

Hypersurfaces of Spin^c Manifolds and Lawson Type Correspondence

Roger Nakad and Julien Roth

March 15, 2012

Abstract

Simply connected 3-dimensional homogeneous manifolds $\mathbb{E}(\kappa, \tau)$, with 4-dimensional isometry group, have a canonical Spin^c structure carrying parallel or Killing spinors. The restriction to any hypersurface of these parallel or Killing spinors allows to characterize isometric immersions of surfaces into $\mathbb{E}(\kappa, \tau)$. As application, we get an elementary proof of a Lawson type correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$. Real hypersurfaces of the complex projective space and the complex hyperbolic space are also characterized via Spin^c spinors.

Keywords: Spin^c structures, Killing and parallel spinors, isometric immersions, Lawson type correspondence, Sasaki hypersurfaces.

Mathematics subject classifications (2010): 58C40, 53C27, 53C40, 53C80.

1 Introduction

It is well-known that a conformal immersion of a surface in \mathbb{R}^3 could be characterized by a spinor field φ satisfying

$$D\varphi = H\varphi, \quad (1)$$

where D is the Dirac operator and H the mean curvature of the surface (see [12] for instance). In [4], Friedrich characterized surfaces in \mathbb{R}^3 in a geometrically invariant way. More precisely, consider an isometric immersion of a surface (M^2, g) into \mathbb{R}^3 . The restriction to M of a parallel spinor of \mathbb{R}^3 satisfies, for all $X \in \Gamma(TM)$, the following relation

$$\nabla_X \varphi = -\frac{1}{2} II X \bullet \varphi, \quad (2)$$

where ∇ is the spinorial Levi-Civita connection of M , “ \bullet ” denotes the Clifford multiplication of M and II is the shape operator of the immersion. Hence, φ is a solution

of the Dirac equation (1) with constant norm. Conversely, assume that a Riemannian surface (M^2, g) carries a spinor field φ , satisfying

$$\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi, \quad (3)$$

where E is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that $E = 2\ell^\varphi$. Here ℓ^φ is a field of symmetric endomorphisms associated with the field of quadratic forms, denoted also by ℓ^φ , called the energy-momentum tensor which is given, on the complement set of zeroes of φ , by

$$\ell^\varphi(X) = \Re \left\langle X \bullet \nabla_X \varphi, \frac{\varphi}{|\varphi|^2} \right\rangle,$$

for any $X \in \Gamma(TM)$. Then, the existence of a pair (φ, E) satisfying (3) implies that the tensor $E = 2\ell^\varphi$ satisfies the Gauss and Codazzi equations and by Bonnet's theorem, there exists a local isometric immersion of (M^2, g) into \mathbb{R}^3 with E as shape operator. Friedrich's result was extended by Morel [16] for surfaces of the sphere \mathbb{S}^3 and the hyperbolic space \mathbb{H}^3 .

Recently, the second author [25] gave a spinorial characterization of surfaces isometrically immersed into 3-dimensional homogeneous manifolds with 4-dimensional isometry group. These manifolds, denoted by $\mathbb{E}(\kappa, \tau)$ are Riemannian fibrations over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ with constant curvature κ and bundle curvature τ . This fibration can be represented by a unit vector field ξ tangent to the fibers.

The manifolds $\mathbb{E}(\kappa, \tau)$ are Spin having a special spinor field ψ . This spinor is constructed using real or imaginary Killing spinors on $\mathbb{M}^2(\kappa)$. If $\tau \neq 0$, the restriction of ψ to a surface gives rise to a spinor field φ satisfying, for every vector field X ,

$$\nabla_X \varphi = -\frac{1}{2}IIX \bullet \varphi + i\frac{\tau}{2}X \bullet \bar{\varphi} - i\frac{\alpha}{2}g(X, T)T \bullet \bar{\varphi} + i\frac{\alpha}{2}fg(X, T)\bar{\varphi}. \quad (4)$$

Here $\alpha = 2\tau - \frac{\kappa}{2\tau}$, f is a real function and T is a vector field on M such that $\xi = T + f\nu$ is the decomposition of ξ into tangential and normal parts (ν is the normal vector field of the immersion). The spinor $\bar{\varphi}$ is given by $\bar{\varphi} := \varphi^+ - \varphi^-$, where $\varphi = \varphi^+ + \varphi^-$ is the decomposition into positive and negative spinors. Up to some additional geometric assumptions on T and f , the spinor φ allows to characterize the immersion of the surface into $\mathbb{E}(\kappa, \tau)$ [25].

In the present paper, we consider Spin^c structures on $\mathbb{E}(\kappa, \tau)$ instead of Spin structures. The manifolds $\mathbb{E}(\kappa, \tau)$ have a canonical Spin^c structure carrying a natural spinor field, namely a real Killing spinor with Killing constant $\frac{\tau}{2}$. The restriction of this Killing spinor to M gives rise to a special spinor satisfying

$$\nabla_X \varphi = -\frac{1}{2}IIX \bullet \varphi + i\frac{\tau}{2}X \bullet \bar{\varphi}.$$

This spinor, with a curvature condition on the auxiliary bundle, allows the characterization of the immersion of M into $\mathbb{E}(\kappa, \tau)$ without any additional geometric assumption on f or T (see Theorem 1). From this characterization, we get an elementary spinorial proof of a Lawson type correspondence for constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ (see Theorem 2).

The second advantage of using Spin^c structures in this context is when we consider hypersurfaces of 4-dimensional manifolds. Indeed, any oriented 4-dimensional Kähler manifold has a canonical Spin^c structure with parallel spinors. In particular, the complex space forms $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Then, using an analogue of Bonnet's Theorem for complex space forms, we prove a spinorial characterization of hypersurfaces of the complex projective space $\mathbb{C}P^2$ and of the complex hyperbolic space $\mathbb{C}H^2$. This work generalizes to the complex case the results of [16] and [13]. Finally, we apply this characterization for Sasaki hypersurfaces.

Acknowledgement: Both authors are grateful to Oussama Hijazi for his encouragements, valuable comments and relevant remarks.

2 Preliminaries

In this section we briefly introduce basic facts about Spin^c geometry of hypersurfaces (see [14, 15, 5, 20, 21]). Then we give a short description of the complex space form $\mathbb{M}_{\mathbb{C}}^2(c)$ of complex dimension 2, the 3-dimensional homogeneous manifolds with 4-dimensional isometry group $\mathbb{E}(\kappa, \tau)$ and their hypersurfaces (see [2, 26]).

2.1 Hypersurfaces and induced Spin^c structures

Spin^c structures on manifolds: Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$ without boundary. We denote by $P_{\text{SO}_n}M$ the SO_n -principal bundle over M of positively oriented orthonormal frames. A Spin^c structure of M is a Spin_n^c -principal bundle $(P_{\text{Spin}_n^c}M, \pi, M)$ and an \mathbb{S}^1 -principal bundle $(P_{\mathbb{S}^1}M, \pi, M)$ together with a double covering given by $\theta : P_{\text{Spin}_n^c}M \rightarrow P_{\text{SO}_n}M \times_M P_{\mathbb{S}^1}M$ such that $\theta(ua) = \theta(u)\xi(a)$, for every $u \in P_{\text{Spin}_n^c}M$ and $a \in \text{Spin}_n^c$, where ξ is the 2-fold covering of Spin_n^c over $\text{SO}_n \times \mathbb{S}^1$. Let $\Sigma M := P_{\text{Spin}_n^c}M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ and $\rho_n : \text{Spin}_n^c \rightarrow \text{End}(\Sigma_n)$ denotes the complex spinor representation. A section of ΣM will be called a spinor field. The spinor bundle ΣM is equipped with a natural Hermitian scalar product denoted by $\langle \cdot, \cdot \rangle$.

Additionally, any connection 1-form $A : T(P_{\mathbb{S}^1}M) \rightarrow i\mathbb{R}$ on $P_{\mathbb{S}^1}M$ and the connection 1-form ω^M on $P_{\text{SO}_n}M$, induce a connection on the $\text{SO}_n \times \mathbb{S}^1$ -principal bundle $P_{\text{SO}_n}M \times_M P_{\mathbb{S}^1}M$, and hence a covariant derivative ∇ on $\Gamma(\Sigma M)$ [5, 21]. The curvature of A is an imaginary valued 2-form denoted by $F_A = dA$, i.e., $F_A = i\Omega$, where Ω

is a real valued 2-form on $P_{\mathbb{S}^1}M$. We know that Ω can be viewed as a real valued 2-form on M [5, 11]. In this case $i\Omega$ is the curvature form of the auxiliary line bundle L . It is the complex line bundle associated with the \mathbb{S}^1 -principal bundle via the standard representation of the unit circle. For every spinor field ψ , the Dirac operator is locally defined by

$$D\psi = \sum_{j=1}^n e_j \cdot \nabla_{e_j} \psi,$$

where $\{e_1, \dots, e_n\}$ is a local oriented orthonormal tangent frame and “ \cdot ” denotes the Clifford multiplication. The Dirac operator is an elliptic, self-adjoint operator with respect to the L^2 -scalar product $(\cdot, \cdot) = \int_M \langle \cdot, \cdot \rangle v_g$ and verifies, for any spinor field ψ , the Schrödinger-Lichnerowicz formula

$$D^2\psi = \nabla^* \nabla \psi + \frac{1}{4} S \psi + \frac{i}{2} \Omega \cdot \psi, \quad (5)$$

where S is the scalar curvature of M , ∇^* is the adjoint of ∇ with respect to (\cdot, \cdot) and $\Omega \cdot$ is the extension of the Clifford multiplication to differential forms. For any $X \in \Gamma(TM)$, the Ricci identity is given by

$$\sum_{k=1}^n e_k \cdot \mathcal{R}(e_k, X)\psi = \frac{1}{2} \text{Ric}(X) \cdot \psi - \frac{i}{2} (X \lrcorner \Omega) \cdot \psi, \quad (6)$$

where Ric is the Ricci curvature of (M^n, g) and \mathcal{R} is the curvature tensor of the spinorial connection ∇ . In odd dimension, the volume form $\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdot \dots \cdot e_n$ acts on ΣM as the identity, i.e., $\omega_{\mathbb{C}} \cdot \psi = \psi$ for any spinor $\psi \in \Gamma(\Sigma M)$. Besides, in even dimension, we have $\omega_{\mathbb{C}}^2 = 1$. We denote by $\Sigma^{\pm}M$ the eigenbundles corresponding to the eigenvalues ± 1 , hence $\Sigma M = \Sigma^+M \oplus \Sigma^-M$ and a spinor field ψ can be written $\psi = \psi^+ + \psi^-$. The conjugate $\bar{\psi}$ of ψ is defined by $\bar{\psi} = \psi^+ - \psi^-$.

