ON COMPACT ANISOTROPIC WEINGARTEN HYPERSURFACES IN EUCLIDEAN SPACE

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ABSTRACT. We show that, up to homotheties and translations, the Wulff shape $\mathcal{W}_F$ is the only compact embedded hypersurface of the Euclidean space satisfying $H_r^F = aH^F + b$ with $a \geqslant 0$, $b > 0$, where H^F and H_r^F are respectively the anisotropic mean curvature and anisotropic r-th mean curvature associated with the function $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$. Further, we show that if the L^2-norm of $H_r^F - aH^F - b$ is sufficiently close to 0 then M is $W^{2,2}$ close to the Wulff shape.

1. INTRODUCTION

The classical Alexandrov theorem [1] says that a closed embedded hypersurface of the Euclidean space must be a round sphere. The hypothesis, that the hypersurface should be embedded, is crucial as proved by the counter examples of Wente [16], Kapouleas [10] or Hsiang-Teng-Yu [8]. Further, this result has been extended to scalar curvature and then higher order mean curvatures by Ros [12, 13] as well as for any concave function of the principal curvatures by Korevaar [11]. For higher order mean curvatures, the necessity of the embedding is still an open question.

Very recently, de Lima [2] proved a comparable result for the so called linear Weingarten hypersurfaces satisfying $H_r = aH + b$ for two real constants $a \geqslant 0$ and $b > 0$, where H and H_r are respectively the mean curvature and the r-th mean curvature of the hypersurfaces. The hypersurfaces are supposed to be embedded in this result too.

In the present paper, we extend this result to higher order anisotropic mean curvatures and then, we study its stability.

Let $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ be a smooth function satisfying the following convexity assumption

$$(1) \qquad\qquad A_F = (\nabla dF + F\mathrm{Id}_{|T_x\mathbb{S}^n})_x > 0,$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F and > 0 means positive definite in the sense of quadratic forms. Now, we consider the following map

$$\begin{aligned} \phi : \quad \mathbb{S}^n &\longrightarrow \quad \mathbb{R}^{n+1} \\ x &\longmapsto \quad F(x)x + (\mathrm{grad}_{|S^n}F)_x. \end{aligned}$$

The image $\mathcal{W}_F = \phi(\mathbb{S}^n)$ is called the Wulff shape of F and is a smooth convex hypersurface of $\mathbb{R}^{n+1}$ due to condition (1). It is to note that if $F = 1$, then the Wulff shape is the sphere $\mathbb{S}^n$.

Let $X : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion of n-dimensional closed,

connected and oriented Riemannian manifold M into $\mathbb{R}^{n+1}$. We denote by ν a normal unit vector field globally defined on M, that is, we have $\nu : M \longrightarrow \mathbb{S}^n$. We set $S_F = -A_F \circ d\nu$, where A_F is defined in (1). The operator S_F is called the F-Weingarten operator or anisotropic shape operator. In this anisotrpic setting, we can define all the corresponding extrinsic quantities like anisotropic principal curvatures, anisotropic mean curvature and higher order mean curvatures (see the preliminaries section for the precise definitions).

All the above mentioned results by Alexandrov and Ros have analogues for anisotropic mean curvatures with the Wulff shape replacing the sphere (see [7]).

The first result of this paper is an isotropic version of the result of de Lima for linear Weingerten hypersurfaces. Namely, we have the following.

Theorem 1.1. *Let $n \geqslant 2$ be an integer, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ a smooth function satisfying the convexity assumption (1) and let M be a closed, connected and embedded hypersurface of $\mathbb{R}^{n+1}$. Assume that the higher order anisotropic mean curvature H_r^F, $r \in \{2, \cdots n\}$ never vanishes and satisfies $H_r^F = aH^F + b$ for some real constants $a \geqslant 0$ and $b > 0$. Then, up to translations and homotheties, M is the Wulff shape $\mathcal{W}_F$.*

Here, a natural question arises related to the stability of the characterization in Theorem 1.1. We can ask, if H_r^F is almost equal to $aH^F + b$ (in some sense to be precise), then the hypersurface will be close to the Wulff shape $\mathcal{W}_F$ or not ?

