Spinorial characterizations of surfaces into 3-dimensional homogeneous manifolds

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Abstract

We give spinorial characterizations of isometrically immersed surfaces into 3-dimensional homogeneous manifolds with 4-dimensional isometry group in terms of existence of a particular spinor field. This generalizes works by T. Friedrich for $\mathbb{R}^3$ and B. Morel for $\mathbb{S}^3$ and $\mathbb{H}^3$. The main argument is the interpretation of the energy-momentum tensor of such a spinor field as the second fundamental form up to a tensor depending on the structure of the ambient space.

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1 Introduction

Over the past years, the spinorial tool has been used successfully in the study of the geometry and the topology of submanifolds of space forms. The spinorial approach allows to solve naturally some problems of geometry of submanifolds. For instance, some simple proofs of the Alexandrov theorem in the Euclidean space ([13]) or in the hyperbolic space ([11]) were given and new results about Einstein manifolds or Lagrangian submanifolds of Kähler manifolds ([12]) were obtained.

In the same time, several results about 3-dimensional homogeneous manifolds with 4-dimensional isometry group were obtained, especially concerning minimal and constant mean curvature surfaces in these spaces, by H. Rosenberg *and al.* ([1, 18] for instance). In this paper, we make use of the spinorial tool in the study of surfaces in these 3-dimensional homogeneous manifolds. As in the case of space forms, we can think that a spinorial approach could yield to solve some open questions in the theory of surfaces in 3-dimensional homogeneous manifolds, as the existence of an Alexandrov type theorem in the Heisenberg group Nil_3.

The first step of such an approach is to understand the spinorial geometry of these surfaces and particularly to give a spinorial characterization of the surfaces which are isometrically immersed into these homogeneous manifolds. Precisely, we characterize these surfaces by the existence of a special spinor field. Then, we show that the existence of such a special spinor field is equivalent to the existence of a spinor field solution of a weaker equation (involving the Dirac operator) with an additional condition on the norm of the spinor field (see Theorems 1 and 2).

The results that we give in this paper are non-trivial generalizations of ([8]) and ([16]) to these homogeneous manifolds.

2 Preliminaries

We begin the paper with a short section of preliminaries in order to recall the basic facts about spin geometry of hypersurfaces. For further details, the reader can refer to [4, 9, 14] for general properties of spin manifolds and [2, 10, 15, 3] for the basics of extrinsic spin geometry.

2.1 Hypersurfaces and induced spin structures

Let (M^3, g) be a 3-dimensional Riemannian spin manifold and ΣM its spinor bundle. We denote by ∇^M the spinorial Levi-Civita connection on ΣM. The Clifford multiplication will be denoted by γ^M and $\langle .,. \rangle$ is the natural Hermitian product on ΣM compatible with ∇^M and γ^M. Finally, we denote by D^M the Dirac operator, locally given by $D^M = \sum_{i=1}^{3} \gamma^M(e_i)\nabla^M_{e_i}$, where $\{e_1, e_2, e_3\}$ is a local orthonormal frame of TM.

Now let N be an orientable surface of M. Since the normal bundle is trivial, the hypersurface N is also spin. Indeed, the existence of a normal unit vector field ν globally defined on N induces a spin structure from the spin structure of M.

Then we can consider the intrinsic spinor bundle ΣN of N and the extrinsic spinor bundle on N defined by $\mathbf{S} := \Sigma M_{|N}$. Since N is odd-dimensional, then there exists a natural identification between these two spinor bundles:

$$\mathbf{S} \equiv \Sigma N, \tag{1}$$

and between their connections and Clifford multiplications:

$$\nabla^{\mathbf{S}} \equiv \nabla^N \quad \text{and} \quad \gamma^{\mathbf{S}} \equiv \gamma^N. \tag{2}$$

The interest of this identification is that we can use restrictions of ambient spinors to study the intrinsic Dirac operator of N, with the following identities

$$\nabla^{\mathbf{S}}_X = \nabla^M_X - \frac{1}{2}\gamma^M(AX)\gamma^M(\nu) \tag{3}$$

$$\gamma^{\mathbf{S}}(X) = \gamma^M(X)\gamma^M(\nu), \tag{4}$$

where A is the shape operator of the immersion of N into M. Equation (3) is called the spinorial Gauss formula.

We finish this section by the following well-known fact. The spinor bundle ΣN decomposes in two subbundles under the action of the complexe volume element $\gamma^N(\omega_2) = i\gamma^N(e_1)\gamma^N(e_2)$, where $\{e_1, e_2\}$ is a local orthonormal frame of TN. Namely, we have $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$ where $\Sigma^\pm N$ is the bundle of eigenspinors for the action of $\gamma^N(\omega_2)$ associated with the eigenvalue ± 1. Hence, any spinor field φ decomposes in the following way: $\varphi = \varphi^+ + \varphi^-$, with $\gamma^N(\omega_2)\varphi^\pm = \pm\varphi^\pm$. We denote $\overline{\varphi} = \gamma^N(\omega_2)\varphi = \varphi^+ - \varphi^-$. Moreover, the Clifford multiplication by a vector exchanges $\Sigma^+ N$ and $\Sigma^- N$. Hence, the Dirac operator D^N also exchanges $\Sigma^+ N$ and $\Sigma^- N$.

2.2 3-dimensional homogeneous manifolds with 4-dimensional isometry group

In this section, we give a description of 3-dimensional homogeneous manifolds with 4-dimensional isometry group. Such a manifold is a Riemannian fibration over a simply connected 2-dimensional manifold with constant curvature κ and such that the fibers are geodesic. We denote by τ the bundle curvature defined below by (5), which measures the default of the fibration to be a Riemannian product. When τ vanishes, we get a product manifold $\mathbb{M}^2(\kappa) \times \mathbb{R}$, where $\mathbb{M}^2(\kappa)$ is the 2-dimensional space form of curvature κ.
When $\tau \neq 0$, these manifolds are of three types: they have the isometry group of the Berger spheres if $\kappa > 0$, of the Heisenberg group Nil_3 if $\kappa = 0$ or of $\widetilde{PSL_2}(\mathbb{R})$ if $\kappa < 0$. In the sequel, we denote these homogenous manifolds by $\mathbb{E}(\kappa, \tau)$. We begin by giving a short description of $\mathbb{E}(\kappa, \tau)$. For further details, one can refer to [6] or [19].

Let $\mathbb{E}(\kappa, \tau)$ be a 3-dimensional homogeneous manifold with 4-dimensional isometry group and assume that $\tau \neq 0$, *i.e.*, $\mathbb{E}(\kappa, \tau)$ is not a product manifold $\mathbb{M}^2(\kappa) \times \mathbb{R}$. As we said, $\mathbb{E}(\kappa, \tau)$ is a Riemannian fibration over a simply connected 2-dimensional manifold with constant curvature κ and such that the fibers are geodesic. Now, let ξ be a unitary vector field tangent to the fibers. We call it the vertical vector field. This vector field is a Killing vector field (corresponding to the translations along the fibers) and satisfying

$$\overline{\nabla}_X \xi = \tau X \wedge \xi, \tag{5}$$

where $\overline{\nabla}$ is the Riemannian connection of $\mathbb{E}(\kappa, \tau)$ and $\wedge$ is the vector product in $\mathbb{E}(\kappa, \tau)$, that is, for any $X, Y, Z \in \Gamma(TM)$,

$$\langle X \wedge Y, Z \rangle = \det{}_{\{e_1, e_2, \xi\}}(X, Y, Z).$$

Note that if $\tau = 0$, then $\xi = \frac{\partial}{\partial t}$ is the unit vector field giving the orientation of $\mathbb{R}$ in the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$.
The manifold $\mathbb{E}(\kappa, \tau)$, with $\tau \neq 0$, admit a local direct orthonormal frame $\{e_1, e_2, e_3\}$ with

$$e_3 = \xi$$

and such that the Christoffel symbols $\overline{\Gamma}_{ij}^k = \langle \overline{\nabla}_{e_i} e_j, e_k \rangle$ are

$$\begin{cases} \overline{\Gamma}_{12}^3 = \overline{\Gamma}_{23}^1 = -\overline{\Gamma}_{21}^3 = -\overline{\Gamma}_{13}^2 = \tau, \\ \overline{\Gamma}_{32}^1 = -\overline{\Gamma}_{31}^2 = \tau - \frac{\kappa}{2\tau}, \\ \overline{\Gamma}_{ii}^i = \overline{\Gamma}_{ij}^i = \overline{\Gamma}_{ji}^i = \overline{\Gamma}_{ii}^j = 0, \quad \forall\, i, j \in \{1, 2, 3\}. \end{cases} \tag{6}$$

Then we have

$$[e_1,e_2]=2\tau e_3,\quad [e_2,e_3]=\frac{\kappa}{2\tau}e_1,\quad [e_3,e_1]=\frac{\kappa}{2\tau}e_2.$$

We will call $\{e_1,e_2,\xi\}$ the canonical frame of $\mathbb{E}(\kappa,\tau)$.

2.3 Compatibility equations

Here, we give the basic identities satisfied by surfaces into $\mathbb{E}(\kappa,\tau)$. Let N be an orientable surface of $\mathbb{E}(\kappa,\tau)$ with shape operator A associated with the unit normal ν. Moreover, we denote $\xi = T + f\nu$ where the function f is the normal component of ξ and T is its tangential part. Then, from Equation (5), we deduce easily the following properties.

Proposition 2.1. *For $X\in\mathfrak{X}(N)$, we have*

$$\nabla_X T = f(AX-\tau JX) \quad \textit{and} \quad df(X) = -\langle AX-\tau JX, T\rangle,$$

where J is the rotation of angle $\frac{\pi}{2}$ on TN.

