LOWER BOUNDS FOR THE EIGENVALUES OF THE Spin^c DIRAC OPERATOR ON SUBMANIFOLDS

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ABSTRACT. We prove lower bounds for the eigenvalues of the Spin^c Dirac operator on submanifolds. These estimates are expressed in terms of extrinsic and intrinsic quantities. We also give estimates involving the Energy-Momentum tensor as well as conformal bounds. The limiting cases of these estimates give rise to particular spinor fields, called *generalized twisted Killing spinors*, which are also studied.

1. INTRODUCTION AND PREMILINARIES

The limiting cases of estimates for eigenvalues of the Dirac operator on compact (with or without boundary) manifolds give rise to examples of special geometries. For instance, equality in the classical inequalitiy of Friedrich [3]

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_M \mathrm{Scal}_M,$$

where n is the dimension of the manifold M and Scal_M its scalar curvature, forces the manifold to be Einstein with positive scalar curvature, due to the fact that the eigenspinor associated with the first eigenvalue $\lambda = \sqrt{\frac{n}{4(n-1)} \inf_M \mathrm{Scal}_M}$ is a real Killing spinor. This is also the case for the conformal inequality of Hijazi [9]

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \mu_1,$$

where μ_1 is the first eigenvalue of the Yamabe operator. Friedrich inequality can be improved by showing that [10]:

$$\lambda^2 \geqslant \frac{1}{4} \inf_M (\mathrm{Scal}_M + |Q_\varphi|^2),$$

where Q_φ is the Energy-Momentum tensor associated with a the eigenspinor φ of the eigenvalue λ.

On the other hand, in the recent years, many estimates have been proved for the eigenvalues of the Spin^c Dirac operator. A great difference between both cases is that the equality cases for Spin^c lower bounds are less and therefore give larger classes of limiting manifolds (see [8, 13, 14] for instance).

In this note, we prove a new lower bound for the eigenvalues of the Spin^c Dirac operator on submanifolds of Spin^c manifolds (see Theorem 2.1). This generalizes for the Spin^c case, previous estimates by Hijazi-Zhang [11, 12] and Ginoux-Morel [5]. The limiting case is characterized by the existence of particular spinor fields called *generalized twisted Killing spinors*. We will study these particular spinor fields and show that under a natural assumption on the dimension and the codimension of the submanifold, they are in fact *twisted Killing spinors* which generalize naturally the usual Killing spinors for twisted spinor bundles. Finally, we also prove conformal estimates (Theorem 2.3) as well as estimates involving the Energy-Momentum tensor (Theorem 2.2).

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We recall briefly some basic facts about Spin^c manifolds and their hypersurfaces (the reader can refer to [1, 2, 4, 5]). Let $(\widetilde{M}^{m+n}, \widetilde{g})$ be a Riemannian Spin^c manifold and M^m a submanifold isometrically immersed into $\widetilde{M}$. We consider $\{e_1, \cdots, e_m, \nu_1, \cdots \nu_n\}$ a positively oriented local orthonormal basis of $T\widetilde{M}|_M$ such that $\{e_1, \cdots, e_m\}$ (respectively $\{\nu_1, \cdots, \nu_n\}$) is a positively oriented local orthonormal basis of TM (respectively NM). Assume that M is also Spin^c and denote by NM the normal bundle of the immersion of M into $\widetilde{M}$. Let $i\widetilde{\Omega}$ (resp. $i\Omega$) be the curvature 2-form associated with a fixed connection on the auxiliary line bundle defining the Spin^c structure on $\widetilde{M}$ (resp. M). Since the manifolds M and $\widetilde{M}$ are Spin^c, there exists a Spin^c structure on the bundle NM. We denote by ΣN the Spin^c bundle of NM and let

$$\Sigma := \begin{cases} \Sigma M \otimes \Sigma N & \text{if } m \text{ or } n \text{ is even,} \\ \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N & \text{otherwise.} \end{cases}$$

It is well known that there is a natural isomorphism between Σ and $\Sigma\widetilde{M}_{|M}$. Moreover, we denote by ∇ the covariant derivative on Σ defined by

$$\nabla := \begin{cases} \nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N} & \text{if } m \text{ or } n \text{ is even,} \\ \nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N} \oplus \nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N} & \text{otherwise.} \end{cases}$$