In order to give an answer to this question, we introduce the following convenient notation. For three positive real numbers r_0, h_1 and h_2, we define $\mathcal{M}(r_0, h_1, h_2)$ as the set of all closed, connected and embedded hypersurface of $\mathbb{R}^{n+1}$ so that the extrinsic radius, that is, the radius of the smallest closed ball containing M, is smaller than r_0 whereas the first and second order anisotropic mean curvatures H^F and H_2^F satisfy

$$\|H^F\|_\infty \leqslant h_1 \quad \text{and} \quad \inf(H_2^F) \geqslant h_2.$$

Now, we can prove the following result.

Theorem 1.2. *Let $n \geqslant 2$ be an integer, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ a smooth function satisfying the convexity assumption (1) and let $M \in \mathcal{M}(r_0, h_1, h_2)$. Assume that the second order anisotropic mean curvature H_2^F never vanishes and satisfies $H_2^F = aH^F + b + \varepsilon$ for some real constants $a \geqslant 0$, $b > 0$ and ε a smooth function. Set $\rho = \left(\frac{V(M)}{V(\mathcal{W}_F)} \right)^{\frac{1}{n}}$. Then there exists a smooth parametrisation $\psi : \mathcal{W}_{\rho F} \longrightarrow M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and an explicit constant K depending on n, F, r_0, h_1 and h_2 so that*

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,2}(\mathcal{W}_{\rho F})} \leqslant K\|\varepsilon\|_2.$$

Here $\mathcal{W}_{\rho F} = \rho\mathcal{W}_F$ is just the image of $\mathcal{W}_F$ by the homothety of center 0 and ratio ρ.

This result is an improvement of the result obtained by the first author in [14] and [15] for almost constant mean curvature and scalar curvature in the L^2-sense. In particular, if F is constant, we get the following corollary for the anisotropic case.

Corollary 1.3. *Let $n \geqslant 2$ be an integer and assume that $M \in \mathcal{M}(r_0, h_1, h_2)$. Assume that the second order anisotropic mean curvature H_2 never vanishes and satisfies $H_2 = aH + b + \varepsilon$ for some constants $a \geqslant 0$, $b > 0$ and ε a smooth function.*

Set $\rho = \left(\frac{V(M)}{V(\mathbb{S}^n)}\right)^{\frac{1}{n}}$ *Then there exists a smooth parametrisation* $\psi : \mathbb{S}^n(\rho) \longrightarrow M$, *a vector* $c_0 \in \mathbb{R}^{n+1}$ *and a constant* K' *depending on* n, r_0, h_1 *and* h_2 *so that*

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,2}(\mathbb{S}^n(\rho))} \leqslant K'\|\varepsilon\|_2.$$

This result has also to be compared with the results of [14] and [15]. In particular for $a = 0$, this gives a stability result associated with the Alexandrov theorem of H_2.x

2. Preliminaries

Here, we recall the basics of anisotropic mean curvatures. These facts are classical, hence, we will not write their proofs. First, let $F : \mathbb{S}^n \longrightarrow \mathbb{R}^*_+$ be a smooth function satisfying the following convexity assumption (1)

$$A_F = (\nabla dF + F\mathrm{Id}_{T_x\mathbb{S}^n})_x > 0$$

at any point $x \in \mathbb{S}^n$, in the sense of quadratic forms and where ∇dF is the Hessian of F. Now, we consider the following map

$$\begin{aligned}
\phi : \mathbb{S}^n &\longrightarrow \mathbb{R}^{n+1} \\
x &\longmapsto F(x)x + (\mathrm{grad}_{\mathbb{S}^n} F)_x.
\end{aligned}$$

The image $\mathcal{W}_F = \phi(\mathbb{S}^n)$, which is also a smooth hypersurface of $\mathbb{R}^{n+1}$, is called the Wulff shape of F. Moreover, from the convexity condition (1), $\mathcal{W}_F$ is convex. Note that if F is a positive constant c, the Wulff shape $\mathcal{W}_F$ is just the sphere of radius c.

