Isometric Immersions of Hypersurfaces in 4-dimensional Manifolds via Spinors

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Abstract

We give a spinorial characterization of isometrically immersed hypersurfaces into 4-dimensional space forms and product spaces $\mathbb{M}^3(\kappa)\times\mathbb{R}$, in terms of the existence of particular spinor fields, called generalized Killing spinors or equivalently solutions of a Dirac equation. This generalizes to higher dimensions several recent results for surfaces by T. Friedrich, B. Morel and the two authors. The main argument is the interpretation of the energy-momentum tensor of a generalized Killing spinor as the second fundamental form, possibly up to a tensor depending on the ambient space. As an application, we deduce some non-existence results for isometric immersions into the 4-dimensional Euclidean space

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1 Introduction

A classical problem in Riemannian geometry is to know when a Riemannian manifold (M^n, g) can be isometrically immersed into a fixed Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$. In this paper, we will focus on the case of hypersurfaces, that is $p = 1$.

The case of space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ is well-known. The Gauss and Codazzi-Mainardi equations are necessary and sufficient conditions. Recently, B. Daniel ([3]) gave an analogous characterization for hypersurfaces in the product spaces $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$.

In low dimensions, namely for surfaces, another necessary and sufficient condition is now well-known, namely the existence of a special spinor field called *generalized Killing spinor field* (see [4, 16, 18, 9, 11]). Note that this condition is not restrictive since any oriented surface is also spin. This approach was first used by T. Friedrich ([4]) for surfaces in $\mathbb{R}^3$ and then extended to other 3-dimensional Riemannian manifolds by ([16, 18]).

More generally, the restriction φ of a parallel spinor field on $\mathbb{R}^{n+1}$ to an oriented Riemannian hypersurface M^n is a solution of a generalized Killing equation

$$\nabla^{\Sigma M}_X\varphi = -\frac{1}{2}\gamma^M(A(X))\varphi, \tag{1}$$

where γ^M and $\nabla^{\Sigma M}$ are respectively the Clifford multiplication and the spin connection on M^n, and A is the Weingarten tensor of the immersion. Conversely, Friedrich proves in [4] that, in the two dimensional case, if there exists a generalized Killing spinor field satisfying equation (1), where A is an arbitrary field of symmetric endomorphisms of TM, then A satisfies the Codazzi-Mainardi and Gauss equations of hypersurface theory and is consequently the Weingarten tensor of a local isometric immersion of M into $\mathbb{R}^3$. Moreover, in this case, the solution φ of the generalized Killing equation is equivalently a solution of the Dirac equation

$$D\varphi = H\varphi, \tag{2}$$

where $|\varphi|$ is constant and H is a real-valued function.

One feature of those spinor representations is that fundamental topological informations can be read off more easily from the spinorial data (see for example [8]).

The question of a spinorial characterization of 3-dimensional manifolds as hypersurfaces into a given 4-dimensional manifold is also of special interest since, again, any oriented 3-dimensional manifold is spin. The case of hypersurfaces of the 4-dimensional Euclidean space has been treated by Morel in [16], when A is a Codazzi tensor. Here, we extend Morel's result to other 4-dimensional space forms and product spaces, that is $\mathbb{S}^4$, $\mathbb{H}^4$ (see Theorem 1), $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ (see Theorem 2).

The techniques we use in this article are different from those in Friedrich and Morel's approach. The main difference is that unlike in the 2-dimensional case, the spinor bundle of a 3-dimensional manifold does not decompose into subbundles of positive and negative half-spinors. In this case, the condition for an isometric immersion is the existence of two particular spinor fields on the manifold instead of one as in the case of surfaces. Moreover, we prove the equivalence between the generalized Killing equation and the Dirac equation for spinor fields of constant norm in the above four cases.

The last paragraph is devoted to an application. We prove in a straightforward way using our results and the existence of special spinors on certain three-dimensional η-Einstein manifolds that they cannot be isometrically immersed into the Euclidean space $\mathbb{R}^4$.

2 Preliminaries

2.1 Hypersurfaces and induced spin structures

We begin by preliminaries on hypersurfaces and induced spin structures. The reader can refer to [12, 5, 2] for basic facts about spin geometry and [1, 15, 7] for the spin geometry of hypersurfaces.

Let (N^{n+1}, g) be a Riemannian spin manifold and ΣN its spinor bundle. We denote by ∇ the Levi-Civita connection on TN, and $\nabla^{\Sigma N}$ the spin connection on ΣN. The Clifford multiplication will be denoted by γ and $\langle .,. \rangle$ is the natural Hermitian product on ΣN, compatible with ∇ and γ. Finally, we denote by D the Dirac operator on N locally given by $D = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}$, where $\{e_1, \cdots, e_{n+1}\}$ is an orthonormal frame of TN.

Now let M be an orientable hypersurface of N. Since the normal bundle is trivial, the hypersurface M is also spin. Indeed, the existence of a normal unit vector field ν globally defined on M induces a spin structure from that on N.

Then we can consider the intrinsic spinor bundle of M denoted by ΣM. We denote respectively by $\nabla^{\Sigma M}$, γ^M and D^M, the Levi-Civita connection, the Clifford multiplication and the intrinsic Dirac operator on M. We can also define an extrinsic spinor bundle on M by $\mathbf{S} := \Sigma N_{|M}$. Then we recall the identification between these two spinor bundles (*cf* [7], [15] or [1] for instance):

$$\mathbf{S} \equiv \begin{cases} \Sigma M & \text{if } n \text{ is even} \\ \Sigma M \oplus \Sigma M & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

The interest of this identification is that we can use restrictions of ambient spinors to study the intrinsic Dirac operator of M. Indeed, we can define an extrinsic connection $\nabla^{\mathbf{S}}$ and a Clifford multiplication $\gamma^{\mathbf{S}}$ on $\mathbf{S}$ by

$$\nabla^{\mathbf{S}} = \nabla + \frac{1}{2}\gamma(\nu)\gamma(A), \tag{4}$$

$$\gamma^{\mathbf{S}} = \gamma(\nu)\gamma, \tag{5}$$

where ν is the exterior normal unit vector field and A the associated Weingarten operator. By the previous identification given by (3), we can also identify connections and Clifford multiplications.

$$\nabla^{\mathbf{S}} \equiv \begin{cases} \nabla^{\Sigma M} & \text{if } n \text{ is even,} \\ \nabla^{\Sigma M} \oplus \nabla^{\Sigma N} & \text{if } n \text{ is odd,} \end{cases} \tag{6}$$

$$\gamma^{\mathbf{S}} \equiv \begin{cases} \gamma^M & \text{if } n \text{ is even,} \\ \gamma^M \oplus -\gamma^M & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Then, we can consider the following extrinsic Dirac operator on M, acting on sections of $\mathbf{S}$, denoted by $\mathbf{D}$ and given locally by

$$\mathbf{D} = \sum_{i=1}^{n} \gamma^{\mathbf{S}}(e_i)\nabla^{\mathbf{S}}_{e_i}, \tag{8}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal local frame of TM. Then, by (4), we have

$$\mathbf{D} = \frac{n}{2}H - \gamma(\nu)\sum_{i=1}^{n}\gamma(e_i)\nabla_{e_i}, \tag{9}$$

that is, for any $\psi \in \Gamma(\mathbf{S})$

$$\mathbf{D}\psi := \frac{n}{2}H\psi - \gamma(\nu)D\psi - \nabla_\nu\psi. \tag{10}$$

Remark 1. *In the sequel, when we are only considering 3-dimensional manifolds, we will denote for the sake of simplicity the Clifford multiplication by a dot.*

We have all the spinorial ingredients, and now, we will give some reminders about surfaces into product spaces.