We have the following identification between the Clifford multiplication

$$X \cdot_M \varphi = (X \cdot \omega_\perp \cdot \psi)|_M, \tag{1}$$

where $\varphi = \psi|_M$, $\psi \in \Gamma(\Sigma\widetilde{M})$, $\omega_\perp := \omega_n$ if n is even and $\omega_\perp = -i\omega_n$ if n is odd, with $\omega_n = i^{\left[\frac{n+1}{2}\right]}\nu_1 \cdots \nu_n$ the complex volume element of the normal bundle. We also recall the Spin^c Gauss formula

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2}\sum_{j=1}^m e_j \cdot B(X, e_j) \cdot \varphi, \tag{2}$$

where $\widetilde{\nabla}$ is the spinorial connection on $\widetilde{M}$ and B is the second fundamental form of M in $\widetilde{M}$. We will denote by H the mean curvature. Now, let us define the following Dirac operators

$$\mathbf{D} = \sum_{j=1}^m e_j \cdot \nabla_{e_j}, \qquad \widetilde{D} := \sum_{j=1}^m e_j \cdot \widetilde{\nabla}_{e_j},$$

and

$$D_H = (-1)^n \omega_\perp \cdot \widetilde{D} = (-1)^n \omega_\perp \cdot \mathbf{D} + \frac{1}{2} H \cdot \omega_\perp \cdot .$$

Clearly, from the Spin^c Gauss formula (2), we have $\widetilde{D} = \mathbf{D} - \frac{1}{2}H\cdot$. Moreover (see [1, Lemma 2.1]), $\mathbf{D}$ and D_H are formally self-adjoint and $D_H^2 = \widetilde{D}^*\widetilde{D}$ where $\widetilde{D}^*$ is the formal adjoint of $\widetilde{D}$ with respect to the L^2-scalar product $\int_M \Re e \langle \cdot, \cdot \rangle \, dv_g = \int_M (\cdot, \cdot) dv_g$. We finish this section of preliminaries by the two following lemmas. The first one generalizes the classical Schrödinger-Lichnerowicz formula in the context of twisted Spin^c spinor bundles. Before stating the lemma, we define on $M_\varphi = \{x \in M \mid \varphi(x) \neq 0\}$, the function R^N_φ associated to a spinor field $\varphi \in \Gamma(\Sigma)$ by $R^N_\varphi := 2\sum_{i,j=1}^n \left\langle e_i \cdot e_j \cdot Id \otimes \mathcal{R}^N_{e_i,e_j}\varphi, \frac{\varphi}{|\varphi|^2} \right\rangle$, where $\mathcal{R}^N_{e_i,e_j}$ stands for the spinorial curvature of the normal bundle NM.

Lemma 1.1 (Twisted Schrödinger-Lichnerowicz formula). *For any spinor field $\varphi \in \Gamma(\Sigma)$, pointwise on M_φ, we have*

$$\left\langle \mathbf{D}^2\varphi, \varphi \right\rangle = \langle \nabla^*\nabla\varphi, \varphi \rangle + \frac{1}{4}(\mathrm{Scal}_M + R^N_\varphi)|\varphi|^2 + \frac{i}{2}\langle \Omega \cdot \varphi, \varphi \rangle . \tag{3}$$

Proof: We give the proof in the case where m or n is even. The other case is similar. We compute the square of the Dirac operator $\mathbf{D}$ acting on $\varphi = \alpha \otimes \sigma$. We have

$$\mathbf{D}^2\varphi = \nabla^*\nabla\varphi + \frac{1}{2}\sum_{i,j=1}^{m} e_i \cdot e_j \cdot \mathcal{R}_{e_i,e_j}\varphi, \tag{4}$$

where ∇^* is the formal adjoint of ∇ and $\mathcal{R}$ is the spinorial curvature associated with the connection ∇. From the definition of ∇, we see easily that

$$\mathcal{R}_{e_i,e_j}\varphi = (\mathcal{R}^M_{e_i,e_j}\sigma) \otimes \alpha + \sigma \otimes (\mathcal{R}^N_{e_i,e_j}\alpha),$$

where $\mathcal{R}^M$ denotes the spinorial curvature on M. Then, a classical computation on each factor gives the desired formula. □