Let (M^n, g) be a closed, connected, oriented Riemannian manifold isometrically immersed into $\mathbb{R}^{n+1}$ by X and denote by ν its Gauss map. The (real-valued) second fundamental form B of the immersion is defined by

$$B(Y, Z) = \left\langle \overline{\nabla}_Y \nu, Z \right\rangle,$$

for any $Y, Z \in \Gamma(TM)$, where $\langle \cdot, \cdot \rangle$ and $\overline{\nabla}$ are respectively the Riemannian metric and the Riemannian connection of $\mathbb{R}^{n+1}$. We also denote by S the Weingarten operator, which is the $(1, 1)$-tensor associated with B via the metric g.

We consider $N_F = \phi(\nu) : M \longrightarrow \mathcal{W}_F$, the anisotropic Gauss map of M. We set $S_F = -dN_F = -A_F \circ d\nu = A_F \circ S$ which is the anisotropic shape operator, also called F-Weingarten operator and its eigenvalues $\kappa_1, \cdots, \kappa_n$ are the anisotropic principal curvatures. Now let us recall that the anisotropic higher order mean curvatures H_r^F are defined by

$$H_r^F = \frac{1}{\binom{n}{r}} \sigma_r(S_F),$$

where $\sigma_r(S_F)$ is the r-th elementary symmetric polynomial with n variables computed for anisotropic principal cruvatures $\kappa_1, \cdots, \kappa_n$.

We denote simply by H^F the anisotropic mean curvature H_1^F. Moreover, for convenience, we set $H_0^F = 1$ and $H_{n+1}^F = 0$ by convention. For the Wulff shape $\mathcal{W}_F$, $\kappa_1 = \kappa_2 = \cdots = \kappa_n$ are constants and so all the anisotropic mean curvatures H_r^F are also constants. Moreover, if $\kappa_1 = \kappa_2 = \cdots = \kappa_n$, then the hypersurface has to be the Wulff shape (up to homotheties and translations).

Lemma 2.1. *Let* $r \in \{1, \cdots n - 1\}$. *If* $H_{r+1} > 0$ *then*

    (1) $H_j^F > 0$ *for all* $j \in \{1, \cdots, r\}$,

    (2) $\left(H_r^F\right)^{\frac{1}{r}} \leqslant \left(H_{r-1}^F\right)^{\frac{1}{r-1}} \leqslant \cdots \leqslant \left(H_2^F\right)^{\frac{1}{2}} \leqslant H^F$. *Moreover, in any of these inequalities, equality occurs at a point p if and only if all the anisotropic principal curvatures at p are equal. Hence, equality occurs everywhere if and only if M is the Wulff shape $\mathcal{W}_F$, up to translations and homotheties.*

We want to point out that the error term in the inequality $\left(H_2^F\right)^{\frac{1}{2}} \leqslant H^F$ is well known and easy to compute since we have

$$(2) \qquad\qquad n(n-1)\left(\left(H^F\right)^2 - H_2^F\right) = ||\tau_F||^2,$$

where $\tau_F = S_F - H^F \mathrm{Id}$ is the anisotropic umbilicity tensor.

    We also recall the anisotropic analogue of the classical Hsiung-Minkowski formulas [9]. The proof can be found in [6] for instance.

Lemma 2.2. *Let $r \in \{0, \cdots n-1\}$. Then, we have*

$$\int_M \left(F(\nu)H_r^F + H_{r+1}^F \langle X, \nu \rangle \right)dv_g.$$

    Note that for the particular case $r = 0$, we get the following

$$(3) \qquad\qquad \int_M H^F \langle X, \nu \rangle dv_g = -\int_M F(\nu)dv_g.$$

For more convenience, we will denote $V_F(M) = \displaystyle\int_M F(\nu)dv_g$.

Finally, we recall these two useful results in the case of hypersurfaces which are embedded and so, bound some domains in $\mathbb{R}^{n+1}$. First, we have this classical identity obtained directly by the divergence formula

$$(4) \qquad\qquad \int_M \langle X, \nu \rangle dv_g = -(n+1)V(\Omega),$$

where $V(\Omega)$ is the volume of the domain Ω bounded by M. Secondly, we have the anisotropic analogue of the well-known Heintze-Karcher inequality.