Every spin manifold has a trivial Spin^c structure [5]. In fact, we choose the trivial line bundle with the trivial connection whose curvature $i\Omega$ is zero. Also every Kähler manifold M of complex dimension m ($n = 2m$) has a canonical Spin^c structure coming from the complex structure J . Let \times be the Kähler form defined by the complex structure J , i.e. $\times(X, Y) = g(JX, Y)$ for all vector fields $X, Y \in \Gamma(TM)$. The complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into

$$T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M,$$

where $T_{1,0}M$ (resp. $T_{0,1}M$) is the i -eigenbundle (resp. $-i$ -eigenbundle) of the complex linear extension of the complex structure. Indeed,

$$T_{1,0}M = \overline{T_{0,1}M} = \{X - iJX \mid X \in \Gamma(TM)\}.$$

Thus, the spinor bundle of the canonical Spin^c structure is given by

$$\Sigma M = \Lambda^{0,*}M = \bigoplus_{r=0}^m \Lambda^r(T_{0,1}^*M),$$

where $T_{0,1}^*M$ is the dual space of $T_{0,1}M$. The auxiliary bundle of this canonical Spin^c structure is given by $L = (K_M)^{-1} = \Lambda^m(T_{0,1}^*M)$, where $K_M = \Lambda^m(T_{1,0}^*M)$ is the canonical bundle of M [5]. This line bundle L has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = -i\rho$, where ρ is the Ricci form given by $\rho(X, Y) = \text{Ric}(JX, Y)$. Hence, this Spin^c structure carries parallel spinors (the constant complex functions) lying in the set of complex functions $\Lambda^{0,0}M \subset \Lambda^{0,*}M$ [17]. Of course, we can define another Spin^c structure for which the spinor bundle is given by $\Lambda^{*,0}M = \bigoplus_{r=0}^m \Lambda^r(T_{1,0}^*M)$ and the auxiliary line bundle by K_M . This Spin^c structure will be called the anti-canonical Spin^c structure [5].

Spin^c hypersurfaces and the Gauss formula: Let N be an oriented $(n+1)$ -dimensional Riemannian Spin^c manifold and $M \subset N$ be an oriented hypersurface. The manifold M inherits a Spin^c structure induced from the one on N , and we have [21]

$$\Sigma M \simeq \begin{cases} \Sigma N|_M & \text{if } n \text{ is even,} \\ \Sigma^+ N|_M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X , tangent to M , is given by

$$X \bullet \phi = (X \cdot \nu \cdot \psi)|_M, \quad (7)$$

where $\psi \in \Gamma(\Sigma N)$ (or $\psi \in \Gamma(\Sigma^+ N)$ if n is odd), ϕ is the restriction of ψ to M , “ \cdot ” is the Clifford multiplication on N , “ \bullet ” that on M and ν is the unit inner normal vector. The connection 1-form defined on the restricted \mathbb{S}^1 -principal bundle ($P_{\mathbb{S}^1}M := P_{\mathbb{S}^1}N|_M, \pi, M$), is given by $A = A^N|_M : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}N)|_M \rightarrow i\mathbb{R}$. Then the curvature 2-form $i\Omega$ on the \mathbb{S}^1 -principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega = i\Omega^N|_M$, which can be viewed as an imaginary 2-form on M and hence as the curvature form of the line bundle L , the restriction of the auxiliary bundle L^N to M . For every $\psi \in \Gamma(\Sigma N)$ ($\psi \in \Gamma(\Sigma^+ N)$ if n is odd), the real 2-forms Ω and Ω^N are related by [21]

$$(\Omega^N \cdot \psi)|_M = \Omega \bullet \phi - (\nu \lrcorner \Omega^N) \bullet \phi. \quad (8)$$

We denote by $\nabla^{\Sigma N}$ the spinorial Levi-Civita connection on ΣN and by ∇ that on ΣM . For all $X \in \Gamma(TM)$, we have the spinorial Gauss formula [21]:

$$(\nabla_X^{\Sigma N} \psi)|_M = \nabla_X \phi + \frac{1}{2} IIX \bullet \phi, \quad (9)$$

where II denotes the Weingarten map of the hypersurface. Moreover, Let D^N and D be the Dirac operators on N and M , after denoting by the same symbol any spinor and its restriction to M , we have

$$\tilde{D}\phi = \frac{n}{2}H\phi - \nu \cdot D^N\phi - \nabla_\nu^{\Sigma N}\phi, \quad (10)$$

where $H = \frac{1}{n}\text{tr}(II)$ denotes the mean curvature and $\tilde{D} = D$ if n is even and $\tilde{D} = D \oplus (-D)$ if n is odd.

2.2 Basic facts about $\mathbb{E}(\kappa, \tau)$ and their surfaces

We denote a 3-dimensional homogeneous manifolds with 4-dimensional isometry group by $\mathbb{E}(\kappa, \tau)$. It is a Riemannian fibration over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ with constant curvature κ and such that the fibers are geodesic. We denote by τ the bundle curvature, which measures the default of the fibration to be a Riemannian product. Precisely, we denote by ξ a unit vertical vector field, that is tangent to the fibers. The vector field ξ is a Killing field and satisfies for all vector field X ,

$$\bar{\nabla}_X \xi = \tau X \wedge \xi,$$

where $\bar{\nabla}$ is the Levi-Civita connection and \wedge is the exterior product. When τ vanishes, we get a product manifold $\mathbb{M}^2(\kappa) \times \mathbb{R}$. If $\tau \neq 0$, these manifolds are of three types: They have the isometry group of the Berger spheres if $\kappa > 0$, of the Heisenberg group Nil_3 if $\kappa = 0$ or of $\widetilde{\text{PSL}}_2(\mathbb{R})$ if $\kappa < 0$.

Note that if $\tau = 0$, then $\xi = \frac{\partial}{\partial t}$ is the unit vector field giving the orientation of \mathbb{R} in the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$. The manifold $\mathbb{E}(\kappa, \tau)$, with $\tau \neq 0$, admits a local direct orthonormal frame $\{e_1, e_2, e_3\}$ with

$$e_3 = \xi,$$

and such that the Christoffel symbols $\bar{\Gamma}_{ij}^k = \langle \bar{\nabla}_{e_i} e_j, e_k \rangle$ are given by

$$\begin{cases} \bar{\Gamma}_{12}^3 = \bar{\Gamma}_{23}^1 = -\bar{\Gamma}_{21}^3 = -\bar{\Gamma}_{13}^2 = \tau, \\ \bar{\Gamma}_{32}^1 = -\bar{\Gamma}_{31}^2 = \tau - \frac{\kappa}{2\tau}, \\ \bar{\Gamma}_{ii}^i = \bar{\Gamma}_{ij}^i = \bar{\Gamma}_{ji}^i = \bar{\Gamma}_{ii}^j = 0, \quad \forall i, j \in \{1, 2, 3\}, \end{cases} \quad (11)$$

We call $\{e_1, e_2, e_3 = \xi\}$ the canonical frame of $\mathbb{E}(\kappa, \tau)$.

Let M be a simply connected orientable surface of $\mathbb{E}(\kappa, \tau)$ with shape operator II associated with the unit inner normal vector ν . Moreover, we denote $\xi = T + f\nu$ where the function f is the normal component of ξ and T is its tangential part. We introduce the following notion of compatibility equations.

Definition 2.1 (Compatibility equations). *We say that $(M, \langle \cdot, \cdot \rangle, E, T, f)$ satisfies the compatibility equations for $\mathbb{E}(\kappa, \tau)$ if and only if for any $X, Y, Z \in \Gamma(TM)$,*

$$K = \det(E) + \tau^2 + (\kappa - 4\tau^2)f^2 \quad (12)$$

$$\nabla_X EY - \nabla_Y EX - W[X, Y] = (\kappa - 4\tau^2)f(\langle Y, T \rangle X - \langle X, T \rangle Y), \quad (13)$$

$$\nabla_X T = f(EX - \tau JX), \quad (14)$$

$$df(X) = -\langle EX - \tau JX, T \rangle, \quad (15)$$

where K is the Gauss curvature of M .

Remark 1. *The relations (12) and (13) are the Gauss and Codazzi equations for an isometric immersion into $\mathbb{E}(\kappa, \tau)$ obtained by a computation of the curvature tensor of $\mathbb{E}(\kappa, \tau)$. Equations (14) and (15) are coming from the fact that $\bar{\nabla}_X \xi = \tau X \wedge \xi$.*

In [1, 2], Daniel proves that these compatibility equations are necessary and sufficient for the existence of an isometric immersion F from M into $\mathbb{E}(\kappa, \tau)$ with shape operator $dF \circ E \circ dF^{-1}$ and so that $\xi = dF(T) + f\nu$.

2.3 Basic facts about $\mathbb{M}_{\mathbb{C}}^2(c)$ and their real hypersurfaces

Let $(\mathbb{M}_{\mathbb{C}}^2(c), J, \bar{g})$ be the complex space form of constant holomorphic sectional curvature $4c \neq 0$ and complex dimension 2, that is for $c = 1$, $\mathbb{M}_{\mathbb{C}}^2(c)$ is the complex projective space $\mathbb{C}P^2$ and if $c = -1$, $\mathbb{M}_{\mathbb{C}}^2(c)$ is the complex hyperbolic space $\mathbb{C}H^2$. It is a well-known fact that the curvature tensor \bar{R} of $\mathbb{M}_{\mathbb{C}}^2(c)$ is given by

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, W) = & c \left\{ \bar{g}(Y, Z)\bar{g}(X, W) - \bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(JY, Z)\bar{g}(JX, W) \right. \\ & \left. - \bar{g}(JX, Z)\bar{g}(JY, W) - 2\bar{g}(JX, Y)\bar{g}(JZ, W) \right\}, \end{aligned}$$

for all X, Y, Z and W tangent vector fields to $\mathbb{M}_{\mathbb{C}}^2(c)$.

Let M^3 be an oriented real hypersurface of $\mathbb{M}_{\mathbb{C}}^2(c)$ endowed with the metric g induced by \bar{g} . We denote by ν a normal unit inner vector globally defined on M and by II the shape operator of this immersion. Moreover, the complex structure J induces on M an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$, where \mathfrak{X} is the $(1, 1)$ -tensor defined by $g(\mathfrak{X}X, Y) = \bar{g}(JX, Y)$ for all $X, Y \in \Gamma(TM)$, $\xi = -J\nu$ is a tangent vector field and η the 1-form associated with ξ , that is so that $\eta(X) = g(\xi, X)$ for all $X \in \Gamma(TM)$. Then, we see easily that the following holds:

$$\mathfrak{X}^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \text{and} \quad \mathfrak{X}\xi = 0. \quad (16)$$

Here, we recall that given an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ one defines a 2-form Θ by $\Theta(X, Y) = g(X, \mathfrak{X}Y)$ for all $X, Y \in \Gamma(TM)$. Now, $(\mathfrak{X}, \xi, \eta, g)$ is said to satisfy the contact condition if $-2\Theta = d\eta$ and if it is the case, $(\mathfrak{X}, \xi, \eta, g)$ is called a contact metric structure on M . A contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ is called a Sasakian structure (and M a Sasaki manifold) if ξ is a Killing vector field (or equivalently, $\mathfrak{X} = \nabla\xi$) and

$$(\nabla_X \mathfrak{X})Y = \eta(Y)X - g(X, Y)\xi, \quad \text{for all } X, Y \in \Gamma(TM).$$

From the relation between the Riemannian connections of $\mathbb{M}_{\mathbb{C}}^2(c)$ and M , $\bar{\nabla}_X Y = \nabla_X Y + g(II X, Y)\nu$, we deduce the two following identities:

$$(\nabla_X \mathfrak{X})Y = \eta(Y)II X - g(II X, Y)\xi, \quad (17)$$

$$\nabla_X \xi = \mathfrak{X}II X. \quad (18)$$

From the expression of the curvature of $\mathbb{M}_{\mathbb{C}}^2(c)$ given above, we deduce the Gauss and Codazzi equations. First, the Gauss equation says that for all $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned} g(R(X, Y)Z, W) = & c \left\{ g(Y, Z)\bar{g}(X, W) - g(X, Z)g(Y, W) + g(\mathfrak{X}Y, Z)g(\mathfrak{X}X, W) \right. \\ & \left. - g(\mathfrak{X}X, Z)g(\mathfrak{X}Y, W) - 2g(\mathfrak{X}X, Y)g(\mathfrak{X}Z, W) \right\} \\ & + g(II Y, Z)g(II X, W) - g(II X, Z)g(II Y, W). \end{aligned} \quad (19)$$

The Codazzi equation is

$$d^\nabla II(X, Y) = c(\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X - 2g(\mathfrak{X}X, Y)\xi). \quad (20)$$

Now, we ask if the Gauss equation (19) and the Codazzi equation (20) are sufficient to get an isometric immersion of (M, g) into $\mathbb{M}_{\mathbb{C}}^2(c)$.

