and a positive C^∞-differentiable function f on M.*

(1) *If $\inf_M \left(2\alpha - \frac{\Delta f}{f}\right)$ is non-positive then M is not f-biharmonic.*
(2) *If $\inf_M \left(2\alpha - \frac{\Delta f}{f}\right)$ is positive and M is proper f-biharmonic then*

$$0 < |H|^2 \leqslant \inf_M \left(\frac{2\alpha - \frac{\Delta f}{f}}{2}\right).$$

Proof: Let M be a f-biharmonic complex surface of $N(\alpha, \beta)$ with non-zero constant mean curvature. Then, by the second assertion of Corollary 3.2, we have

$$-\Delta^\perp H + \tfrac{\Delta f}{f} H + \operatorname{tr}\left(B(\cdot, A_H \cdot)\right) - 2\alpha H = 0, \;\; \textit{and} \;\; A_H \operatorname{grad} f = 0.$$

Replacing $2\alpha + 3\beta$ by 2α in the proof of Proposition 3.7, we have the required result. □

Remark 3.9. *Note that we can obtain analogues of all the results of this section for submanifolds of the complex space forms $M^n_{\mathbb{C}}(4c)$ directly form Corollary 3.2. We do not write them here for briefness. However, there is no analogue for complex submanifolds since any complex submanifold of $M^n_{\mathbb{C}}(4c)$ in fact minimal.*

4. f-Biharmonic submanifolds of generalized Sasakian space forms

Now, we consider f-biharmonic submanifolds of generalized Sasakian space forms and give the following theorm for its characterization.

Theorem 4.1. *Let M^p be a submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$, with second fundamental form B, shape operator A, mean curvature H and a positive C^∞-differentiable function f on M. Then M is f-biharmonic submanifold of $\widetilde{M}(f_1, f_2, f_3)$ if and only if the following two equations are satisfied*

$$-\Delta^\perp H + \operatorname{tr} B(\cdot, A_H \cdot) + \frac{\Delta f}{f} H + 2\nabla^\perp_{\operatorname{grad}(\ln f)} H = pf_1 H - f_2|\xi^\top|^2 H - pf_2\eta(H)\xi^\perp - 3f_3 NsH$$

and

$$\frac{p}{2}\operatorname{grad}|H|^2 + 2\operatorname{tr} A_{\nabla^\perp H}(\cdot) - 2A_H \operatorname{grad}(\ln f) = -2f_2(p-1)\eta(H)\xi^\top - 6f_3 PsH.$$

Proof: At first, we calculate the curvature tensor of generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. From equation (20), we have

$$\begin{aligned} R^\star(X,Y)Z &= f_1R_1^\star(X,Y)Z + f_2R_1^\star(X,Y)Z + f_3R_2^\star(X,Y)Z \\ &= f_1\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\} \\ &+ f_2\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \tilde{g}(X,Z)\eta(Y)\xi - \tilde{g}(Y,Z)\eta(X)\xi\} \\ &+ f_3\{\tilde{g}(X,\phi Z)\phi Y - \tilde{g}(Y,\phi Z)\phi X + 2\tilde{g}(X,\phi Y)\phi Z\}. \end{aligned}$$

Let us consider $\{e_1, e_2, ..., e_p\}$ an orthogonal basis of the tangent space of M. Then, we have

$$\begin{aligned} R^\star(e_i,H)e_i &= f_1\{\tilde{g}(H,e_i)e_i - \tilde{g}(e_i,e_i)H\} + f_2\{\eta(e_i)\eta(e_i)H - \eta(H)\eta(e_i)e_i + \tilde{g}(e_i,e_i)\eta(H)\xi\} \\ &+ f_3\{\tilde{g}(e_i,\phi e_i)\phi H - \tilde{g}(H,\phi e_i)\phi e_i + 2\tilde{g}(e_i,\phi H)\phi e_i\}. \end{aligned}$$