Lemma 2.3 ([6])**.** *If M is embedded (so bounds a domain Ω) and H^F is everywehere positive, then the following inequality holds*

$$\int_M \frac{F(\nu)}{H^F}dv_g \geqslant (n+1)V(\Omega),$$

with equality if and only if M is the Wulff shape $\mathcal{W}_F$ (up to translations and homotheties).

    Finally, we add that we use the following convention for the L^p-norms. For a smooth function f defined on M, the L^p-norm of f is defined by

$$\|f\|_p = \left(\frac{\int_M |f|^p dv_g}{V(M)}\right)^{\frac{1}{p}}.$$

3. Proof of Theorem 1.1

Since M is compact and embedded, it bounds a domain in $\mathbb{R}^{n+1}$. We denote this domain by Ω. Moreover, always by compactness, M has at least an elliptic point. Since H_r^F never vanishes, we can assume by connectedness and the positivity of A_F that H_r^F is positive everywhere (by choosing ν as the inner normal unit vector). Now, from the anisotropic Hsiung-Minkowski formula, we have

$$
\begin{aligned}
0 &= \int_M \left(H_{r-1}^F F(\nu) + H_r^F \langle X, \nu \rangle \right) dv_g \\
&= \int_M \left(H_{r-1}^F F(\nu) + a H^F \langle X, \nu \rangle + b \langle X, \nu \rangle \right) dv_g,
\end{aligned}
\tag{5}
$$

by the assumption $H_r^F = a H^F + b$. Moreover, by the Hsiung-Minkowski formula again for H^F and the divergence theorem, we get that

$$
\int_M H^F \langle X, \nu \rangle dv_g = -V_F(M), \quad and \quad \int_M \langle X, \nu \rangle dv_g = -(n+1)V(\Omega),
$$

which gives after substitution in (5)

$$
\int_M H_{r-1}^F F(\nu) dv_g - a V_F(M) - b(n+1)V(\Omega) = 0.
\tag{6}
$$

Since H_r^F is positive everywhere, by Lemma 2.1, H_{r-1}^F is also positive everywhere and $\left(H_{r-1}^F \right)^{\frac{1}{r-1}} \geqslant \left(H_r^F \right)^{\frac{1}{r}}$ with equality if and only if the point is anisotropically umbilical. Hence, we get

$$
a V_F(M) + b(n+1)V(\Omega) \geqslant \int_M F(\nu) \left(H_r^F \right)^{\frac{r-1}{r}} dv_g.
$$

Writing $\left(H_r^F \right)^{\frac{r-1}{r}} = H_r^F \left(H_r^F \right)^{\frac{-1}{r}}$ and using $H_r^F = a H^F + b$ again, we obtain

$$
a V_F(M) + b(n+1)V(\Omega) \geqslant a \int_M F(\nu) H^F \left(H_r^F \right)^{-\frac{1}{r}} dv_g + b \int_M F(\nu) \left(H_r^F \right)^{-\frac{1}{r}} dv_g.
$$

Since a, b are nonnegative numbers and F is a positive function, we get, from $\left(H_r^F \right)^{\frac{1}{r}} \leqslant H^F$, the following

$$
a V_F(M) + b(n+1)V(\Omega) \geqslant a V_F(M) + b \int_M \frac{F(\nu)}{H^F} dv_g.
$$

Finally, we use the anisotropic version of the Heintze-Karcher inequality

$$
\int_M \frac{F(\nu)}{H^F} dv_g \geqslant (n+1)V(\Omega)
$$

to get

$$
a V_F(M) + b(n+1)V(\Omega) \geqslant a V_F(M) + b(n+1)V(\Omega),
$$

which means that all the previous inequalities are in fact equalities. In particular, we have $\left(H_{r-1}^F \right)^{\frac{1}{r-1}} \geqslant \left(H_r^F \right)^{\frac{1}{r}}$ at every point and so M is totally anisotropically umbilical. Hence, up to homotheties and translations, M is the Wulff shape $\mathcal{W}_F$. This concludes the proof. $\qquad \square$

4. PROOF OF THEOREM 1.2

For the sake of clarity, we first proof the following lemma.