2.2 Basic facts about product spaces

In this section, we recall some basic facts on the product spaces $\mathbb{M}^n(\kappa) \times \mathbb{R}$ and their hypersurfaces. More details can be found in [3] for instance. In the sequel, we will denote by $\mathbb{M}^n(\kappa)$ the n-dimensional simply connected space form of constant sectional curvature κ. That is,

$$\mathbb{M}^n(\kappa) = \begin{cases} \mathbb{S}^n(\kappa) & \text{if } \kappa > 0 \\ \mathbb{R}^n & \text{if } \kappa = 0 \\ \mathbb{H}^n(\kappa) & \text{if } \kappa < 0. \end{cases}$$

We denote by $\overline{\nabla}$ and $\overline{R}$ the Levi-Civita connection and the curvature tensor of $\mathbb{M}^n(\kappa) \times \mathbb{R}$. Finally, let $\frac{\partial}{\partial t}$ be the unit vector field giving the orientation of $\mathbb{R}$ in the product $\mathbb{M}^n(\kappa) \times \mathbb{R}$.

Now, let M be an orientable hypersurface of $\mathbb{M}^n(\kappa) \times \mathbb{R}$ and ν its unit normal vector. Let T be the projection of the vector $\frac{\partial}{\partial t}$ on the tangent bundle TM. Moreover, we consider the function f defined by:

$$f := \left\langle \nu, \frac{\partial}{\partial t} \right\rangle.$$

It is clear that

$$\frac{\partial}{\partial t} = T + f\nu.$$

Since $\frac{\partial}{\partial t}$ is a unit vector field, we have:

$$||T||^2 + f^2 = 1.$$

Let's compute the curvature tensor of $\mathbb{M}^n(\kappa) \times \mathbb{R}$ for tangent vectors to M.

Proposition 2.1. *[3, 19] For all $X, Y, Z, W \in \Gamma(TM)$, we have:*

$$\begin{aligned} \left\langle \overline{R}(X,Y)Z, W \right\rangle &= \kappa\big(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle \\ &\quad -\langle Y, T\rangle\langle W, T\rangle\langle X, Z\rangle - \langle X, T\rangle\langle Z, T\rangle\langle Y, W\rangle \\ &\quad +\langle X, T\rangle\langle W, T\rangle\langle Y, Z\rangle + \langle Y, T\rangle\langle Z, T\rangle\langle X, W\rangle \big), \end{aligned}$$

and

$$\left\langle \overline{R}(X,Y)\nu, Z \right\rangle = \kappa f \big(\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle \big).$$

The fact that $\dfrac{\partial}{\partial t}$ is parallel implies the following two identities

Proposition 2.2. *[3, 19] For $X \in \Gamma(TM)$, we have*

$$\nabla_X T = fA(X), \tag{11}$$

and

$$df(X) = -\langle A(X), T\rangle. \tag{12}$$

Proof: We know that $\overline{\nabla}_X \frac{\partial}{\partial t} = 0$ and $\frac{\partial}{\partial t} = T + f\nu$, so

$$\begin{aligned} 0 &= \overline{\nabla}_X T + df(X)\nu + f\overline{\nabla}_X \nu \\ &= \nabla_X T + \langle A(X), T\rangle\nu + df(X)\nu - fA(X). \end{aligned}$$

Now, it is sufficient to consider the normal and tangential parts to obtain the above identities. □

Definition 2.3 (Compatibility Equations)**.** *We sA(Y) that* $(M, \langle ., . \rangle, A, T, f)$ *satisfies the compatibility equations for* $\mathbb{M}^n(\kappa)\times\mathbb{R}$ *if and only if for any* $X, Y, Z \in \Gamma(TM)$ *the two equations*

$$
\begin{aligned}
(13) \qquad R(X,Y)Z =& \langle A(X), Z\rangle A(Y) - \langle A(Y), Z\rangle A(X) \\
&+ \kappa\Big(\langle X, Z\rangle Y - \langle Y, Z\rangle X - \langle Y, T\rangle\langle X, Z\rangle T \\
&- \langle X, T\rangle\langle Z, T\rangle Y + \langle X, T\rangle\langle Y, Z\rangle T + \langle Y, T\rangle\langle Z, T\rangle X\Big),
\end{aligned}
$$

$$
(14) \qquad \nabla_X A(Y) - \nabla_Y A(X) - A[X,Y] = \kappa f(\langle Y, T\rangle X - \langle X, T\rangle Y)
$$

and equations (11) *and* (12) *are satisfied.*

Remark 2. *The relations (13) and (14) are the Gauss and Codazzi-Mainardi equations for an isometric immersion into* $\mathbb{M}^n(\kappa)\times\mathbb{R}$*.*

Finally, we recall a result of B. Daniel ([3]) which gives a necessary and sufficient condition for the existence of an isometric immersion of an oriented, simply connected surface M into $\mathbb{S}^n(\kappa)\times\mathbb{R}$ or $\mathbb{H}^n(\kappa)\times\mathbb{R}$.

Theorem (Daniel [3])**.** *Let* $(M, \langle ., . \rangle)$ *be an oriented, simply connected Riemannian manifold and* ∇ *its Riemannian connection. Let* A *be a field of symmetric endomorphisms* $A_y : T_yM \longrightarrow T_yM$*,* T *a vector field on* M *and* f *a smooth function on* M*, such that* $||T||^2 + f^2 = 1$*. If* $(M, \langle ., . \rangle, A, T, f)$ *satisfies the compatibility equations for* $\mathbb{M}^n(\kappa)\times\mathbb{R}$*, then, there exists an isometric immersion*

$$F : M \longrightarrow \mathbb{M}^n(\kappa)\times\mathbb{R}$$

so that the Weingarten operator of the immersion related to the normal ν *is*

$$dF \circ A \circ dF^{-1}$$

and such that

$$\frac{\partial}{\partial t} = dF(T) + f\nu.$$

Moreover, this immersion is unique up to a global isometry of $\mathbb{M}^n(\kappa)\times\mathbb{R}$ *which preserves the orientation of* $\mathbb{R}$*.*

3 Isometric immersions via spinors

3.1 Generalized Killing spinors

The case of space forms We introduce the notion of generalized Killing spinors corresponding to hypersurfaces of the space forms $\mathbb{M}^n(\kappa)$. These spinors are obtained by restriction (using (4)) of a parallel (*resp.* real Killing or imaginary Killing) spinor field of the ambient space $\mathbb{R}^n$ (*resp.* $\mathbb{S}^n(\kappa)$ or $\mathbb{H}^n(\kappa)$). If n is odd, they are the restriction of the positive part of the ambient spinor fields. We set $\eta \in \mathbb{C}$ such that $\kappa = 4\eta^2$.

Definition 3.1. *A generalized Killing spinor on a Riemannian spin manifold M with spin connection $\nabla^{\Sigma M}$ is a solution φ of the generalized Killing equation*

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}A(X)\cdot\varphi + \eta X\cdot\omega_n^{\mathbb{C}}\cdot\varphi, \tag{15}$$

for all $X \in \Gamma(TM)$, where A is a field of g-symmetric endomorphisms and $\eta \in \mathbb{C}$. Here, $\omega_n^{\mathbb{C}}$ stands for the complex volume element and "$\cdot$" is the Clifford multiplication on M.

Remark 3. *Note that the complex number η must be either real or purely imaginary because of the following well-known property of Killing spinors. If φ satisfies*

$$\nabla_X^{\Sigma M}\varphi = \eta X\cdot\varphi,$$

for all $X \in \Gamma(TM)$ then η is either real or purely imaginary.

The norm of a generalized Killing spinor field satisfies the following

Lemma 3.2. *Let φ be a generalized Killing spinor. Then*

1. *If $\eta \in \mathbb{R}$, we have $|\varphi| = Const.$*
2. *If $\eta \in i\mathbb{R}$, we have $X|\varphi|^2 = -2i\eta\langle iX\cdot\omega_n^{\mathbb{C}}\cdot\varphi,\varphi\rangle$, for all $X \in \Gamma(TM)$*

Proof : First, we recall the well-known following lemma.