Definition 2.2 (Compatibility equations). *Let (M^3, g) be a simply connected oriented Riemannian manifold endowed with an almost contact metric structure $(\mathfrak{X}, \xi, \eta)$ and E be a field of symmetric endomorphisms on M . We say that $(M, g, E, \mathfrak{X}, \xi, \eta)$ satisfies the compatibility equations for $\mathbb{M}_{\mathbb{C}}^2(c)$ if and only if for any $X, Y, Z, W \in \Gamma(TM)$, we have*

$$\begin{aligned} g(R(X, Y)Z, W) = & c \left\{ g(Y, Z)\bar{g}(X, W) - g(X, Z)g(Y, W) + g(\mathfrak{X}Y, Z)g(\mathfrak{X}X, W) \right. \\ & \left. - g(\mathfrak{X}X, Z)g(\mathfrak{X}Y, W) - 2g(\mathfrak{X}X, Y)g(\mathfrak{X}Z, W) \right\} \\ & + g(EY, Z)g(EX, W) - g(EX, Z)g(EY, W), \end{aligned} \quad (21)$$

$$d^\nabla E(X, Y) = c(\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X - 2g(\mathfrak{X}X, Y)\xi). \quad (22)$$

$$(\nabla_X \mathfrak{X})Y = \eta(Y)EX - g(EX, Y)\xi, \quad (23)$$

$$\nabla_X \xi = \mathfrak{X}EX. \quad (24)$$

In [23], P. Piccione and D. V. Tausk proves that the Gauss equation (21) and the Codazzi equation (22) together with (23) and (24) are necessary and sufficient for the existence of an isometric immersion from M into $\mathbb{M}_{\mathbb{C}}^2(c)$ such that the complex structure of $\mathbb{M}_{\mathbb{C}}^2(c)$ over M is given by $J = \mathfrak{X} + \eta(\cdot)\nu$.

3 Isometric immersions into $\mathbb{E}(\kappa, \tau)$ via spinors

The manifold $\mathbb{E}(\kappa, \tau)$ has a Spin^c structure carrying a Killing spinor with Killing constant $\frac{\tau}{2}$. The restriction of this Spin^c structure to any surface M defines a Spin^c structure on M with a special spinor field. This spinor field characterizes the isometric immersion of M into $\mathbb{E}(\kappa, \tau)$.

3.1 Special spinors fields on $\mathbb{E}(\kappa, \tau)$ and their surfaces

On Spin^c manifolds, A. Moroianu defined projectable spinors for arbitrary Riemannian submersions of Spin^c manifolds with 1-dimensional totally geodesic fibers [19, 18]. These spinors will be used to get a Killing spinor on $\mathbb{E}(\kappa, \tau)$.

Proposition 3.1. *The canonical Spin^c structure on $\mathbb{M}^2(\kappa)$ induces a Spin^c structure on $\mathbb{E}(\kappa, \tau)$ carrying a Killing spinor with Killing constant $\frac{\tau}{2}$.*

Proof: By enlargement of the group structures, the two-fold covering $\theta : P_{\text{Spin}_2^c} \mathbb{M} \longrightarrow P_{\text{SO}_2} \mathbb{M} \times_{\mathbb{M}} P_{\mathbb{S}^1} \mathbb{M}$, gives a two-fold covering

$$\theta : P_{\text{Spin}_3^c} \mathbb{M} \longrightarrow P_{\text{SO}_3} \mathbb{M} \times_{\mathbb{M}} P_{\mathbb{S}^1} \mathbb{M},$$

which, by pull-back through π , gives rise to a Spin^c structure on $\overline{\mathbb{M}} := \mathbb{E}(\kappa, \tau)$ [18, 19] and the following diagram commutes

$$\begin{array}{ccc} P_{\text{Spin}_3^c} \overline{\mathbb{M}} & \longrightarrow & P_{\text{Spin}_3^c} \mathbb{M} \\ \downarrow \pi^* \theta & & \downarrow \theta \\ P_{\text{SO}_3} \overline{\mathbb{M}} \times_{\overline{\mathbb{M}}} P_{\mathbb{S}^1} \overline{\mathbb{M}} & \longrightarrow & P_{\text{SO}_3} \mathbb{M} \times_{\mathbb{M}} P_{\mathbb{S}^1} \mathbb{M} \end{array}$$

The next step is to relate the covariant derivatives of spinors on \mathbb{M} and $\overline{\mathbb{M}}$. We point out an important detail: Since we are actually interested to get a Killing spinor on $\overline{\mathbb{M}}$, the connection on $P_{\mathbb{S}^1} \overline{\mathbb{M}}$ (which defines the covariant derivative of spinors on $\overline{\mathbb{M}}$) that we will consider will be the pull-back connection if $\tau = 0$ and will not be the pull-back connection if $\tau \neq 0$. Hence, when $\tau = 0$, the connection A_0 on $P_{\mathbb{S}^1} \overline{\mathbb{M}}$ is given by

$$A_0((\pi^* s)_*(X^*)) = A(s_* X) \quad \text{and} \quad A_0((\pi^* s)_* \xi) = 0.$$

Now, if $\tau \neq 0$, we consider a connection A_0 on $P_{\mathbb{S}^1} \overline{\mathbb{M}}$ given by

$$A_0((\pi^* s)_*(X^*)) = A(s_* X) \quad \text{and} \quad A_0((\pi^* s)_* \xi) = i(2\tau - \frac{\kappa}{2\tau}),$$

where $e_3 = \xi$ is the vertical vector field on $\mathbb{E}(\kappa, \tau)$ if $\tau \neq 0$ or $e_3 = \partial t$ if $\tau = 0$, X^* is the horizontal lift of a vector field X on \mathbb{M} , A is the connection defined on $P_{\mathbb{S}^1} \mathbb{M}$ and s a local section of $P_{\mathbb{S}^1} \mathbb{M}$. Recall that we have an identification of the pull back $\pi^* \Sigma \mathbb{M}$ with $\Sigma \overline{\mathbb{M}}$ [18, 19], and with respect to this identification, if X is a vector field and ψ a spinor field on \mathbb{M} , then

$$X^* \cdot \pi^* \psi = \pi^*(X \cdot \psi) \quad \text{and} \quad \xi \cdot \pi^* \psi = i\pi^*(\overline{\psi}). \quad (25)$$

The sections of $\Sigma \overline{\mathbb{M}}$ which can be written as pull-back of sections of $\Sigma \mathbb{M}$ are called projectable spinors [18, 19]. Now, we relate the covariant derivative $\nabla^{\mathbb{E}(\kappa, \tau)}$ of projectable spinors on $\mathbb{E}(\kappa, \tau)$ to the covariant derivative ∇ of spinors on \mathbb{M} . In fact, any

spinor field ψ is locally written as $\psi = [\widetilde{b \times s}, \sigma]$, where $b = (e_1, e_2)$ is a base of $\mathbb{M}^2(\kappa)$, $s : U \rightarrow P_{\mathbb{S}^1}\mathbb{M}$ is a local section of $P_{\mathbb{S}^1}\mathbb{M}$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \rightarrow P_{SO_2}\mathbb{M} \times_{\mathbb{M}} P_{\mathbb{S}^1}\mathbb{M}$ by the 2-fold covering. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*(\widetilde{b \times s}), \pi^*\sigma]$. It is easy to see that the projection $\pi^*(\widetilde{b \times s})$ onto $P_{SO_3}\overline{\mathbb{M}}$ is the canonical frame $(e_1^*, e_2^*, e_3 = \xi)$ and its projection onto $P_{\mathbb{S}^1}\overline{\mathbb{M}}$ is just $\pi^*\sigma$. We have

$$\begin{aligned}
\nabla_{e_1^*}^{\mathbb{E}(\kappa, \tau)} \pi^*\psi &= [\pi^*(\widetilde{b \times s}), e_1^*(\pi^*\sigma)] + \frac{1}{2}\bar{g}(\nabla_{e_1^*} e_1^*, e_2^*)e_1^* \cdot e_2^* \cdot \pi^*\psi \\
&\quad + \frac{1}{2} \sum_{j=1}^2 \bar{g}(\nabla_{e_1^*} e_j^*, e_3)e_j^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_*e_1^*)\pi^*\psi \\
&\stackrel{(11)}{=} [\pi^*(\widetilde{b \times s}), \pi^*(e_1(\sigma))] + \frac{1}{2}g(\nabla_{e_1} e_1, e_2)\pi^*(e_1 \cdot e_2 \cdot \psi) \\
&\quad + \frac{\tau}{2}e_2^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A(s_*X)\pi^*\psi \\
&= \pi^*\left([\widetilde{b \times s}, (e_1(\sigma))]\right) + \frac{1}{2}g(\nabla_{e_1} e_1, e_2)e_1 \cdot e_2 \cdot \psi \\
&\quad + \frac{\tau}{2}e_1^* \cdot \psi + \frac{1}{2}A(s_*X)\psi \\
&= \pi^*(\nabla_{e_1}\psi) + \frac{\tau}{2}e_1 \cdot \pi^*\psi.
\end{aligned}$$

The same holds for e_2^* . Similarly, if $\tau \neq 0$ we have

$$\begin{aligned}
\nabla_{e_3}^{\mathbb{E}(\kappa, \tau)} \pi^*\psi &= [\pi^*(\widetilde{b \times s}), e_3(\pi^*\sigma)] + \frac{1}{2}\bar{g}(\nabla_{e_3} e_1^*, e_2^*)e_1^* \cdot e_2^* \cdot \pi^*\psi \\
&\quad + \frac{1}{2} \sum_{j=1}^2 \bar{g}(\nabla_{e_3} e_j^*, e_3)e_j^* \cdot e_3 \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_*e_3)\pi^*\psi \\
&\stackrel{(11)}{=} \frac{1}{2}\left(\frac{\kappa}{2\tau} - \tau\right)e_1^* \cdot e_2^* \cdot \pi^*\psi + \frac{i}{2}\left(2\tau - \frac{\kappa}{2\tau}\right)\pi^*\psi \\
&= \frac{1}{2}\left(\frac{\kappa}{2\tau} - \tau\right)e_3 \cdot \pi^*\psi + \frac{1}{2}\left(2\tau - \frac{\kappa}{2\tau}\right)e_3 \cdot \pi^*\bar{\psi}.
\end{aligned}$$

Now, the canonical Spin^c structure on $\mathbb{M}^2(\kappa)$ carries a parallel spinor $\psi \in \Gamma(\Sigma_0\mathbb{M}) \subset \Gamma(\Sigma^+\mathbb{M})$, so $\bar{\psi} = \psi$. For this canonical Spin^c structure, the determinant line bundle corresponding to $P_{\mathbb{S}^1}\mathbb{M}$ is $K_{\mathbb{M}}^{-1}$ and the connection 1-form A on $P_{\mathbb{S}^1}\mathbb{M}$ is the connection for the Levi-Civita connection extended to $K_{\mathbb{M}}^{-1}$. Hence, the spinor $\pi^*\psi$ is a Killing spinor field on $\mathbb{E}(\kappa, \tau)$, because

$$\nabla_{e_j^*}^{\mathbb{E}(\kappa, \tau)} \pi^*\psi = \frac{\tau}{2}e_j^* \cdot \pi^*(\psi), \quad \text{for } j = 1, 2 \quad \text{and} \quad \nabla_{\xi}^{\mathbb{E}(\kappa, \tau)} \pi^*\psi = \frac{\tau}{2}\xi \cdot \pi^*\psi.$$

Now, if $\tau = 0$, a same computation of $\nabla_{e_3}^{\mathbb{E}(\kappa, \tau)} \pi^*\psi$ gives that $\pi^*\psi$ is a parallel spinor field on $\mathbb{E}(\kappa, \tau)$.