Lemma 4.1. *Let $n \geqslant 2$ be an integer, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ be a smooth function satisfying the convexity assumption (1) and let M be a closed connected and embedded hypersurface of $\mathbb{R}^{n+1}$. Assume that the second order anisotropic mean curvature H_2^F never vanishes and satisfies $H_2^F = aH^F + b + \varepsilon$ for some real constants $a \geqslant 0$, $b > 0$ and ε a smooth function. Then we have*

$$\|\tau_F\|_2^2 \leqslant 2n(n-1) \left(\frac{\sup(F(\nu))}{\inf\left((H_2^F)\right)^{\frac{1}{2}}} + R \right) \frac{\|H^F\|_\infty}{\inf(F(\nu))} \|\varepsilon\|_2^2.$$

Proof: By the Hsiung-Minkowski formula for H_2^F and H_F

$$(7) \qquad \int_M \left(F(\nu)H_1^f + \langle X, \nu \rangle \right) dv_g = 0$$

and using the assumption $H_2^F = aH^F + b + \varepsilon$, we have

$$
\begin{aligned}
0 &= \int_M \left(F(\nu)H^F + aH^F\langle X,\nu\rangle + b\langle X,\nu\rangle + \varepsilon\langle X,\nu\rangle \right) dv_g \\
(8) \qquad &= \int_M F(\nu)H^F dv_g - aV_F(M) - b(n+1)V(\Omega) + \int_M \varepsilon\langle X,\nu\rangle dv_g,
\end{aligned}
$$

where we have used both relations (3) and (4).
Now, from (2), we have

$$H^F = \left(H_2^F\right)^{\frac{1}{2}} + \frac{\|\tau_F\|^2}{n(n-1)\left(H^F + \left(H_2^F\right)^{\frac{1}{2}}\right)}.$$

Using this in (8) gives

$$
\begin{aligned}
0 &= \int_M \left(H_2^F\right)^{\frac{1}{2}} F(\nu) dv_g + \frac{1}{n(n-1)} \int_M \frac{F(\nu)\|\tau_F\|^2}{H^F + \left(H_2^F\right)^{\frac{1}{2}}} dv_g \\
(9) \qquad &\quad - aV_F(M) - b(n+1)V(\Omega) + \int_M \varepsilon\langle X,\nu\rangle dv_g.
\end{aligned}
$$

Now, we consider the first term of the right hand side. We have

$$
\begin{aligned}
\int_M \left(H_2^F\right)^{\frac{1}{2}} F(\nu) dv_g &= \int_M H_2^F \left(H_2^F\right)^{-\frac{1}{2}} F(\nu) dv_g \\
&= a\int_M F(\nu)H^F \left(H_2^F\right)^{-\frac{1}{2}} dv_g + b\int_M F(\nu)\left(H_2^F\right)^{-\frac{1}{2}} dv_g \\
&\quad + \int_M \varepsilon F(\nu)\left(H_2^F\right)^{-\frac{1}{2}} dv_g \\
&\geqslant aV_F(M) + b\int_M \frac{F(\nu)}{H^F} dv_g + \int_M \varepsilon F(\nu)\left(H_2^F\right)^{-\frac{1}{2}} dv_g \\
(10) \qquad &\geqslant aV_F(M) + b(n+1)V(\Omega) + \int_M \varepsilon F(\nu)\left(H_2^F\right)^{-\frac{1}{2}} dv_g
\end{aligned}
$$

where we have used successively the assumption $H_2^F = aH^F + b + \varepsilon$, the fact that $\left(H_2^F\right)^{-\frac{1}{2}} \geqslant \frac{1}{H^F}$ and the Heintze-Karcher inequality.

Thus, from (9) and (10), we get

$$\frac{1}{n(n-1)} \int_M \frac{F(\nu)\|\tau_F\|^2}{H^F + \left(H_2^F\right)^{\frac{1}{2}}} dv_g \quad \leqslant \quad -\int_M \varepsilon F(\nu)\left(H_2^F\right)^{-\frac{1}{2}} dv_g - \int_M \varepsilon \langle X, \nu \rangle dv_g$$

$$\text{(11)} \qquad\qquad\qquad \leqslant \quad \left(\frac{\sup(F(\nu))}{\inf((H_2^F))^{\frac{1}{2}}} + R\right) V(M)\|\varepsilon\|_2^2$$

where R is the extrinsic radius of M, that is, the radius of the smallest closed ball containing M.