We have this second elementary lemma:

Lemma 1.2. *For any spinor field $\varphi \in \Gamma(\Sigma)$, we have*

$$\langle i\Omega \cdot \varphi, \varphi\rangle \geq -\frac{c_m}{2}|\Omega||\varphi|^2, \tag{5}$$

where $c_m = 2[\frac{m}{2}]^{\frac{1}{2}}$.

Proof: If m or n is even, then a spinor $\varphi \in \Sigma$ can be written $\varphi = \sum_{j=1}^{p} \sigma_j \otimes \alpha_j$. Without loss of generality, we can assume that $p \leqslant 2^{[\frac{n}{2}]}$ and that the α_j are orthogonal with respect to the inner product of ΣN. Hence, we have

$$\begin{aligned}
\langle i\Omega \cdot \varphi, \varphi\rangle &= \langle i\Omega \cdot (\sum_{j=1}^{p} \sigma_j \otimes \alpha_j), \sum_{k=1}^{p} \sigma_k \otimes \alpha_k\rangle = \sum_{j,k=1}^{p} \langle i(\Omega \underset{M}{\cdot} \sigma_j) \otimes \alpha_j, \sigma_k \otimes \alpha_k\rangle \\
&= \sum_{j,k=1}^{p} \langle i(\Omega \underset{M}{\cdot} \sigma_j), \sigma_k\rangle\langle\alpha_j, \alpha_k\rangle \\
&= \sum_{j=1}^{p}\langle i(\Omega \underset{M}{\cdot} \sigma_j), \sigma_j\rangle|\alpha_j|^2 \geq -\sum_{j=1}^{p} \frac{c_m}{2}|\Omega||\sigma_j|^2|\alpha_j|^2 \geq -\frac{c_m}{2}|\Omega||\varphi|^2.
\end{aligned}$$

Note that we use the fact that the scalar product on Σ is the product of the scalar products on ΣM and ΣN and the that the α_j are orthogonal. We also use the classical estimate on M, that is $\langle i\Omega \cdot \sigma_j, \sigma_j\rangle \geq -\frac{c_m}{2}|\Omega||\sigma_j|^2$ (see [8]). If m and n are odd, then $\Sigma = \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N$ and a spinor $\varphi \in \Sigma$ is of the form $\varphi = \psi \oplus \psi'$ with $\psi = \sum_{j=1}^{p} \sigma_j \otimes \alpha_j$ and $\psi' = \sum_{k=1}^{q} \sigma'_k \otimes \alpha'_k$. Thus, by the same computation as above on each factor, we have

$$\langle i\Omega \cdot \varphi, \varphi\rangle = \langle i\Omega \cdot \psi, \psi\rangle + \langle i\Omega \cdot \psi', \psi'\rangle \geq -\frac{c_m}{2}|\Omega|\left(|\psi|^2 + |\psi'|^2\right) \geq -\frac{c_m}{2}|\Omega||\varphi|^2.$$

This concludes the proof. □

2. EIGENVALUE ESTIMATES FOR SUBMANIFOLDS

Now, we have all the ingredients to state the eigenvalue estimates. We begin by the following basic estimates involving intrinsic terms (scalar curvature, curvature of the line bundle over M) and extrinsic terms (mean curvature and spinorial normal curvature). This result generalizes in the Spin^c setting the estimate of Hijazi-Zhang [11] (extended to any codimension by Ginoux-Morel [5]).