Remark 2. *Every Sasakian manifold has a canonical Spin^c structure: In fact, giving a Sasakian structure on a manifold (M^n, g) is equivalent to give a Kähler structure on the cone over M . The cone over M is the manifold $M \times_{r,2} \mathbb{R}^+$ equipped with the metric $r^2g + dr^2$. Moreover, there is a 1-1-correspondence between Spin^c structures on M and that on its cone [17]. Hence, every Sasakian manifold has a canonical (resp. anti-canonical) Spin^c structure coming from the canonical one (resp. anti-canonical one) on its cone.*

In [17], A. Moroianu classified all complete simply connected Spin^c manifolds carrying real Killing spinors and he proved that the only complete simply connected Spin^c manifolds carrying real Killing spinors (other than the Spin manifolds) are the non-Einstein Sasakian manifolds endowed with their canonical (or anti-canonical) Spin^c structure.

The manifold $\mathbb{E}(\kappa, \tau)$ is a complete simply connected non-Einstein manifold and hence the only Spin^c structure carrying a Killing spinor is the canonical one (or the anti-canonical). Hence, the Spin^c structure on $\mathbb{E}(\kappa, \tau)$ described above, (i.e. the one coming from $\mathbb{M}^2(\kappa)$) is nothing than the canonical Spin^c structure coming from the Sasakian structure.

We point out that, in a similar way, the anti-canonical Spin^c structure on $\mathbb{M}^2(\kappa)$ (carrying a parallel spinor field lying in $\Sigma^-\mathbb{M}$) induces also on $\mathbb{E}(\kappa, \tau)$ the anti-canonical Spin^c structure with a Killing spinor $\pi^*\psi$ of Killing constant $\frac{\tau}{2}$ if $\tau \neq 0$ and a parallel spinor $\pi^*\psi$ if $\tau = 0$. In both cases, we have $\xi \cdot \pi^*\psi = i\pi^*\bar{\psi} = -i\pi^*\psi$. For $\tau \neq 0$, the connection A_0 is chosen to be

$$A_0((\pi^*s)_*(X^*)) = A(s_*X) \quad \text{and} \quad A_0((\pi^*s)_*\xi) = -i(2\tau - \frac{\kappa}{2\tau}).$$

When $\tau = 0$, it is the pull-back connection.

From now, we will denote the Killing spinor field $\pi^*\psi$ on $\mathbb{E}(\kappa, \tau)$ by ψ . Since, it is a Killing spinor, we have

$$(\nabla^{\mathbb{E}(\kappa, \tau)})^*\nabla^{\mathbb{E}(\kappa, \tau)}\psi = \frac{3\tau^2}{4}\psi \quad \text{and} \quad D^{\mathbb{E}(\kappa, \tau)}\psi = -\frac{3\tau}{2}\psi.$$

By the Schrödinger-Lichnerowicz formula, we get

$$\frac{i}{2}\Omega^{\mathbb{E}(\kappa, \tau)} \cdot \psi = \frac{3\tau^2}{2}\psi - \frac{(\kappa - \tau^2)}{2}\psi,$$

where $i\Omega^{\mathbb{E}(\kappa, \tau)}$ is the curvature 2-form of the auxiliary line bundle associated with the Spin^c structure. Finally,

$$\Omega^{\mathbb{E}(\kappa, \tau)} \cdot \psi = i(\kappa - 4\tau^2)\psi. \tag{26}$$

3.2 Spinorial characterization of surfaces of $\mathbb{E}(\kappa, \tau)$

Let $\kappa, \tau \in \mathbb{R}$ with $\kappa - 4\tau^2 \neq 0$ and M be a simply connected oriented Riemannian surface immersed into $\mathbb{E}(\kappa, \tau)$. The vertical vector field ξ is written $\xi = T + f\nu$ where T be a vector field on M and f a real-valued function on M so that $f^2 + \|T\|^2 = 1$. We endowed $\mathbb{E}(\kappa, \tau)$ with the Spin^c structure described above, carrying a Killing spinor of Killing constant $\frac{\tau}{2}$.

Lemma 3.2. *The restriction φ of the Killing spinor ψ on $\mathbb{E}(\kappa, \tau)$ is a solution of the following equation*

$$\nabla_X \varphi + \frac{1}{2} IIX \bullet \varphi - i\frac{\tau}{2} X \bullet \bar{\varphi} = 0, \quad (27)$$

called the restricted Killing spinor equation. Moreover, $f = \frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$ and the curvature 2-form of the connection on the auxiliary line bundle associated with the induced Spin^c structure is given by $\Omega(t_1, t_2) = -(\kappa - 4\tau^2)f$, in any local orthonormal frame $\{t_1, t_2\}$.

Proof: We restrict the Spin^c structure on $\mathbb{E}(\kappa, \tau)$ to M . By the Gauss formula (9), the restriction φ of the Killing spinor ψ on $\mathbb{E}(\kappa, \tau)$ satisfies

$$\nabla_X \varphi + \frac{1}{2} IIX \bullet \varphi - \frac{\tau}{2} X \cdot \psi|_M = 0.$$

Let $\{t_1, t_2, \nu\}$ be a local orthonormal frame of $\mathbb{E}(\kappa, \tau)$ such that $\{t_1, t_2\}$ is a local orthonormal frame of M and ν a unit normal vector field of the surface. The action of the volume forms on M and $\mathbb{E}(\kappa, \tau)$ gives

$$\begin{aligned} X \bullet \bar{\varphi} &= i(X \bullet t_1 \bullet t_2 \bullet \varphi) \\ &= i(X \cdot \nu \cdot t_1 \cdot t_2 \cdot \psi)|_M \\ &= -i(X \cdot \psi)|_M, \end{aligned}$$

which gives Equation (27). The vector field T splits into $T = \nu_1 + h\xi$ where ν_1 is a vector field generated by e_1 and e_2 and h a real function. The scalar product of T by $\xi = T + f\nu$ and the scalar product of $T = \nu_1 + h\xi$ by ξ gives $\|T\|^2 = h$ which means that $h = 1 - f^2$. Hence, the normal vector field ν can be written as $\nu = f\xi - \frac{1}{f}\nu_1$. As we mentioned before, the Spin^c structure on $\mathbb{E}(\kappa, \tau)$ induces a Spin^c structure on M with induced auxiliary line bundle. Next, we want to prove that the curvature 2-form of the connection on the auxiliary line bundle of M is equal to $i\Omega(t_1, t_2) = -i(\kappa - 4\tau^2)f$. Since the spinor ψ is Killing, the equality (6) gives, for all $X \in T(\mathbb{E}(\kappa, \tau))$

$$\text{Ric}^{\mathbb{E}(\kappa, \tau)}(X) \cdot \psi - i(X \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)}) \cdot \psi = 2\tau^2 X \cdot \psi, \quad (28)$$

Where Ric is the Ricci tensor of $\mathbb{E}(\kappa, \tau)$. Therefore, we compute,

$$\begin{aligned} (\nu \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)}) \bullet \varphi &= (\nu \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)}) \cdot \nu \cdot \psi|_M \\ &= i(2\tau^2 \psi + \nu \cdot \text{Ric}^{\mathbb{E}(\kappa, \tau)} \nu \cdot \psi)|_M. \end{aligned}$$

But we have $\text{Ric}^{\mathbb{E}(\kappa, \tau)} e_3 = 2\tau^2 e_3$, $\text{Ric}^{\mathbb{E}(\kappa, \tau)} e_1 = (\kappa - 2\tau^2) e_1$ and $\text{Ric}^{\mathbb{E}(\kappa, \tau)} e_2 = (\kappa - 2\tau^2) e_2$. Hence,

$$\begin{aligned} \text{Ric}^{\mathbb{E}(\kappa, \tau)} \nu &= f \text{Ric}^{\mathbb{E}(\kappa, \tau)} e_3 - \frac{1}{f} \text{Ric}^{\mathbb{E}(\kappa, \tau)} \nu_1 = 2\tau^2 f e_3 - \frac{1}{f} (\kappa - 2\tau^2) \nu_1 \\ &= 2\tau^2 f e_3 + (\kappa - 2\tau^2) (\nu - f e_3) \\ &= -(\kappa - 4\tau^2) f e_3 + (\kappa - 2\tau^2) \nu. \end{aligned}$$

We conclude using Equation (25) that

$$(\nu \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)}) \bullet \varphi = -i(\kappa - 4\tau^2) \varphi - (\kappa - 4\tau^2) f (\nu \cdot \psi)|_M.$$

By Equation (8), we get that $\Omega \bullet \varphi = -(\kappa - 4\tau^2) f (\nu \cdot \psi)|_M$. The scalar product of the last equality with $t_1 \bullet t_2 \bullet \varphi$ gives

$$\Omega(t_1, t_2) |\varphi|^2 = f(\kappa - 4\tau^2) (\psi, t_1 \cdot t_2 \cdot \nu \cdot \psi)|_M = -f(\kappa - 4\tau^2) |\varphi|^2.$$

We write in the frame $\{t_1, t_2, \nu\}$

$$\Omega^{\mathbb{E}(\kappa, \tau)}(t_1, t_2) t_1 \cdot t_2 \cdot \psi + \Omega^{\mathbb{E}(\kappa, \tau)}(t_1, \nu) t_1 \cdot \nu \cdot \psi + \Omega^{\mathbb{E}(\kappa, \tau)}(t_2, \nu) t_2 \cdot \nu \cdot \psi = i(\kappa - 4\tau^2) \psi. \quad (29)$$

But we know that $\Omega^{\mathbb{E}(\kappa, \tau)}(t_1, t_2) = \Omega(t_1, t_2) = -(\kappa - 4\tau^2) f$. For the other terms, we compute

$$\Omega^{\mathbb{E}(\kappa, \tau)}(t_1, \nu) = \Omega^{\mathbb{E}(\kappa, \tau)}(t_1, \frac{1}{f} e_3 - \frac{1}{f} T) = -\frac{1}{f} g(T, t_2) \Omega^{\mathbb{E}(\kappa, \tau)}(t_1, t_2) = (\kappa - 4\tau^2) g(T, t_2),$$

where the term $\Omega^{\mathbb{E}(\kappa, \tau)}(t_1, e_3)$ vanishes since by Equation (28) we have $e_3 \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)} = 0$. Similarly, we find that $\Omega^{\mathbb{E}(\kappa, \tau)}(t_2, \nu) = -(\kappa - 4\tau^2) g(T, t_1)$. By substituting these values into (29) and taking Clifford multiplication with $t_1 \cdot t_2$, we get

$$T \bullet \varphi = -f \varphi + \bar{\varphi}.$$

Finally, take the real part of the scalar product of the last equation by φ , we get $f = \frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$.