Moreover, using again the fact that $\left(H_2^F\right)^{\frac{1}{2}} \leqslant H^F$, we obtain the following

$$\text{(12)} \qquad \|\tau_F\|_2^2 \quad \leqslant \quad 2n(n-1)\left(\frac{\sup(F(\nu))}{\inf\left((H_2^F)\right)^{\frac{1}{2}}} + R\right) \frac{\|H^F\|_\infty}{\inf(F(\nu))}\|\varepsilon\|_2^2,$$

which achieves the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we will combine Lemma 4.1 with the following result of de Rosa and Gioffrè to prove Theorem 1.2.

Theorem 4.2 (De Rosa-Gioffrè [3, 4])**.** *Let $n > 2$, $p \in (1, +\infty)$ and $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ satisfying the convexity assumption* (1)*. There exists a constant $\delta_0 = \delta_0(n, p, F) > 0$ such that if M is closed hypersurface into $\mathbb{R}^{n+1}$ satisfying*

$$Vol(M) = V(\mathcal{W}_F) \quad and \quad \int_M \|\tau_F\|^p dv_g \leqslant \delta_0$$

then there exists a smooth parametrisation $\psi : \mathcal{W}_F \longrightarrow M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant C depending on n, p and F so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,p}(\mathcal{W}_F)} \leqslant C\|\tau_F\|_p.$$

Moreover, if $p \in (1, n]$, then the condition $\int_M \|\tau_F\|^p dv_g \leqslant \delta_0$ can be dropped.

Here, it is important to mention that the volume of M is supposed to be equal to $V(\mathcal{W}_F)$. If we do not assume this, the same holds replacing $\mathcal{W}_F$ by the homothetic of $\mathcal{W}_F$ of volume equal to $V(M)$, that is for $\mathcal{W}_{\rho F}$ for $\rho = \left(\frac{V(M)}{V(\mathcal{W}_F)}\right)^{\frac{1}{n}}$. The authors first prove this result for convex hypersurfaces in [3] and then explain how to remove convexity in [4].

5. One last result

We finish this paper by proving a final result in the same spirit of Theorem 1.2. It is well known that a closed connected immersed hypersurface of the Euclidean space, with constant mean curvature, is a round sphere if the scalar product $\langle X, \nu \rangle$ between the position vector and the normal vector has fixed sign. Of course, the hypersurface is not supposed to be embedded because in this case, the assumption $\langle X, \nu \rangle$ has fixed sign is superfluous. An anisotropic version of this result has also been proved in [6]. We prove stability result for this characterization. Before stating the result, we introduce $\mathcal{M}'(r_0, r_1, h_1)$ as the set of all closed, connected and embedded hypersurface of $\mathbb{R}^{n+1}$ so that the support function satisfies $\inf |\langle X, \nu \rangle| \geqslant r_0$, the extrinsic radius is smaller than r_0 and the L^∞ norm of the anisotropic mean curvature is smaller than h_1. Namely, we have:

Theorem 5.1. *Let $n \geqslant 2$ be an integer, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+^*$ a smooth function satisfying the convexity assumption (1) and let $M \in \mathcal{M}'(r_0, r_1, h_1)$. Assume that the support function $\langle X, \nu \rangle$ has fixed sign and there exists a non-zero constant h_0 and a smooth function ε so that $H^F = h_0 + \varepsilon$. Set $\rho = \left(\frac{V(M)}{V(\mathcal{W}_F)} \right)^{\frac{1}{n}}$. Then there exists a smooth parametrisation $\psi : \mathcal{W}_{\rho F} \longrightarrow M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and an explicit constant K_1 depending on n, r_0, r_1, h_1, F so that*

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,2}(\mathcal{W}_{\rho F})} \leqslant K_1 \|\varepsilon\|_1^{\frac{1}{2}}.$$