Lemma 3.3. *Let ψ be a spinor field and β a real 1-form or 2-form. Then*

$$\Re e\,\langle\beta\cdot\psi,\psi\rangle = 0.$$

Now, from this lemma, we deduce easily the proof of Lemma 3.2

1. If $\eta \in \mathbb{R}$, we have,

   $$X|\varphi|^2 = 2\langle\nabla_X^{\Sigma N}\varphi,\varphi\rangle = 2\langle\eta X\cdot_N\varphi,\varphi\rangle = -2\eta\langle\varphi, X\cdot_N\varphi\rangle = 0$$

   and consequently $|\varphi| = Const.$

2. If $\eta \in i\mathbb{R}$, we have

   $$X|\varphi|^2 = 2\langle\eta X\cdot\omega_n^{\mathbb{C}}\varphi,\varphi\rangle + \langle A(X)\cdot\varphi,\varphi\rangle = -i2\eta\langle iX\cdot\omega_n^{\mathbb{C}}\varphi,\varphi\rangle.$$

□

The case of product spaces We give the following definition of the generalized Killing spinor fields corresponding to hypersurfaces of $\mathbb{M}^n(\kappa)\times\mathbb{R}$. These spinors are obtained by restriction of particular spinor fields on $\mathbb{M}^n(\kappa)\times\mathbb{R}$ playing the role of Killing spinors on space forms (see [18] for details). We set $\eta \in \mathbb{C}$ such that $\kappa = 4\eta^2$.

Definition 3.4. *A spinor field which satisfies the equation*

$$\nabla_X^{\Sigma M}\varphi = -\frac{1}{2}A(X)\cdot\varphi + \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi + \eta\,\langle X,T\rangle\,\varphi, \tag{16}$$

for all $X \in \Gamma(TM)$ where "$\cdot$" stands for the Clifford multiplication on M, T is a vector field over M and f a smooth function on M. Such a spinor field is called a generalized Killing spinor on $\mathbb{M}^n(\kappa)\times\mathbb{R}$.

These spinor fields satisfy the following properties

Proposition 3.5. *1. If $\eta \in \mathbb{R}$, then the norm of a generalized Killing spinor is constant.*

2. If $\eta \in i\mathbb{R}$, then the norm of a generalized Killing spinor satisfies for any $X \in \Gamma(TM)$:

$$X|\varphi|^2 = \Re e \langle iX \cdot T \cdot \varphi + ifX \cdot \varphi, \varphi \rangle .$$

Proof: We need to compute $X|\varphi|^2$ for $X \in \Gamma(TM)$. We have

$$X|\varphi|^2 = 2\Re e \left\langle \nabla^{\Sigma M}_X \varphi, \varphi \right\rangle .$$

We replace $\nabla^{\Sigma M}_X \varphi$ by the expression given by (16), and we use Lemma 3.3 to conclude that

$$\Re e \langle A(X) \cdot \varphi, \varphi \rangle = 0,$$

and

$$\Re e \langle fX \cdot \varphi, \varphi \rangle = 0.$$

By this lemma again, we see that

$$\Re e \langle X \cdot T \cdot \varphi, \varphi \rangle + \Re e \langle \langle X, T \rangle \varphi, \varphi \rangle = 0.$$

So $X|\varphi|^2 = 0$ and then φ has constant norm.
If $\eta \in i\mathbb{R}$, an analogous computation yields the result. $\square$

Remark 4. *In the case $\eta \in i\mathbb{R}$, the norm of φ is not constant. Nevertheless, we can show that φ never vanishes.*

3.2 The main results

Here, we state the main results of this paper. The first result gives a characterization of hypersurfaces in 4-dimensional space forms assuming the existence of two generalized Killing spinor fields which are equivalently solutions of two Dirac equations. Part of this result can be found in the thesis of the first author [10].

Theorem 1. *Let (M^3, g) be a 3-dimensional simply connected spin manifold, $H : M \longrightarrow \mathbb{R}$ a real valued function and A a field of symmetric endomorphisms on TM. The following statements are equivalent:*

1. *The spinor fields φ_j, $j = 1, 2$, are non-vanishing solutions of the Dirac equations:*

$$\begin{cases} D\varphi_1 = (\frac{3}{2}H + 3\eta)\varphi_1, \\ \\ D\varphi_2 = -(\frac{3}{2}H + 3\eta)\varphi_2, \end{cases}$$

with $\begin{cases} |\varphi_j| = Const \text{ if } \eta \in \mathbb{R}, \\ X|\varphi_j|^2 = 2\Re e \langle \eta X \cdot \varphi_j, \varphi_j \rangle \quad \text{if} \quad \eta \in i\mathbb{R}. \end{cases}$

2. *The spinor fields φ_j, $j = 1, 2$, are non-trivial solutions of the generalized Killing equations*

$$\begin{cases} \nabla_X^{\Sigma M}\varphi_1 = \frac{1}{2}A(X)\cdot\varphi_1 - \eta X\cdot\varphi_1 \\ \\ \nabla_X^{\Sigma M}\varphi_2 = -\frac{1}{2}A(X)\cdot\varphi_2 + \eta X\cdot\varphi_2, \end{cases}$$

with $\frac{1}{2}\mathrm{tr}\,(A) = H$.

Moreover both statements imply that

3. *there exists an isometric immersion $F : M \hookrightarrow \mathbb{M}^4(\kappa)$ into the 4-dimensional space form of curvature $\kappa = 4\eta^2$ with mean curvature H and Weingarten tensor $dF \circ A \circ dF^{-1}$.*

Remark 5. *Note that in the case of $\mathbb{R}^4$, Assertion 3. is equivalent to Assertions 1. and 2. (see [16])*

Now, we state the second result which gives a characterization of hypersurfaces into the 4-dimensional product spaces $\mathbb{M}^3(\kappa)\times\mathbb{R}$.

Theorem 2. *Let (M^3, g) be a 3-dimensional simply connected spin manifold, f, $H : M \longrightarrow \mathbb{R}$ two real valued functions, T a vector field and A a field of symmetric endomorphisms on TM, such that*

$$\begin{cases} ||T||^2 + f^2 = 1, \\ \nabla_X T = fA(X), \\ df(X) = -\langle A(X), T\rangle. \end{cases}$$

The following statements are equivalent:

1. *The spinor fields φ_j, $j = 1, 2$, are non-vanishing solutions of the generalized Dirac equations*

$$\begin{cases} D\varphi_1 = \frac{3}{2}H\varphi_1 - 2\eta T\cdot\varphi_1 - 3\eta f\varphi_1, \\ \\ D\varphi_2 = -\frac{3}{2}H\varphi_2 - 2\eta T\cdot\varphi_2 + 3\eta f\varphi_2, \end{cases}$$

with constant norm if $\eta \in \mathbb{R}$ or satisfying $X|\varphi|^2 = \Re e\big(iX\cdot T\cdot\varphi + ifX\cdot\varphi, \varphi\big)$ if $\eta \in i\mathbb{R}$.

2. *The spinor fields φ_j, $j = 1, 2$, are non-trivial solutions of the generalized Killing equations*

$$\begin{cases} \nabla_X^{\Sigma M}\varphi_1 = -\frac{1}{2}A(X)\cdot\varphi_1 + \eta X\cdot T\cdot\varphi_1 + \eta fX\cdot\varphi_1 + \eta\,\langle X, T\rangle\,\varphi_1, \\ \\ \nabla_X^{\Sigma M}\varphi_2 = \frac{1}{2}A(X)\cdot\varphi_2 + \eta X\cdot T\cdot\varphi_2 - \eta fX\cdot\varphi_2 + \eta\,\langle X, T\rangle\,\varphi_2. \end{cases}$$

Moreover, both statements imply

3. *There exists an isometric immersion F from M into $\mathbb{S}^3(\kappa)\times\mathbb{R}$ (resp. $\mathbb{H}^3(\kappa)\times\mathbb{R}$, with $\kappa = 4\eta^2$) of mean curvature H such that the Weingarten tensor related to the normal ν is given by*

$$dF \circ A \circ dF^{-1}$$

and such that

$$\frac{\partial}{\partial t} = dF(T) + f\nu.$$

Remark 6. *As we will see in the proof (Lemma 4.3), the condition of the existence of the two spinor fields φ_1 and φ_2 is equivalent to the existence of only one generalized Killing spinor field with A a Codazzi tensor field.*

4 Proof of the theorems

We will prove Theorems 1 and 2 jointly. For this, we need three general lemmas.

4.1 Three main lemmas

First, we establish the following lemma which gives the Gauss equation from a generalized Killing spinor.