Proof : On one hand, we have

$$\begin{aligned}\overline{\nabla}_X\xi &= \overline{\nabla}_X(T+f\nu)\\ &= \overline{\nabla}_X T + df(X)\nu + f\overline{\nabla}_X\nu\\ &= \nabla_X T + \langle AX,T\rangle\,\nu + df(X)\nu - fAX,\end{aligned}$$

and on the other hand

$$\begin{aligned}\overline{\nabla}_X\xi &= \tau X\wedge\xi\\ &= \tau X\wedge(T+f\nu)\\ &= \tau\big(\langle JX,T\rangle\,\nu - fJX\big).\end{aligned}$$

We get the two identities by taking the normal and tangential parts. □

Definition 2.2 (Compatibility equations)**.** *We say that $(N,\langle .,.\rangle,A,T,f)$ satisfies the compatibility equations for $\mathbb{E}(\kappa,\tau)$ if and only if for any $X,Y,Z\in\mathfrak{X}(N)$,*

$$K=\det(A)+\tau^2+(\kappa-4\tau^2)f^2 \tag{7}$$

$$\nabla_X AY-\nabla_Y AX - A[X,Y] = (\kappa-4\tau^2)f(\langle Y,T\rangle X-\langle X,T\rangle Y), \tag{8}$$

$$\nabla_X T = f(AX-\tau JX),\ \textit{and} \tag{9}$$

$$df(X) = -\langle AX-\tau JX,T\rangle, \tag{10}$$

where K is the Gauss curavture of N.

Remark 1. *The relations (7) and (8) are the Gauss and Codazzi equations for an isometric immersion into $\mathbb{E}(\kappa,\tau)$ obtained by a computation of the curvature tensor of $\mathbb{E}(\kappa,\tau)$.*

In [5, 6], Daniel proves that these compatibility equations are necessary and sufficient for the existence of an isometric immersion F from N into $\mathbb{E}(\kappa,\tau)$ with shape operator $dF\circ A\circ dF^{-1}$ and so that $\xi = dF(T)+f\nu$.

3 Special spinor fields on surfaces of $\mathbb{E}(\kappa,\tau)$

Now, we describe the special spinor fields which will be involved in our characterization of surfaces into $\mathbb{E}(\kappa,\tau)$.

3.1 Special spinor fields on surfaces of $\mathbb{M}^2(\kappa)\times\mathbb{R}$

3.1.1 Spinor bundle of $\mathbb{M}^2(\kappa)\times\mathbb{R}$

It is a well-known fact that since $\mathbb{M}^2(\kappa)$ is spin, then $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is also spin and the spin structures of $\mathbb{M}^2(\kappa)$ and $\mathbb{M}^2(\kappa)\times\mathbb{R}$ are in a one-to-one correspondence (*cf* [3] for more details).
We will explain explictly the spinor bundle of $\mathbb{M}^2(\kappa)\times\mathbb{R}$. We have seen in Section 2.1 that

$$\Sigma(\mathbb{M}^2(\kappa)\times\mathbb{R})_{|\mathbb{M}^2}\equiv\Sigma\mathbb{M}^2(\kappa).$$

So, if $\varphi\in\Gamma(\Sigma(\mathbb{M}^2(\kappa)\times\mathbb{R}))$, then for any $t\in\mathbb{R}$: $\varphi(.,t)\in\Gamma(\Sigma\mathbb{M}^2(\kappa))$. Hence a section of $\Sigma(\mathbb{M}^2(\kappa)\times\mathbb{R})$ is a $\mathcal{C}^\infty$-map

$$\begin{array}{rccl}\varphi: & \mathbb{R} & \longrightarrow & \Gamma(\Sigma(\mathbb{M}^2(\kappa)))\\ & t & \longmapsto & \varphi_t.\end{array}$$

3.1.2 Restriction to a surface

In the sequel, we will consider some particular and interesting sections, precisely the sections which do not depend on t. We have the spinorial Gauss formula,

$$\overline{\nabla}_X\varphi=\nabla_X\varphi+\frac{1}{2}\gamma(AX)\gamma(\nu)\varphi,$$

where $\overline{\nabla}$ is the spinorial connection on $\mathbb{M}^2(\kappa)\times\mathbb{R}$, the spinorial connection on $\mathbb{M}^2(\kappa)$ is ∇, the Clifford multiplication on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is γ and A is the Weingarten operator of the immersion of $\mathbb{M}^2(\kappa)$ into $\mathbb{M}^2(\kappa)\times\mathbb{R}$. Since $\mathbb{M}^2(\kappa)$ is totally geodesic in the product $\mathbb{M}^2(\kappa)\times\mathbb{R}$, if we take $\varphi_t=\varphi_0$ a Killing spinor on $\mathbb{M}^2(\kappa)$, *i.e.* $\nabla_X\varphi_0=\eta\gamma^{\mathbb{M}^2(\kappa)}(X)\varphi_0$, with $\kappa=4\eta^2$, then we have

$$\overline{\nabla}_X\varphi=\eta\gamma^{\mathbb{M}^2(\kappa)}(X)\varphi=\eta\gamma(X)\gamma(\frac{\partial}{\partial t})\varphi$$

On the other hand, the complexe volume form $\omega_3^{\mathbb{C}}=-e_1\cdot e_2\cdot\frac{\partial}{\partial t}$ acts as identity, so we have

$$\gamma(e_1)\gamma\left(\frac{\partial}{\partial t}\right)\varphi=-\gamma(e_2)\varphi,\ \text{and}\ \gamma(e_2)\gamma\left(\frac{\partial}{\partial t}\right)\varphi=\gamma(e_1)\varphi.$$

Hence we deduce that

$$\left\{\begin{array}{l}\overline{\nabla}_{e_1}\varphi=-\eta\gamma(e_2)\varphi,\\[2mm] \overline{\nabla}_{e_2}\varphi=\eta\gamma(e_1)\varphi,\\[2mm] \overline{\nabla}_{\frac{\partial}{\partial t}}\varphi=0.\end{array}\right.\tag{11}$$

These particular spinor fields will play the same role as Killing spinor fields in the work of Morel [16] for $\mathbb{S}^3$ or $\mathbb{H}^3$.

Now, let $(N,\langle\cdot,\cdot\rangle)$ be a surface of $\mathbb{M}^2(\kappa)\times\mathbb{R}$, oriented by ν. Since $(N,\langle\cdot,\cdot\rangle)$ is oriented, it could be equipped with a spin structure induced from the spin structure of $\mathbb{M}^2(\kappa)\times\mathbb{R}$ and we have the following identification between the spinor bundles

$$\Sigma\left(\mathbb{M}^2(\kappa)\times\mathbb{R}\right)_{|N}\cong\Sigma N.$$

For any $X\in\mathfrak{X}(N)$ and any $\psi\in\Gamma(\Sigma\left(\mathbb{M}^2(\kappa)\times\mathbb{R}\right))$, we have from (3)

$$\begin{aligned}\left(\overline{\nabla}_X\psi\right)_{|N} &= \nabla_X\left(\psi_{|N}\right)+\frac{1}{2}\gamma^N(AX)\psi_{|N}\\ &= \nabla_X\left(\psi_{|N}\right)+\frac{1}{2}\gamma(AX)\gamma(\nu)\psi_{|N}.\end{aligned}$$

If we use this forumla for the particular spinor field on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ given by (11), we get

$$\begin{aligned}\nabla_X\varphi &= \overline{\nabla}_X\varphi-\frac{1}{2}\gamma^N(AX)\varphi\\ &= \eta\gamma(X_t)\gamma\left(\frac{\partial}{\partial t}\right)\varphi-\frac{1}{2}\gamma^N(AX)\varphi,\end{aligned}$$

where X_t is the part of X tangent to $\mathbb{M}^2(\kappa)$, that is,

$$\begin{aligned}X_t &= X-\left\langle X,\frac{\partial}{\partial t}\right\rangle\frac{\partial}{\partial t}\\ &= X-\langle X,T\rangle T-f\langle X,T\rangle\nu.\end{aligned}$$

So, we deduce that

$$\begin{aligned}\nabla_X\varphi &= \eta\gamma(X)\gamma\left(\frac{\partial}{\partial t}\right)\varphi-\eta\left\langle X,\frac{\partial}{\partial t}\right\rangle\gamma\left(\frac{\partial}{\partial t}\right)\gamma\left(\frac{\partial}{\partial t}\right)\varphi-\frac{1}{2}\gamma^N(AX)\varphi\\ &= \eta\gamma(X)\gamma\left(\frac{\partial}{\partial t}\right)\varphi+\eta\left\langle X,\frac{\partial}{\partial t}\right\rangle\varphi-\frac{1}{2}\gamma^N(AX)\varphi\\ &= \eta\gamma(X)\gamma(T)\varphi+\eta f\gamma(X)\gamma(\nu)\varphi+\eta\langle X,T\rangle\varphi-\frac{1}{2}\gamma^N(AX)\varphi.\end{aligned}$$

On the other hand, if we denote by $\omega=e_1\underset{N}{\cdot}e_2$ the real volume element on N, we have the following relations

$$\begin{cases}\gamma(X)=-\gamma^N(X)\gamma^N(\omega),\\ \gamma(\nu)=\gamma^N(\omega).\end{cases}$$

By using these two identities, the fact that $\omega^2=-1$ and that ω anti-commuts with vector fields tangent to N, we get

$$\begin{aligned}\nabla_X\varphi &= \eta\gamma^N(X)\gamma^N(\omega)\gamma^N(T)\gamma^N(\omega)\varphi+\eta\langle X,T\rangle\varphi-\eta f\gamma^N(X)\gamma^N(\omega)\gamma^N(\omega)\varphi\\ &\quad-\frac{1}{2}\gamma^N(AX)\varphi\end{aligned}$$

Now, we can rewrite this equation in an intrinsic way as follows

$$\nabla_X\varphi = \eta X\cdot T\cdot\varphi+\eta fX\cdot\varphi+\eta\langle X,T\rangle\varphi-\frac{1}{2}AX\cdot\varphi. \tag{12}$$

where "$\cdot$" stands for the Clifford multiplication on N. This spinor field satisfies the following property

Proposition 3.1. *Let φ be a solution of Equation (12)*

i) If $\eta = \frac{1}{2}$, then the norm of φ is constant.

ii) If $\eta = \frac{i}{2}$, then the norm of φ satisfies for any $X \in \mathfrak{X}(N)$:

$$X|\varphi|^2 = \Re e \left\langle iX \cdot T \cdot \varphi + ifX \cdot \varphi, \varphi \right\rangle.$$

Proof: We need to compute $X|\varphi|^2$ for $X \in \mathfrak{X}(N)$. We have,

$$X|\varphi|^2 = 2\Re e \left\langle \nabla_X \varphi, \varphi \right\rangle.$$

We replace $\nabla_X \varphi$ by the expression given by (12), and we use the compatibility of the Clifford multiplication with the scalar product.