Theorem 2.1. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor* *field* $\varphi \in \Gamma(\Sigma)$ *for the submanifold Dirac operator* D_H*, i.e.* $D_H\varphi = \lambda\varphi$*. Assume that* $m \geq 2$ *and*

$$\mathrm{Scal}_M + R^E_\varphi - c_m|\Omega^M| > \frac{m-1}{m}||H||^2 > 0$$

on M_φ*, then, we have*

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\varphi}\left(\sqrt{\frac{m}{m-1}(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|)} - ||H||\right)^2.$$

Moreover, if equality holds, then φ *is a twisted generalized Killing spinor.*

Proof: Let λ be an eigenvalue of D_H and q a smooth function, nowhere equal to $\frac{1}{m}$. We consider the following modified connection $\nabla^{\lambda,q}$ defined by

$$\nabla^{\lambda,q}_X\psi = \nabla_X\psi + \frac{1-q}{2(1-mq)}X\cdot H\cdot\psi + q\lambda X\cdot\omega_\perp\cdot\psi, \tag{6}$$

for any spinor field $\psi \in \Gamma(\Sigma)$. Let φ be an eigenspinor for D_H associated with the eigenvalue λ. Using the Twisted Schrödinger-Lichnerowicz formula (3), we can easily compute

$$\begin{aligned}\int_M |\nabla^{\lambda,q}\varphi|^2\mathrm{v}_g &= \int_M (1+mq^2-2q)\Big[\lambda^2 - \frac{1}{4}\Big(\frac{\mathrm{Scal}_M + R^N_\varphi}{1+mq^2-2q} - \frac{(m-1)\|H\|^2}{(1-mq)^2}\Big)|\varphi|^2 \\ &\quad - \frac{i}{2(1+mq^2-2q)} < \Omega\cdot\varphi,\varphi > \Big]\mathrm{v}_g. \end{aligned} \tag{7}$$

Then, using Inequality (5) and by assuming $m(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|) > (m-1)\|H\|^2 > 0$, we can choose q so that

$$(1-mq)^2 = \frac{(m-1)\|H\|}{\sqrt{\frac{m}{m-1}(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|)} - \|H\|}, \tag{8}$$

on M_φ. Inserting (8) in (7), and since the complement of M_ψ in M is of measure 0, we conclude because the left member of (7) is nonnegative. If equality occurs, we have $\nabla^{\lambda,q}_X\varphi = 0$. This implies

$$\mathbf{D}\varphi = \frac{m(1-q)}{2(1-mq)}H\cdot\varphi + mq\lambda\omega_\perp\cdot\varphi.$$

On the other hand, since φ is an eigenspinor for D_H, we get

$$\begin{aligned} 0 &= \lambda\omega_\perp\cdot\varphi + \frac{H}{2}\cdot\varphi - \frac{m(1-q)}{2(1-mq)}H\cdot\varphi - mq\lambda\omega_\perp\cdot\varphi \\ &= (1-mq)^2\lambda\omega_\perp\cdot\varphi - (m-1)\frac{H}{2}\cdot\varphi. \end{aligned}$$

Since the function $1-mq$ never vanishes and λ is a positive eigenvalue, we have

$$\omega_\perp\cdot\varphi = \frac{(m-1)}{2(1-mq)^2\lambda}H\cdot\varphi, \tag{9}$$

and so Equation (6) becomes

$$\nabla_X\varphi = -\frac{1-q}{2(1-mq)}X\cdot H\cdot\varphi - \frac{q(m-1)}{2(1-mq)^2}X\cdot H\cdot\varphi.$$

Therefore, we get

$$X|\varphi|^2 = 2\Re e\,\langle\nabla_X\varphi,\varphi\rangle = \left(-\frac{1-q}{(1-mq)} - \frac{q(m-1)}{(1-mq)^2}\right)\Re e\,\langle X\cdot H\cdot\varphi,\varphi\rangle = 0,$$

because $\langle\alpha\cdot\varphi,\varphi\rangle$ is imaginary if α is a real 2-form. Hence, φ has constant norm and so $M_\varphi = M$. Moreover, equality also implies $2|\lambda| = \sqrt{\frac{m}{m-1}(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|)} - ||H||$. From the expression of q and Equation (9), we have $\omega_\perp\cdot\varphi = \mathrm{sgn}(\lambda)\frac{H}{||H||}\cdot\varphi$ and thus φ satisfies $\nabla_X\varphi = -\frac{f}{m}X\cdot\omega_\perp\cdot\varphi$, with $f = \frac{\mathrm{sgn}(\lambda)}{2}\sqrt{\frac{m}{m-1}(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|)}$. That is,