Lemma 4.1. *Let (M^3, g) be a 3-dimensional spin manifold. Assume that there exists a non-trivial spinor field φ solution of the following equation*

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}A(X)\cdot\varphi + \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi + \eta\langle X, T\rangle\varphi, \tag{17}$$

where A, T and f satisfy

$$\nabla_X T = fA(X), \quad df(X) = -\langle A(X), T\rangle \quad \text{and}$$
$$d^\nabla A(X,Y) = 4\eta^2 f\big(\langle Y,T\rangle X - \langle X,T\rangle Y\big),$$

then the curvature tensor R of (M,g) is given by

$$\begin{aligned} R(X,Y)Z =& \langle A(X),Z\rangle A(Y) - \langle A(Y),Z\rangle A(X) \\ &+ \kappa\Big(\langle X,Z\rangle Y - \langle Y,Z\rangle X - \langle Y,T\rangle\langle X,Z\rangle T \\ &- \langle X,T\rangle\langle Z,T\rangle Y + \langle X,T\rangle\langle Y,Z\rangle T + \langle Y,T\rangle\langle Z,T\rangle X\Big). \end{aligned} \tag{18}$$

Proof: We compute the spinorial curvature $\mathcal{R}(X,Y)\varphi = \nabla_X^{\Sigma M}\nabla_Y^{\Sigma M}\varphi - \nabla_Y^{\Sigma M}\nabla_X^{\Sigma M}\varphi - \nabla_{[X,Y]}^{\Sigma M}\varphi$. From [19, 18], we now that

$$\begin{aligned} \nabla_X^{\Sigma M}\nabla_Y^{\Sigma M}\varphi = & \underbrace{\eta f Y\cdot A(X)\cdot\varphi}_{\alpha_1(X,Y)} + \underbrace{\eta^2 Y\cdot T\cdot X\cdot T\cdot\varphi}_{\alpha_2(X,Y)} + \underbrace{\eta^2 fY\cdot T\cdot X\cdot\varphi}_{\alpha_3(X,Y)} \\ & \underbrace{-\frac{\eta}{2}Y\cdot T\cdot A(X)\cdot\varphi}_{-\alpha_4(X,Y)} \underbrace{-\eta\langle A(X),T\rangle Y\cdot\varphi}_{-\alpha_5(X,Y)} + \underbrace{\eta^2 fY\cdot X\cdot T\cdot\varphi}_{\alpha_6(X,Y)} \\ & + \underbrace{\eta^2\langle X,T\rangle Y\cdot T\cdot\varphi}_{\alpha_7(X,Y)} + \underbrace{\eta^2 f^2 Y\cdot X\cdot\varphi}_{\alpha_8(X,Y)} + \underbrace{\eta^2 f\langle X,T\rangle Y\cdot\varphi}_{\alpha_9(X,Y)} \\ & \underbrace{-\frac{\eta}{2}fY\cdot A(X)\cdot\varphi}_{-\alpha_{10}(X,Y)} + \underbrace{\eta f\langle Y, A(X)\rangle\varphi}_{\alpha_{11}(X,Y)} + \underbrace{\eta^2\langle Y,T\rangle X\cdot T\cdot\varphi}_{\alpha_{12}(X,Y)} \\ & + \underbrace{\eta^2 f\langle Y,T\rangle X\cdot\varphi}_{\alpha_{13}(X,Y)} + \underbrace{\eta^2\langle X,T\rangle\langle Y,T\rangle\varphi}_{\alpha_{14}(X,Y)} \underbrace{-\frac{\eta}{2}\langle Y,T\rangle A(X)\cdot\varphi}_{-\alpha_{15}(X,Y)} \\ & \underbrace{-\frac{1}{2}\nabla_X^{\Sigma M}(A(Y))\cdot\varphi}_{-\alpha_{16}(X,Y)} \underbrace{-\frac{\eta}{2}A(Y)\cdot X\cdot T\cdot\varphi}_{-\alpha_{17}(X,Y)} \underbrace{-\frac{\eta}{2}fA(Y)\cdot X\cdot\varphi}_{-\alpha_{18}(X,Y)} \end{aligned}$$

$$
\underbrace{-\frac{\eta}{2}\langle X,T\rangle A(Y)\cdot\varphi}_{-\alpha_{19}(X,Y)}+\underbrace{\frac{1}{4}A(Y)\cdot A(X)\cdot\varphi}_{\alpha_{20}(X,Y)}+\underbrace{\eta\nabla^{\Sigma M}_X Y\cdot T\cdot\varphi}_{\alpha_{21}(X,Y)}
+\underbrace{\eta f\nabla^{\Sigma M}_X Y\cdot\varphi}_{\alpha_{22}(X,Y)}+\underbrace{\eta\left\langle\nabla^{\Sigma M}_X Y,T\right\rangle\varphi}_{\alpha_{23}(X,Y)}.
$$

That is,

$$
\nabla^{\Sigma M}_X\nabla^{\Sigma M}_Y\varphi=\sum_{i=1}^{23}\alpha_i(X,Y).
$$

By symmetry, it is obvious that

$$
\nabla^{\Sigma M}_Y\nabla^{\Sigma M}_X\varphi=\sum_{i=1}^{23}\alpha_i(Y,X).
$$

On the other hand, we have

$$
\begin{aligned}
\nabla^{\Sigma M}_{[X,Y]}\varphi &= \underbrace{\eta[X,Y]\cdot T\cdot\varphi}_{\beta_1([X,Y])}+\underbrace{\eta f[X,Y]\cdot\varphi}_{\beta_2([X,Y])}\\
&\quad+\underbrace{\eta\langle[X,Y],T\rangle\varphi}_{\beta_3([X,Y])}\underbrace{-\frac{1}{2}A[X,Y]\cdot\varphi}_{-\beta_4([X,Y])}.
\end{aligned}
$$

Since the connection ∇ is torsion-free, we have

$$
\begin{aligned}
\alpha_{21}(X,Y)-\alpha_{21}(Y,X)-\beta_1([X,Y])&=0,\\
\alpha_{22}(X,Y)-\alpha_{22}(Y,X)-\beta_2([X,Y])&=0,\\
\alpha_{23}(X,Y)-\alpha_{23}(Y,X)-\beta_3([X,Y])&=0.
\end{aligned}
$$

Moreover, lots of terms vanish by symmetry, namely α_1, α_4, α_5, α_{10}, α_{11}, α_{14}, α_{15}, α_{17}, α_{18} and α_{19}.
On the other hand, the terms α_2, α_7, α_8 and α_{12} can be combined. Indeed, if we set

$$
\alpha=\alpha_2+\alpha_7+\alpha_8+\alpha_{12},
$$

then

$$
\begin{aligned}
\alpha(X,Y)-\alpha(Y,X) &= \eta^2\Big[f^2(Y\cdot X-X\cdot Y)+Y\cdot T\cdot X\cdot T-X\cdot T\cdot Y\cdot T\Big]\cdot\varphi\\
&= \eta^2\Big[f^2(Y\cdot X-X\cdot Y)+||T||^2(Y\cdot X-X\cdot Y)\Big]\cdot\varphi\\
&\quad -2\eta^2\left(\langle X,T\rangle Y\cdot T-\langle Y,T\rangle X\cdot T\right)\cdot\varphi.
\end{aligned}
$$

If we set

$$
\beta=\alpha_3+\alpha_6+\alpha_9+\alpha_{13},
$$

we obtain

$$
\beta(X,Y)-\beta(Y,X)=\eta^2 f\left(\langle Y,T\rangle X-\left\langle X,T\right\rangle Y\right)\cdot\varphi.
$$

Finally, we get

$$\begin{aligned}\mathcal{R}(X,Y)\varphi &= \frac{1}{4}\left(A(Y)\cdot A(X)-A(X)\cdot A(Y)\right)\cdot\varphi-\frac{1}{2}d^{\nabla}A(X,Y)\cdot\varphi\\ &+\eta^2 f\left(\langle Y,T\rangle X-\langle X,T\rangle Y\right)\cdot\varphi+\eta^2\left(Y\cdot X-X\cdot Y\right)\cdot\varphi\\ &-2\eta^2\left(\langle X,T\rangle Y\cdot T-\langle Y,T\rangle X\cdot T\right)\cdot\varphi.\end{aligned}$$

Since we assume that A satisfies the following Codazzi equation

$$d^{\nabla}A(X,Y)=4\eta^2 f\big(\langle Y,T\rangle X-\langle X,T\rangle Y\big),$$

we have

$$\begin{aligned}(19)\quad \mathcal{R}(X,Y)\varphi &= \frac{1}{4}\left(A(Y)\cdot A(X)-A(X)\cdot A(Y)\right)\cdot\varphi\\ &+\eta^2 f\left(\langle Y,T\rangle X-\langle X,T\rangle Y\right)\cdot\varphi+\eta^2\left(Y\cdot X-X\cdot Y\right)\cdot\varphi\end{aligned}$$

Now, let $X=e_i$ and $Y=e_j$ with $i\neq j$. The Ricci identity sA(Y)s that:

$$(20)\qquad \mathcal{R}(e_i,e_j)\cdot\varphi=\frac{1}{2}[R_{ijik}e_j-R_{ijij}e_k-R_{ijjk}e_i]\cdot\varphi,$$

where (i,j,k) is any cyclic permutation of $(1,2,3)$.