Moreover, if $\|\varepsilon\|_\infty \leqslant \alpha h_0$ for some $\alpha \in (0,1)$, then, there exists another explicit constant K_2 depending on n, r_0, F so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,2}(\mathcal{W}_{\rho F})} \leqslant K_2 \|\varepsilon\|_\infty^{\frac{1}{2}}.$$

Remark 5.2. *This last theorem says that if the support function has constant sign, and H^F is almost constant, then the hypersurface is $W^{2,2}$-close to the Wulff shape and either the L^1 or L^∞-norm of the error term ε quantify this closeness. But for a control by the L^1-norm of the error, the counterpart is that the constant K_1 depends on more geometric quantities as for a control by the L^∞-norm.*

Proof: First, we compute

$$\int_M \langle X, \nu \rangle \left(\left(H^F \right)^2 - H_2^F \right) dv_g = h_0 \int_M H^F \langle X, \nu \rangle dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g - \int_M H_2^F \langle X, \nu \rangle dv_g$$

$$= -h_0 V_F(M) + \int_M F(\nu) H^F dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g$$

where we have used the first and second Hsiung-Minkowski formulas. Replacing H^F by $h_0 + \varepsilon$ in the second term of the right hand side, we get

$$\int_M \langle X, \nu \rangle \left(\left(H^F \right)^2 - H_2^F \right) dv_g = -h_0 V_F(M) + h_0 V_F(M) + \int_M \varepsilon F(\nu) dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g$$

$$= \int_M \varepsilon F(\nu) dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g.$$

Since $\left(H^F \right)^2 - H_2^F = \dfrac{1}{n(n-1)} \|\tau_F\|^2$ and from the assumptions that $\langle X, \nu \rangle$ has fixed sign and $\inf |\langle X, \nu \rangle| \geqslant r_0$, we obtain

$$(13) \qquad \frac{r_0}{n(n-1)} \int_M \|\tau_F\|^2 dv_g \quad \leqslant \quad \int_M \varepsilon F(\nu) dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g.$$

First, we deduce

$$(14) \qquad \|\tau_F\|_2^2 \leqslant \frac{n(n-1)}{r_0} \left(\sup F(\nu) + R\|H^F\|_\infty \right) \|\varepsilon\|_1,$$

where we have used that $|\langle X, \nu \rangle| \leqslant R$. Combining (14) with the result of de Rosa and Gioffrè, we get the first point of the theorem by setting

$$K_1 = C \left[\frac{n(n-1)}{r_0} \left(\sup F(\nu) + R\|H^F\|_\infty \right) \right]^{\frac{1}{2}},$$

where C is the constant of de Rosa and Gioffrè's result which depends only on n and F, since here $p = 2$. Second, from (13), we get

$$\|\tau_F\|_2^2 V(M) \;\;\leqslant\;\; \frac{n(n-1)}{r_0} \left(\int_M \varepsilon F(\nu) dv_g + \int_M \varepsilon H^F \langle X, \nu \rangle dv_g \right)$$

$$(15) \qquad\qquad \leqslant\;\; \frac{n(n-1)}{r_0} \left(\sup F(\nu) V(M) + \int_M |H^F \langle X, \nu \rangle| dv_g \right) \|\varepsilon\|_\infty.$$

Moreover, if there exists $\alpha \in (0, 1)$ so that $\|\varepsilon\|_\infty \leqslant \alpha h_0$, there $|H_F| \geqslant (1 - \alpha)|h_0| > 0$. Hence, since both H^F and $\langle X, \nu \rangle$ have fixed sign, we have

$$\int_M |H^F \langle X, \nu \rangle| dv_g = \left| \int_M H^F \langle X, \nu \rangle dv_g \right| = \int_M F(\nu) dv_g$$

by the first Hsiung-Minkowski formula.

From this and (15), we obtain

$$\|\tau_F\|_2^2 \leqslant \frac{2n(n-1) \sup F(\nu)}{r_0}$$

and we conclude again by the result of de Rosa and Gioffrè by setting

$$K_2 = C \left[\frac{2n(n-1) \sup F(\nu)}{r_0} \right].$$

Note that K_2 depends only on n, F and r_0. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

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