3.2 Special spinor fields on surfaces of $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$

3.2.1 Special spinor fields on $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$

We will give here the expression of special spinor fields on $\mathbb{E}(\kappa, \tau)$ which will replace parallel or Killing spinors that we have in space forms. For this, let us endow $\mathbb{E}(\kappa, \tau)$ with its trivial spin structure. We can consider a constant section φ of the spinor bundle. From the local expression of the spinorial connection and of the Christoffel symbols Γ_{ij}^k given by (6), we can compute $\overline{\nabla}_X \varphi$. We recall that the complex volume element $\omega_3^{\mathbb{C}} = -e_1 \cdot e_2 \cdot \xi$ acts as identity, which implies

$$e_1 \cdot e_2 \cdot \varphi = \xi \cdot \varphi, \quad e_2 \cdot \xi \cdot \varphi = e_1 \cdot \varphi, \quad \text{and} \quad \xi \cdot e_1 \cdot \varphi = e_2 \cdot \varphi$$

The expression of the spinorial connection is for any $X \in \mathfrak{X}(\mathbb{E}(\kappa, \tau))$

$$\overline{\nabla}_X \varphi = X(\varphi) + \frac{1}{4} \sum_{i,j=1}^{3} \left\langle \overline{\nabla}_X e_i, e_j \right\rangle \gamma(e_i) \gamma(e_j) \varphi,$$

So, we deduce from (6) that for a constant section φ,

$$\text{(13)} \qquad \left\{ \begin{array}{l} \overline{\nabla}_{e_1} \varphi = \dfrac{1}{2} \tau e_1 \cdot \varphi, \\[2ex] \overline{\nabla}_{e_2} \varphi = \dfrac{1}{2} \tau e_2 \cdot \varphi, \\[2ex] \overline{\nabla}_{\xi} \varphi = \dfrac{1}{2} \left(\dfrac{\kappa}{2\tau} - \tau \right) \xi \cdot \varphi. \end{array} \right.$$

3.2.2 Restriction to a surface

In this section, we give the restriction of these spinors to a surface of $\mathbb{E}(\kappa, \tau)$. By a computation similar to the one of Section 3.1.2, we obtain, after restriction to N, a spinor field satisying

$$\text{(14)} \quad \nabla_X \varphi = -\frac{\tau}{2} X \cdot \omega \cdot \varphi + \frac{\alpha}{2} \left\langle X, T \right\rangle T \cdot \omega \cdot \varphi - \frac{\alpha}{2} f \left\langle X, T \right\rangle \omega \cdot \varphi - \frac{1}{2} AX \cdot \varphi,$$

where we have set $\alpha = 2\tau - \frac{\kappa}{2\tau}$. From this equation, we get the following consequence for the norm of a solution.

Proposition 3.2. *The norm of a spinor field solution of (14) is constant.*

4 Isometric immersions into $\mathbb{M}^2(\kappa)\times\mathbb{R}$

4.1 Statement of the result

Here, we state the main result of the present paper which give a spinorial characterization of surfaces isometrically immersed into the product spaces $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$. Before stating the theorem, we want to recall that the energy-momentum tensor Q_φ associated with a spinor field φ is the symmetric 2-tensor defined by

$$Q_\varphi(X,Y)=\frac{1}{2}\Re e\left\langle X\cdot\nabla_Y\varphi+Y\cdot\nabla_X\varphi,\frac{\varphi}{|\varphi|^2}\right\rangle, \tag{15}$$

on the complementary set of zeros of φ.

Theorem 1. *Let $\eta\in\mathbb{R}$ or $i\mathbb{R}$, $\eta\neq0$ and $\kappa=4\eta^2$. Let $(N,\langle.,.\rangle)$ be a connected, oriented and simply connectd Riemannian surface. Let T be a vector field, f and H two real functions on N satisfying $f^2+||T||^2=1$. Let A be a field of endomorphisms on TN with $\mathrm{tr}\,(A)=2H$. The following three datas are equivalent:*

i) an isometric immersion F from N into $\mathbb{S}^2(\kappa)\times\mathbb{R}$ (resp. $\mathbb{H}^2(\kappa)\times\mathbb{R}$) of mean curvature H *such that the Weingarten operator related to the normal ν is given by*

$$dF\circ A\circ dF^{-1}$$

and such that

$$\frac{\partial}{\partial t}=dF(T)+f\nu.$$

ii) a non-trivial spinor field φ satisfying

$$\nabla_X\varphi=\eta X\cdot T\cdot\varphi+\eta fX\cdot\varphi+\eta\langle X,T\rangle\varphi-\frac{1}{2}AX\cdot\varphi,$$

where A satisfies

$$\nabla_XT=fAX,\quad\text{and}\quad df(X)=-\langle AX,T\rangle.$$

iii) a nowhere vanishing spinor field φ satisfying

$$D\varphi=H\varphi-\eta T\cdot\varphi-2\eta f\varphi,$$

of constant norm $\Big($resp. satisfying $X|\varphi|^2=\Re e\big(iX\cdot T\cdot\varphi+ifX\cdot\varphi,\varphi\big)\Big)$, and such that

$$\nabla_XT=2fQ_\varphi(X)-fB(X)\quad\text{and}\quad df=-2Q_\varphi(T)+B(T),$$

where Q_φ is the energy-momentum tensor associated with the spinor field φ and B is the symmetric 2-tensor defined by

$$\begin{aligned}B(X,Y)\;=\;&-2\Re e\left\langle\eta\langle X,Y\rangle T\cdot\varphi,\frac{\varphi}{|\varphi|^2}\right\rangle-2\Re e\left\langle\eta f\langle X,Y\rangle\varphi,\frac{\varphi}{|\varphi|^2}\right\rangle\\&-\Re e\left\langle\eta\left(\langle X,T\rangle Y+\langle Y,T\rangle X\right)\cdot\varphi,\frac{\varphi}{|\varphi|^2}\right\rangle.\end{aligned}$$

Note that

- if $\eta = \frac{1}{2}$, then $B = f\mathrm{Id}$.
- if $\eta = \frac{i}{2}$, then B satisfies

$$\begin{cases} B_{11} = -\Re e \left\langle iT\cdot\varphi, \varphi\right\rangle - \Re e \left\langle iT_1 e_1\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle \\ B_{12} = B_{21} = \frac{1}{2}\Re e \left\langle i(T_1e_2 + T_2e_1)\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle \\ B_{22} = -\Re e \left\langle iT\cdot\varphi, \varphi\right\rangle - \Re e \left\langle iT_2 e_2\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle \end{cases}$$

4.2 Special spinor fields and compatibility equations

In a first time, we show that the existence of a solution of Equation (12) implies the Gauss and Codazzi equations. We will see that the integrability conditions for this equation are precisely the Gauss and Codazzi equations.

Proposition 4.1. *Let $(N, \langle\cdot,\cdot\rangle)$ be an oriented surface with a non-trivial solution of Equation (12) and such that Equations (9) and (10) are satisfied. Then, the Gauss and Codazzi equations for $\mathbb{M}^2(\kappa)\times\mathbb{R}$ are also satisfied.*

Proof: The proof of this proposition is based on the computation of the spinorial curvature applied to the spinor field φ solution of Equation (12), *i.e.*

$$\mathcal{R}(X,Y)\varphi = \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi,$$

for $X, Y \in \mathfrak{X}(N)$. Using the expression given by (12), Equations (9) and (10), the fact that $\omega^2 = -1$ and that ω anticommuts with the vector fields tangent to N, we get

$$\begin{aligned}\nabla_X\nabla_Y\varphi \;=\; & \underbrace{\eta fY\cdot AX\cdot\varphi}_{\alpha_1(X,Y)} + \underbrace{\eta^2 Y\cdot T\cdot X\cdot T\cdot\varphi}_{\alpha_2(X,Y)} + \underbrace{\eta^2 fY\cdot T\cdot X\cdot\varphi}_{\alpha_3(X,Y)} \\ & \underbrace{-\frac{\eta}{2}Y\cdot T\cdot AX\cdot\varphi}_{-\alpha_4(X,Y)} \underbrace{-\eta\left\langle AX,T\right\rangle Y\cdot\varphi}_{-\alpha_5(X,Y)} + \underbrace{\eta^2 fY\cdot X\cdot T\cdot\varphi}_{\alpha_6(X,Y)}\end{aligned}$$

$$\begin{aligned} & + \underbrace{\eta^2\left\langle X,T\right\rangle Y\cdot T\cdot\varphi}_{\alpha_7(X,Y)} + \underbrace{\eta^2 f^2 Y\cdot X\cdot\varphi}_{\alpha_8(X,Y)} + \underbrace{\eta^2 f\left\langle X,T\right\rangle Y\cdot\varphi}_{\alpha_9(X,Y)} \\ & \underbrace{-\frac{\eta}{2}fY\cdot AX\cdot\varphi}_{-\alpha_{10}(X,Y)} + \underbrace{\eta f\left\langle Y,AX\right\rangle\varphi}_{\alpha_{11}(X,Y)} + \underbrace{\eta^2\left\langle Y,T\right\rangle X\cdot T\cdot\varphi}_{\alpha_{12}(X,Y)} \\ & + \underbrace{\eta^2 f\left\langle Y,T\right\rangle X\cdot\varphi}_{\alpha_{13}(X,Y)} + \underbrace{\eta^2\left\langle X,T\right\rangle\left\langle Y,T\right\rangle\varphi}_{\alpha_{14}(X,Y)} \underbrace{-\frac{\eta}{2}\left\langle Y,T\right\rangle AX\cdot\varphi}_{-\alpha_{15}(X,Y)} \\ & \underbrace{-\frac{1}{2}\nabla_X(AY)\cdot\varphi}_{-\alpha_{16}(X,Y)} \underbrace{-\frac{\eta}{2}AY\cdot X\cdot T\cdot\varphi}_{-\alpha_{17}(X,Y)} \underbrace{-\frac{\eta}{2}fAY\cdot X\cdot\varphi}_{-\alpha_{18}(X,Y)}\end{aligned}$$

$$-\underbrace{\frac{\eta}{2}\langle X,T\rangle AY\cdot\varphi}_{-\alpha_{19}(X,Y)}+\underbrace{\frac{1}{4}AY\cdot AX\cdot\varphi}_{\alpha_{20}(X,Y)}+\underbrace{\eta\nabla_XY\cdot T\cdot\varphi}_{\alpha_{21}(X,Y)}$$
$$+\underbrace{\eta f\nabla_XY\cdot\varphi}_{\alpha_{22}(X,Y)}+\underbrace{\eta\langle\nabla_XY,T\rangle\varphi}_{\alpha_{23}(X,Y)}$$