Further with a simple computation we find

$$\begin{aligned}A(e_j)\cdot A(e_i)-A(e_i)\cdot A(e_j) &= 2(A_{ik}A_{jj}-A_{ij}A_{jk})e_i\\ &\quad -2(A_{ik}A_{ji}-A_{ii}A_{jk})e_j+2(A_{ij}A_{ji}-A_{ii}A_{jk})e_k.\end{aligned}$$

With the integrability condition (19) this yields

$$\begin{aligned}(\nabla_{e_j}A)(e_i)-(\nabla_{e_i}A)(e_j) &= (R_{ijjk}-(A_{ik}A_{jj}-A_{ij}A_{jk})+\kappa f^2)e_i\\ &\quad -(R_{ijik}-(A_{ik}A_{ji}-A_{ii}A_{jk})+\kappa f^2)e_j\\ &\quad +(R_{ijij}-(A_{ij}A_{ji}-A_{ii}A_{jk})+\kappa f^2)e_k\\ &\quad +\kappa f\left(\langle e_i,T\rangle e_j-\langle e_i,T\rangle e_i\right),\end{aligned}$$

which proves that, if A is a Codazzi tensor, it satisfies the Gauss equation too. This observation was made by Morel ([16]) in the Riemannian case for a parallel tensor A. We point out that the converse is also true. □

Now, we state a second lemma which will give the equivalence between the Dirac equation and the Killing equation (up to a condition on the norm of the spinor field).

Lemma 4.2. *Let (M^3,g) be a 3-dimensional spin manifold. Assume that there exists a non-trivial spinor field φ, solution of the following equation*

$$(21)\qquad D\varphi=\frac{3}{2}H\varphi-2\eta T\cdot\varphi-3\eta f\varphi,$$

where the norm of φ satisfies for all $X\in\Gamma(TM)$

$$X|\varphi|^2=2\Re e\left\langle \eta X\cdot T\cdot\varphi+\eta f X\cdot\varphi,\varphi\right\rangle.$$

Then φ is a solution of the following generalized Killing spinors equation

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}A(X)\cdot\varphi + \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi + \eta\left\langle X, T\right\rangle\varphi. \tag{22}$$

Proof: The 3-dimensional complex spinor space is $\Sigma_3 \cong \mathbb{C}^2$. The complex spin representation is then real 4-dimensional. We now define the map

$$\begin{array}{rcl} f:\ \mathbb{R}^3\oplus\mathbb{R} & \longrightarrow & \Sigma_3 \\ (v,\ r) & \longmapsto & v\cdot\varphi + r\varphi, \end{array}$$

where φ is a given non-vanishing spinor.
Obviously f is an isomorphism. Then for all $\psi \in \Sigma_3$ there is a unique pair $(v,\ r) \in (\mathbb{R}^3\oplus\mathbb{R}) \cong T_pM^3\oplus\mathbb{R}$, such that $\psi = v\cdot\varphi + r\varphi$.
Consequently $\left(\nabla_X^{\Sigma M}\varphi\right)_p \in \Gamma(T_p^*M\otimes\Sigma_3)$ can be expressed as follows:

$$\nabla_X^{\Sigma M}\varphi = B(X)\cdot\varphi + \omega(X)\varphi,$$

for all $p\in M$ and for all vector fields X, with ω a 1-form and B a (1,1)-tensor field.
Moreover we have

$$X|\varphi|^2 = 2\Re e\langle\nabla_X^{\Sigma M}\varphi,\ \varphi\rangle = 2\langle\omega(X)\varphi,\ \varphi\rangle \Rightarrow \omega(X) = \frac{d(|\varphi|^2)}{2|\varphi|^2}(X).$$

which yields $\omega(X) = \Re e\left\langle \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle$.
Now, let $B = S+U$ with S the symmetric and U the skew-symmetric part of B. Let $\{e_i\}$ be an orthonormal basis of TM and φ a solution of the Dirac equation (21). We have

$$\begin{aligned} D\varphi &= \sum_{i=1}^{3} e_i\cdot\nabla_{e_i}^{\Sigma M}\varphi = \sum_{i,j=1}^{3} e_i\cdot B_{ij}e_j\cdot\varphi + \sum_{j=1}^{3}\omega(e_j)e_j\cdot\varphi \\ &= \sum_{i=1}^{3} U_{ij}e_i\cdot e_j\cdot\varphi + \sum_{i=1}^{3} S_{ii}e_i\cdot e_i\cdot\varphi + \sum_{i\neq j}^{3}\underbrace{S_{ij}}_{sym.}\ \underbrace{e_i\cdot e_j}_{skew-sym.}\cdot\varphi + W\cdot\varphi, \end{aligned}$$

where W is the vector field defined by $W := \sum_{j=1}^{3}\omega(e_j)e_j$. Then,

$$\begin{aligned} D\varphi &= -2\sum_{i<j}^{3} U_{ij}e_i\cdot e_j\cdot\varphi + \sum_{i=1}^{3} S_{ii}e_i\cdot e_i\cdot\varphi + W\cdot\varphi \\ &= -2(U_{12}e_1\cdot e_2 + U_{13}e_1\cdot e_3 + U_{23}e_2\cdot e_3)\cdot\varphi - \mathrm{tr}\,(B)\varphi + W\cdot\varphi \end{aligned}$$

We recall that the complex volume element $\omega_3^{\mathbb{C}} = -e_1\cdot e_2\cdot e_3$ acts as the identity on ΣM, where $\{e_1, e_2, e_3\}$ is a local orthonormal frame of TM. So we deduce that for any spinor field on M, $e_i\cdot e_j\cdot\varphi = e_k\cdot\varphi$, where (i,j,k) is a cyclic permutation of $(1,2,3)$. From this fact, we get

$$D\varphi = -2(U_{12}e_3 - U_{13}e_2 + U_{23}e_1)\cdot\varphi - \mathrm{tr}\,(B)\varphi + W\cdot\varphi.$$

On the other hand, we have

$$D\varphi = \frac{3}{2}H\varphi - 2\eta T\cdot\varphi - 3\eta f\varphi.$$

Note that $\Re e\langle (U_{12}e_3 - U_{13}e_2 + U_{23}e_1)\varphi, \varphi\rangle = 0$ and $\Re e\,\langle W\cdot\varphi, \varphi\rangle = 0$. It follows that

$$\frac{3}{2}H|\varphi|^2 - 2\Re e\,\langle \eta T\cdot\varphi, \varphi\rangle - 3\Re e\,\langle \eta f\varphi, \varphi\rangle = -\mathrm{tr}\,(B)|\varphi|^2.$$

Moreover, since $\left\{\frac{\varphi}{|\varphi|}, \frac{e_1\cdot\varphi}{|\varphi|}, \frac{e_2\cdot\varphi}{|\varphi|}, \frac{e_3\cdot\varphi}{|\varphi|}\right\}$ is an orthonormal frame of $\Sigma_p M$ for the real scalar product $\langle .,.\rangle$, we deduce that

$$\begin{aligned} -2\big(U_{12}e_3 - U_{13}e_2 + U_{23}e_1\big)\cdot\varphi &= -3\eta f\varphi - W\cdot\varphi - 2\eta T\cdot\varphi + 2\Re e\,\langle \eta T\cdot\varphi, \varphi\rangle\,\varphi \\ &\quad +3\Re e\,\langle \eta f\varphi, \varphi\rangle\,\varphi. \end{aligned}$$