Taking the trace and using (21) in the above equation, we get

$$\begin{aligned} tr\big(R^\star(\cdot,H)\cdot\big) &= -f_1pH + f_2\sum_i\{\eta(e_i)^2H - \eta(H)\eta(e_i)e_i + |e_i|^2\eta(H)\xi\} \\ &+ f_3\sum_i\{tr(P)\phi H - \tilde{g}(H,Ne_i)\phi e_i + 2\tilde{g}(e_i,sH)\phi e_i\} \\ &= -f_1pH + f_2\{|\xi^\top|^2H - \eta(H)\xi^\top + p\eta(H)\xi\} \\ &+ f_3\sum_i\{tr(P)sH + tr(P)tH - \tilde{g}(H,Ne_i)Pe_i - \tilde{g}(H,Ne_i)Ne_i \\ &+ 2\tilde{g}(e_i,sH)Pe_i + 2\tilde{g}(e_i,sH)Ne_i\}. \end{aligned}$$

It implies that

$$tr\big(R^\star(\cdot,H)\cdot\big) = -f_1pH + f_2\{|\xi^\top|^2H - \eta(H)\xi^\top + p\eta(H)\xi\} + 3f_3(PsH + NsH),$$

by considering the anti-symmetry property of ϕ, $tr(P) = 0$ and $\tilde{g}(H, Ne_i) = -\tilde{g}(tH, e_i)$.

Now, from value of $tr\big(R^\star(\cdot,H)\cdot\big)$ and equations (24), (26), we have result of the theorem by considering the tangential and normal parts. □

Now, we have the following corollary if we assume different particular cases in Theorem 4.1.

Corollary 4.2. *Let M^p be a submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.*

(1) *If M is invariant then M is f-biharmonic if and only if*

$$-\Delta^\perp H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^\perp_{\mathrm{grad}(\ln f)}H = pf_1H - f_2|\xi^\top|^2H - pf_2\eta(H)\xi^\perp$$

and

$$\frac{p}{2}\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^\perp H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = -2f_2(p-1)\eta(H)\xi^\top - 6f_3PsH.$$

(2) *If M is anti-invariant then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H = pf_1H - f_2|\xi^{\top}|^2H - pf_2\eta(H)\xi^{\perp} - 3f_3NsH$$

and

$$\frac{p}{2}\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = -2f_2(p-1)\eta(H)\xi^{\top}.$$

(3) *If ξ is normal to M then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H = pf_1H - pf_2\eta(H)\xi - 3f_3NsH$$

and

$$\frac{p}{2}\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = 0.$$

(4) *If ξ is tangent to M then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H = pf_1H - f_2H - 3f_3NsH$$

and

$$\frac{p}{2}\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = -6f_3PsH.$$

(5) *If M is a hypersurface then M is f-biharmonic if and only if*

$$-\Delta^{\perp}H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H = (2nf_1 + 3f_3)H - f_2|\xi^{\top}|^2H - (2nf_2 + 3f_3)\eta(H)\xi^{\perp}$$

and

$$n\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = -(2(2n-1)f_1 + 6f_3)\eta(H)\xi^{\top}.$$

Proof. The proof is a direct consequence of Theorem 4.1 using the following facts.

(1) If M is invariant then $P = 0$.
(2) If M is anti-invariant then $N = 0$.
(3) If ξ is normal then $\eta(\mathrm{grad} f) = 0$ and M is anti-invariant which implies $P = 0$.
(4) If ξ is tangent then $\eta(H) = 0$.
(5) If M is a hypersurface then $sH = 0$.

Analogously to the case of generalized complex space forms (Proposition 3.6), we can obtain some curvature properties in some special cases by using characterizations of f-biharmonic submanifolds of generalized Sasakian space forms.