That is,

$$\nabla_X\nabla_Y\varphi=\sum_{i=1}^{23}\alpha_i(X,Y).$$

Obviously, by symmetry, we have

$$\nabla_Y\nabla_X\varphi=\sum_{i=1}^{23}\alpha_i(Y,X).$$

On the other hand, we have

$$\begin{aligned}\nabla_{[X,Y]}\varphi &= \underbrace{\eta[X,Y]\cdot T\cdot\varphi}_{\beta_1([X,Y])}+\underbrace{\eta f[X,Y]\cdot\varphi}_{\beta_2([X,Y])}\\ &\quad+\underbrace{\eta\langle[X,Y],T\rangle\varphi}_{\beta_3([X,Y])}-\underbrace{\frac{1}{2}A[X,Y]\cdot\varphi}_{-\beta_4([X,Y])}.\end{aligned}$$

Since the connection ∇ is torsion-free, we get $\nabla_XY-\nabla_YX-[X,Y]=0$, which implies that

$$\begin{aligned}\alpha_{21}(X,Y)-\alpha_{21}(Y,X)-\beta_1([X,Y])=0,\\ \alpha_{22}(X,Y)-\alpha_{22}(Y,X)-\beta_2([X,Y])=0,\\ \alpha_{23}(X,Y)-\alpha_{23}(Y,X)-\beta_3([X,Y])=0.\end{aligned}$$

Moreover, by symmetry, we have

$$\alpha_{11}(X,Y)-\alpha_{11}(Y,X)=0\quad\text{and}\quad\alpha_{14}(X,Y)-\alpha_{14}(Y,X)=0.$$

Other terms vanish by symmetry. Namely,

$$\alpha_1(X,Y)+\alpha_{10}(X,Y)+\alpha_{18}(X,Y)-\alpha_1(Y,X)-\alpha_{10}(Y,X)-\alpha_{18}(Y,X)=0,$$

$$\alpha_3(X,Y)+\alpha_6(X,Y)-\alpha_3(Y,X)-\alpha_6(Y,X)=0,$$

and

$$\begin{aligned}\alpha_4(X,Y)+\alpha_5(X,Y)+\alpha_{15}(X,Y)+\alpha_{17}(X,Y)+\alpha_{19}(X,Y)\\ -\alpha_4(Y,X)-\alpha_5(Y,X)-\alpha_{15}(Y,X)-\alpha_{17}(X,Y)-\alpha_{19}(X,Y)=0.\end{aligned}$$

The terms α_2, α_7, α_8 et α_{12} can be combined. Indeed, if we set

$$\alpha=\alpha_2+\alpha_7+\alpha_8+\alpha_{12},$$

then

$$\begin{aligned}\alpha(X,Y)-\alpha(Y,X) &= \eta^2\Big[f^2\left(Y\cdot X-X\cdot Y\right)+Y\cdot T\cdot X\cdot T-X\cdot T\cdot Y\cdot T\Big]\cdot\varphi\\ &= \eta^2\Big[f^2\left(Y\cdot X-X\cdot Y\right)+||T||^2\left(Y\cdot X-X\cdot Y\right)\Big]\cdot\varphi\\ &\quad -2\eta^2\left(\langle X,T\rangle Y\cdot T-\langle Y,T\rangle X\cdot T\right)\cdot\varphi,\end{aligned}$$

by taking X and Y as a direct orthonormal frame $\{e_1,e_2\}$, we have

$$\begin{aligned}\alpha(e_1,e_2)-\alpha(e_2,e_1) &= 2\eta^2\Big[-T_1e_2\cdot(T_1e_1+T_2e_2)+T_2e_1\cdot(T_1e_1+T_2e_2)\Big]\cdot\varphi\\ &\quad -2\eta^2e_1\cdot e_2\cdot\varphi\\ &= -2\eta^2\omega\cdot\varphi+2\eta^2\left(T_1^2\omega\cdot+T_1T_2-T_1T_2+T_2^2\omega\cdot\right)\varphi\\ &= -2\eta^2f^2\omega\cdot\varphi.\end{aligned}$$

Always for $X=e_1$ and $Y=e_2$,

$$\alpha_9(e_1,e_2)+\alpha_{13}(e_1,e_2)-\alpha_9(e_2,e_1)-\alpha_{13}(e_2,e_1)=2T\cdot\omega\cdot\varphi=-2J(T)\cdot\varphi,$$

where J is the rotation of positive angle $\frac{\pi}{2}$ on TN. Finally, we get

$$\begin{aligned}\mathcal{R}(e_1,e_2)\varphi &= \frac{1}{4}\left(Ae_2\cdot Ae_1-Ae_1\cdot Ae_2\right)\cdot\varphi-2\eta^2f^2\omega\cdot\varphi\\ &\quad -\frac{1}{2}d^{\nabla}A(e_1,e_2)\cdot\varphi-2J(T)\cdot\varphi.\end{aligned}$$

Using the Ricci identity

$$\mathcal{R}(e_1,e_2)\varphi=-\frac{1}{2}R_{1212}e_1\cdot e_2\cdot\varphi,$$

and since $\kappa=4\eta^2$, we have

$$\Big(\underbrace{R_{1212}-\det(A)-\kappa f^2}_{G}\Big)\,\omega\cdot\varphi=\Big(\underbrace{d^{\nabla}A(e_1,e_2)+\kappa^2fJ(T)}_{C}\Big)\cdot\varphi,$$

that is,

$$G\omega\cdot\varphi=C\cdot\varphi,$$

where G is a function and C a vector field. Since, $\omega\cdot\varphi=-i\overline{\varphi}$, we obtain

$$C\cdot\varphi^{\pm}=\pm iG\varphi^{\mp}.$$

Thus,

$$||C||^2\varphi^{\pm}=-G^2\varphi^{\pm}.$$

Since φ is a non-trivial solution of (12), by Proposition 3.1, φ^+ et φ^- cannot vanish at the same point. Then $C=0$ and $G=0$. But $G=0$ is the Gauss equation and $C=0$ is the Codazzi equation, which achieves the proof. □

Hence, by Daniel's result, there exists an isometric immersion from N into $\mathbb{E}(\kappa,\tau)$.

4.3 Special spinor fields and Dirac equation

If a spinor field φ is a solution Equation (12), then, we deduce immediately that φ is also a solution of the corresponding Dirac equation

$$D\varphi = H\varphi - \eta T\cdot\varphi - 2\eta f\varphi. \tag{16}$$

In this section, we will see that this Dirac equation, which is weaker than Equation (12), is in fact equivalent for spinors with a norm satisfying the condition of Proposition 3.1. We write $\varphi = \varphi^+ + \varphi^-$ with $\varphi^\pm \in \Sigma^\pm$ and then we have

$$D\varphi^\pm = H\varphi^\mp - \eta T\cdot\varphi^\pm - 2\eta f\varphi^\mp. \tag{17}$$

Now, we define the following endomorphisms

$$Q^\pm_\varphi(X,Y) = \Re e\left\langle \nabla_X\varphi^\pm, Y\cdot\varphi^\mp\right\rangle,$$

and

$$\begin{array}{rcl} B^\pm(X,Y) &=& -\Re e\left\langle \eta X\cdot T\cdot\varphi^\pm, Y\cdot\varphi^\mp\right\rangle - \Re e\left\langle \eta\left\langle X,T\right\rangle\varphi^\pm, Y\cdot\varphi^\mp\right\rangle \\ && -\Re e\left\langle \eta f X\cdot\varphi^\mp, Y\cdot\varphi^\mp\right\rangle. \end{array}$$

On the other hand, we set

$$A^\pm = Q^\pm_\varphi + B^\pm, \tag{18}$$

and

$$W = \frac{A^+}{|\varphi^-|^2} - \frac{A^-}{|\varphi^+|^2}.$$

From now, we will divide our study in two parts, $\eta = \frac{1}{2}$ and $\eta = \frac{i}{2}$. We give all the details for the case $\eta = \frac{1}{2}$ and only specify the small differences for the case $\eta = \frac{i}{2}$.

4.3.1 The case $\eta = \frac{1}{2}$

In this section, we assume that the norm of φ is constant and we will show that W is identically zero. For that, we will show that W is symmetric, trace-free with rank less or equal to 1. First, we give, this elementary lemma.