Further we compute

$$\langle U(e_j)\cdot\varphi,\ e_i\cdot\varphi\rangle = \sum_k^3 U_{kj} \underbrace{\langle e_k\cdot\varphi,\ e_i\cdot\varphi\rangle}_{=-\langle e_i\cdot e_k\cdot\varphi,\ \varphi\rangle=0,\ i\neq k} = U_{ij}|\varphi|^2.$$

Consequently, for $i, j \in \{1,2,3\}$, we have

$$\begin{aligned} -2\sum_{k<l}^3 U_{lk}\,\langle e_k\cdot e_l\cdot\varphi, e_i\cdot e_j\cdot\varphi\rangle &= -3\,\langle \eta f\varphi, e_i\cdot e_j\cdot\varphi\rangle - \langle W\cdot\varphi, e_i\cdot e_j\cdot\varphi\rangle \\ &\quad -2\,\langle \eta T\cdot\varphi, e_i\cdot e_j\cdot\varphi\rangle + 2\,\langle \eta T\cdot\varphi, \varphi\rangle\,\langle \varphi, e_i\cdot e_j\cdot\varphi\rangle \\ &\quad +3\,\langle \eta f\varphi, \varphi\rangle\,\langle \varphi, e_i\cdot e_j\cdot\varphi\rangle\,. \end{aligned}$$

Moreover, in the 3-dimensional case at most three of the four indices could be distinct. Then, for $m \neq n$, $\langle e_m\cdot e_n\cdot\varphi,\ \varphi\rangle = 0$ holds and as the trace of a skew-symmetric tensor vanishes, we have: $\langle e_k\cdot e_l\cdot\varphi,\ e_j\cdot e_i\cdot\varphi\rangle \neq 0 \Leftrightarrow k = i, l = j$ or $k = j, l = i$, $i \neq j$, which yield

$$\begin{aligned} -2U_{ij}|\varphi|^2 &= -2\langle U(e_j)\cdot\varphi, e_i\cdot\varphi\rangle \\ &= -3\,\langle \eta f\varphi, e_i\cdot e_j\cdot\varphi\rangle - \langle W\cdot\varphi, e_i\cdot e_j\cdot\varphi\rangle - 2\,\langle \eta T\cdot\varphi, e_i\cdot e_j\cdot\varphi\rangle \\ &\quad +3\,\langle \eta f\varphi, \varphi\rangle\,\langle e_j\cdot\varphi, e_i\cdot\varphi\rangle + 2\,\langle \eta T\cdot\varphi, \varphi\rangle\,\langle e_j\cdot\varphi, e_i\cdot\varphi\rangle\,. \end{aligned}$$

Then, we deduce that

$$\begin{aligned} -2U(X) &= X\cdot W\cdot\varphi - \langle X\cdot W\cdot\varphi, \varphi\rangle\,\frac{\varphi}{|\varphi|^2} - 2\eta X\cdot T\cdot\varphi \\ &\quad +2\,\langle \eta X\cdot T\cdot\varphi, \varphi\rangle\,\frac{\varphi}{|\varphi|^2} + 3\left\langle \eta f\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle X\cdot\varphi \\ &\quad +2\left\langle \eta T\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle X\cdot\varphi - 3\eta f X\cdot\varphi + 3\,\langle \eta f X\cdot\varphi, \varphi\rangle\,\frac{\varphi}{|\varphi|^2}. \end{aligned} \tag{23}$$

From now on, we will consider separately the cases $\eta \in \mathbb{R}$ and $\eta \in i\mathbb{R}$.

The case $\eta \in \mathbb{R}$

Since η is real, the norm of φ is constant and so $\omega(X) = 0$ for any vector field X. Consequently, using Lemma 3.3, we get

$$\begin{aligned} U(X)\cdot\varphi &= \eta X\cdot T\cdot\varphi - \eta\left\langle X\cdot T\cdot\varphi,\varphi\right\rangle\frac{\varphi}{|\varphi|^2} \\ &= \eta X\cdot T\cdot\varphi + \eta\left\langle X,T\right\rangle\varphi. \end{aligned}$$

Moreover,

$$\begin{aligned} Q_\varphi(e_i,e_j) &= \frac{1}{2}\left\langle e_i\cdot\nabla^{\Sigma M}_{e_j}\varphi + e_j\cdot\nabla^{\Sigma M}_{e_i}\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle \\ &= \frac{1}{2}\left\langle \sum_k^3 S_{jk}e_i\cdot e_k\cdot\varphi + \sum_k^3 S_{ik}e_j\cdot e_k\cdot\varphi, \frac{\varphi}{|\varphi|^2}\right\rangle \\ &= -S_{ij}|\varphi|^2 \quad \Rightarrow S(X) = -Q_\varphi(X). \end{aligned}$$

Now, we set

$$A(X) := 2Q_\varphi(X) + 2\eta f X.$$

Finally, we obtain

$$\nabla^{\Sigma M}_X\varphi = \frac{1}{2}A(X)\cdot\varphi + \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi + \left\langle X,T\right\rangle\varphi, \tag{24}$$

which achieves the proof in the case $\eta \in \mathbb{R}$.

The case $\eta \in i\mathbb{R}$

Here, η is not real and so the norm of φ is not constant but satisfies

$$X|\varphi|^2 = 2\Re e\left\langle \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi,\varphi\right\rangle.$$

Then

$$\omega(X) = \frac{X|\varphi|^2}{2|\varphi|^2} = \frac{1}{2|\varphi|^2}\Re e\left\langle \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi,\varphi\right\rangle. \tag{25}$$

Like in the case $\eta \in \mathbb{R}$, we have $S(X) = -Q_\varphi(X)$ and we set

$$A(X) := 2Q_\varphi(X) + V(X),$$

where $V(X)$ is the symmetric endomorphism field defined by

$$\begin{aligned} V(X,Y) &= 2\Re e\left\langle \eta\left\langle X,Y\right\rangle T\cdot\varphi,\varphi\right\rangle + 2\Re e\left\langle \eta f\left\langle X,Y\right\rangle\varphi,\varphi\right\rangle \\ &\quad + \Re e\left\langle \eta\left(\left\langle X,T\right\rangle Y + \left\langle Y,T\right\rangle X\right)\cdot\varphi,\varphi\right\rangle. \end{aligned} \tag{26}$$

Since

$$\nabla^{\Sigma M}_X\varphi = S(X)\cdot\varphi + U(X)\cdot\varphi + \omega(X)\varphi,$$

we deduce from (25), (23) and (26) that

$$\nabla^{\Sigma M}_X\varphi = \frac{1}{2}A(X)\cdot\varphi + \eta X\cdot T\cdot\varphi + \eta f X\cdot\varphi + \eta\left\langle X,T\right\rangle\varphi. \tag{27}$$

□

Now, we give a final lemma which will allow us to use Lemma 4.1 for the proof of Theorems 1 and 2. Indeed, in Theorems 1 and 2, we do not suppose anything about the symmetric tensor A. Nevertheless, the existence of two generalized Killing spinor fields implies that A is Codazzi.