Remark 3. Using also the Equation $T \bullet \varphi = -f \varphi + \bar{\varphi}$, we can deduce that

$$g(T, t_1) = \Re \left\langle it_2 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \right\rangle \quad \text{and} \quad g(T, t_2) = -\Re \left\langle it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \right\rangle.$$

Proposition 3.3. Let (M^2, g) be an oriented Spin^c surface carrying a non-trivial solution φ of the following equation

$$\nabla_X \varphi + \frac{1}{2} EX \bullet \varphi - i \frac{\tau}{2} X \bullet \bar{\varphi} = 0,$$

where E denotes a symmetric tensor field defined on M . Moreover, assume that the curvature 2-form of the associated auxiliary bundle satisfies $i\Omega(t_1, t_2) = -(\kappa - 4\tau^2)f = -(\kappa - 4\tau^2)\frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$ in any local orthonormal frame $\{t_1, t_2\}$ of M . Then, there exists an isometric immersion of (M^2, g) into $\mathbb{E}(\kappa, \tau)$ with shape operator E , mean curvature H and such that, over M , the vertical vector is $\xi = dF(T) + f\nu$, where ν is the unit normal vector to the surface and T is the tangential part of ξ given by

$$g(T, t_1) = \Re \left\langle it_2 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \right\rangle \quad \text{and} \quad g(T, t_2) = -\Re \left\langle it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} \right\rangle.$$

Proof: We compute the action of the spinorial curvature tensor \mathcal{R} on φ . We have

$$\begin{aligned} \nabla_{t_1} \nabla_{t_2} \varphi &= -\frac{1}{2} \nabla_{t_1} E(t_2) \bullet \varphi + \frac{1}{4} E(t_2) \bullet E(t_1) \bullet \varphi - \frac{\tau}{4} E(t_2) \bullet t_2 \bullet \varphi \\ &\quad - \frac{\tau}{2} \nabla_{t_1} t_1 \bullet \varphi + \frac{\tau}{4} t_1 \bullet E(t_1) \bullet \varphi - \frac{\tau^2}{4} t_1 \bullet t_2 \bullet \varphi. \end{aligned}$$

As well as

$$\begin{aligned} \nabla_{t_2} \nabla_{t_1} \varphi &= -\frac{1}{2} \nabla_{t_2} E(t_1) \bullet \varphi + \frac{1}{4} E(t_1) \bullet E(t_2) \bullet \varphi - \frac{\tau}{4} E(t_1) \bullet t_1 \bullet \varphi \\ &\quad - \frac{\tau}{2} \nabla_{t_2} t_2 \bullet \varphi + \frac{\tau}{4} t_2 \bullet E(t_2) \bullet \varphi - \frac{\tau^2}{4} t_2 \bullet t_1 \bullet \varphi. \end{aligned}$$

So, taking into account that $[t_1, t_2] = \nabla_{t_1} t_2 - \nabla_{t_2} t_1$, a straightforward computation gives

$$\mathcal{R}(t_1, t_2)\varphi = -\frac{1}{2}(d^\nabla E)(t_1, t_2) \bullet \varphi - \frac{1}{2} \det E t_1 \bullet t_2 \bullet \varphi - \frac{\tau^2}{2} t_1 \bullet t_2 \bullet \varphi.$$

On the other hand, it is well known that

$$R(t_1, t_2)\varphi = -\frac{1}{2} R_{1212} t_1 \bullet t_2 \bullet \varphi + \frac{i}{2} \Omega(t_1, t_2)\varphi.$$

Therefore, we have

$$(R_{1212} - \det E - \tau^2)t_1 \bullet t_2 \bullet \varphi = (d^\nabla E(t_1, t_2) - if(\kappa - 4\tau^2))\varphi. \quad (30)$$

Now, let T a vector field of M given by

$$g(T, t_1)|\varphi|^2 = \Re \langle it_2 \bullet \varphi, \varphi \rangle \quad \text{and} \quad g(T, t_2)|\varphi|^2 = -\Re \langle it_1 \bullet \varphi, \varphi \rangle.$$

It is easy to check that $T \bullet \varphi = -f\varphi + \bar{\varphi}$ and hence $f^2 + \|T\|^2 = 1$. In the following, we will prove that the spinor field $\theta := i\varphi - if\bar{\varphi} + JT \bullet \varphi$ is zero. For this, it is sufficient to prove that its norm vanishes. Indeed, we compute

$$|\theta|^2 = |\varphi|^2 + f^2|\varphi|^2 + \|T\|^2|\varphi|^2 - 2\Re \langle i\varphi, if\bar{\varphi} \rangle + 2\Re \langle i\varphi, JT \bullet \varphi \rangle \quad (31)$$

Therefore, Equation (31) becomes

$$\begin{aligned}
|\theta|^2 &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2\Re \langle i\varphi, JT \bullet \varphi \rangle \\
&= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, t_1)\Re \langle i\varphi, t_1 \bullet \varphi \rangle + 2g(JT, t_2)\Re \langle i\varphi, t_2 \bullet \varphi \rangle \\
&= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, t_1)g(T, t_2)|\varphi|^2 - 2g(JT, t_2)g(T, t_1)|\varphi|^2 \\
&= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2g(T, t_2)^2|\varphi|^2 - 2g(T, t_1)^2|\varphi|^2 \\
&= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2\|T\|^2|\varphi|^2 = 0.
\end{aligned}$$

Thus, we deduce $if\varphi = -f^2t_1 \bullet t_2 \bullet \varphi - fJT \bullet \varphi$, where we use the fact that $\bar{\varphi} = it_1 \bullet t_2 \bullet \varphi$. In this case, Equation (30) can be written as

$$(R_{1212} - \det E - \tau^2 - (\kappa - 4\tau^2)f^2)t_1 \bullet t_2 \bullet \varphi = (d^\nabla E(t_1, t_2) + (\kappa - 4\tau^2)JT) \bullet \varphi.$$

This is equivalent to say that both terms $R_{1212} - \det E - \tau^2 - (\kappa - 4\tau^2)f^2$ and $d^\nabla E(t_1, t_2) + (\kappa - 4\tau^2)JT$ are equal to zero. In fact, these are the Gauss-Codazzi equations in Definition 2.1. In order to obtain the two other equations, we simply compute the derivative of $T \bullet \varphi = -f\varphi + \bar{\varphi}$ in the direction of X in two ways. First, using that $iX \bullet \bar{\varphi} = JX \bullet \varphi$, we have

$$\begin{aligned}
\nabla_X T \bullet \varphi + T \bullet \nabla_X \varphi &= \nabla_X T \bullet \varphi - \frac{1}{2}T \bullet EX \bullet \varphi + i\frac{\tau}{2}T \bullet X \bullet \bar{\varphi} \\
&= \nabla_X T \bullet \varphi - \frac{1}{2}T \bullet EX \bullet \varphi + \frac{\tau}{2}T \bullet JX \bullet \varphi. \quad (32)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\nabla_X(T \bullet \varphi) &= -X(f)\varphi - f\nabla_X \varphi + \nabla_X \bar{\varphi} \\
&= -X(f)\varphi + \frac{1}{2}fEX \bullet \varphi + \frac{1}{2}EX \bullet \bar{\varphi} - i\frac{\tau}{2}fX \bullet \bar{\varphi} - \frac{i}{2}\tau X \bullet \varphi \\
&= -X(f)\varphi + \frac{1}{2}fEX \bullet \varphi - \frac{1}{2}f\tau JX \bullet \varphi \\
&\quad + \frac{1}{2}EX \bullet (T \bullet \varphi + f\varphi) - \frac{i}{2}\tau X \bullet \varphi \\
&= -X(f)\varphi + \frac{1}{2}fEX \bullet \varphi + \frac{1}{2}EX \bullet (T \bullet \varphi + f\varphi) \\
&\quad - \frac{i}{2}\tau X \bullet \varphi - \frac{1}{2}f\tau JX \bullet \varphi. \quad (33)
\end{aligned}$$

Take Equation (33) and subtract (32) to get

$$-X(f)\varphi + fEX \bullet \varphi - g(T, EX)\varphi - \nabla_X T \bullet \varphi - \frac{\tau}{2}T \bullet JX \bullet \varphi = 0.$$

Taking the real part of the scalar product of the last equation with φ and using that $\langle iX \bullet \varphi, \varphi \rangle = -g(T, JX)|\varphi|^2$, we get

$$X(f) = -g(T, EX) + \tau g(JX, T).$$

The imaginary part of the same scalar product gives $\nabla_X T = f(EX - \tau JX)$, which gives that there exists an immersion F from M into $\mathbb{E}(\kappa, \tau)$ with shape operator $dF \circ E \circ dF^{-1}$ and $\xi = dF(T) + f\nu$.

Now, we state the main result of this section, which characterize any isometric immersion of a surface (M, g) into $\mathbb{E}(\kappa, \tau)$.

Theorem 1. *Let $\kappa, \tau \in \mathbb{R}$ with $\kappa - 4\tau^2 \neq 0$. Consider (M^2, g) a simply connected oriented Riemannian surface. We denote by E a field of symmetric endomorphisms of TM , with trace equal to $2H$. The following statements are equivalent:*

1. *There exists an isometric immersion F of (M^2, g) into $\mathbb{E}(\kappa, \tau)$ with shape operator E , mean curvature H and such that, over M , the vertical vector is $\xi = dF(T) + f\nu$, where ν is the unit normal vector to the surface, f is a real function on M and T the tangential part of ξ .*
2. *There exists a Spin^c structure on M carrying a non-trivial spinor field φ satisfying*

$$\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi + i\frac{\tau}{2}X \bullet \bar{\varphi}.$$

Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame $\{t_1, t_2\}$, by $\Omega(t_1, t_2) = -(\kappa - 4\tau^2)f = -(\kappa - 4\tau^2)\frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$.

3. *There exists a Spin^c structure on M carrying a non-trivial spinor field φ of constant norm satisfying*

$$D\varphi = H\varphi - i\tau\bar{\varphi}.$$

Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame $\{t_1, t_2\}$, by $\Omega(t_1, t_2) = -(\kappa - 4\tau^2)f = -(\kappa - 4\tau^2)\frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$.