Lemma 4.2. *The endomorphisms $Q^\pm_\varphi$ and $B^\pm$ satisfy*

1. $\mathrm{tr}\,(Q^\pm_\varphi) = -H|\varphi^\mp|^2 + \frac{1}{2}\Re e\left\langle T\cdot\varphi^\pm, \varphi^\mp\right\rangle + f|\varphi^\mp|^2$,
2. $\mathrm{tr}\,(B^\pm) = -\frac{1}{2}\Re e\left\langle T\cdot\varphi^\pm, \varphi^\mp\right\rangle - f|\varphi^\mp|^2$,
3. $Q^\pm_\varphi(e_1,e_2) = Q^\pm_\varphi(e_2,e_1) + \frac{1}{2}\Re e\left\langle \omega\cdot T\cdot\varphi^\pm, \varphi^\mp\right\rangle$,
4. $B^\pm(e_1,e_2) = B^\pm(e_2,e_1) - \frac{1}{2}\Re e\left\langle \omega\cdot T\cdot\varphi^\pm, \varphi^\mp\right\rangle$.

Proof : We begin by the computation of the trace of $B^\pm$ and $Q^\pm_\varphi$. We have

$$\begin{array}{rcl} \mathrm{tr}\,(Q^\pm_\varphi) &=& Q^\pm_\varphi(e_1,e_1) + Q^\pm_\varphi(e_2,e_2) \\ &=& \Re e\left\langle \nabla_{e_1}\varphi^\pm, e_1\cdot\varphi^\mp\right\rangle + \left\langle \nabla_{e_2}\varphi^\pm, e_2\cdot\varphi^\mp\right\rangle \\ &=& -\Re e\left\langle D\varphi^\pm, \varphi^\mp\right\rangle \\ &=& -H\Re e\left\langle \varphi^\mp, \varphi^\mp\right\rangle + \dfrac{1}{2}\Re e\left\langle T\cdot\varphi^\pm, \varphi^\mp\right\rangle + \Re e\left\langle f\varphi^\mp, \varphi^\mp\right\rangle \\ &=& -H|\varphi^\mp|^2 + \dfrac{1}{2}\Re e\left\langle T\cdot\varphi^\pm, \varphi^\mp\right\rangle + f|\varphi^\mp|^2. \end{array}$$

and

$$\begin{aligned}\mathrm{tr}\,(B^{\pm}) &= B^{\pm}(e_1,e_1)+B^{\pm}(e_2,e_2)\\ &= -\Re e\left\langle T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle+\frac{1}{2}\Re e\left\langle T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle-\Re e\left\langle f\varphi^{\mp},\varphi^{\mp}\right\rangle\\ &= -\frac{1}{2}\Re e\left\langle T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle-f|\varphi^{\mp}|^2.\end{aligned}$$

Then, we compute $Q^{\pm}_{\varphi}(e_1,e_2)$.

$$\begin{aligned}Q^{\pm}_{\varphi}(e_1,e_2) &= \Re e\left\langle\nabla_{e_1}\varphi^{\pm},e_2\cdot\varphi^{\mp}\right\rangle\\ &= \Re e\left\langle e_1\cdot\nabla_{e_1}\varphi^{\pm},e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle\\ &= \Re e\left\langle D\varphi^{\pm},\omega\cdot\varphi^{\mp}\right\rangle-\Re e\left\langle e_2\cdot\nabla_{e_2}\varphi^{\pm},e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle\\ &= \Re e\left\langle D\varphi^{\pm},\omega\cdot\varphi^{\mp}\right\rangle+Q^{\pm}_{\varphi}(e_2,e_1).\end{aligned}$$

Using (17), we obtain

$$\begin{aligned}\Re e\left\langle D\varphi^{\pm},\omega\cdot\varphi^{\mp}\right\rangle &= \Re e\left\langle H\varphi^{\mp},\omega\cdot\varphi^{\mp}\right\rangle-\frac{1}{2}\Re e\left\langle T\cdot\varphi^{\pm},\omega\cdot\varphi^{\mp}\right\rangle-f\Re e\left\langle\varphi^{\mp},\omega\cdot\varphi^{\mp}\right\rangle\\ &= -\frac{1}{2}\Re e\left\langle T\cdot\varphi^{\pm},\omega\cdot\varphi^{\mp}\right\rangle\\ &= \frac{1}{2}\Re e\left\langle\omega\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle.\end{aligned}$$

Finally, we compute $B^{\pm}(e_1,e_2)-B^{\pm}(e_2,e_1)$.

$$\begin{aligned}B^{\pm}(e_1,e_2)-B^{\pm}(e_2,e_1) &= -\frac{1}{2}\Re e\left\langle e_1\cdot T\cdot\varphi^{\pm},e_2\cdot\varphi^{\mp}\right\rangle-\frac{1}{2}\Re e\left\langle T_1\varphi^{\pm},e_2\cdot\varphi^{\mp}\right\rangle\\ &\quad-\frac{1}{2}\Re e\left\langle fe_1\cdot\varphi^{\mp},e_2\cdot\varphi^{\mp}\right\rangle+\frac{1}{2}\Re e\left\langle e_2\cdot T\cdot\varphi^{\pm},e_1\cdot\varphi^{\mp}\right\rangle\\ &\quad+\frac{1}{2}\Re e\left\langle T_2\varphi^{\pm},e_1\cdot\varphi^{\mp}\right\rangle+\frac{1}{2}\Re e\left\langle fe_2\cdot\varphi^{\mp},e_1\cdot\varphi^{\mp}\right\rangle\\ &= \frac{1}{2}\Re e\left\langle e_2\cdot e_1\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle-\frac{1}{2}\Re e\left\langle e_1\cdot e_2\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle\\ &\quad+\frac{1}{2}\Re e\left\langle(T_1e_2-T_2e_1)\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle\\ &= -\Re e\left\langle\omega\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle+\frac{1}{2}\Re e\left\langle\omega\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle\\ &= -\frac{1}{2}\Re e\left\langle\omega\cdot T\cdot\varphi^{\pm},\varphi^{\mp}\right\rangle.\end{aligned}$$

□

This lemma implies immediately that W is symmetric and trace-free. Now, we give this last lemma which shows that the rank of W is at most 1.

Lemma 4.3. *The endomorphism field W satisfies*

$$\Re e\left\langle W(X)\cdot\varphi^{-},\varphi^{+}\right\rangle=0$$

for any tangent vector field X.

Proof: First, since φ has constant norm, for any tangent vector field X, we have

$$\begin{aligned} 0 &= X|\varphi|^2 \\ &= X\left(|\varphi^+|^2 + |\varphi^-|^2\right) \\ &= 2\Re e\left(\nabla_X\varphi^+, \varphi^+\right) + 2\Re e\left(\nabla_X\varphi^-, \varphi^-\right) \\ &= 2\Re e\left(U(X)\cdot\varphi^-, \varphi^+\right), \end{aligned} \tag{19}$$

where $U(X) := \dfrac{Q^+_\varphi(X)}{|\varphi^-|^2} - \dfrac{Q^-_\varphi(X)}{|\varphi^+|^2}$. In order to simplify the notations, we set

$$U^+(X) := \frac{Q^+_\varphi(X)}{|\varphi^-|^2} \quad \text{and} \quad U^-(X) := \frac{Q^-_\varphi(X)}{|\varphi^+|^2}.$$

On the other hand, we denote

$$V^+(X) := \frac{B^+(X)}{|\varphi^-|^2} \quad , \quad V^-(X) := \frac{B^-(X)}{|\varphi^+|^2} \quad \text{and} \quad V = V^+ - V^-.$$

Let us compute $\Re e\left(V(X)\cdot\varphi^-, \varphi^+\right)$.

$$\begin{aligned} \Re e\left(V(X)\cdot\varphi^-, \varphi^+\right) &= \Re e\left(V^+(X)\cdot\varphi^-, \varphi^+\right) - \Re e\left(V^-(X)\cdot\varphi^-, \varphi^+\right) \\ &= \Re e\left(V^+(X)\cdot\varphi^-, \varphi^+\right) + \Re e\left(V^-(X)\cdot\varphi^+, \varphi^-\right). \end{aligned}$$

But ,

$$\Re e\left(V^+(X)\cdot\varphi^-, \varphi^+\right) = \Re e\left(V^+(X,e_1)e_1\cdot\varphi^-, \varphi^+\right) + \Re e\left(V^+(X,e_2)e_2\cdot\varphi^-, \varphi^+\right).$$

Now, we compute the term $V^+(X,e_j)$ for $j = 1, 2$.

$$\begin{aligned} V^+(X,e_j) &= -\frac{1}{2}\Re e\left\langle X\cdot T\cdot\varphi^+, e_j\cdot\frac{\varphi^-}{|\varphi^-|^2}\right\rangle - \frac{1}{2}\Re e\left\langle X,T\right\rangle\left\langle\varphi^+, e_j\cdot\frac{\varphi^-}{|\varphi^\mp|^2}\right\rangle \\ &\quad -\frac{1}{2}\Re e\left\langle fX\cdot\varphi^-, e_j\cdot\frac{\varphi^-}{|\varphi^-|^2}\right\rangle \end{aligned}$$

Since $\left\{e_1\cdot\frac{\varphi^-}{|\varphi^-|}, e_2\cdot\frac{\varphi^-}{|\varphi^-|}\right\}$ is a local orthonormal frame of $\Gamma(\Sigma^+N)$ for the scalar product $\Re e\left\langle .,.\right\rangle$, we deduce the following

$$V^+(X)\cdot\varphi^- = -\frac{1}{2}X\cdot T\cdot\varphi^+ - \frac{1}{2}\left\langle X,T\right\rangle\varphi^+ - \frac{1}{2}fX\cdot\varphi^-.$$

Then, we conclude that

$$\begin{aligned} \Re e\left\langle V^+(X)\cdot\varphi^-, \varphi^+\right\rangle &= -\frac{1}{2}\Re e\left\langle X\cdot T\cdot\varphi^+, \varphi^+\right\rangle - \frac{1}{2}\Re e\left\langle\left\langle X,T\right\rangle\varphi^+, \varphi^+\right\rangle \\ &\quad -\frac{1}{2}\Re e\left\langle fX\cdot\varphi^+, \varphi^-\right\rangle. \end{aligned}$$