Proposition 4.3. (1) *Let M^{2n} be a hypersurface of generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with non zero constant mean curvature H and ξ is tangent to M. Then M is proper f-biharmonic if and only if*

$$|B|^2 = 2nf_1 - f_2 + 3f_3 - \tfrac{\Delta f}{f}, \quad \textit{and} \quad A\,\mathrm{grad} f = 0,$$

or equivalently if and only if

$$\mathrm{Scal}_M = 2n(2n-2)f_1 + (4n-1)f_2 - (2n-4)f_3 + (2n-1)H^2 + \tfrac{\Delta f}{f}H \quad \textit{and} \quad A\,\mathrm{grad} f = 0.$$

(2) *There exists no proper f-biharmonic hypersurfaces with constant mean curvature and constant scalar curvature so that ξ is tangent.*

Proof. Let M be a f-biharmonic hypersurface of $\widetilde{M}(f_1, f_2, f_3)$ with non zero constant mean curvature and ξ tangent to M. Then, from Corollary 4.2, we have

$$\begin{cases} -\Delta^{\perp}H + \mathrm{tr}B(\cdot, A_H\cdot) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f))}H \\ \\ = (pf_1 + 3f_3)H - f_2|\xi^{\top}|^2H - (2nf_2 + 3f_3)\eta(H)\xi^{\perp}, \\ \\ n\mathrm{grad}|H|^2 + 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot) - 2A_H\mathrm{grad}(\ln f) = 0. \end{cases}$$

Now, as per assumption, ξ is tangent to M which gives $\eta(H) = \eta(\nu) = 0$. Therefore, we have

$$\phi^2\nu = -\nu + \eta(\nu)\xi = -\nu.$$

On the other hand, we have

$$\begin{aligned} \phi^2\nu &= \phi(s\nu + t\nu) \\ &= Ps\nu + Ns\nu + st\nu + t^2\nu. \end{aligned}$$

Hence, we get

$$-\nu = Ps\nu + Ns\nu + st\nu + t^2\nu. \tag{28}$$

Moreover, since $\langle \phi\nu, \nu\rangle = \Omega(\nu, \nu) = 0$, we have that $\phi\nu$ is tangent, i.e., $t\nu = 0$. Thus, Equation (28) becomes

$$-\nu = Ps\nu + Ns\nu,$$

and so $Ps = 0$ and $Ns = -\mathrm{Id}$ by identification of tangential and normal parts. Using these results in the above f-biharmonic condition for the hypersurfaces of generalized Sasakian space forms, we have

$$\begin{cases} \mathrm{tr}B(\cdot, A_H\cdot) = (2nf_1 + 3f_3)H - f_2|\xi^{\top}|^2H - \frac{\Delta f}{f}H, \\ \\ A_H\mathrm{grad}(\ln f) = 0. \end{cases}$$

Hence, the second equation is trivial and the first becomes

$$\mathrm{tr}B(\cdot, A_H\cdot) = 2nf_1H - f_2H + 3f_3H - \frac{\Delta f}{f}H,$$

or equivalently

$$|B|^2 = 2nf_1 - f_2 + 3f_3 - \frac{\Delta f}{f},$$

since $\mathrm{tr}B(\cdot, A_H\cdot) = |B|^2H$ and H is a non zero constant.

Similarly, using Gauss formula for second part, we have

$$\begin{aligned} Scal_M &= \sum_{i,j} \tilde{g}(R^{\star}(e_i, e_j)e_j, e_i) - |B|^2 - pH^2 \\ &= \sum_{i,j} f_1\{\tilde{g}(e_j, e_j)\tilde{g}(e_i, e_i) - \tilde{g}(e_i, e_j)\tilde{g}(e_j, e_i)\} + \sum_{i,j} f_2\{\eta(e_i)\eta(e_j)\tilde{g}(e_j, e_i) \\ &- \eta(e_j)\eta(e_j)\tilde{g}(e_i, e_i) + \tilde{g}(e_i, e_j)\eta(e_j)\tilde{g}(\xi, e_i) - \tilde{g}(e_j, e_j)\eta(e_i)\tilde{g}(\xi, e_i)\} \\ &+ \sum_{i,j} f_3\{\tilde{g}(e_i, \phi e_j)\tilde{g}(\phi e_j, e_i) - \tilde{g}(e_j, \phi e_j)\tilde{g}(\phi e_i, e_i) + 2\tilde{g}(e_i, \phi e_j)\tilde{g}(\phi e_j, e_i)\} \\ &- |B|^2 - pH^2 = 2n(2n-1)f_1 + 2(2n-1)f_2 - (2n-1)f_3 - |B|^2 - pH^2. \end{aligned}$$