ψ is a generalized twisted Killing spinor. Note that here f is *a priori* a function. We will see in Section 3 that under some assumptions on the dimensions m and n, the function f is constant and from $2|\lambda| = \sqrt{\frac{m}{m-1}(\mathrm{Scal}_M + R^N_\varphi - c_m|\Omega|)} - ||H|| = 2|f| - ||H||$, we will deduce that $||H||$ is also constant. □

Now, we define the Energy-Momentum tensor associated with a spinor field $\psi \in \Gamma(\Sigma)$ on M_ψ by

$$Q^\psi_{ij} = \frac{1}{2}(e_i \cdot \omega_\perp \cdot \nabla_{e_j}\psi + e_j \cdot \omega_\perp \cdot \nabla_{e_i}\psi, \frac{\psi}{|\psi|^2}).$$

Note that

$$Q^\psi_{ij} = \frac{1}{2}(e_i \cdot_M \nabla_{e_j}\psi + e_j \cdot_M \nabla_{e_i}\psi, \frac{\psi}{|\psi|^2}),$$

so it is the intrinsic Energy-Momentum tensor and it is the one appearing in the Einstein Dirac equation. We have the following estimate involving the Energy-Momentum tensor.

Theorem 2.2. *Let (M^m, g) be a compact Riemannian* Spin^c *manifold isometrically immersed into a Riemannian* Spin^c *manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor field $\varphi \in \Gamma(\Sigma)$ for the submanifold Dirac operator D_H, i.e. $D_H\varphi = \lambda\varphi$. Assume that $m \geq 2$ and*

$$\mathrm{Scal}_M + R^N_\varphi + 4|Q^\varphi|^2 - c_m|\Omega^M| > ||H||^2 > 0$$

on M_φ, then, we have

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\varphi}\left(\mathrm{Scal}_M + R^N_\varphi + 4|Q^\varphi|^2 - c_m|\Omega|) - ||H||\right)^2.$$

Moreover, if equality holds, then φ is a twisted (symmetric) EM-spinor.

Proof: For any real function q that never vanishes, consider the modified covariant derivative defined on $\Gamma(\Sigma)$ by

$$\nabla^Q_{e_i}\psi = \nabla_{e_i}\psi - \frac{1}{2mq}e_i \cdot H \cdot \psi + (-1)^{n+1}q\lambda e_i \cdot \omega_\perp \cdot \psi + \sum_j Q^\psi_{ij}e_j \cdot \omega_\perp \cdot \psi.$$

Again, we can compute, for an eigenspinor φ

$$\begin{aligned}\int_M |\nabla^Q\varphi|^2 \mathrm{v}_g \quad &= \quad \int_M (1+mq)^2\Big[\lambda^2 - \frac{1}{4}\Big(\frac{\mathrm{Scal}_M + R^N_\varphi + 4|Q^\varphi|^2}{(1+mq^2)} - \frac{\|H\|^2}{mq^2}\Big)\Big]|\varphi|^2\mathrm{v}_g \\ &\quad -\frac{1}{4}\int_M (1+mq^2)\Big[\frac{2}{mq(1+mq^2)}\Big(\|H\|^2 - \frac{< H\cdot\varphi, \omega_\perp\cdot\varphi >^2}{|\varphi|^4}\Big)\Big]|\psi|^2\mathrm{v}_g \\ (10) \qquad &\quad -\frac{i}{2}\int_M < \Omega\cdot\varphi, \varphi > \mathrm{v}_g.\end{aligned}$$

Now, we use again (5) and if moreover, $\mathrm{Scal}_M + R^N_\varphi - c_M|\Omega| + 4|Q^\varphi|^2 > \|H\|^2 > 0$, we take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{\mathrm{Scal}_M + R^N_\varphi - c_M|\Omega| + 4|Q^\varphi|^2} - \|H\|)}},$$

and then by the Cauchy Schwarz inequality, we have

$$\|H\|^2 - \frac{< H\cdot\varphi, \omega_\perp\cdot\varphi >^2}{|\varphi|^4} \geq 0.$$