We can easily see that for any vector field X,

$$\Re e\left\langle X\cdot T\cdot\varphi^+, \varphi^+\right\rangle + \Re e\left\langle\left\langle X,T\right\rangle\varphi^+, \varphi^+\right\rangle = 0,$$

which yields

$$\Re e\left\langle V^+(X)\cdot\varphi^-, \varphi^+\right\rangle = -\frac{1}{2}\Re e\left\langle fX\cdot\varphi^+, \varphi^-\right\rangle.$$

By the same way, we show

$$\Re e \left\langle V^-(X) \cdot \varphi^+, \varphi^- \right\rangle = -\frac{1}{2} \Re e \left\langle fX \cdot \varphi^-, \varphi^+ \right\rangle .$$

Finally, we conclude that

$$\Re e \left\langle V(X) \cdot \varphi^-, \varphi^+ \right\rangle = 0. \tag{20}$$

Since $W = U + V$, we deduce from (19) and (20) that

$$\Re e \left\langle W(X) \cdot \varphi^-, \varphi^+ \right\rangle = 0.$$

This achieves the proof of the lemma. □

The fact that

$$\Re e \left\langle W(X) \cdot \varphi^-, \varphi^+ \right\rangle = 0$$

implies that W has rank at most 1. Since, by Lemma 4.2, W is also symmetric and trace-free, we deduce the following

Proposition 4.4. *The endomorphisms field W is identically zero.*

Now, we can state the following proposition

Proposition 4.5. *If φ is a non-trivial solution of the Dirac equation (16) with constant norm, then φ is also a solution of Equation (12).*

Proof: We set $F := A^+ + A^-$. The fact that $W = 0$ implies

$$\frac{F}{|\varphi|^2} = \frac{A^+}{|\varphi^-|^2} = \frac{A^-}{|\varphi^+|^2}. \tag{21}$$

On the other hand, we have seen that for any vector field X,

$$\begin{aligned} \nabla_X \varphi &= U^+(X) \cdot \varphi^- + U^-(X) \cdot \varphi^+ \\ &= \frac{A^+}{|\varphi^-|^2} \cdot \varphi^- + \frac{A^-}{|\varphi^+|^2} \cdot \varphi^+ - V^+(X) \cdot \varphi^- - V^-(X) \cdot \varphi^+, \end{aligned}$$

which gives with (21)

$$\nabla_X \varphi = \frac{F(X)}{|\varphi|^2} \cdot \varphi - V^+(X) \cdot \varphi^- - V^-(X) \cdot \varphi^+. \tag{22}$$

The tensor F is symmetric because A^+ and A^- are symmetric. Then, by setting $A := -2F$, Equation (22) gives

$$\nabla_X \varphi = \frac{1}{2} X \cdot T \cdot \varphi + \frac{1}{2} fX \cdot \varphi + \frac{1}{2} \left\langle X, T \right\rangle \varphi - \frac{1}{2} AX \cdot \varphi.$$

This concludes the proof. □

4.3.2 The case $\eta = \frac{i}{2}$

The scheme of the proof is exactly the same as for $\eta = \frac{1}{2}$. The two minor differences are that the expression of the spinor field solution of the Dirac equation is different and the norm of this spinor is not constant but satisfies

$$X|\varphi|^2 = \Re e \left\langle iX \cdot T \cdot \varphi + ifX \cdot \varphi, \varphi \right\rangle.$$

We do not give the details but all the lemmas of last section have an analogue in the case $\eta = \frac{i}{2}$, and we obtain the following proposition.

Proposition 4.6. *If φ is a non-trivial solution of the Dirac equation (16) with its norm satisfying*

$$X|\varphi|^2 = \Re e \left\langle iX \cdot T \cdot \varphi + ifX \cdot \varphi, \varphi \right\rangle,$$

then φ is also a solution of Equation (12).

Indeed, as in the case $\eta = \frac{1}{2}$, we have this elementary lemma.

Lemma 4.7. *The endomorphisms $Q^{\pm}_{\varphi}$ and $B^{\pm}$ satisfy*

1. $\mathrm{tr}\,(Q^{\pm}_{\varphi}) = -H|\varphi^{\mp}|^2 + \frac{1}{2}\Re e \left\langle iT \cdot \varphi^{\pm}, \varphi^{\mp} \right\rangle.$
2. $\mathrm{tr}\,(B^{\pm}) = -\frac{1}{2}\Re e \left\langle iT \cdot \varphi^{\pm}, \varphi^{\mp} \right\rangle.$
3. $Q^{\pm}_{\varphi}(e_1, e_2) = Q^{\pm}_{\varphi}(e_2, e_1) \mp \frac{1}{2}\Re e \left\langle \omega \cdot T \cdot \varphi^{\pm}, \varphi^{\mp} \right\rangle \pm f|\varphi^{\mp}|^2,$
4. $B^{\pm}(e_1, e_2) = B^{\pm}(e_2, e_1) \pm \frac{1}{2}\Re e \left\langle \omega \cdot T \cdot \varphi^{\pm}, \varphi^{\mp} \right\rangle \mp f|\varphi^{\mp}|^2.$

The proof of this lemma is straightforward.

Now, we have all ingredients to prove Theorem 1.

4.4 Proof of Theorem 1

The proof is the same for the two cases.
The equivalence between (i) and (ii) have been proved in Section 4.2. On the other hand, it is clear that (ii) implies (iii). Finally, we have to show that (iii) implies (ii). We have seen in Section 4.3 that a spinor field solution of the Dirac equation (16) with the norm assumption is also solution of Equation (12). We just have to show that

$$\nabla_X T = fAX \quad \text{and} \quad df(X) = -\left\langle AX, T \right\rangle.$$

These two identities just come form the assumptions

$$\nabla_X T = 2fQ_{\varphi}(X) - fB(X) \quad \text{and} \quad df = -2Q_{\varphi}(T) + B(T),$$

together with the expression of $A = -2F = -2\frac{A^+ + A^-}{|\varphi|^2}$, with $A^{\pm}$ given by (18). This achieves the proof. ☐

5 Isometric immersions into $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$

In this section, we give a spinorial characterization for surfaces into $\mathbb{E}(\kappa,\tau)$ with $\tau \neq 0$.

5.1 Statement of the result

First, we state the main result about spinorial characterization of surfaces isometrically immersed into $\mathbb{E}(\kappa,\tau)$.

Theorem 2. *Let $\kappa,\tau \in \mathbb{R}$, $\tau \neq 0$ such that $\kappa \neq 4\tau^2$. We set $\alpha = 2\tau - \frac{\kappa}{2\tau}$. Let $(N,\langle .,.\rangle)$ be a connected, simply connected and oriented Riemannian surface. Let T be a vector field, f and H two functions on N satisying $f^2 + ||T||^2 = 1$. Let A be a field of symmetric endomorphisms on TN with $\mathrm{tr}\,(A) = 2H$. The following three statements are equivalent.*

i) There exists an isometric immersion F from N into $\mathbb{E}(\kappa,\tau)$ with mean curvature H such that the Weingarten operator related to the normal ν is given by

$$dF \circ A \circ dF^{-1}$$

and such that

$$\xi = dF(T) + f\nu.$$

ii) There exists a non-trivial spinor field φ solution of the equation

$$\nabla_X\varphi = -\frac{\tau}{2}X\cdot\omega\cdot\varphi + \frac{\alpha}{2}\langle X,T\rangle T\cdot\omega\cdot\varphi - \frac{\alpha}{2}f\langle X,T\rangle\omega\cdot\varphi - \frac{1}{2}AX\cdot\varphi,$$

where A satisfies

$$\nabla_X T = f(AX - \tau JX), \quad \text{and} \quad df(X) = -\langle AX - \tau JX, T\rangle.$$

iii) There exists a non-trivial spinor field φ solution of the Dirac equation

$$D\varphi = H\varphi + \tau\omega\cdot\varphi - \frac{\alpha}{2}||T||^2\omega\cdot\varphi - \frac{\alpha}{2}fT\cdot\omega\varphi,$$

with constant norm and such that

$$\nabla_X T = 2fQ_\varphi(X) + fB(T) + f\tau J(T)$$

and

$$df = -2Q_\varphi(T) - B(T) - \tau J(T),$$

where Q_φ is the energy-momentum tensor associated with φ and B is the tensor defined in an orthonormal frame $\{e_1,e_2\}$ by the following matrix

$$\begin{pmatrix} \alpha T_1T_2 & \frac{\alpha}{2}\left(T_1^2 - T_2^2\right) \\ \frac{\alpha}{2}\left(T_1^2 - T_2^2\right) & -\alpha T_1T_2 \end{pmatrix},$$

with $T_i = \langle T, e_i\rangle$.

The proof of Theorem 2 uses the same arguments as in the proof of Theorem 1 and Daniel's theorem [6] for isometric immersions into $\mathbb{E}(\kappa,\tau)$.