Using the value of $|B|^2$ obtain in the first part of the proof, we get the required result, that is,

$$\mathrm{Scal}_M = 2n(2n-2)f_1 + (4n-1)f_2 - (2n-4)f_3 + (2n-1)H^2 + \frac{\Delta f}{f}H.$$

Moreover, since $\mathrm{grad}(\ln f) = \frac{1}{f}\mathrm{grad}f$ and $A_H = HA$ with H is a positive constant, the equation $A_H \mathrm{grad}(\ln f) = 0$ reduces to $A\,\mathrm{grad}f = 0$. This concludes the proof. □

Now, from this proposition, we can prove the following non-existence result.

Corollary 4.4. *Let M^{2n} be a constant mean curvature hypersurface of generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with ξ tangent. If the functions f_1, f_2, f_3 satisfy the inequality $2nf_1 - f_2 + 3f_3 \leqslant \frac{(\Delta f)}{f}$ on M then M is not biharmonic.*
In particular, there exists no proper f-biharmonic CMC hypersurface with ξ tangent and f satisfying

- $\tilde{c} \leqslant \frac{4}{2n+2}[\frac{\Delta f}{f} - \frac{6n-2}{4}]$ *in a Sasakian space form* $\widetilde{M}^{2n+1}_S(\tilde{c})$.
- $\tilde{c} \leqslant \frac{4}{2n+2}[\frac{\Delta f}{f} + \frac{6n-2}{4}]$ *in a Kenmotsu space form* $\widetilde{M}^{2n+1}_K(\tilde{c})$.
- $\tilde{c} \leqslant \frac{4}{2n+2}\frac{\Delta f}{f}$ *in a cosymplectic space form* $\widetilde{M}^{2n+1}_C(\tilde{c})$.

Proof: As per assumption, M is a hypersurface of $\widetilde{M}(f_1, f_2, f_3)$ with non zero constant mean curvature H and ξ tangent to M. From Proposition 4.3, M is f-biharmonic if and only if its second fundamental form B satisfies $|B|^2 = 2nf_1 - f_2 + 3f_3 - \frac{\Delta f}{f}$. In other words, this is not possible if

$$2nf_1 - f_2 + 3f_3 \leqslant \frac{\Delta f}{f}. \tag{29}$$

Now, $f_1 = \frac{\tilde{c}+3}{4}$ and $f_2 = f_3 = \frac{\tilde{c}-1}{4}$ if $\widetilde{M}(f_1, f_2, f_3)$ is a Sasakian space form where $\tilde{c}$ is ϕ-sectional curvature. Therefore, the inequality $2nf_1 - f_2 + 3f_3 \leqslant \frac{\Delta f}{f}$ reduces to $\tilde{c} \leqslant \frac{4}{2n+2}[\frac{\Delta f}{f} - \frac{6n-2}{4}]$. Similarly, we have $f_1 = \frac{\tilde{c}-3}{4}$ and $f_2 = f_3 = \frac{\tilde{c}+1}{4}$ (*resp.* $f_1 = f_2 = f_3 = \frac{\tilde{c}}{4}$) for the Kenmotsu (*resp.* cosymplectic) case and the inequality $2nf_1 - f_2 + 3f_3 \leqslant \frac{\Delta f}{f}$ reduces to $\tilde{c} \leqslant \frac{4}{2n+2}[\frac{\Delta f}{f} + \frac{6n-2}{4}]$ (*resp.* $\tilde{c} \leqslant \frac{4}{2n+2}\frac{\Delta f}{f}$). □

Now, we have the following proposition analogous to complex case.