If equality holds, then $\nabla^Q\varphi = 0$ and equality occurs in the Cauchy-Schwarz inequality, that is, $\|H\|^2 - \frac{< H\cdot\varphi, \omega_\perp\cdot\varphi >^2}{|\varphi|^4} = 0$. Thus, proceeding as in the proof of Theorem 2.1, we deduce that $\nabla_X\varphi = -Q(X)\cdot\omega_\perp\cdot\varphi$, that is ψ is a twisted (symmetric) Energy-Momentum spinor (EM-spinor). □

Note that, by a completely similar computation, we can obtain a lower bound involving both symmetric and skew-symmetric Energy-Momentum tensors as in [7]. We do not write it in this note. Finally, following the idea of Hijazi [9], we consider a conformal change of the metric $\bar{g} = e^{2u}\widetilde{g}$. Let $\Sigma \longrightarrow \bar{\Sigma}$, $\psi \longrightarrow \bar{\psi}$ be the corresponding isometry between the two spinor bundles. Recall that for 2 spinors ψ and φ on Σ and for any vector field X on $\widetilde{M}$, we have

$$\langle\varphi,\psi\rangle = \langle\bar{\varphi},\bar{\psi}\rangle_{\bar{g}} \quad \text{and} \quad \bar{X}\bar{\cdot}\bar{\psi} = \overline{X\cdot\psi}$$

Note that we have

$$\overline{\mathbf{D}}(e^{\frac{-(m-1)}{2}u}\overline{\psi}) = e^{\frac{-(m+1)}{2}u}\overline{\mathbf{D}\psi}$$

where $\overline{\mathbf{D}}$ denotes the Dirac operator w.r.t the metric $\overline{g}$. Moreover, the corresponding mean curvature is given by

$$\overline{H} = e^{-2u}(H - m\,\mathrm{grad}^N u)$$

Now, assume that $\mathrm{grad}^N u = 0$, then D_H is also conformally covariant and we have

$$\overline{D_H}(e^{\frac{-(m-1)}{2}u}\overline{\psi}) = e^{\frac{-(m+1)}{2}u}\overline{D_H\psi}$$

From now on, we will consider regular conformal change of the metric g, i.e., $\mathrm{grad}^N u = 0$.

Theorem 2.3. *Let (M^m, g) be a compact Riemannian Spin^c manifold isometrically immersed into a Riemannian Spin^c manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Consider a non-trivial eigenspinor field $\varphi \in \Gamma(\Sigma)$ for the submanifold Dirac operator D_H, i.e. $D_H\psi = \lambda\psi$. For any regular conformal change of metric $\bar{g} = e^{2u}\widetilde{g}$, assume that $m \geq 3$ and*

$$\overline{\mathrm{Scal}}_M\ e^{2u} + R^N_\varphi + 4|Q^\psi|^2 - c_m|\Omega^M| > ||H||^2 > 0$$

on M_φ, then, we have

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\psi}\left(\sqrt{\overline{\mathrm{Scal}}_M\ e^{2u} + R^N_\psi + 4|Q^\psi|^2 - c_m|\Omega|}) - ||H||\right)^2.$$

Proof: For $\psi \in \Gamma(\Sigma)$ an eigenspinor of D_H with eigenvalue λ, let $\overline{\varphi} := e^{\frac{-(n-1)u}{2}}\overline{\psi}$. Then, we have $\overline{D_H}\overline{\varphi} = \lambda e^{-u}\overline{\varphi}$. Recall that $\overline{\nabla}_{\overline{e_i}}\overline{\psi} = \overline{\nabla_{e_i}\psi} - \frac{1}{2}\overline{e_i\cdot du\cdot\psi} - \frac{1}{2}e_i(u)\overline{\psi}$ and $\overline{e_i} = e^{-u}e_i$. Now, it is straightforward to get $\overline{Q}^{\overline{\varphi}}_{\bar{i},\bar{j}} = e^{-u}Q^\psi_{i,j}$, hence $|\overline{Q}^{\overline{\varphi}}|^2 = e^{-2u}|Q^\psi|^2$. Equation (10) is also true on $(\widetilde{M}, \overline{g})$. If, we apply it to $\overline{\varphi}$, we get