5.2 Special Spinor field and compatibility equations

Proposition 5.1. *If* $(N, \langle ., . \rangle)$ *adimts a spinor field solution of Equation* (14) *and if Equations* (9) *and* (10) *are fulfiled, then, the Gauss and Codazzi equations for* $\mathbb{E}(\kappa, \tau)$ *are satisfied.*

Proof : We compute the spinorial curvature of the spinor field φ solution of (14).

$$
\begin{aligned}
\nabla_X \nabla_Y \varphi \quad = \quad & \underbrace{-\frac{\tau}{2}(\nabla_X Y) \cdot \omega \cdot \varphi}_{\alpha_1(X,Y)} + \underbrace{\frac{\tau^2}{4} Y \cdot \ X \cdot \varphi}_{\alpha_2(X,Y)} - \underbrace{\frac{\alpha\tau}{4} \langle X, T \rangle Y \cdot T \cdot \varphi}_{\alpha_3(X,Y)} \\
& \underbrace{- \frac{\alpha\tau}{4} f \langle X, T \rangle Y \cdot \varphi}_{\alpha_4(X,Y)} - \underbrace{\frac{\tau}{4} Y \cdot AX \cdot \omega \cdot \varphi}_{\alpha_5(X,Y)} + \underbrace{\frac{\alpha}{2} \langle \nabla_X Y, T \rangle T \cdot \omega \cdot \varphi}_{\alpha_6(X,Y)} \\
& \underbrace{+ \frac{\alpha}{2} f \langle Y, AX \rangle T \cdot \omega \cdot \varphi}_{\alpha_7(X,Y)} - \underbrace{\frac{\alpha\tau}{2} f \langle Y, JX \rangle T \cdot \omega \cdot \varphi}_{\alpha_8(X,Y)} \\
& \underbrace{+ \frac{\alpha}{2} f \langle Y, T \rangle AX \cdot \omega \cdot \varphi}_{\alpha_9(X,Y)} - \underbrace{\frac{\alpha\tau}{2} f \langle Y, T \rangle JX \cdot \omega \cdot \varphi}_{\alpha_{10}(X,Y)} \\
& \underbrace{- \frac{\alpha\tau}{2} \langle Y, T \rangle T \cdot X \cdot \varphi}_{\alpha_{11}(X,Y)} - \underbrace{\frac{\alpha^2}{4} \langle X, T \rangle \langle Y, T \rangle ||T||^2 \varphi}_{\alpha_{12}(X,Y)} \\
& \underbrace{+ \frac{\alpha^2}{4} f \langle X, T \rangle \langle Y, T \rangle T \cdot \varphi}_{\alpha_{13}(X,Y)} + \underbrace{\frac{\alpha}{4} \langle Y, T \rangle T \cdot AX \cdot \omega \cdot \varphi}_{\alpha_{14}(X,Y)} \\
& \underbrace{+ \frac{\alpha}{2} \langle Y, T \rangle \langle AX, T \rangle \omega \cdot \varphi}_{\alpha_{15}(X,Y)} - \underbrace{\frac{\alpha\tau}{2} \langle Y, T \rangle \langle JX, T \rangle \omega \cdot \varphi}_{\alpha_{16}(X,Y)} \\
& \underbrace{- \frac{\alpha}{2} \langle \nabla_X Y, T \rangle \omega \cdot \varphi}_{\alpha_{17}(X,Y)} - \underbrace{\frac{\alpha}{2} f^2 \langle Y, AX \rangle \omega \cdot \varphi}_{\alpha_{18}(X,Y)} \\
& \underbrace{+ \frac{\alpha\tau}{2} f^2 \langle Y, JX \rangle \omega \cdot \varphi}_{\alpha_{19}(X,Y)} + \underbrace{\frac{\alpha\tau}{4} f \langle Y, T \rangle X \cdot \varphi}_{\alpha_{20}(X,Y)}
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\frac{\alpha^2}{4}f\langle X,T\rangle\langle Y,T\rangle T\cdot\varphi}_{\alpha_{21}(X,Y)}-\underbrace{\frac{\alpha^2}{4}f^2\langle X,T\rangle\langle Y,T\rangle\varphi}_{\alpha_{22}(X,Y)}\\
&-\underbrace{\frac{\alpha}{4}f\langle Y,T\rangle AX\cdot\omega\cdot\varphi}_{\alpha_{23}(X,Y)}-\underbrace{\frac{\tau}{2}\big(\nabla_X AY\big)\cdot\varphi}_{\alpha_{24}(X,Y)}\\
&+\underbrace{\frac{\tau}{4}AY\cdot X\cdot\omega\cdot\varphi}_{\alpha_{25}(X,Y)}-\underbrace{\frac{\alpha}{4}\langle X,T\rangle AY\cdot T\cdot\omega\cdot\varphi}_{\alpha_{26}(X,Y)}\\
&+\underbrace{\frac{\alpha}{4}\langle X,T\rangle AY\cdot\omega\cdot\varphi}_{\alpha_{27}(X,Y)}+\underbrace{\frac{1}{4}AY\cdot AX\cdot\varphi}_{\alpha_{28}(X,Y)}
\end{aligned}
$$

As in the previous case, lots of terms vanish by symmetry and for $X=e_1$ and $Y=e_2$, we have:

$$
\begin{aligned}
\mathcal{R}(e_1,e_2) &= \frac{1}{4}(Ae_2\cdot Ae_1-Ae_1\cdot Ae_2)\cdot\varphi-\frac{\tau^2}{2}\omega\cdot\varphi-\frac{1}{2}\mathrm{d}^\nabla(e_1,e_2)\\
&\quad +2\alpha\tau fT\cdot\omega\cdot\varphi+\alpha\tau f^2\omega\cdot\varphi.
\end{aligned}
$$

Using the fact that $\alpha\tau=2\tau^2-\frac{\kappa}{2}$ and the Ricci identity, we get

$$
\Big(\underbrace{R_{1212}-\det(A)-\tau^2-(\kappa-4\tau^2)f^2}_{G}\Big)\omega\cdot\varphi=\Big(\underbrace{\mathrm{d}^\nabla(e_1,e_2)+(\kappa-4\tau^2)fJT}_{C}\Big)\cdot\varphi,
$$

that is,

$$
G\omega\cdot\varphi=C\cdot\varphi,
$$

where G is a real function and C is a vector field. Moreover, since $\omega\cdot\varphi=-i\overline{\varphi}$, we deduce that

$$
C\cdot\varphi^\pm=\pm iG\varphi^\mp,
$$

and so

$$
||C||^2\varphi^\pm=-G^2\varphi^\pm.
$$

Since φ is a solution of (14), its norm is constant. Therefore, φ^+ and φ^- can not vanish at the same point which yields $C=0$ and $G=0$. But $G=0$ is the Gauss equation and $C=0$ is the Codazzi equation. This achieves the proof. □

This proposition proves together with Daniel's result that *ii*) implies *i*) in Theorem 2. Note that *i*) implies *ii*) is obvious from the section of preliminaries. Now, we will show that *iii*) implies *ii*).

5.3 Special spinor fields and Dirac equation

If a spinor field φ is a solution of Equation (14), then, we deduce immediately that φ is also a solution of the corresponding Dirac equation

$$
D\varphi=H\varphi+\tau\omega\cdot\varphi-\frac{\alpha}{2}||T||^2\omega\cdot\varphi-2\eta fT\omega\cdot\varphi. \tag{23}
$$

In this section, we will see that this Dirac equation is equivalent to Equation (14) for spinors of constant norm. We set $\varphi=\varphi^++\varphi^-$, with $\varphi^\pm\in\Sigma^\pm N$. Then, we have

$$D\varphi^\pm=H\varphi^\mp+\tau\omega\cdot\varphi^\mp-\frac{\alpha}{2}||T||^2\omega^\mp\cdot\varphi-2\eta fT\omega\cdot\varphi^\pm. \tag{24}$$

Now, we define the following endomorphisms

$$Q^\pm_\varphi(X,Y)=\Re e\left\langle\nabla_X\varphi^\pm,Y\cdot\varphi^\mp\right\rangle,$$

and

$$B^\pm(X,Y)=\mp\Re e\left(i\frac{\alpha}{2}<X,T>T\cdot\varphi^\mp,Y\cdot\varphi^\mp\right)\mp\Re e\left(i\frac{\alpha}{2}f<X,T>\cdot\varphi^\pm,Y\cdot\varphi^\mp\right)$$

On the other hand, we set

$$A^\pm=Q^\pm_\varphi+B^\pm, \tag{25}$$

and

$$W=\frac{A^+}{|\varphi^-|^2}-\frac{A^-}{|\varphi^+|^2}.$$

From Equation (24), we deduce easily that

Lemma 5.2. *1.* $\mathrm{tr}\left(Q^\pm_\varphi\right)=-H|\varphi^\pm|^2\pm\frac{\alpha}{2}f\Re e\left(iT\cdot\varphi^\pm,\varphi^\mp\right)$,

2. $Q^\pm_\varphi(e_1,e_2)=Q^\pm_\varphi(e_2,e_1)+\left(\tau-\frac{\alpha}{2}||T||^2\right)|\varphi^\mp|^2+\frac{\alpha}{2}f\Re e\left(T\cdot\varphi^\pm,\varphi^\mp\right)$,

3. $\mathrm{tr}\left(B^\pm\right)=\mp\frac{\alpha}{2}f\Re e\left(iT\cdot\varphi^\pm,\varphi^\mp\right)$,

4. $B^\pm(e_1,e_2)=B^\pm(e_2,e_1)+\frac{\alpha}{2}||T||^2|\varphi^\mp|^2-\frac{\alpha}{2}f\Re e\left(T\cdot\varphi^\pm,\varphi^\mp\right)$.

Proof : The proof is similar to the proof of Lemma 4.2 but we use here Equation (24). □

Thus, we deduce immediately the following:

Lemma 5.3. *The endomorphisms $A^\pm$ satisfy*

1. $\mathrm{tr}\left(A^\pm\right)=-H|\varphi^\mp|^2$,

2. $A^\pm(e_1,e_2)=A^\pm(e_2,e_1)+\tau|\varphi^\mp|^2$.

Now, we give a next lemma showing that W is of rank at most 1.