Proof: Proposition 3.3 and Lemma 3.2 give the equivalence between the first two statements. If the statement (2) holds, it is easy to check that in this case the Dirac operator acts on φ to give $D\varphi = H\varphi - i\tau\bar{\varphi}$. Moreover, for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} X(|\varphi|^2) &= 2\Re \langle \nabla_X \varphi, \varphi \rangle \\ &= \Re \langle i\tau X \bullet \bar{\varphi}, \varphi \rangle = 0. \end{aligned}$$

Hence φ is of constant norm. Now, consider a non-trivial spinor field φ of constant length, which satisfies $D\varphi = H\varphi - i\tau\bar{\varphi}$. Define the following 2-tensors on (M^2, g)

$$T_{\pm}^{\varphi}(X, Y) = \Re \langle \nabla_X \varphi^{\pm}, Y \bullet \varphi^{\mp} \rangle.$$

First note that

$$\text{tr}T_{\pm}^{\varphi} = -\Re \langle D\varphi^{\pm}, \varphi^{\mp} \rangle = -H|\varphi^{\mp}|^2. \quad (34)$$

Moreover, we have the following relations [16]

$$T_{\pm}^{\varphi}(t_1, t_2) = \tau|\varphi^{\mp}|^2 + T_{\pm}^{\varphi}(t_2, t_1), \quad (35)$$

$$\nabla_X \varphi^+ = \frac{T_+^\varphi(X)}{|\varphi^-|^2} \bullet \varphi^-, \quad (36)$$

$$\nabla_X \varphi^- = \frac{T_-^\varphi(X)}{|\varphi^+|^2} \bullet \varphi^+, \quad (37)$$

$$|\varphi^+|^2 T_+^\varphi = |\varphi^-|^2 T_-^\varphi, \quad (38)$$

where the vector field $T_+^\varphi(X)$ is defined by $g(T_+^\varphi(X), Y) = T_+^\varphi(X, Y)$ for $Y \in \Gamma(TM)$. Now let $F^\varphi := T_+^\varphi + T_-^\varphi$. Thus, we have

$$\frac{F^\varphi}{|\varphi|^2} = \frac{T_+^\varphi}{|\varphi_-^\varphi|^2} = \frac{T_-^\varphi}{|\varphi^+|^2}.$$

Hence $F^\varphi/|\varphi|^2$ is well defined on the whole surface M , and

$$\nabla_X \varphi = \nabla_X \varphi^+ + \nabla_X \varphi^- = \frac{F^\varphi(X)}{|\varphi|^2} \bullet \varphi, \quad (39)$$

where the vector field $F^\varphi(X)$ is defined by $g(F^\varphi(X), Y) = F^\varphi(X, Y)$, for all $Y \in \Gamma(TM)$. Note that by Equation (35), the 2-tensor F^φ is not symmetric. Define now the symmetric 2-tensor

$$T^\varphi(X, Y) = -\frac{1}{2|\varphi|^2} (F^\varphi(X, Y) + F^\varphi(Y, X)).$$

It is straightforward to show that

$$T^\varphi(t_1, t_1) = -F^\varphi(t_1, t_1)/|\varphi|^2, \quad T^\varphi(t_2, t_2) = -F^\varphi(t_2, t_2)/|\varphi|^2,$$

$$T^\varphi(t_1, t_2) = -F^\varphi(t_1, t_2)/|\varphi|^2 + \frac{\tau}{2} \quad \text{and} \quad T^\varphi(t_2, t_1) = -F^\varphi(t_2, t_1)/|\varphi|^2 - \frac{\tau}{2}.$$

Taking into account these last relations in Equation (39), we conclude

$$\nabla_X \varphi = -T^\varphi(X) \bullet \varphi + i\frac{\tau}{2} X \bullet \bar{\varphi}.$$

3.3 Application: a spinorial proof of Daniel correspondance

In [2], B. Daniel gave a Lawson type correspondance for constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$. Namely, he proved the following

Theorem 2. *Let $\mathbb{E}(\kappa_1, \tau_1)$ and $\mathbb{E}(\kappa_2, \tau_2)$ be two 3-dimensional homogeneous manifolds with four dimensional isometry group and assume that $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$. Consider ξ_1 and ξ_2 be the vertical vectors of $\mathbb{E}(\kappa_1, \tau_1)$ and $\mathbb{E}(\kappa_2, \tau_2)$ respectively and (M^2, g) a simply connected surface isometrically immersed into $\mathbb{E}(\kappa_1, \tau_1)$ with constant mean curvature H_1 so that $H_1^2 \geq \tau_2^2 - \tau_1^2$. We denote by ν_1 be the unit inner normal of the*

immersion, T_1 the tangential projection of ξ_1 and $f = \langle \nu_1, \xi_1 \rangle$.

Let $H_2 \in \mathbb{R}$ and $\theta \in \mathbb{R}$ so that

$$H_2^2 + \tau_2^2 = H_1^2 + \tau_1^2, \quad \text{and} \quad \tau_2 + iH_2 = e^{i\theta}(\tau_1 + iH_1).$$

Then, there exists an isometric immersion F from (M^2, g) into $\mathbb{E}(\kappa_2, \tau_2)$ with mean curvature H_2 and so that over M

$$\xi_2 = dF(T_2) + f\nu_2,$$

where ν_2 is the unit normal inner vector of the immersion and T_2 the tangential part of ξ_2 . Moreover, the respective shape operator E_1 and E_2 are related by the following

$$E_2 - H_2\text{Id} = e^{\theta J}(E_1 - H_1\text{Id}).$$

With the help of Theorem 1, we give an alternative proof of this results using spinors.

Proof of Theorem 2: Since M^2 is isometrically immersed into $\mathbb{E}(\kappa_1, \tau_1)$ there exists a spinor field φ_1 of constant norm (say $|\varphi_1| = 1$) satisfying

$$D\varphi_1 = H_1\varphi_1 - i\tau_1\bar{\varphi}_1,$$

associated with the Spin^c structure whose line bundle has a connection of curvature given by $\Omega = -(\kappa - 4\tau^2)f$, where $f = \frac{\langle \varphi, \bar{\varphi} \rangle}{|\varphi|^2}$. We deduce that

$$\begin{aligned} D\varphi_1^+ &= H_1\varphi_1^- + i\tau_1\varphi_1^- \\ D\varphi_1^- &= H_1\varphi_1^+ - i\tau_1\varphi_1^+. \end{aligned}$$

Now, we define $\varphi_2 = \varphi_1^+ + e^{i\theta}\varphi_1^-$. First, we have

$$\begin{aligned} D\varphi_2 &= D\varphi_1^+ + e^{i\theta}D\varphi_1^- \\ &= (H_1 + i\tau_1)\varphi_1^- - ie^{i\theta}(\tau_1 + iH_1)\varphi_1^+ \end{aligned}$$

Since $\tau_2 + iH_2 = e^{i\theta}(\tau_1 + iH_1)$, we deduce that $H_1 + i\tau_1 = e^{i\theta}(H_2 + i\tau_2)$ and so $D\varphi_2 = H_2\varphi_2 - i\tau_2\bar{\varphi}_2$. Secondly,

$$\frac{\langle \varphi_1, \bar{\varphi}_1 \rangle}{|\varphi_1|^2} = \frac{\langle \varphi_2, \bar{\varphi}_2 \rangle}{|\varphi_2|^2}.$$

Now, since $\kappa_1 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2$, the considered Spin^c structure on M is given by $i\Omega = -i(\kappa_2 - 4\tau_2^2)f$ and hence, by Theorem 1, there exists an isometric immersion F from (M^2, g) into $\mathbb{E}(\kappa_2, \tau_2)$ with mean curvature H_2 and so that $\xi_2 = dF(T_2) + f\nu_2$, where ν_2 is the unit normal inner vector of the surface and T_2 the tangential part of ξ_2 .

Remark 4. By the proof of Proposition 3.3, we have that

$$g(T_2, t_1)|\varphi_2|^2 = \Re \langle it_2 \bullet \varphi_2, \varphi_2 \rangle \quad \text{and} \quad g(T_2, t_2)|\varphi_2|^2 = -\Re \langle it_1 \bullet \varphi_2, \varphi_2 \rangle.$$

So, it is easy to see that $T_2 = e^{\theta J}(T_1)$.

4 Isometric immersions into $\mathbb{M}_{\mathbb{C}}^2(c)$ via spinors

In this section, we consider the canonical Spin^c structure on $\mathbb{M}_{\mathbb{C}}^2(c)$ carrying a parallel spinor field ψ lying in $\Sigma^+(\mathbb{M}_{\mathbb{C}}^2(c))$. The restriction of this Spin^c structure to any hypersurface M^3 defines a Spin^c structure on M with a special spinor field. This spinor field characterizes the isometric immersion of M into $\mathbb{M}_{\mathbb{C}}^2(c)$.

4.1 Special spinors fields on $\mathbb{M}_{\mathbb{C}}^2(c)$ and their surfaces

Assume that there exists an isometric immersion of (M^3, g) into $\mathbb{M}_{\mathbb{C}}^2(c)$ with shape operator II . By section 2.3, we know that M has an almost contact metric structure $(\mathfrak{X}, \xi, \eta)$ such that $\mathfrak{X}X = JX - \eta(X)\nu$ for every $X \in \Gamma(TM)$.

Lemma 4.1. *The restriction φ of the parallel spinor ψ on $\mathbb{M}_{\mathbb{C}}^2(c)$ is a solution of the generalized Killing equation*

$$\nabla_X \varphi + \frac{1}{2} II X \bullet \varphi = 0, \quad (40)$$

Moreover, φ satisfies $\xi \bullet \varphi = -i\varphi$. The curvature 2-form of the auxiliary line bundle associated with the induced Spin^c structure is given by $\Omega(X, Y) = -6c \times (X, Y)$, where \times is the Kähler form of $\mathbb{M}_{\mathbb{C}}^2(c)$ given by $\times(X, Y) = \bar{g}(JX, Y)$.

Proof: First, since ψ is parallel, we have $D^{\mathbb{M}_{\mathbb{C}}^2(c)}\psi = (\nabla^{\mathbb{M}_{\mathbb{C}}^2(c)})^*\nabla^{\mathbb{M}_{\mathbb{C}}^2(c)}\psi = 0$. Hence, by the Schrödinger-Lichnerowicz formula, we get

$$\Omega^{\mathbb{M}_{\mathbb{C}}^2(c)} \cdot \psi = 12ci\psi. \quad (41)$$

By the Gauss formula (9), the restriction φ of the parallel spinor ψ on $\mathbb{M}_{\mathbb{C}}^2(c)$ satisfies

$$\nabla_X \varphi = -\frac{1}{2} II X \bullet \varphi.$$

Since the spinor ψ is parallel, Equality (6) gives

$$\text{Ric}^{\mathbb{M}_{\mathbb{C}}^2(c)}(X) \cdot \psi = i(X \lrcorner \Omega^{\mathbb{E}(\kappa, \tau)}) \cdot \psi$$

Where Ric is the Ricci tensor of $\mathbb{M}_{\mathbb{C}}^2(c)$. Therefore, we compute,

$$\begin{aligned} (\nu \lrcorner \Omega^{\mathbb{M}_{\mathbb{C}}^2(c)}) \bullet \varphi &= (\nu \lrcorner \Omega^{\mathbb{M}_{\mathbb{C}}^2(c)}) \cdot \nu \cdot \psi|_M \\ &= -\nu \cdot (\nu \lrcorner \Omega^{\mathbb{M}_{\mathbb{C}}^2(c)}) \cdot \psi|_M \\ &= i\nu \cdot \text{Ric}^{\mathbb{M}_{\mathbb{C}}^2(c)} \nu \cdot \psi|_M \\ &= -6ci\varphi. \end{aligned}$$

By Equation (8), we get that

$$\Omega \bullet \varphi = 6ci\varphi. \quad (42)$$

Now, for any $X, Y \in \Gamma(TM)$, we have

$$\Omega(X, Y) = \Omega^{\mathbb{M}_\mathbb{C}^2(c)}(X, Y) = -\rho(X, Y) = -\text{Ric}(JX, Y) = -6c\bar{g}(JX, Y).$$

Let e_1 be a unit vector field tangent to M such that $\{e_1, e_2 = Je_1, \xi\}$ is an orthonormal basis of TM . In this basis, we have

$$\Omega \bullet \varphi = \Omega(e_1, e_2) e_1 \bullet e_2 \bullet \varphi + \Omega(e_1, \xi) e_1 \bullet \xi \bullet \varphi + \Omega(e_2, \xi) e_2 \bullet \xi \bullet \varphi.$$

But,

$$\Omega(e_1, e_2) = -6c \quad \text{and} \quad \Omega(e_1, \xi) = \Omega(e_2, \xi) = 0.$$

Finally, $\Omega \bullet \varphi = -6ce_1 \bullet e_2 \bullet \varphi$. Using (42) and the fact that $e_1 \bullet e_2 \bullet \xi \bullet \varphi = -\varphi$, we conclude that $\xi \bullet \varphi = -i\varphi$.