Theorem 4.5. *Let M^q be a submanifold of Sasakian (Kenmotsu or cosymplectic) space form $\widetilde{M}^{2n+1}_S(\tilde{c})$ (resp. $\widetilde{M}^{2n+1}_K(\tilde{c})$ or $\widetilde{M}^{p+1}_C(\tilde{c})$) with constant mean curvature H so that ξ and ϕH are tangent. Further, we consider $F(f, q, \tilde{c})$ the function defined on M by*

$$F(f,q,\tilde{c}) = qf_1 - f_2 + 3f_3 - \frac{\Delta f}{f} = \begin{cases} \frac{(q+2)\tilde{c}}{4} + \frac{(3q-2)}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}^{p+1}_S(\tilde{c}), \\ \frac{(q+2)\tilde{c}}{4} - \frac{(3q-2)}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}^{p+1}_K(\tilde{c}), \\ \frac{(q+2)\tilde{c}}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}^{p+1}_C(\tilde{c}). \end{cases}$$

Then we have the following observations.

(1) *If $\inf_M F(f, q, \tilde{c})$ is non-positive then M is not f-biharmonic.*

(2) *If* $\inf_M F(f,q,\tilde{c})$ *is positive and* M *is proper* f*-biharmonic then*

$$0<|H|^2\leqslant \frac{1}{q}\inf_M F(f,q,\tilde{c}).$$

Proof: As M is proper f-biharmonic submanifold with constant mean curvature H and ξ tangent to M, so we get form Corollary 4.2 that

$$\begin{cases} -\Delta^{\perp}H+\mathrm{tr}B(\cdot,A_H\cdot)+\frac{2}{f}\nabla^{\perp}_{\mathrm{grad}f}H+\frac{\Delta f}{f}H=qf_1H-f_2H-3f_3NtH,\\ \\ 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot)-2A_H\mathrm{grad}(\ln f)=-6f_3PtH.\end{cases}$$

Now, considering ϕH is tangent implies that $sH=0$. Again applying ϕ gives that $\phi^2H=PtH+NtH$. But from $\phi^2H=-H+\eta(H)\xi$ and ξ is tangent, we have $\phi^2H=-H$. Therefore, comparing tangential and normal parts, we get $PtH=0$ and $NtH=-H$. Using these facts in the above equation, we get

$$\begin{cases} -\Delta^{\perp}H+\mathrm{tr}B(\cdot,A_H\cdot)=qf_1H-f_2H+3f_3H-\frac{\Delta f}{f}H,\\ \\ 2\mathrm{tr}A_{\nabla^{\perp}H}(\cdot)-2A_H\mathrm{grad}(\ln f)=0.\end{cases}$$

Now, considering ν as an real eigenvalue of the eigenfunction f corresponding to Laplacian operator Δ, i.e., $\frac{\Delta f}{f}=\nu$, from first equation, we have

$$\begin{aligned} -\Delta^{\perp}H+\mathrm{tr}B(\cdot,A_H\cdot) &= qf_1H-f_2H+3f_3H-\nu H\\ &= F(f,q,\tilde{c})H.\end{aligned}$$

Taking scalar product by H, we get

$$-\left\langle\Delta^{\perp}H,H\right\rangle+\langle\mathrm{tr}B(\cdot,A_H\cdot),H\rangle=F(f,q,\tilde{c})|H|^2.$$

Using the facts $\langle\mathrm{tr}B(\cdot,A_H\cdot),H\rangle=|A_H|^2$, $|H|$ is a constant and the Böchner formula, i.e., $\frac{1}{2}\Delta|H|^2=\left\langle\Delta^{\perp}H,H\right\rangle-|\nabla^{\perp}H|^2$ in the above equation, we have

$$|A_H|^2+|\nabla^{\perp}H|^2=F(f,q,\tilde{c})|H|^2.$$

Now, this equation reduces to

$$F(f,q,\tilde{c})|H|^2=|A_H|^2+|\nabla^{\perp}H|^4\geqslant q|H|^2+|\nabla^{\perp}H|^2\geqslant q|H|^4,$$

by considering the Cauchy-Schwarz inequality $|A_H|^2\geqslant\frac{1}{q}\mathrm{tr}(A_H)=q|H|^4$. It implies that

$$F(f,q,\tilde{c})\geqslant q|H|^2,$$

as $|H|$ is a positive constant. This proves the two assertions of the theorem. □