Lemma 5.4. *The endomorphism field W satisfies*

$$\Re e\left\langle W(X)\cdot\varphi^+,\varphi^-\right\rangle=0.$$

Proof : First of all, since φ is of constant norm, for any tangent vector field X, we have

$$\begin{aligned}0&=X|\varphi|^2\\&=X\left(|\varphi^+|^2+|\varphi^-|^2\right)\\&=2\Re e\left(\nabla_X\varphi^+,\varphi^+\right)+2\Re e\left(\nabla_X\varphi^-,\varphi^-\right)\\&=2\Re e\left(U(X)\cdot\varphi^-,\varphi^+\right),\end{aligned} \tag{26}$$

where $U(X) := \dfrac{Q^+_\varphi(X)}{|\varphi^-|^2} - \dfrac{Q^-_\varphi(X)}{|\varphi^+|^2}$. So we set

$$U^+(X) := \frac{Q^+_\varphi(X)}{|\varphi^-|^2} \quad \text{et} \quad U^-(X) := \frac{Q^-_\varphi(X)}{|\varphi^+|^2},$$

and

$$V^+(X) := \frac{B^+(X)}{|\varphi^-|^2}, \quad V^-(X) := \frac{B^-(X)}{|\varphi^+|^2} \quad \text{et} \quad V = V^+ - V^-.$$

Now, we compute $\Re e\,(V(X)\cdot\varphi^-,\varphi^+)$.

$$\begin{aligned}\Re e\left(V(X)\cdot\varphi^-,\varphi^+\right) &= \Re e\left(V^+(X)\cdot\varphi^-,\varphi^+\right) - \Re e\left(V^-(X)\cdot\varphi^-,\varphi^+\right)\\ &= \Re e\left(V^+(X)\cdot\varphi^-,\varphi^+\right) + \Re e\left(V^-(X)\cdot\varphi^+,\varphi^-\right).\end{aligned}$$

But,

$$\Re e\left(V^\pm(X)\cdot\varphi^\mp,\varphi^\pm\right) = \Re e\left(V^\pm(X,e_1)e_1\cdot\varphi^\mp,\varphi^\pm\right) + \Re e\left(V^\pm(X,e_2)e_2\cdot\varphi^\mp,\varphi^\pm\right).$$

Moreover, we have for $j = 1, 2$

$$V^\pm(X,e_j) = \mp\frac{\alpha}{2}\Re e\left(i < X,T > T\cdot\varphi^\mp, e_j\cdot\frac{\varphi^\mp}{|\varphi^\mp|^2}\right) - \frac{\alpha}{2}\Re e\left(if < X,T > \cdot\varphi^\pm, e_j\frac{\varphi^\mp}{|\varphi^\mp|^2}\right)$$

Since $\left\{e_1\cdot\frac{\varphi^\mp}{|\varphi^\mp|}, e_2\cdot\frac{\varphi^\mp}{|\varphi^\mp|}\right\}$ is a local orthonormal frame of $\Gamma(\Sigma^\pm N)$ for the inner product $\Re e\,\langle .,.\rangle$, we deduce that

$$V^\pm(X)\cdot\varphi^\mp = \mp i\frac{\alpha}{2}\langle X,T\rangle T\cdot\varphi^\mp \mp i\frac{\alpha}{2}f\langle X,T\rangle\varphi^\pm.$$

Thus, we deduce easily that

$$\Re e\left\langle V^\pm(X)\cdot\varphi^\mp,\varphi^\pm\right\rangle = \mp\frac{\alpha}{2}\Re e\left\langle\langle X,T\rangle T\cdot\varphi^\mp,\varphi^\pm\right\rangle.$$

So, we conclude that

$$\Re e\left(V(X)\cdot\varphi^-,\varphi^+\right) = 0 \tag{27}$$

Since $W = U + V$, we deduce from (26) and (27) que

$$\Re e\left(W(X)\cdot\varphi^-,\varphi^+\right) = 0.$$

This achieves the proof. □

Now, since W is a symmetric 2-tensor with rank at most 1 and trace-free then W vanishes identically.

We conclude this section with this last proposition.

Proposition 5.5. *If φ is a non-trivial solution of the Dirac equation (23) with constant norm, then φ is a solution of Equation (14).*

Preuve: We set $F := A^+ + A^-$. The fact that $W = 0$ implies

$$\frac{F}{|\varphi|^2} = \frac{A^+}{|\varphi^-|^2} = \frac{A^-}{|\varphi^+|^2}. \tag{28}$$

Moreover, we have already seen that for any vector field X,

$$\begin{array}{rcl}\nabla_X\varphi &=& U^+(X)\cdot\varphi^- + U^-(X)\cdot\varphi^+\\ &=& \dfrac{A^+}{|\varphi^-|^2}\cdot\varphi^- + \dfrac{A^-}{|\varphi^+|^2}\cdot\varphi^+ - V^+(X)\cdot\varphi^- - V^-(X)\cdot\varphi^+,\end{array}$$

which from (28) yields

$$\nabla_X\varphi = \frac{F(X)}{|\varphi|^2}\cdot\varphi - V^+(X)\cdot\varphi^- - V^-(X)\cdot\varphi^+. \tag{29}$$

We note that the tensor F is not symmetric, so we consider the following symmetric $(2,0)$-tensor S:

$$S(X,Y) := -\frac{1}{2|\varphi|^2}\left(F(X,Y) + F(Y,X)\right).$$

Moreover, from Lemma 5.3, we get

$$\left\{\begin{array}{l} S(e_1,e_1) = -F(e_1,e_1)/|\varphi|^2,\\ S(e_2,e_2) = -F(e_2,e_2)/|\varphi|^2,\\ S(e_1,e_2) = -F(e_1,e_2)/|\varphi|^2 + \frac{\tau}{2},\\ S(e_2,e_1) = -F(e_2,e_1)/|\varphi|^2 - \frac{\tau}{2}.\end{array}\right. \tag{30}$$

From this relations and (29), we deduce

$$\nabla_X\varphi = -\frac{\tau}{2}X\cdot\omega\cdot\varphi + \frac{\alpha}{2}\langle X,T\rangle T\cdot\omega\cdot\varphi - \frac{\alpha}{2}f\langle X,T\rangle\omega\cdot\varphi - SX\cdot\varphi,$$

that is, φ is solution of (14) if we set $A = 2S$. □

5.4 Proof of Theorem 2

Now, we have all ingredients to finish the proof of Theorem 2. Equivalence between points (*i*) and (*ii*) has been proved in Section 5.2 and clearly, (*ii*) implies (*iii*) by taking the trace. The last thing to prove is that (*iii*) implies (*ii*). We have proved in Section 5.3 that a spinor field solution of the Dirac equation (23) and of constant norm is solution of Equation (14). The last points to check is that

$$\nabla_X T = f(AX - \tau JX) \quad\text{and}\quad df(X) = -\langle AX - \tau JX, T\rangle.$$

Here again, from the expression of $A = 2S$ given by (30) and (25) and the assumptions

$$\nabla_X T = 2fQ_\varphi(X) + fB(T) + f\tau J(T) \quad\text{and}\quad df = -2Q_\varphi(T) - B(T) - \tau J(T),$$

we get the wanted identities. □

6 Two final remarks

6.1 Other special spinor fields on surfaces of $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$

We want to point out that the choice of our special spinor fields in $\mathbb{E}(\kappa,\tau)$ with $\tau \neq 0$ is not the only possible choice. Indeed, we could have proceed as in the case of products, that is $\tau = 0$, by considering a Killing spinor on the base of the fibration and lift it to the ambient space. One can refer to [17] for the study of spinors on Riemannian submersions.
This choice gives us a unitary approach for any τ. Nevertheless, with this approach, we obtain special spinor fields on surfaces of $\mathbb{E}(\kappa,\tau)$, $\tau \neq$ with more complicated derivatives. Indeed, we have the following spinor field on $\mathbb{E}(\kappa,\tau)$:

$$(31) \qquad \begin{cases} \overline{\nabla}_{e_1}\varphi = -\eta e_2 \cdot \varphi, \\ \\ \overline{\nabla}_{e_2}\varphi = \eta e_1 \cdot \varphi, \\ \\ \overline{\nabla}_{\xi}\varphi = \frac{\tau}{2}\xi \cdot \varphi, \end{cases}$$

Note that for $\tau = 0$, we have the spinor field on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ given in (11). Thus, after restriction to a surface of $\mathbb{E}(\kappa,\tau)$, we obtain the following spinor field:

$$(32) \qquad \begin{aligned} \nabla_X\varphi \;=\; & -\frac{1}{2}A(X)\cdot\varphi + \eta X\cdot\ T\cdot\varphi + \eta f X\cdot\varphi + \eta f < X,T > \varphi \\ & -\frac{\tau}{2} < X,T > T\cdot\omega\cdot\varphi + \frac{\tau}{2}f < X,T > \omega\cdot\varphi, \end{aligned}$$

and therefore

$$(33) \qquad D\varphi = H\varphi - 2\eta T\cdot\varphi - 2\eta f\varphi + \eta f T\cdot\varphi + \frac{\tau}{2}||T||^2\omega\cdot\varphi + \frac{\tau}{2}fT\cdot\omega\cdot\varphi,$$

with $4\eta^2 = \kappa$. Note that if $\tau = 0$, we have exactly Equations (12) and (16) for surfaces of $\mathbb{M}^2(\kappa) \times \mathbb{R}$.
In the case where $\tau \neq 0$, the computation of the spinorial curvature of a spinor field solution of (32) is very long and the equivalence between Equations (32) and (33) with a norm assumption is very technical.
Thus, although we are conviced that both cases $\tau = 0$ and $\tau \neq 0$ can be treated in a unified way, we have decided, for a sake of clarity, to work with spinors obtained by constant sections to make the proof more readable in the case $\tau \neq 0$.

6.2 The Lawson correspondence in 3-homogeneous manifolds

In [6], Daniel gives a Lawson correspondence for cmc-surfaces into 3-homogeneous manifolds. The Lawson correspondence is a local isometric correspondence between constant mean curvature surfaces into two different 3-homogeneous manifolds. Precsely, let $\mathbb{E}(\kappa_1,\tau_1)$ and $\mathbb{E}(\kappa_2,\tau_2)$ two 3-dimensional homogeneous manifolds with 4-dimensional isometry group with $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$. Then, a surface of constant mean curvature H_1 in $\mathbb{E}(\kappa_1,\tau_1)$ has a sister surface of constant mean curvature H_2 in $\mathbb{E}(\kappa_2,\tau_2)$ with $\tau_1^2 + H_1^2 = \tau_2^2 + H_2^2$. In particular, this gives a local isometric correspondence between minimal surfaces in the

Heisenberg group and cmc $\frac{1}{2}$ surfaces into $\mathbb{H}^2 \times \mathbb{R}$.
This result is a generalization of the Lawson correspondence for cmc surfaces in space forms. It would be very interesting to understand these two correspondences in terms of spinors. Note that the classical Lawson correspondence (for cmc-surfaces in space forms) is still not well understood via spinors.

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