Lemma 4.2. *Let E be a field of symmetric endomorphisms on a Spin^c manifold M^3 of dimension 3, then*

$$\begin{aligned} E(e_i) \bullet E(e_j) - E(e_j) \bullet E(e_i) &= 2(a_{j3}a_{i2} - a_{j2}a_{i3})e_1 \\ &\quad + 2(a_{i3}a_{j1} - a_{i1}a_{j3})e_2 \\ &\quad + 2(a_{i1}a_{j2} - a_{i2}a_{j1})e_3, \end{aligned} \quad (43)$$

where $(a_{ij})_{i,j}$ is the matrix of E written in any local orthonormal frame of TM .

Proposition 4.3. *Let (M^3, g) be a Riemannian Spin^c manifold endowed with an almost contact metric structure $(\mathfrak{X}, \xi, \eta)$. Assume that there exists a non-trivial spinor φ satisfying*

$$\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi \quad \text{and} \quad \xi \bullet \varphi = -i\varphi,$$

where E is a field of symmetric endomorphisms on M . We suppose that the curvature 2-form of the connection on the auxiliary line bundle associated with the Spin^c structure is given by $\Omega(e_1, e_2) = -6c$ and $\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$. Hence, the Gauss equation for $\mathbb{M}_\mathbb{C}^2(c)$ is satisfied if and only if the Codazzi equation for $\mathbb{M}_\mathbb{C}^2(c)$ is satisfied.

Proof: We compute the spinorial curvature \mathcal{R} on φ , we get

$$\mathcal{R}_{X,Y} \varphi = -\frac{1}{2}d^\nabla E(X, Y) \bullet \varphi + \frac{1}{4}(EY \bullet EX - EX \bullet EY) \bullet \varphi.$$

In the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$, the Ricci identity (6) gives that

$$\begin{aligned} \frac{1}{2}\text{Ric}(X) \bullet \varphi - \frac{i}{2}(X \lrcorner \Omega) \bullet \varphi &= \frac{1}{4} \sum_{k=1}^3 e_k \bullet (EX \bullet Ee_k - Ee_k \bullet EX) \bullet \varphi \\ &\quad - \frac{1}{2} \sum_{k=1}^3 e_k \bullet d^\nabla E(e_k, X) \bullet \varphi. \end{aligned}$$

By Lemma 4.2 and for $X = e_1$, the last identity becomes

$$\begin{aligned}
& (R_{1221} + R_{1331} - a_{11}a_{33} - a_{11}a_{22} + a_{13}^2 + a_{12}^2 - 5c)e_1 \bullet \varphi \\
& + (R_{1332} - a_{12}a_{33} + a_{32}a_{13})e_2 \bullet \varphi \\
& + (R_{1223} - a_{22}a_{13} + a_{32}a_{12})e_3 \bullet \varphi \\
= & -e_2 \bullet d^\nabla E(e_2, e_1) \bullet \varphi - e_3 \bullet d^\nabla E(e_3, e_1) \bullet \varphi \\
& + ce_1 \bullet \varphi. \tag{44}
\end{aligned}$$

Since $|\varphi|$ is constant ($|\varphi| = 1$), the set $\{\varphi, e_1 \bullet \varphi, e_2 \bullet \varphi, e_3 \bullet \varphi\}$ is an orthonormal frame of ΣM with respect to the real scalar product $\Re \langle \cdot, \cdot \rangle$. Hence, from Equation (44) we deduce

$$\begin{aligned}
R_{1221} + R_{1331} - (a_{11}a_{33} + a_{11}a_{22} - a_{13}^2 - a_{12}^2 + 5c) &= g(d^\nabla E(e_1, e_3), e_3) - g(d^\nabla E(e_1, e_3), e_2) + c \\
R_{1332} - (a_{12}a_{33} - a_{32}a_{13}) &= g(d^\nabla E(e_1, e_3), e_1) \\
R_{1223} - (a_{22}a_{13} - a_{32}a_{12}) &= g(d^\nabla E(e_1, e_2), e_1) \\
g(d^\nabla E(e_1, e_2), e_2) &= -g(d^\nabla E(e_1, e_3), e_3)
\end{aligned}$$

The same computation holds for the unit vector fields e_2 and e_3 and we get

$$\begin{aligned}
R_{2331} - (a_{12}a_{33} - a_{13}a_{23}) &= -g(d^\nabla E(e_2, e_3), e_2) \\
R_{2332} + R_{2112} - (a_{22}a_{33} + a_{22}a_{11} - a_{13}^2 - a_{12}^2 + 5c) &= g(d^\nabla E(e_2, e_3), e_1) + g(d^\nabla E(e_1, e_2), e_3) + c \\
R_{2113} - (a_{23}a_{11} - a_{12}a_{13}) &= -g(d^\nabla E(e_1, e_2), e_2) \\
g(d^\nabla E(e_1, e_2), e_1) &= g(d^\nabla E(e_2, e_3), e_3) \\
R_{3221} - (a_{13}a_{22} - a_{23}a_{21}) &= -g(d^\nabla E(e_2, e_3), e_3) \\
R_{3112} - (a_{32}a_{11} - a_{31}a_{12}) &= g(d^\nabla E(e_1, e_3), e_3) \\
R_{3113} + R_{3223} - (a_{22}a_{33} - a_{11}a_{33} + a_{13}^2 + a_{23}^2) &= g(d^\nabla E(e_2, e_3), e_1) - g(d^\nabla E(e_1, e_3), e_2) \\
g(d^\nabla E(e_2, e_3), e_2) &= -g(d^\nabla E(e_1, e_3), e_1)
\end{aligned}$$

The last twelve equations imply that the Gauss equation for $\mathbb{M}_{\mathbb{C}}^2(c)$ is satisfied if and only if the Codazzi equation for $\mathbb{M}_{\mathbb{C}}^2(c)$ is satisfied.

4.2 Spinorial characterization of hypersurfaces of $\mathbb{M}_{\mathbb{C}}^2(c)$

Now, we give the main result of this section:

Theorem 3. *Let (M^3, g) be a simply connected oriented Riemannian manifold endowed with an almost contact metric structure $(\mathfrak{X}, \xi, \eta)$. Let E be a field of symmetric endomorphisms on M with trace equal to $3H$. Assume that the Gauss or the Codazzi equation for $\mathbb{M}_{\mathbb{C}}^2(c)$ is satisfied. Then, the following statements are equivalent:*

1. *There exists an isometric immersion of (M^3, g) into $\mathbb{M}_{\mathbb{C}}^2(c)$ with shape operator E , mean curvature H and so that, over M , the complex structure of $\mathbb{M}_{\mathbb{C}}^2(c)$ is given by $J = \mathfrak{X} + \eta(\cdot)\nu$, where ν is the unit normal vector of the immersion.*

2. There exists a Spin^c structure on M carrying a non-trivial spinor φ satisfying

$$\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi \quad \text{and} \quad \xi \bullet \varphi = -i\varphi.$$

The curvature 2-form of the connection on the auxiliary bundle associated with the Spin^c structure is given by $\Omega(e_1, e_2) = -6c$ and $\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$.

3. There exists a Spin^c structure on M carrying a non-trivial spinor φ of constant norm and satisfying

$$D\varphi = \frac{3}{2}H\varphi \quad \text{and} \quad \xi \bullet \varphi = -i\varphi.$$

The curvature 2-form of the connection on the auxiliary bundle associated with the Spin^c structure is given by $\Omega(e_1, e_2) = -6c$ and $\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$.

Proof: By Lemma 4.1, the first statement implies the second one. Using Proposition 4.3, to show that 2 \implies 1, it suffices to show that $\nabla_X \xi = \mathfrak{X}EX$. In fact, we simply compute the derivative of $\xi \bullet \varphi = -i\varphi$ in the direction of $X \in \Gamma(TM)$ to get

$$\nabla_X \xi \bullet \varphi = \frac{i}{2}EX \bullet \varphi + \frac{1}{2}\xi \bullet EX \bullet \varphi$$

Using that $-ie_2 \bullet \varphi = e_1 \bullet \varphi$, the last equation reduces to

$$\nabla_X \xi \bullet \varphi - g(EX, e_1)e_2 \bullet \varphi + g(EX, e_2)e_1 \bullet \varphi = 0.$$

Finally $\nabla_X \xi \bullet \varphi = \mathfrak{X}EX$. Now, we compute the derivative of $-ie_2 \bullet \phi = e_1 \bullet \phi$ in the direction of e_1 to get

$$\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi - \frac{1}{2}e_2 \bullet Ee_1 \bullet \phi = i\nabla_{e_1}e_1 \bullet \phi - \frac{i}{2}e_1 \bullet Ee_1 \bullet \phi.$$

But, using that $\xi \bullet \phi = -i\phi$, we have

$$\frac{1}{2}e_2 \bullet Ee_1 \bullet \phi - \frac{i}{2}e_1 \bullet Ee_1 \bullet \phi = -a_{11}\xi \bullet \phi - a_{12}\phi.$$

Denoting by Γ_{ij}^k the Christoffel symbols of $\{e_1, \mathfrak{X}e_1, \xi\}$, we have $\nabla_{e_1}e_1 = \Gamma_{11}^1e_1 + \Gamma_{11}^2e_2 + \Gamma_{11}^3e_3$. Moreover, using that $\nabla_{e_1}e_3 = \mathfrak{X}Ee_1$, we get

$$\Gamma_{11}^3 = g(\nabla_{e_1}e_1, e_3) = -g(e_1, \nabla_{e_1}e_3) = a_{12}.$$

Hence, $\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi = -a_{11}\xi \bullet \phi + \Gamma_{11}^1e_2 \bullet \phi + \Gamma_{11}^2e_2 \bullet \phi$. Finally

$$\nabla_{e_1}(\mathfrak{X}e_1) \bullet \phi - \mathfrak{X}(\nabla_{e_1}e_1) \bullet \phi = -a_{11}\xi \bullet \phi,$$

which is Equation (23) for $X = Y = e_1$. Similarly, we compute the derivative of $-ie_2 \bullet \phi = e_1 \bullet \phi$ in the direction of e_2 and ξ to get Equation (23) for any $X, Y \in \Gamma(TM)$. It is easy to see that the assertion 2 implies the assertion 3. For $3 \Rightarrow 2$, since φ is of constant norm ($|\varphi| = 1$), the set $\{\varphi, e_1 \bullet \varphi, e_2 \bullet \varphi, e_3 \bullet \varphi\}$ is a local orthonormal frame of ΣM with respect to the real scalar product $\Re \langle \cdot, \cdot \rangle$. Hence, for every $X \in \Gamma(TM)$, we have

$$\nabla_X \varphi = \eta(X)\varphi + \ell(X) \bullet \varphi, \quad (45)$$

where η is a 1-form and ℓ is a $(1, 1)$ -tensor field. Moreover it is easy to check that $\eta = \frac{d(|\varphi|^2)}{2|\varphi|^2}$ and $\ell(X) = -\ell^\varphi(X)$. Since φ is of constant norm we have $\eta = 0$. Moreover, $\ell(X) = -\ell^\varphi(X)$ is symmetric of trace $\frac{3}{2}H$. It suffices to consider $E = 2\ell^\varphi$ to get the second assertion.