Now, we have the analogous result replacing the assumption that ϕH is tangent by ϕH is normal. Namely, we have:

Proposition 4.6. *Let* $\psi:M^q\to\widetilde{M}^{p+1}_S(\tilde{c})$ *(resp.* $\widetilde{M}^{p+1}_K(\tilde{c})$ *or* $\widetilde{M}^{p+1}_C(\tilde{c})$*) be a submanifold of Sasakian (Kenmotsu or cosymplectic) space form with constant mean curvature* H *so that* ξ *is* *tangent and ϕH is normal. Further, we consider $F(f,q,\tilde{c})$ the function defined on M by*

$$G(f,q,\tilde{c}) = qf_1 - f_2 - \frac{\Delta f}{f} = \begin{cases} \frac{(q-1)\tilde{c}}{4} + \frac{(3q+1)}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}_S^{p+1}(\tilde{c}), \\ \frac{(q-1)\tilde{c}}{4} - \frac{(3q+1)}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}_K^{p+1}(\tilde{c}), \\ \frac{(q-1)\tilde{c}}{4} - \frac{\Delta f}{f} & \text{for } \widetilde{M}_C^{p+1}(\tilde{c}). \end{cases}$$

Then we have the following observations.

(1) *If $\inf_M G(f,q,\tilde{c})$ is non-positive then M is not f-biharmonic.*

(2) *If $\inf_M G(f,q,\tilde{c})$ is positive and M is proper f-biharmonic then*

$$0 < |H|^2 \leqslant \frac{1}{q}\inf_M G(f,q,\tilde{c}).$$

Proof: Now, in this case, M is proper f-biharmonic submanifold with ξ is tangent and ϕH is normal. Normality of ϕH implies that $sH = 0$. Therefore, from Corollary 4.2, we have

$$\begin{aligned} -\Delta^{\perp} H + \mathrm{tr}B(\cdot, A_H \cdot) &= qf_1 H - f_2 H \\ &= G(f,q,\tilde{c})H. \end{aligned}$$

Similarly, as in the previous theorem, taking the scalar product by H and using the Böchner formula and then with the help of the Cauchy-Schwarz inequality, we get

$$G(f,q,\tilde{c})|H|^2 = |A_H|^2 + |\nabla^{\perp} H|^4 \geqslant q|H|^2 + |\nabla^{\perp} H|^2 \geqslant q|H|^4.$$

It easily provides the inequality $G(f,q,\tilde{c}) \geqslant q|H|^2$, since $|H|$ is a positive constant. We get $0 < |H|^2 \leqslant \frac{1}{q}\inf_M G(f,q,\tilde{c})$, which concludes the proof. □

5. f-biharmonic Legendre curves in (α,β)-trans-Sasakian generalized Sasakian space forms

In this last section, we will focus on f-biharmonic curves in (α,β)-trans-Sasakian generalized Sasakian space forms.

First, we briefly give some recalls about (α,β)-trans-Sasakian manifold. This class of contact metric manifold has been introduced by Oubina [27] as a generalization of both Sasakian and Kenmostu manfiolds. An almost contact metric manifold (M,g,ϕ,ξ,η) is called trans-Sasakian of type (α,β) or (α,β)-trans-Sasakian if the following holds

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi),$$

for any tangent vector field X and α, β two smooth functions on M. In [21], Marrero proved that every trans-Sasakian manifold of dimension greater or equal to 5 is either α-Sasakian, β-Kenmotsu or cosymplectic. Moreover, Alegre and Carriazo [2] proved that if $\widetilde{M}(f_1,f_2,f_3)$ is a connected α-Sasakian generalized Sasakian space form, then the functions f_1, f_2 and f_3 are constant and

(1) if $\alpha = 0$, then $f_1 - f_2 = f_3$ and so $M(f_1,f_2,f_3)$ is a cosymplectic space form,
(2) if $\alpha \neq 0$, then α is constant and $f_1 - \alpha^2 = f_2 = f_3$.