Lemma 4.3. *Let (M^3, g) a 3-dimensional spin manifold. Assume that there exist two non-trivial spinor fields φ_1 and φ_2 such that*

$$\nabla^{\Sigma}_X \varphi_1 = \frac{1}{2} A(X) \cdot \varphi_1 + \eta X \cdot T \cdot \varphi_1 + \eta f X \cdot \varphi_1 + \langle X, T \rangle \varphi_1, \tag{28}$$

and

$$\nabla^{\Sigma M}_X \varphi_2 = -\frac{1}{2} A(X) \cdot \varphi_2 + \eta X \cdot T \cdot \varphi_2 - \eta f X \cdot \varphi_2 + \langle X, T \rangle \varphi_2, \tag{29}$$

where A, T and f satisfy

$$\nabla^{\Sigma M}_X T = f A(X), \quad df(X) = -\langle A(X), T \rangle,$$

then the tensor A satisfies the Codazzi-Mainardi equations, that is

$$d^{\nabla} A(X, Y) = 4\eta^2 f \big(\langle Y, T \rangle X - \langle X, T \rangle Y \big).$$

Proof : From the proof of Lemma 4.1, we know that the equation satisfied by φ_1 implies

$$\begin{aligned} (\nabla_{e_j} A)(e_i) - (\nabla_{e_i} A)(e_j) \;=\; & (R_{ijjk} - (A_{ik}A_{jj} - A_{ij}A_{jk}) + \kappa f^2) e_i \\ & -(R_{ijik} - (A_{ik}A_{ji} - A_{ii}A_{jk}) + \kappa f^2) e_j \\ & +(R_{ijij} - (A_{ij}A_{ji} - A_{ii}A_{jk}) + \kappa f^2) e_k \\ & +\kappa f \left(\langle e_i, T \rangle e_j - \langle e_i, T \rangle e_i \right). \end{aligned} \tag{30}$$

On the other hand, by an analogous computation for the spinor field φ_2, we get

$$\begin{aligned} -(\nabla_{e_j} A)(e_i) + (\nabla_{e_i} A)(e_j) \;=\; & (R_{ijjk} - (A_{ik}A_{jj} - A_{ij}A_{jk}) + \kappa f^2) e_i \\ & -(R_{ijik} - (A_{ik}A_{ji} - A_{ii}A_{jk}) + \kappa f^2) e_j \\ & +(R_{ijij} - (A_{ij}A_{ji} - A_{ii}A_{jk}) + \kappa f^2) e_k \\ & -\kappa f \left(\langle e_i, T \rangle e_j - \langle e_i, T \rangle e_i \right). \end{aligned}$$

If we combine the last two equalities, we get

$$\begin{cases} R_{ijjk} - (A_{ik}A_{jj} - A_{ij}A_{jk}) + \kappa f^2 = 0, \\ R_{ijik} - (A_{ik}A_{ji} - A_{ii}A_{jk}) + \kappa f^2 = 0, \\ R_{ijij} - (A_{ij}A_{ji} - A_{ii}A_{jk}) + \kappa f^2 = 0, \end{cases}$$

that is exactly the Gauss equation. Then, we get immediately from equation (30) that A also satisfies the Codazzi equation

$$d^{\nabla} A(X, Y) = 4\eta^2 f \big(\langle Y, T \rangle X - \langle X, T \rangle Y \big),$$

for all vector fields X and Y. □

4.2 Proof of the Theorems

The proof of the theorems follows easily from Lemmas 4.1, 4.2 and 4.3 with

$$\begin{cases} \eta = 0 & \text{for } \mathbb{R}^4, \\ \eta = \frac{1}{2},\ T = 0,\ f = 1 & \text{for } \mathbb{S}^4, \\ \eta = \frac{i}{2},\ T = 0,\ f = 1 & \text{for } \mathbb{H}^4, \\ \eta = \frac{1}{2} & \text{for } \mathbb{S}^3 \times \mathbb{R}, \\ \eta = \frac{i}{2} & \text{for } \mathbb{H}^3 \times \mathbb{R}. \end{cases}$$

Indeed, Lemma 4.2 gives the equivalence between Assertions 1. and 2. of the theorems, that is, between the existence of a generalized Killing spinor and a Dirac spinor satisfying an additional norm condition.

The proof of 2. $\Longrightarrow$ 3. is an immediate consequence of Lemmas 4.1 and 4.3. From Lemma 4.3, the problem is reduced to the case of only one generalized Killing spinor field, but with A a Codazzi tensor. Now, if the tensor A satisfies the Codazzi-Mainardi equation, then by Lemma 4.1, it satisfies also the Gauss equation. It is well-known that if the Gauss and Codazzi-Mainardi equations are satisfied for a simply connected manifold, then it can be immersed isometrically in the corresponding space form. For the case of product spaces, by the result of Daniel ([3]), to get an isometric immersion, the two additional conditions (11) and (12) are needed. □

Remark 7. *Conversely, the existence of one generalized Killing spinor field φ_1 with Codazzi tensor field A implies the existence of a second spinor field φ_2. Indeed, as we just saw, M is isometrically immersed into $\mathbb{M}^4(\kappa)$ or $\mathbb{M}^3(\kappa) \times \mathbb{R}$. Then, one just defines φ_2 as $\nu \cdot \varphi_1$, where ν is the normal unit vector field. Thus, if φ_1 satisfies*

$$\nabla^{\Sigma M}_X \varphi_1 = -\frac{1}{2} A(X) \cdot \varphi_1 + \eta X \cdot T \cdot \varphi_1 + \eta f X \cdot \varphi_1 + \eta \langle X, T \rangle \varphi_1,$$

then, by a straightforward computation, φ_2 satisfies

$$\nabla^{\Sigma M}_X \varphi_2 = \frac{1}{2} A(X) \cdot \varphi_2 + \eta X \cdot T \cdot \varphi_2 - \eta f X \cdot \varphi_2 + \eta \langle X, T \rangle \varphi_2.$$

5 Application: Non-existence of isometric immersions for 3-dimensional geometries

In [17] and [13], for instance, it is shown that there exist no isometric immersions for certain 3-dimensional homogeneous spaces into the Euclidean 4-space. As an application of Theorem 1 we give a short non-spinorial proof of the non-existence of such immersions for certain three-dimensional $\eta-$manifolds including the above homogeneous spaces.

5.1 Preliminaries the some 3-dimensional geometries

In this section, we will give some basic facts about 3-dimensional homogeneous manifolds. A complete description can be found in [20]. Let (M^3, g) be a 3-dimensional Riemannian homogeneous manifold. We denote by d the dimension of its isometry group. The possible values of d are 3, 4 and 6. If d is equal to 6, then M is a space form $\mathbb{M}^3(\kappa)$. There is only one geometry with d equal to 3, namely, the solvable group Sol_3. Finally, if $d = 4$, then, there are 5 possible models.

5.1.1 The manifolds $\mathbb{E}(\kappa, \tau)$ with $\tau \neq 0$

Such manifolds are Riemannian fibrations over 2-dimensional space forms. They are denoted by $\mathbb{E}(\kappa, \tau)$ where κ is the curvature of the base of the fibration and τ is the bundle curvature, that is the defect for the fibration to be a product. Note that $\kappa \neq 4\tau^2$, if not, the manifold is a space form. Table 1. gives the classification of these possible geometries.

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2(\kappa) \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2(\kappa) \times \mathbb{R}$
$\tau \neq 0$	$(\mathbb{S}^3, g_{Berger})$	Nil_3	$\widetilde{PSL_2}(\mathbb{R})$

Table 1: *Classification of* $\mathbb{E}(\kappa, \tau)$

From now on, we will focus on the non-product case, *i.e.*, $\tau \neq 0$. In this case, $\mathbb{E}(\kappa, \tau)$ carries a unitary Killing vector field ξ tangent to the fibers and satisfying $\nabla_X \xi = \tau X \wedge \xi$. Moreover, there exits a direct local orthonormal frame $\{e_1, e_2, e_3\}$ with $e_3 = \xi$ and such that the Christoffel symbols are

$$(31) \qquad \begin{cases} \Gamma^3_{12} = \Gamma^1_{23} = -\Gamma^3_{21} = -\Gamma^2_{13} = \tau, \\ \Gamma^1_{32} = -\Gamma^2_{31} = \tau - \dfrac{\kappa}{2\tau}, \\ \Gamma^i_{ii} = \Gamma^i_{ij} = \Gamma^i_{ji} = \Gamma^j_{ii} = 0, \quad \forall\, i, j \in \{1, 2, 3\}. \end{cases}$$

In particular, we deduce from these Christoffel symbols that $\mathbb{E}(\kappa, \tau)$ is η-Einstein. Precisely, we have

$$Ric = \begin{pmatrix} \kappa - 2\tau^2 & 0 & 0 \\ 0 & \kappa - 2\tau^2 & 0 \\ 0 & 0 & 2\tau^2 \end{pmatrix}$$

in the local frame $\{e_1, e_2, \xi\}$. Moreover, from this and the local expression of the spinorial Levi-Civita connection, we deduce that there exists on $\mathbb{E}(\kappa, \tau)$ a spinor field φ satisfying

$$(32) \qquad \nabla_{e_1} \varphi = \frac{1}{2} \tau e_1 \cdot \varphi, \ \nabla_{e_2} \varphi = \frac{1}{2} \tau e_2 \cdot \varphi, \ \nabla_{\xi} \varphi = \frac{1}{2} \left(\frac{\kappa}{2\tau} - \tau \right) \xi \cdot \varphi.$$

One can refer to [18] for details.