$$\begin{aligned}
\int_M |\overline{\nabla}^{\overline{Q}}\overline{\varphi}|^2 \mathrm{v}_{\overline{g}} \quad = \quad & \int_M (1+mq)^2\Big[(\lambda e^{-u})^2 - \frac{1}{4}\Big(\frac{\overline{\mathrm{Scal}}_M + \overline{R}^N_{\overline{\varphi}} + 4|\overline{Q}^{\overline{\varphi}}|^2}{(1+mq^2)} - \frac{\|\tilde{H}\|^2_{\overline{g}}}{mq^2}\Big)\Big]|\overline{\varphi}|^2_{\overline{g}}\mathrm{v}_{\overline{g}} \\
& -\frac{1}{4}\int_M (1+mq^2)\Big[\frac{2}{mq(1+mq^2)}\Big(\|\tilde{H}\|^2_{\overline{g}} - \frac{<\tilde{H}\bar{\cdot}\overline{\varphi}, \overline{\omega}_\perp\bar{\cdot}\overline{\varphi}>^2_{\overline{g}}}{|\varphi|^4_{\overline{g}}}\Big)\Big]|\overline{\varphi}|^2_{\overline{g}}\mathrm{v}_{\overline{g}} \\
& -\frac{i}{2}\int_M \langle\Omega\bar{\cdot}\overline{\varphi}, \overline{\varphi}\rangle_{\overline{g}}\,\mathrm{v}_{\overline{g}}. \qquad (11)
\end{aligned}$$

Since $\tilde{H} = e^{-u}\overline{H}$ and $\overline{R}^N_{\overline{\varphi}} = e^{-2u}R^N_\psi$, we have

$$\begin{aligned}
\int_M |\overline{\nabla}^{\overline{Q}}\overline{\varphi}|^2 \mathrm{v}_{\overline{g}} \quad = \quad & \int_M (1+mq)^2 e^{-2u}\Big[(\lambda)^2 - \frac{1}{4}\Big(\frac{\overline{\mathrm{Scal}}_M e^{2u} + R^N_\psi + 4|Q^\psi|^2}{(1+mq^2)} - \frac{\|H\|^2}{mq^2}\Big)\Big]|\overline{\varphi}|^2\mathrm{v}_{\overline{g}} \\
& -\frac{1}{4}\int_M (1+mq^2)e^{-2u}\Big[\frac{2}{mq(1+mq^2)}\Big(\|H\|^2 - \frac{<H\cdot\psi, \omega_\perp\cdot\psi>^2}{|\psi|^4}\Big)\Big]|\overline{\varphi}|^2\mathrm{v}_{\overline{g}} \\
& -\frac{i}{2}\int_M <\Omega\bar{\cdot}\overline{\varphi}, \overline{\varphi}>_{\overline{g}}\,\mathrm{v}_{\overline{g}}. \qquad (12)
\end{aligned}$$

Since that $\overline{\mathrm{Scal}}_M\, e^{2u} + R^N_\psi - c_M|\Omega| + 4|Q^\psi|^2 > \|H\|^2 > 0$, we take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{\overline{\mathrm{Scal}}_M e^{2u} + R^N_\psi - c_M|\Omega| + 4|Q^\psi|^2} - \|H\|)}}.$$

Then using (5) and the Cauchy Schwarz inequality $\|H\|^2 - \frac{<H\cdot\psi, \omega_\perp\cdot\psi>^2}{|\psi|^4} \geq 0$, we get the desired result. □

By taking u as eigenfunction of the Yamabe operator on M corresponding to the first eigenvalue μ_1, we get the following corollary.