4.3 Characterization of Sasaki hypersurfaces

Theorem 3 characterizes isometric immersions of almost contact metric manifolds into $\mathbb{M}_{\mathbb{C}}^2(c)$ providing that the shape operator E satisfies the Gauss or the Codazzi equation for $\mathbb{M}_{\mathbb{C}}^2(c)$. In this subsection, we eliminate this restriction and we replace it by some geometric conditions on the almost contact metric manifold.

In Section 3, we showed that the 3-dimensional homogeneous manifolds $\mathbb{E}(\kappa, \tau)$ ($\tau \neq 0, \kappa - 4\tau^2 \neq 0$), which are Sasaki, have a Spin^c structure (the anti-canonical Spin^c structure) carrying a Killing spinor field φ of Killing constant $\frac{\tau}{2}$. Moreover $\xi \cdot \varphi = -i\varphi$ and

$$\Omega^{\mathbb{E}(\kappa, \tau)}(e_1, e_2) = -(\kappa - 4\tau^2) \text{ and } \Omega^{\mathbb{E}(\kappa, \tau)}(e_i, e_j) = 0, \quad (46)$$

in the basis $\{e_1, \mathfrak{X}e_1 = e_2, e_3 = \xi\}$. Hence, the statement (2) of Theorem 3 is satisfied for $E = -\tau \text{Id}$ and $c = \frac{\kappa - 4\tau^2}{6} \neq 0$. But $\mathbb{E}(\kappa, \tau)$ cannot be immersed into $\mathbb{M}_{\mathbb{C}}^2(c)$ ($c = \frac{\kappa - 4\tau^2}{6} \neq 0$) with second fundamental form $E = -\tau \text{Id}$ because we know that totally umbilic hypersurfaces in $\mathbb{M}_{\mathbb{C}}^2(c)$ cannot exist. Moreover, the Codazzi equation is not satisfied. In fact, it is easy to check that $d^\nabla E(e_1, e_2) = 0$, and

$$c\{\eta(e_1)\mathfrak{X}e_2 - \eta(e_2)\mathfrak{X}e_1 + 2g(e_1, \mathfrak{X}e_2)\xi\} = -2c\xi \neq 0.$$

From this example, it is clear that the condition “ E satisfies the Gauss equation or the Codazzi equation” is a necessary condition to immerse in $\mathbb{M}_{\mathbb{C}}^2(c)$ an almost contact metric manifold M satisfying the statement (2) of Theorem 3 and even if the manifold M is Sasaki. However, we can state the following:

Theorem 4. *Let (M^3, g) be a simply connected oriented Riemannian manifold endowed with a Sasakian structure $(\mathfrak{X}, \xi, \eta)$. Then, the following statements are equivalent:*

1. *There exists an isometric immersion of (M^3, g) into $\mathbb{M}_{\mathbb{C}}^2(c)$ with mean curvature H and so that, over M , the complex structure of $\mathbb{M}_{\mathbb{C}}^2(c)$ is given by $J = \mathfrak{X} + \eta(\cdot)\nu$, where ν is the unit normal vector of the immersion.*
2. *There exists a Spin^c structure on M carrying a non-trivial spinor φ satisfying*

$$\nabla_X \varphi = -\frac{1}{2}X \bullet \varphi - \frac{i}{2}c \eta(X)\varphi \quad \text{and} \quad \xi \bullet \varphi = -i\varphi.$$

The curvature 2-form of the connection on the auxiliary bundle associated with the Spin^c structure is given by $\Omega(e_1, e_2) = -6c$ and $\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$.

In this case, M is of constant mean curvature $H = \frac{3-c}{3}$ and the shape operator E is given by $E = \text{Id} - c\eta(\cdot)\xi$.

Proof: Assume that (M^3, g) is a Sasaki manifold immersed into $\mathbb{M}_{\mathbb{C}}^2(c)$ with shape operator E . Since ξ is a Killing vector field, Equation (18) implies that $\mathfrak{X}(E\xi) = \nabla_{\xi}\xi = 0$ and hence $E\xi = f\xi$, where f is a real function on M . Also, from Equation (18) and since $\nabla_X \xi = \mathfrak{X}X$, we get $\mathfrak{X}(EX - X) = 0$, for all $X \in \Gamma(TM)$. Then,

$$EX - X = g(EX - X, \xi)\xi.$$

But, $g(EX - X, \xi) = (f - 1)g(X, \xi)$ which gives that $EX = X + (f - 1)g(X, \xi)\xi$. It is straightforward to check that

$$\begin{aligned} (\nabla_X E)(Y) - (\nabla_Y E)(X) &= -(f - 1)\{\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X + 2g(X, \mathfrak{X}Y)\xi\} \\ &\quad + \{df(X)\eta(Y) - df(Y)\eta(X)\}\xi, \end{aligned}$$

for all vectors $X, Y \in \Gamma(TM)$. Comparing the last equation with (20), we get $f - 1 = -c$. This gives $EX = X - c\eta(X)\xi$ and by Theorem 3, we get the statement (2). Now, we assume that the statement (2) holds, i.e., we have on M a Spin^c structure carrying a non-trivial spinor φ satisfying

$$\nabla_X \varphi = -\frac{1}{2}X \bullet \varphi - \frac{i}{2}c \eta(X)\varphi \quad \text{and} \quad \xi \bullet \varphi = -i\varphi. \quad (47)$$

The curvature 2-form of the connection on the auxiliary bundle associated with the Spin^c structure is given by $\Omega(e_1, e_2) = -6c$ and $\Omega(e_i, e_j) = 0$ elsewhere in the basis $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$. We denote by E the endomorphism given for all $X \in \Gamma(TM)$, by $EX = X - c\eta(X)\xi$. From (47), we have $\nabla_X \varphi = -\frac{1}{2}EX \bullet \varphi$ and we can check that $E = \text{Id} - c\eta(\cdot)\xi$ satisfies, for all vectors $X, Y \in \Gamma(TM)$,

$$(\nabla_X E)(Y) - (\nabla_Y E)(X) = c\{\eta(X)\mathfrak{X}Y - \eta(Y)\mathfrak{X}X + 2g(X, \mathfrak{X}Y)\xi\},$$

which is the Codazzi equation (22). By Theorem 3, M is immersed into $\mathbb{M}_{\mathbb{C}}^2(c)$ with shape operator E . Additionally, since $EX = X - c\eta(X)\xi$, we have $H = \frac{3-c}{3}$.

Remark 5. From the above example, $\mathbb{E}(\kappa, \tau)$ with $\tau \neq 0$ endowed with their anti-canonical Spin^c structure cannot be immersed into $\mathbb{M}_{\mathbb{C}}^2(c)$ for $c = \frac{\kappa - 4\tau^2}{6} \neq 0$. In fact, the Killing spinor of Killing constant $\frac{\tau}{2}$ does not satisfy assertion (2) of Theorem 4 because for example, when $\tau = -1$, the endomorphism $E = \text{Id}$ is not of the form $E = \text{Id} - c \eta(\cdot)\xi$. On the other side, it is known that there exists an isometric embedding of $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, into $\mathbb{M}^2(\frac{\kappa}{4} - \tau^2)$ of constant mean curvature $H = \frac{\kappa - 16\tau^2}{12\tau}$ [27]. In a recent work [22], the authors used the canonical and the anti-canonical Spin^c structures on $\mathbb{E}(\kappa, \tau)$, to define another Spin^c structure on $\mathbb{E}(\kappa, \tau)$ satisfying assertion (2) of Theorem 4 and hence allowing to immerse $\mathbb{E}(\kappa, \tau)$ into $\mathbb{M}_{\mathbb{C}}^2(c)$. Other geometric applications are also given.

References

- [1] B. Daniel, *Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces*, Trans. Amer. Math. Soc. 361 (2009), no 12, 6255-6282.
- [2] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. 82 (2007), 87-131.
- [3] B. Daniel, L. Hauswirth and P. Mira, *Constant mean curvature surfaces in homogeneous 3-manifolds*, Lectures Notes of the 4th KIAS Workshop on Diff. Geom.
- [4] T. Friedrich, *On the spinor representation of surfaces in Euclidean 3-space*, J. Geom. Phys. 28 (1998), no 1-2, 143-157.
- [5] Th. Friedrich, *Dirac operator's in Riemannian geometry*, Graduate studies in mathematics, Volume 25, American Mathematical Society.
- [6] G. Habib and R. Nakad, *The Energy-Momentum tensor on low dimensional Spin^c manifolds*, Preprint (2010).
- [7] M. Herzlich et A. Moroianu, *Generalized Killing spinors and conformal eigenvalue estimates for Spin^c manifold*, Ann. Glob. Anal. Geom. 17 (1999), 341-370.
- [8] O. Hijazi, *Eigenvalues of the Dirac operator on compact Kähler manifolds*, Commun. Math. Phys. 160, 563-579 (1994).
- [9] O. Hijazi, S. Montiel and F. Urbano, *Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds*, Math. Z. 253, Number 4 (2006) 821–853.
- [10] K. D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Glob. analysis and geometry, vol. 4, No.3, 291-325 (1986).

- [11] S. Kobayashi et K. Nomizu, *Foundations of differential geometry*, Volume 1, Wiley Classics library Edition Published (1996).
- [12] R. Kusner, N. Schmitt, *The spinor representation of surfaces in space*, ArXiv:dg-ga/9610005.
- [13] M.A. Lawn and J. Roth, *Isometric immersions of hypersurfaces into 4-dimensional manifolds via spinors*, Diff. Geom. Appl., 28 n°2 (2010), 205-219.
- [14] B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, 1989.
- [15] S. Montiel, *Using spinors to study submanifolds*, Roma 2004 - Nancy 2005.
- [16] B. Morel, *Surfaces in \mathbb{S}^3 and \mathbb{H}^3 via spinors*, Sémin. Théor. Spectr. Géom., Univ Grenoble 1, 23 (2005), 131-144.
- [17] A. Moroianu, *Parallel and Killing spinors on Spin^c manifolds*, Commun. Math. Phys. 187 (1997), 417-428.
- [18] A. Moroianu, *Spin^c manifolds and complex contact structures*, Commun. Math. Phys. 193 (1998), 661-673.
- [19] A. Moroianu, *Opérateur de Dirac et submersions riemanniennes*, Ph.D thesis, Ecole polytechnique, 1996.
- [20] R. Nakad, *Lower bounds for the eigenvalues of the Dirac operator on Spin^c manifolds*, J. Geom. Phys. 60 (2010), 1634-1642.
- [21] R. Nakad, *The Energy-Momentum tensor on Spin^c manifolds*, IJGMMP Vol. 8, No. 2, 2011.
- [22] R. Nakad and J. Roth, *Eigenvalue estimates for the Dirac operator on Spin^c hypersurfaces*, submitted 2012.
- [23] P. Piccione and D.V. Tausk, *An existence theorem for G -structure preserving affine immersions*, Indiana Univ. Math. J. 57 n°3 (2008), 1431-1465.
- [24] J. Roth, *Rigidité des hypersurfaces en géométrie riemannienne et spinorielle: aspect extrinsèque et intrinsèque*, Ph.D. thesis, Institut Élie Cartan, 2006.
- [25] J. Roth, *Spinorial characterization of surfaces into 3-dimensional homogeneous manifolds*, J. Geom. Phys. 60 (2010), 1045-106.
- [26] P. Scott, *The geometry of 3-manifolds*, Bull. Lond. Math. Soc. 15 (5) (1983), 401-487.
- [27] F. Torralbo and F. Urbano, *Compact stable mean curvature surfaces in homogeneous 3-manifolds*, Indiana U. Math. J. (2011).