5.1.2 The Lie group Sol_3

The solvable Lie group Sol_3 is the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by the transformation $(x, y) \longrightarrow (e^t x, e^t y)$. Then, we identify Sol_3 with

$\mathbb{R}^3$ and the group multiplication is defined by

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z').$$

The frame $e_1 = e^{-z}\partial_x$, $e_2 = e^{z}\partial_y$, $e_3 = \partial_z$ is orthonormal for the left-invariant metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

We easily check that in the frame $\{e_1, e_2, e_3\}$, the Christoffel symbols are

$$\Gamma^3_{11} = \Gamma^2_{23} = -\Gamma^1_{13} = -\Gamma^3_{22} = -1,$$

and the other identically vanish. So, we deduce the existence of a special spinor field φ on Sol_3 satisfying

$$\overline{\nabla}_{e_1}\varphi = \frac{1}{2}e_2 \cdot \varphi,\ \overline{\nabla}_{e_2}\varphi = \frac{1}{2}e_1 \cdot \varphi,\ \overline{\nabla}_{\xi}\varphi = 0, \tag{33}$$

and the Ricci curvature in the frame $\{e_1, e_2, e_3\}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Details can be found in [6].

5.1.3 The hyperbolic fibration $\mathbb{T}^3_B$

This last example is the hyperbolic fibration defined in [14]. Let B be a matrix of $\mathrm{SL}_2(\mathbb{Z})$, which can be considered as a diffeomorphism of the flat torus $\mathbb{T}^2$ and admit two eigenvalues α and $\frac{1}{\alpha}$. Now let $\mathbb{T}^3_B$ be the 3-dimensional manifold defined by $\mathbb{T}^3_B = \mathbb{T}^2 \times \mathbb{R}/\equiv$, where $\equiv$ is the equivalence relation defined by $(x, y) \equiv (B(x), y + 1)$. We denote by b the slope of the eigenvector associated to the eigenvalue $\frac{1}{\alpha}$. Thus, $\mathbb{T}^3_B$ is a compact manifold of universal covering $\mathbb{R}^3$ equipped with a Riemannian metric for which the base $\{e_1, e_2, e_3\}$ defined as follows is orthonormal

$$e_1 = \alpha^{-z}(-b\partial_x + \partial_y),\ e_2 = \alpha^{z}(\partial_x + b\partial_y),\ e_3 = \partial_z.$$

One can easily check that

$$[e_1, e_2] = 0,\ [e_1, e_3] = \ln(\alpha)e_1,\ [e_2, e_3] = -\ln(\alpha)e_2,$$

and that the Christoffel symbols are given by

$$\Gamma^3_{11} = \Gamma^2_{23} = -\Gamma^1_{13} = -\Gamma^3_{22} = -\ln(\alpha),$$

with the other identically zero. The Ricci curvature is given by the following matrix in the frame $\{e_1, e_2, e_3\}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\ln(\alpha)^2 \end{pmatrix}.$$

From the expression of the Christoffel symbols, there exists a spinor field φ satisfying

$$\nabla_{e_1}\varphi = \frac{1}{2}\ln(\alpha)e_2 \cdot \varphi,\ \nabla_{e_2}\varphi = \frac{1}{2}\ln(\alpha)e_1 \cdot \varphi,\ \nabla_{e_3}\varphi = 0. \tag{34}$$

5.2 A non-existence result

Here is the main result of this section.

Proposition 5.1. *The 3-dimensional manifolds Nil_3, Sol_3, $\widetilde{PSl_2}(\mathbb{R})$, the Berger spheres and the tori $\mathbb{T}^3_B$ cannot be isometrically immersed into $\mathbb{R}^4$, even locally.*

We start by giving the following

Lemma 5.2. *Let (M^3, g) be an oriented Riemannian manifold which is $\eta-$Einstein, i.e. $Ric = \lambda g + \eta\xi \otimes \xi$, with $\eta \neq 0$. Assume that there exists a non-trivial spinor field φ such that $\nabla^{\Sigma M}_X \varphi = -\frac{1}{2}A(X) \cdot \varphi$, where A is a symmetric endomorphism field. Then,*

1. *If $\lambda \neq -\eta$, and A is Codazzi, then*

$$A = \pm \begin{pmatrix} \sqrt{\frac{|\lambda+\eta|}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{|\lambda+\eta|}{2}} & 0 \\ 0 & 0 & \frac{\lambda-\eta}{\sqrt{2|\lambda+\eta|}} \end{pmatrix}$$

   in an orthonormal frame $\{e_1, e_2, \xi\}$.

2. *If $\lambda = -\eta$, then A cannot be Codazzi.*

3. *If $\lambda = 0$ and $\eta < 0$, then A cannot be Codazzi.*

Proof : Using the fact that A is Codazzi, a simple calculation shows

$$R^{\Sigma M}(X, Y) \cdot \varphi = \frac{1}{4}(A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi.$$

Then the Ricci identity (20) yields

$$Ric(X) \cdot \varphi = \mathrm{tr}\,(A)A(X) \cdot \varphi - A^2(X) \cdot \varphi.$$

Now if the manifold is $\eta-$Einstein, we get

$$\Big(\lambda X + \eta\langle X, \xi\rangle\xi - \mathrm{tr}\,(A)A(X) + A^2(X)\Big) \cdot \varphi = 0.$$

Since φ is a non-trivial generalized Killing spinor, it never vanishes. Consequently

$$\lambda X + \eta\langle X, \xi\rangle\xi - \mathrm{tr}\,(A)A(X) + A^2(X) = 0. \tag{35}$$

Let $\{e_1, e_2, e_3\}$ be a diagonalizing frame of A, then from equation (35) e_3 can always be chosen to be ξ and e_1, e_2 orthogonal to ξ. Now denote by a_1, a_2, a_3 the respective eigenvalues. Then equation (35) leads to

$$\begin{cases} a_1a_2 = \frac{\lambda+\eta}{2}, \\ a_2a_3 = \frac{\lambda-\eta}{2}, \\ a_1a_3 = \frac{\lambda-\eta}{2}. \end{cases}$$

If $\lambda = -\eta$, then this system has no solutions. If $\lambda = 0$ and $\eta < 0$, then we have $a_1 = a_2$, and so $a_1^2 = \frac{\eta}{2} < 0$, which is not possible because a_1 is a real number.

Thus, in these two cases, A cannot be Codazzi. If $\lambda \neq -\eta$ simple computations yield the result. $\square$

Proof of Proposition 5.1: Let $M = Nil_3$, Sol_3, $\widetilde{PSl_2}(\mathbb{R})$, $\mathbb{T}^3_B$ or a Berger sphere and assume that M is isometrically immersed in $\mathbb{R}^4$. Then there exists a spinor φ on M verifying $\nabla^{\Sigma M}_X \varphi = -\frac{1}{2} A(X) \cdot \varphi$, where A is shape operator of the immersion and hence Codazzi. Moreover, all these manifolds are η-Einstein. For Sol_3 and $\mathbb{T}^3_B$, we have $\lambda = 0$ and $\eta < 0$, so from Lemma 5.2, A cannot be Codazzi and such a spinor cannot exist. This leads to a contradiction. In the case of Nil_3, $\widetilde{PSl_2}(\mathbb{R})$ and Berger spheres, we have $\lambda = \kappa - 2\tau^2$ and $\eta = 2\tau^2$. Since $\kappa \neq 4\tau^2$, then $\lambda \neq -\eta$ and A is as in part 1 of Lemma 5.2. Finally, a simple computation shows that A is not Codazzi, which is again a contradiction. Thus all these manifolds cannot be immersed isometrically into the 4-dimensional Euclidean space. $\square$

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