Alegre and Carriazo also showed in [3] that if $\widetilde{M}(f_1, f_2, f_3)$ is a β-Sasakian generalized Sasakian space form, then the functions f_1, f_2 and f_3 depend only on the direction of ξ. Examples are given in [1]. We mention finally that examples of 3-dimensional (α, β)-trans-Sasakian with both α and β non-zero are given in [21]. We add that, under the condition that α and β depend only on the direction of ξ, then a 3-dimensional (α, β)-trans-Sasakian is a generalized Sasakian space form with $f_2 = 0$ and f_1, f_3 explicitely given from α, β and the scalar curvature of the manifold (see [3, Theorem 4.7] for the result and [9] for an explicit example).

Let (M^n, g) be a Riemannian manifold and $\gamma : I \longrightarrow M$ be a curve parametrized by arc length. We set $T = \gamma'$. We say that γ is a Frenet curve of osculation order r, $1 \leqslant r \leqslant n$, if there exists orthormal vector fields $E_1 = T$, $E_2, \cdots, E_r$ along γ so that

$$\left\{\begin{array}{lcl} \nabla_T T &=& \kappa_1 E_2, \\ \nabla_T E_2 &=& -\kappa_1 E_1 + \kappa_2 E_3, \\ && \cdots \\ \nabla_T E_{r-1} &=& -\kappa_{r-2} E_{r-2} + \kappa_{r-1} E_r, \\ \nabla_T E_r &=& -\kappa_{r-1} E_{r-1}, \end{array}\right. \tag{30}$$

where $\kappa_1, \cdots, \kappa_{r-1}$ are positive fonctions.
Moreover, a curve γ parametrized by arc length is a f-biharmonic curve if and only if it satisfies Equation (2) which becomes the following for curves

$$f(\nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T) + 2f' \nabla_T \nabla_T T + f'' \nabla_T T = 0. \tag{31}$$

Now, we assume that $M = \widetilde{M}(f_1, f_2, f_3)$ is a (α, β)-trans-Sasakian generalized Sasakian space form and that γ is a Legendre curve of osculating order r. We recall that a Legendre curve is a curve so that the tangent vector field T is orthogonal to the contact vector field ξ, that is so that $\eta(T) = 0$. First, from (30), we have

$$\nabla_T \nabla_T T = -\kappa_1^2 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \tag{32}$$

and

$$\nabla_T \nabla_T \nabla_T T = -3\kappa_1 \kappa_1' E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2) E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4. \tag{33}$$

Moreover, we have the following elementary lemma.

Lemma 5.1. *Assume that $\kappa_1 > 0$, then, we have*

$$\eta(E_2) = -\frac{\beta}{\kappa_1}.$$

Proof: Since M is a (α, β)-trans-Sasakian, we have $\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi)$. Then, using this, $\eta(T) = 0$, $\phi T \perp T$ and $\nabla_T T = \kappa_1 E_2$, we get

$$\begin{array}{rcl} \eta(E_2) &=& \dfrac{1}{\kappa_1} \langle \nabla_T T, \xi \rangle \\ &=& -\dfrac{1}{\kappa_1} \langle T, \nabla_X \xi \rangle \\ &=& -\dfrac{1}{\kappa_1} \langle T, -\alpha \phi T + \beta(T - \eta(T)\xi) \rangle \\ &=& -\dfrac{\beta}{\kappa_1}. \end{array}$$

Hence, from this lemma, $\nabla_T T = \kappa_1 E_2$ and the expression of the curvature (20), we get immediately

$$R(T, \nabla_T T)T = -\kappa_1 f_1 E_2 - \beta f_2 \xi - 3\kappa_1 f_3 \langle \varphi T, E_2 \rangle \varphi T. \tag{34}$$

Thus, we obtain the following characterization of biharmonic Legendre curves in (α, β)-trans-Sasakian generalized Sasakian space forms.