Corollary 2.4. *Under the assumptions of Theorem 2.3 and if*

$$\mu_1 + R^N_\psi + 4|Q^\psi|^2 - c_m|\Omega^M| > ||H||^2 > 0$$

on M_φ, then, we have

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\psi}\left(\sqrt{\mu_1 + R^N_\psi + 4|Q^\psi|^2 - c_m|\Omega|}) - ||H||\right)^2.$$

3. Generalized twisted Killing spinors

We have seen in Theorem 2.1 that if equality occurs, then, the eigenspinor φ is in fact a generalized twisted Killing spinor, that is satisfies the equation

$$\nabla_X\varphi = fX\cdot\omega_\perp\cdot\varphi = fX \underset{M}{\cdot} \varphi, \tag{13}$$

where f is a real function. We prove the following:

Proposition 3.1. *Let (M^m, g) be a compact Riemannian* Spin^c *manifold isometrically immersed into a Riemannian* Spin^c *manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. Let $\varphi \in \Gamma(\Sigma)$ be a generalized twisted Killing spinor with real-valued function f. If $m > n+4$, then f is constant.*

Proof: We define the following forms for $p \in \{1, \cdots, m\}$,

$$\omega_p(X_1, \cdots, X_p) = \left\langle (X_1 \wedge X_2 \wedge \cdots \wedge X_p) \underset{M}{\cdot} \varphi, \varphi \right\rangle,$$

We have the following easy facts (see [6, 8] for instance). For any $k \geqslant 0$, the forms ω_{4k+1} and ω_{4k+2} are imaginary-valued whereas the forms ω_{4k+3} and ω_{4k} are real-valued. Moreover, we have for any $p \geqslant 0$

$$\mathrm{d}\omega_p = ((-1)^p f - f)\omega_{p+1}.$$

In particular, we have for any $k \geqslant 1$

$$\mathrm{d}f \wedge \omega_{2k} = 0. \tag{14}$$

Assume that f is not constant and let $x \in M$ such that $\mathrm{d}f \neq 0$, on a neighborhood V of x. Hence $\mathrm{d}f^\perp$ is of dimension $m-1$ and we consider $\{e_1, \cdots, e_{m-1}\}$ an orthonormal frame of $\mathrm{d}f^\perp$. From this, we have

$$\omega_{2k}(e_{i_1}, \cdots, e_{i_{2k}}) = 0,$$

for any subset $\{i_1, \cdots, i_{2k}\}$ of $\{1, \cdots, m-1\}$. Thus, for $l = \left[\frac{m-1}{2}\right]$, we deduce that the spinor fields φ, $e_{i_1} \underset{M}{\cdot} e_{i_2} \underset{M}{\cdot} \varphi, \cdots$ and $e_{i_1} \underset{M}{\cdot} e_{i_2} \cdots e_{i_{2l}} \underset{M}{\cdot} \varphi$ are orthonormal on V. Consequently, the space spanned by these spinors is a vector subspace of Σ_x of complex dimension

$$1 + \binom{m-1}{2} + \binom{m-1}{4} + \cdots + \binom{m-1}{2l} = 2^{m-2}.$$

Since the complex dimension of Σ is

$$d(m,n) = \begin{cases} 2^{\frac{m+n}{2}} & \text{if } m+n \text{ is even,} \\ 2^{\frac{m+n-1}{2}} & \text{if } m+n \text{ is odd,} \end{cases}$$

we conclude that $2^{m-2} \leqslant d(m,n)$. Therefore, f is constant if $2^{m-2} > d(m,n)$ which corresponds to the condition expressed in the statement of the Proposition. Indeed, on one hand, if $m+n$ is even, then $2^{m-2} > d(m,n)$ is equivalent to $m-2 > \frac{n+m}{2}$, that is $m > n+4$. On the other hand, if f $m+n$ is odd, then $2^{2m-2} > d(m,n)$ is equivalent to $m-2 > \frac{n+m-1}{2}$, that is $m > n+3$. But since $m+n$ is odd, the dimension m cannot be equal to $n+4$ and hence we also have $m > n+4$. □

Hence, we deduce the following:

Corollary 3.2. *Let (M^m, g) be a compact Riemannian* Spin^c *manifold isometrically immersed into a Riemannian* Spin^c *manifold $(\widetilde{M}^{m+n}, \widetilde{g})$. if $m > n+4$ and equality occurs in Theorem 2.1, then M admits a twisted Killing spinor and $||H||$ is constant.*

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