Proposition 5.2. *Let γ be a non-geodesic Legendre Frenet curve of osculating order r in a (α, β)-trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then γ is f-biharmonic if and only if*

$$\begin{aligned} 0 \;=\; & \left(-3\kappa_1\kappa_1' - 2\kappa_1^2 \frac{f'}{f}\right) E_1 \\ & + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 - \kappa_1 f_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f}\right) E_2 \\ & + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2' + 2\kappa_1\kappa_2 \frac{f'}{f}\right) E_3 \\ & + \kappa_1\kappa_2\kappa_3 E_4 - \beta f_2 \xi - 3\kappa_1 f_3 \langle \varphi T, E_2 \rangle \varphi T \end{aligned}$$

□

Remark 5.3. *If γ is of osculating order 3, which is in particular the case if $\widetilde{M}(f_1, f_2, f_3)$ is 3-dimensional, then the term with $\kappa_1\kappa_2\kappa_3 E_4$ disappears.*

Now, we will consider some special cases. First, we focus on α-Sasakian manifolds, that is, with $\beta = 0$. We have the following proposition which generalizes the result of [15] obtained for Sasakian space forms.

Proposition 5.4. *There exists no proper f-biharmonic Legendre curves in a α-Sasakian generalized Sasakian space form with $\varphi T \parallel E_2$ and constant α. In particular, there exists no proper f-biharmonic Legendre curves in 3-dimensional α-Sasakian generalized Sasakian space forms with constant α.*

Proof: Assume that γ is a f-biharmonic Legendre curve of a α-Sasakian generalized Sasakian space forms. Then, from the assumption $\varphi T \parallel E_2$, Proposition 5.2 gives

$$3\kappa_1' + 2\kappa_1 \frac{f'}{f} = 0, \tag{35}$$

$$\kappa_1^2 + \kappa_2^2 + f_1 + 3f_3 = \frac{\kappa_1''}{\kappa_1} + \frac{f''}{f} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f}, \tag{36}$$

$$\kappa_2' + 2\kappa_2 \frac{f'}{f} + 2\frac{\kappa_1'}{\kappa_1}, \tag{37}$$

$$\kappa_2\kappa_3 = 0. \tag{38}$$

□

From $\varphi T \parallel E_2$, we deduce that κ_2 is constant. Indeed, we have $E_2 = \pm\varphi T$, since both E_2 and

φT are unit vectors and so

$$\begin{aligned}\nabla_T E_2 &= \pm\nabla\varphi T \\ &= \pm\left(\nabla\varphi\right)(T) \mp \varphi(\nabla_T T) \\ &= \pm\alpha\left(\langle T,T\rangle\xi - \eta(T)T\right) \mp \varphi(\kappa_1 E_2) \\ &= \pm\alpha\xi \mp \varphi E_2 \\ &= \pm\alpha\xi - \kappa_1 T.\end{aligned}$$

This says that, maybe up to minus sign, ξ has to be E_3 and $\kappa_2 = \alpha$. Hence, since α is constant, then κ_2 is also constant and (35) implies that f is constant. Therefore, there cannot have proper f-biharmonic legendre curves. □

We finish with this last proposition in the 3-dimensional case, which is an immediate application of Proposition 5.2.

Proposition 5.5. *Let γ be a non-geodesic Legendre Frenet curve in a 3-dimensional (α,β)-trans-Sasakian manifold M so that α and β depend only on the direction of ξ so that M is a generalized Sasakian space form associated with functions f_1, $f_2 = 0$ and f_3. Then γ is f-biharmonic if and only if the following equations are satisfied*

$$3\kappa_1\kappa_1' + 2\kappa_1^2\frac{f'}{f} = 0,$$

$$\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 - \kappa_1 f_1 + 2\kappa_1'\frac{f'}{f} + \kappa_1\frac{f''}{f} - 3\kappa_1 f_3\langle\varphi T, E_2\rangle^2 = 0,$$

$$2\kappa_1'\kappa_2 + \kappa_1\kappa_2' + 2\kappa_1\kappa_2\frac{f'}{f} - 3\kappa_1 f_3\langle\varphi T, E_2\rangle\langle\varphi T, E_3\rangle = 0.$$

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